



Bounding clique-width by means of split-decompositions, for directed and undirected graphs

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Graph decompositions are useful

for FPT algorithms and
graph structure theory.

Among the many notions we have

tree-decompositions \rightarrow parameter tree-width ($\text{tw}(G)$)

clique-width decompositions \rightarrow param. clique-width ($\text{cw}(G)$)

modular decomposition (Gallai, 1967)

split decomposition (Cunningham, 1981)

Computations

NP-complete for tree-width and clique-width,

Good approximation algorithms for tree-width,

Linear time algorithms for modular and split decompositions
(Gioan, Habib, Paul *et al.*).

Some algorithmic uses

Clique-width decompositions are useful for checking MSO properties.

Also for MSO₂ properties (MSO using edge set quantification) because MSO₂ = MSO on incidence graph $\text{Inc}(G)$ and :

$$\text{cwd}(\text{Inc}(G)) \leq \text{tw}(G) + 3 \quad \text{for undirected graph } G$$

$$\text{cwd}(\text{Inc}(G)) \leq 2 \cdot \text{tw}(G) + 4 \quad \text{for directed graph } G$$

Tree-width must be bounded for MSO₂ model-checking by XP algorithms (Kreutzer, 2010 ; exact statement is very technical)

How to obtain clique-width decompositions ?

For $\text{Inc}(G)$: from a tree-decomposition of G

For G : from a tree-decomposition of G if G is sparse
in the following senses :

degree $\leq d$: $\text{cwd}(G) \leq d \cdot (\text{tw}(G)+1) + 2$

planar : $\text{cwd}(G) < 6 \cdot \text{tw}(G)$

p -planar, $\text{cwd}(G) < 6 \cdot \text{tw}(G) \cdot (2p+1)$

in general, $\text{cwd}(G)$ is not polynomially bounded in $\text{tw}(G)$,
for G , directed or not.

From the modular decomposition because $\text{cwd}(G)$ is the maximum $\text{cwd}(H)$ where H is a component of the modular decomposition.

Computing the **modular decomposition** may be a preliminary step for the computation of a clique-width decomposition.

We will consider in a similar way the **split-decomposition** (Cunningham, 1981), for directed and undirected graphs.

Modular decomposition, also called **substitution decomposition**

is based on the following notion of substitution of H for a vertex in K (undirected graphs) :

$$G = K[x \leftarrow H]$$

where H replaces x in K and all vertices of H are made adjacent to the neighbours of x.

The corresponding decomposing step expresses G as $K[x \leftarrow H]$ if possible, and decomposes similarly K and H etc.

One obtains a tree that can be seen as an **algebraic term** defining the given graph, whose operations are substitutions in « prime » graphs, those that cannot be decomposed. That such a tree is unique in a certain sense is not important for our purposes.

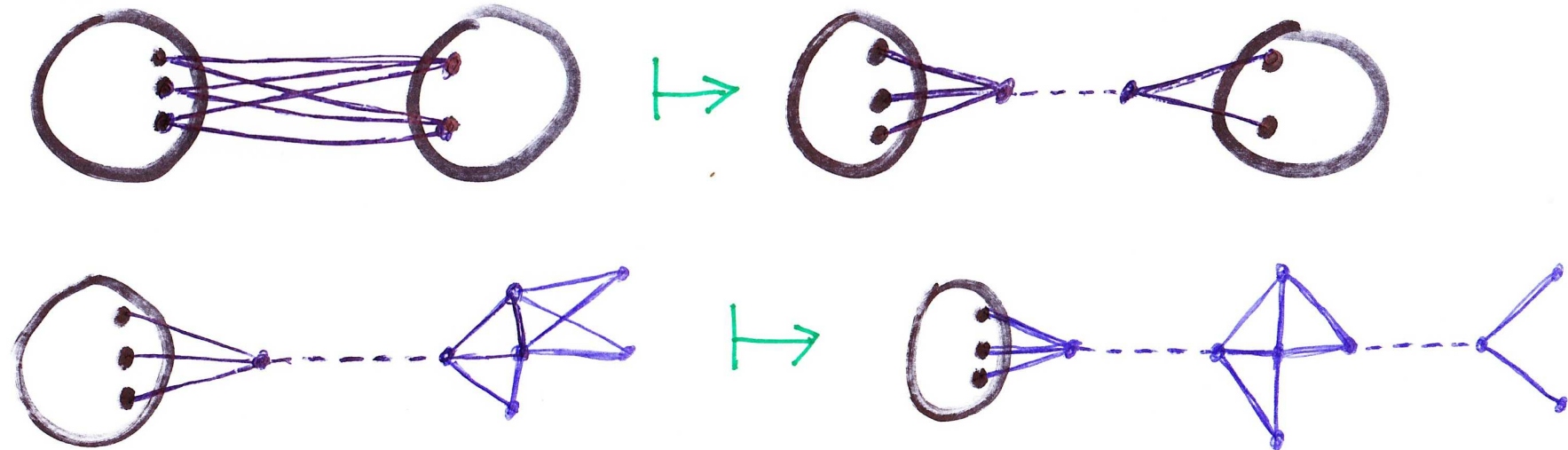
Clique-width fits well with modular decomposition, because from terms with $< k$ labels defining K and H , one gets, by a substitution of terms (and a relabelling) a term with $< k$ labels that defines $K[x \leftarrow H]$.

