Rank-width of countable graphs

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Graph decompositions of finite graphs: useful for FPT algorithms and graph structure (Graph Minor Theorem, perfect graphs). **Width measures** are frequently associated with decompositions: tree-width, path-width, rank-width, linear rank-width, clique-width.
Graph decompositions and widths of countable graphs: Definitions?

Compactness results: is \( \text{wd}(G) \) the least upper bound of \( \text{wd}(H) \), for all \( H \), finite subgraphs of \( G \)?

Yes for tree-width by Kriz & Thomas.
This talk: rank-width of countable graphs.

Two notions of linear rank-width based on two different linear orders: \( \mathbb{Q} \) and \( \mathbb{Z} \).

Two notions of rank-width based on two types of trees: quasi-trees (cf. \( \mathbb{Q} \)) and trees (cf. \( \mathbb{Z} \)).

Different compactness results:

<table>
<thead>
<tr>
<th>Rank-width</th>
<th>(quasi-trees)</th>
<th>Yes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discrete rank-with</td>
<td>(trees)</td>
<td>Yes with gap: ( n \rightarrow 2n )</td>
</tr>
<tr>
<td>Linear rank-width</td>
<td>(( \mathbb{Q} ))</td>
<td>Yes</td>
</tr>
<tr>
<td>Discrete linear rwd</td>
<td>(( \mathbb{Z} ))</td>
<td>No</td>
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</table>
Rank-width : Countable, simple, undirected graphs.

If $X$ and $Y$ are disjoint sets of vertices, $A[X,Y]$ is the adjacency matrix of $G$ between vertices in $X$ and in $Y$.

The rank over $GF(2)$ of a countable matrix $A[X,Y]$ is the lub (least upper bound) of ranks of its finite submatrices.
A **layout** of a graph $G$ is a pair $(T, f)$:

- $T$ is a tree (without root) of degree at most 3,
- $f$ : injective mapping : $V(G) \rightarrow \text{Leaves}(T)$

Each edge $e$ of $T$ yields a bipartition $(X_e, X_e^c)$ of $V(G)$ ($X_e$ or $X_e^c$ may be empty).

$rk(e) := \text{rank of } A[X_e, X_e^c] \text{ over } GF(2)$

$rk(T,f) := \text{lub} \{rk(e) / e, \text{ edge of tree } T\}.$

$rwd(G) := \text{minimal } rk(T,f) \text{ over all layouts } (T,f).$
**Property**: (1) \( \text{rwd}(H) \leq \text{rwd}(G) \) if \( H \subseteq_i G \) (induced subgraph)

(2) If, between \( X \) and \( X^c \), there are \( k \) pairwise disjoint edges, then \( \text{rk}( A[X,X^c]) \geq k \).

**Examples**: Trees have rank-width 1, cycles have rank-width 2; \( n \times n \) square grids have rank-width \( n-1 \) (Jelinek, difficult proof).
Linear rank-width

Defined from layouts of the special form

Equivalent to $a < b < c < d < e < f$: a linear order on vertices. Edge bipartitions $(X_e, X_{e^c})$ are replaced by

*Dedekind cuts*: bipartitions $(X, X^c)$ such that $X < X^c$
These definitions work for countable graphs.

**Terminology**: discrete rank-width, $\text{drwd}(G)$; the good notion of “rank-width” will be different (for countable graphs).

**Theorem**: Discrete rank-width has compactness, but “only” with gap:

$$\text{drwd}(G) \leq 2. \sup\{\text{rwd}(H) \mid H \subseteq_i G, H \text{ finite}\}$$
Proof: Compactness with gap: will come later.

No compactness. Counter-example:

$G$ has vertex set $\mathbb{Q}$ partitioned into two dense subsets $A$ and $B$.

If $x < y$, then $x \rightarrow y$ (edge) if and only if $y \in A$.

Every finite induced subgraph of $G$ is a threshold graph, hence of rank-width 1. But $\text{drwd}(G) \geq 2$ (actually $\text{drwd}(G) = 2$ by the first fact).

Wanted: a definition of rank-width giving exact compactness.
Linear rank-width as warming up.

Same as rank-width with layout = linear order on $V(G)$. Bipartitions are defined from Dedekind cuts $(X, X^c)$ (such that $X < X^c$).

There are 3 types of layout:
- $\mathbb{N}$, $\mathbb{Z}$, suborder of $\mathbb{Q}$ (arbitrary linear order).

Hence 3 notions of linear rank-width with:

$$Lrwd_\mathbb{Q}(G) \leq Lrwd_\mathbb{Z}(G) \leq Lrwd_\mathbb{N}(G) \leq 2.Lrwd_\mathbb{Z}(G)$$
Lrwd_N(G) is called *discrete linear rank-width*, denoted by dLrwd(G).

Lrwd(G) called *linear rank-width* is based on arbitrary linear orders.

*Theorem* : (1) Lrwd has the compactness property.

(2) dLrwd has not, even with a gap function.
Proofs: (1) Using Koenig’s lemma.