Cographs have simple modular decompositions, from which we get the bound 2 for their clique-width.

Modular decomposition also works for directed graphs.

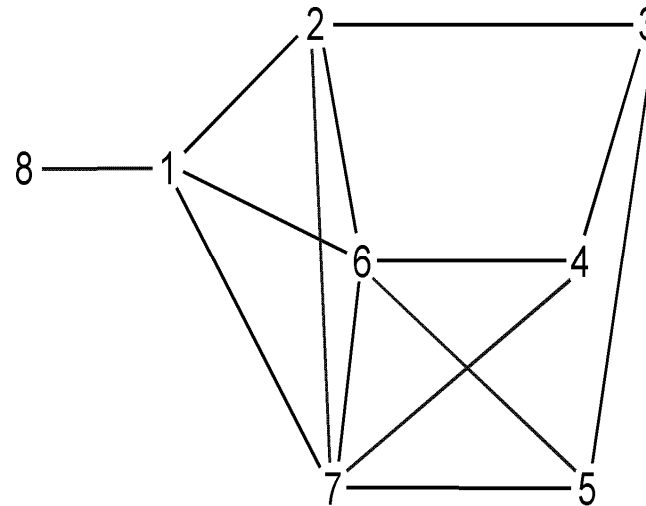
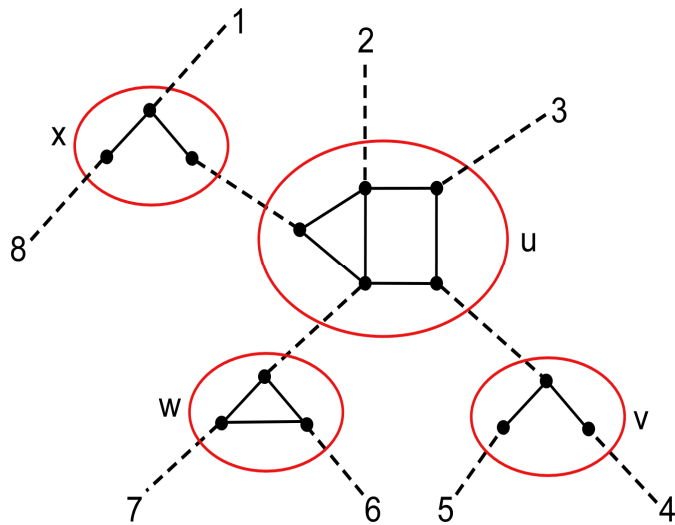
Split decomposition for undirected graphs

The basic decomposition can be iterated.



We obtain an unrooted tree (the tree of modular decomposition is a term, hence a rooted tree), called a *graph-labelled tree* (Gioan, Paul 2012). By some restrictions to decomposing steps, one obtains a unique tree, hence a canonical decomposition.

Example of a graph labelled tree and the associated graph :



Leaves \rightarrow vertices ; alternating paths \rightarrow edges :

between 1 and 8, 1 and 2, 2 and 3, etc.

None between 1 and 3, 1 and 4, etc.

Objectives : Graph labelled tree \rightarrow rooted graph labelled tree \rightarrow
 term over appropriate substitutions \rightarrow bound on clique-width.

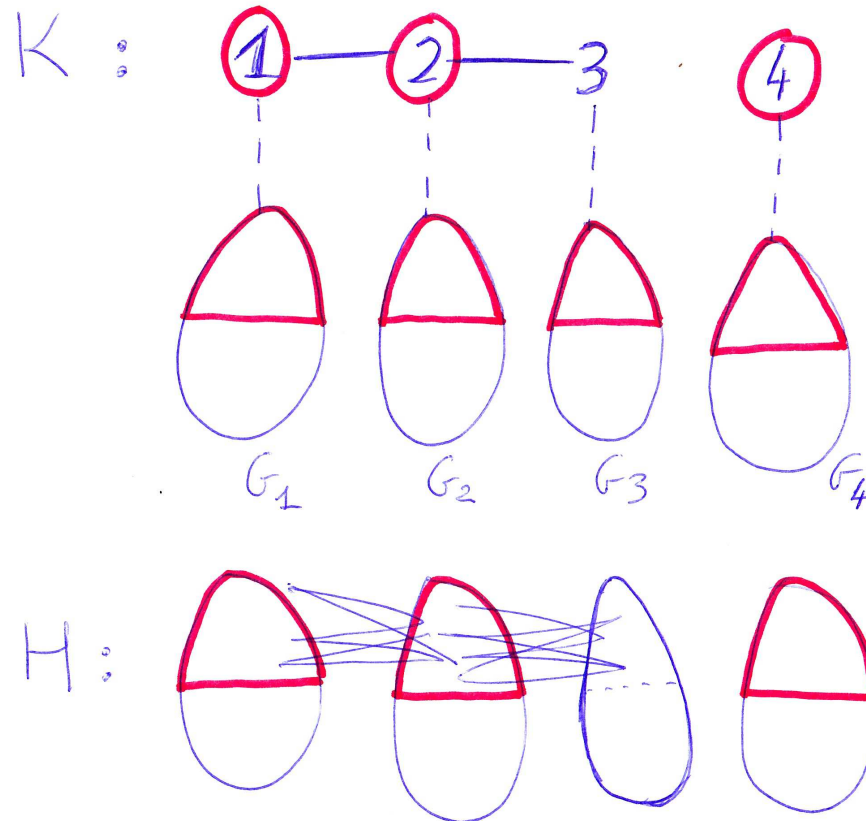
Substitution for graphs with dead vertices.