(2) Counter-example:

\( P = \) the infinite path isomorphic to \( \mathbb{N} \)

\( G = \) the union of \( \omega \) disjoint copies of \( P \)

\( \text{Lrwd}(G) = 1 \) for order \( \mathbb{N} + \mathbb{N} + \mathbb{N} + \ldots \)

\( \text{dLrwd}(G) = \omega. \)
Rank-width based on quasi-trees.

In order to apply Koenig’s lemma, we need a notion of tree closed under countably many insertions of nodes on a path, hence has least upper bounds for topological ordering.

We will define quasi-trees whose “paths” between two nodes can have countably many nodes.
**Betweeness**: If $T$ is a tree, let $B_T(x, y, z)$ mean that $y$ is on the path that links $x$ and $z$. The following properties hold:

A1: $B(x, y, z) \Rightarrow x \neq y \neq z \neq x$.
A2: $B(x, y, z) \Rightarrow B(z, y, x)$.
A3: $B(x, y, z) \Rightarrow \neg B(x, z, y)$.
A4: $B(x, y, z) \land B(y, z, u) \Rightarrow B(x, y, u) \land B(x, z, u)$.
A5: $B(x, y, z) \land B(x, u, y) \Rightarrow B(x, u, z) \land B(u, y, z)$.
A6: $B(x, y, z) \land B(x, u, z) \Rightarrow y = u \lor (B(x, u, y) \land B(u, y, z)) \lor (B(x, y, u) \land B(y, u, z))$.
A7: $x \neq y \neq z \neq x \Rightarrow B(x, y, z) \lor B(x, z, y) \lor B(y, x, z) \lor (\exists u. B(x, u, y) \land B(y, u, z) \land B(x, u, z))$. 
Definition: A quasi-tree is a pair \((N,B)\) where \(N\) is a set (its nodes) and \(B\) a ternary relation that satisfies Properties A1-A7. It is discrete if for each \(x, z\), the set of nodes \(y\) between \(x\) and \(z\) (i.e., such that \(B(x,y,z)\) holds) is finite. Then \(B = B_T\) for some tree \(T\).

An increasing sequence of finite quasi-trees has a lub that is a quasi-tree.

A cut is a partition \((X,X^c)\) such that \(X\) and \(X^c\) are convex; \(X\) is convex if \(B(x,y,z) \land x \in X \land z \in X \Rightarrow y \in X\).
Degree. Nodes y and z are in the same direction relative to node x if:

\[ y = z \lor B(y,z,x) \lor B(z,y,x) \lor \exists u \ ( B(y,u,x) \land B(z,u,x) \land B(y,u,z) ). \]

This is an equivalence relation. Its classes are the directions relative to x.

The degree of x is the number of directions relative to it. A leaf has degree 1 (it is not between two nodes).
A *layout of* $G$ is a quasi-tree $S = (N, B)$ whose nodes have degree $\leq 3$ and such that $V(G) \subseteq$ Leaves($S$).

If $(X, X^c)$ is a cut of $S$:

$$rk(X, X^c) := rk(A[X \cap V(G), X^c \cap V(G)])$$

$$rk(S) := \text{Sup}\{ rk(X, X^c) / (X, X^c) \text{ is a cut } \}$$

$$\text{rwd}(G) := \text{Min}\{ rk(S) / S \text{ is a layout } \}$$

$$\text{drwd}(G) := \text{Min}\{ rk(S) / S \text{ is a discrete layout } \}$$

$$= \text{Min}\{ rk(T, f) / (T, f) \text{ is a layout, } T \text{ tree } \}$$
Theorem: (1) \( \text{rwd} \) has the compactness property.

\[ (2) \text{drwd}(G) \leq 2. \text{rwd}(G). \]

Proof: (1) With Koenig’s lemma, as for \( \text{Lrwd} \).
(2) $\text{drwd}(G) \leq 2 \cdot \text{rwd}(G)$.

First: $\text{drwd}(G) \leq 2 \cdot \text{Lrwd}(G)$.

A countable linear order is described by a rooted binary tree.
Let $\leq$ witness that $\text{Lrwd}(G) \leq k$. We describe $\leq$ by a binary tree. We get as layout of $G$ a tree of degree $\leq 3$. An edge $e$ yields a bipartition of $V(G)$ of the form $(X \cup Z, Y)$. We have:

\[
\text{rk}(A[X \cup Z, Y]) \leq \text{rk}(A[X, Y]) + \text{rk}(A[Z, Y]) \\
\leq \text{rk}(A[X, Y \cup Z]) + \text{rk}(A[X \cup Z, Y]) \leq 2k.
\]
For proving that $\text{drwd}(G) \leq 2 \cdot \text{rwd}(G)$, we use essentially this idea: we encode a layout $S$ by a binary rooted tree.

**Structuring** of a quasi tree $S$.

Choose a leaf $r$.

Maximal *lines* $N_0, N_1, \ldots$ containing $r$.

$U_0 := N_0$,

$U_1 := N_1 - U_0$,

$U_2 := N_2 - (U_0 \cup U_1)$, etc.
Describe each line $U_i$ by a binary tree $T_i$.

Connect trees $T_0, T_1, T_2, \ldots$ into a single tree.
This tree is a layout of $G$ of rank $\leq 2.k$ where $k = r_k(S)$. 