$G = (V, \text{edg}, L, D)$,

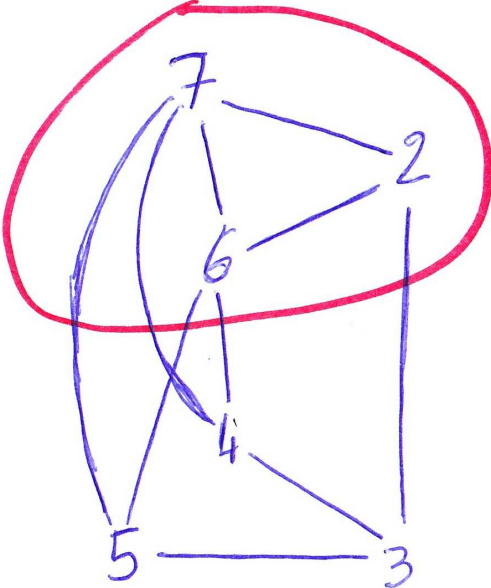
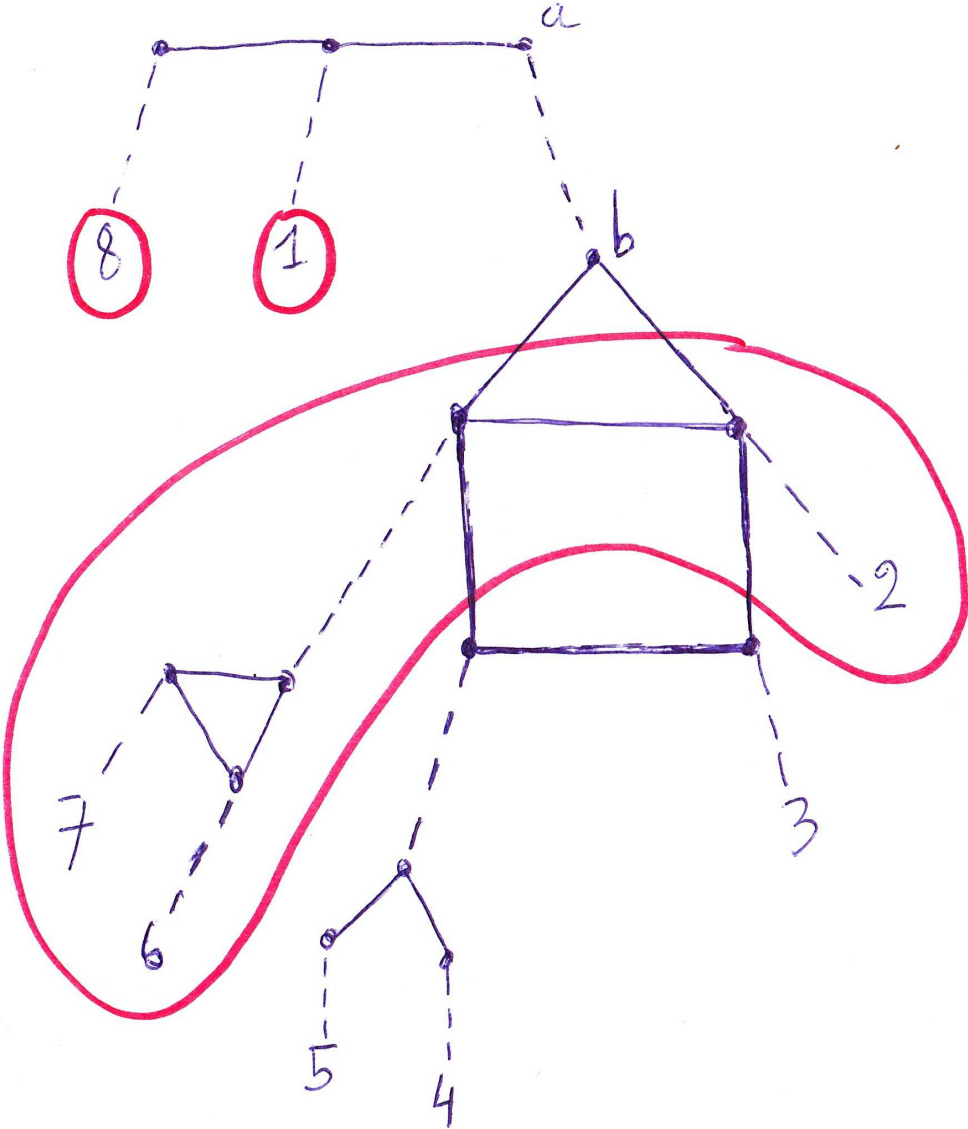
$L =$ live vertices,

$D =$ dead vertices.

$H = K[G_1, G_2, G_3, G_4]$



Application to the example page 10 :

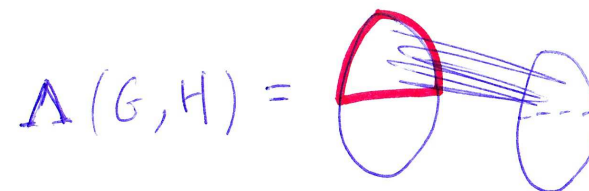
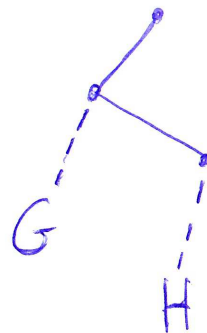
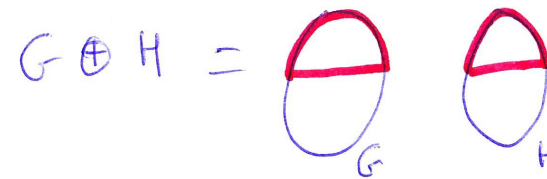
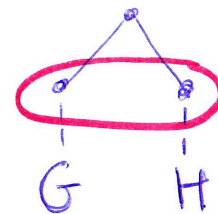
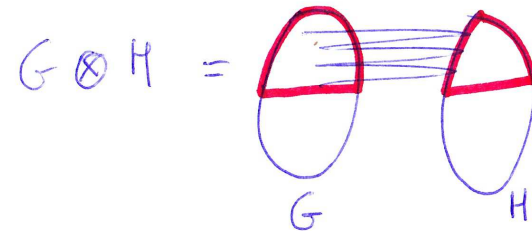
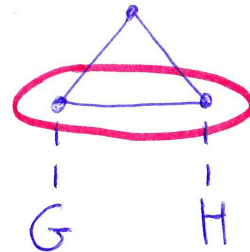


Distance hereditary graphs :

Definition : The distance in any connected induced subgraph is the same as in the considered graph.

Characterization : The components of a split decomposition are stars and cliques. We need only K_3 and S_3 that yield the operations

\otimes , \oplus and Λ :



Distance hereditary graphs are defined by the grammar :

$$G ::= * \mid G \otimes G \mid G \oplus G \mid \Lambda(G,G)$$

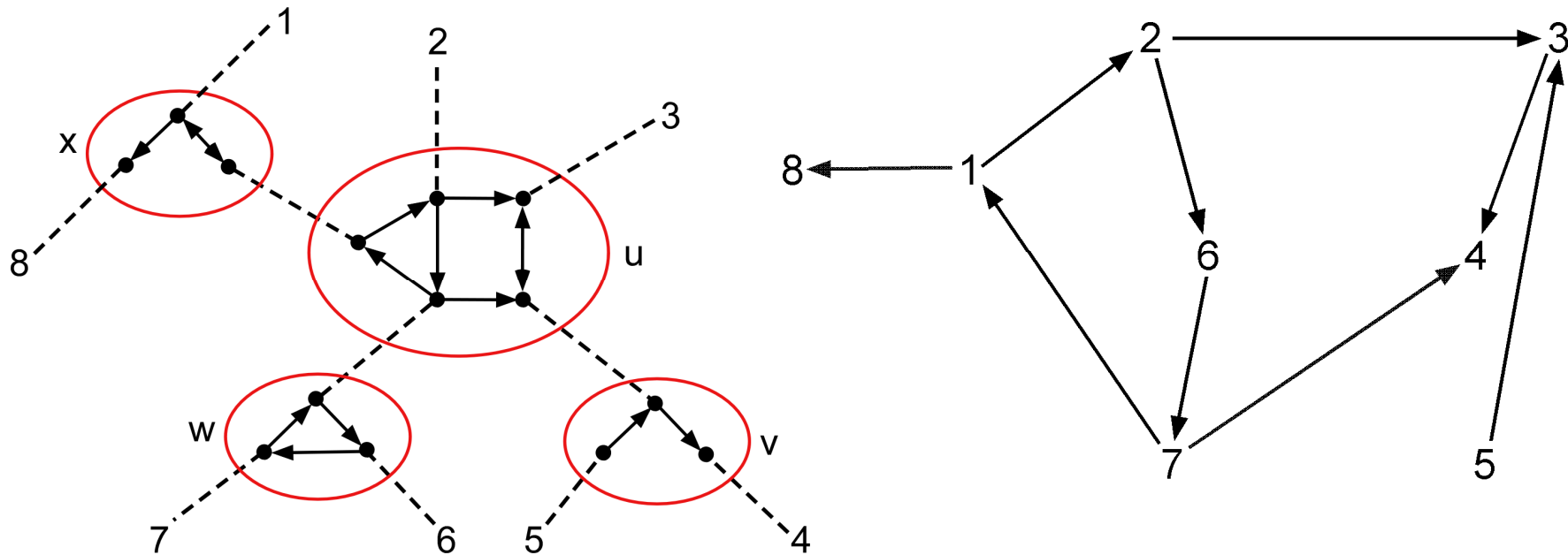
where $*$ is an isolated **live vertex**.

A similar grammar using substitution operations, say $K[. , . , . , .]$, is obtained for the class of graphs defined by graph-labelled trees whose components are cliques, stars and finitely many graphs K .

Theorem : (1) If G is defined by a graph-labelled tree whose components are cliques, stars and graphs K of $\text{cwd}(K) \leq m$, then $\text{cwd}(G) \leq 2m + 1$.

(2) If in addition, K has degree $\leq d < m$, then $\text{cwd}(G) \leq m + d + 1$.

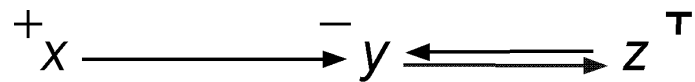
Graph-labelled trees for directed graphs



Directed edges are defined from *directed alternating paths*.

Theorem : If G is defined by a *directed* graph-labelled tree whose components have clique-width $\leq m$, then $\text{cwd}(G) \leq 8.m + 1$.

The **live vertices** are of 3 types : T , + and - ; edges are directed
 $+ \rightarrow -$ and T means + or - . Dead vertices are labelled by \perp .
 Substitution : $G = K[x \leftarrow X, y \leftarrow Y, z \leftarrow Z]$.



K

0 \perp

1 +

2 -

3 T

X

4 +

5 -

6 T

Y

7 +

8 -

9 T

Z

\perp 0

+ 1

\perp 2

+ 3

\perp 4

- 5

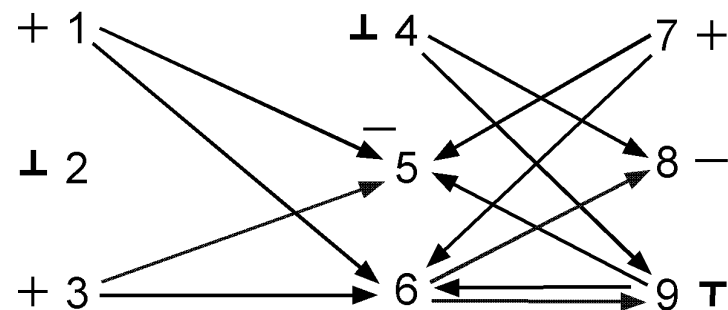
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G

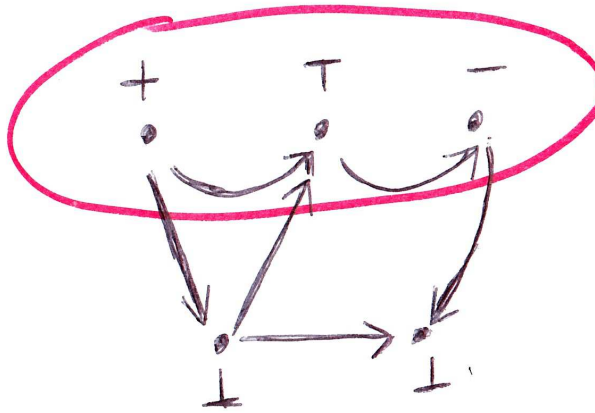
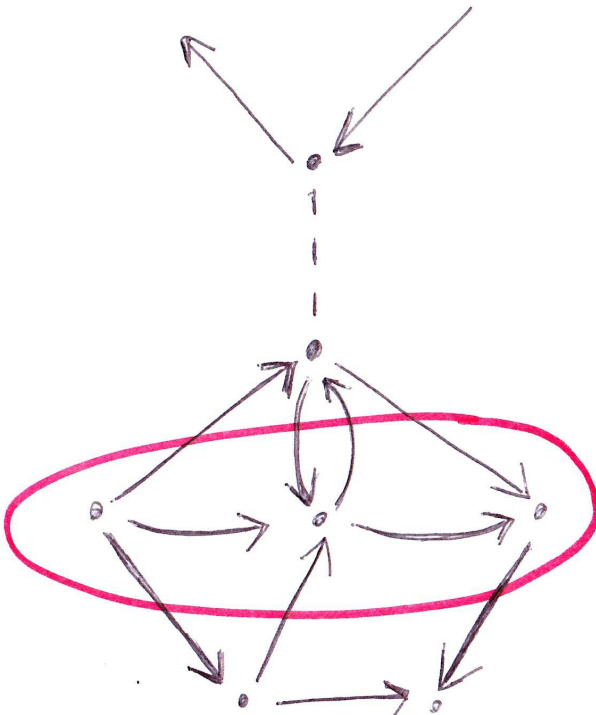
7 +

8 -

9 T



Why these 3 labels ?



Conclusion

Modified notions of substitution.

Split decompositions for undirected or directed graphs (not necessarily strongly connected) expressed by terms over substitution operations.

Good bounds on clique-width (from the clique-widths of components).

Graph grammars. If unambiguous, they can be used for counting or random generation (cf. Chauve, Fusy, Lumbroso for distance hereditary graphs)