

Completions of ordered magmas

BRUNO COURCELLE
University of Bordeaux-I

JEAN-CLAUDE RAOULT
IRIA, Parisq XI

Received March 29, 1978

AMS Categories : 06A75, 18A40, 68A05

Abstract. We give a completion theorem for ordered magmas (i.e. ordered algebras with monotone operations) in a general form. Particular instances of this theorem are already known, and new results follow. The semantics of programming languages is the motivation of such investigations.

Key words complete partial orders, semantics of programming languages.

Introduction. When defining recursive functions by systems of equations (Kleene [5]), one introduces an order relation which means that a partial result approximates another one. This partial order is *complete* (i.e. every ascending chain admits a least upper bound), thus allowing a minimal solution to be defined for the system.

This matter has been rebuilt by Scott, and many authors after him, within the framework of *complete lattices*; that last theory has been developed for its own sake by several authors, among which Birkhoff [1].

Frequently, the lattice structure does not seem necessary and creates instead additional troubles (Plotkin [9], Milner [8] for instance). The notion of complete partial order is good enough, and fits better to the most common instances.

This algebraic framework is suitable for studying program schemes ([2], [3]). We then need distinguish between the *base functions* and the *program-defined functions*, with the help of base functions and various control structures (recursive call, iteration, etc...).

Thus, our domains will be *ordered magmas*, i.e. partial orders equipped with *monotone* operators (no information is lost during a computation). And we shall be concerned with *completeness* (the operators being supposed *continuous*).

More precisely, we shall study the possible embeddings of an ordered magma into a complete ordered magma. Some of the ascending chains may keep their l.u.b., or may be added a new one; this gives different *completions*, each characterized by a universal property. We shall thus define the Γ -*completion* as the completion which preserves the l.u.b. which already exist in a set Γ of subsets of the magma. From this general theorem is derived the “ideal completion” of [9], [1], [2], [4], the “chain-completion” of [8], and the existence of factor objects in the category of ordered magmas. The above mentioned authors would use neither operators nor magmas, but only partial orders (except [4]). But the “chain-completion” in the category of partial orders need not be a complete ordered magma (cf. Corollary 2).

Definitions and the main theorem. Let F be a set of operators with arity. An F -magma M is a domain D_M together with a function $f_M : D_M^k \rightarrow D_M$ for each $f \in F$ with arity k . The homomorphisms of F -magmas, or F -morphisms, shall be compatible with the operators: $\varphi : M \rightarrow M'$ is a F -morphism when

$$\varphi(f_M(a_1, \dots, a_k)) = f_{M'}(\varphi(a_1), \dots, \varphi(a_k))$$

for all $f \in F$ with arity k , and all $a_1, \dots, a_k \in D_M$.

In this paper, we shall only consider *ordered magmas* (therefore “magma” will mean “ordered magma”), with a partial order denoted by \leq_M , a least element Ω_M (associated with the symbol Ω of arity 0 which is always supposed to be an element of F), and monotone operators f_M . An F -morphism between ordered magmas must be monotone.

Finally, if M is a magma, a *sub-F-magma* N of M is a F-magma such that $D_N \subseteq D_M$, and that the inclusion $i : N \rightarrow M$ is a (monotone) F-morphism. It is *full* when $d \leq_M d'$ implies $d \leq_N d'$ for every $d, d' \in D_N$.

Let Γ denote a set of non-empty subsets of D_M . The F-magma M is said to be Γ -*complete* when
— every $A \in \Gamma$ admits a l.u.b. in D_M , denoted by $\sup A$ or $\bigvee_{x \in A} x$;
— $f_m(A_1, \dots, A_k) = \{f(a_1, \dots, a_k); a_1 \in A_1, \dots, a_k \in A_k\} \in \Gamma$ and $\sup f_M(A_1, \dots, A_k) = f_M(\sup A_1, \dots, \sup A_k)$ for all f with arity k and all $A_1, \dots, A_k \in \Gamma$.

A monotone function $\varphi : M^k \rightarrow N$ where M is Γ -complete and N is an arbitrary magma is Γ -*continuous* if for all $A_1, \dots, A_k \in \Gamma$, $\varphi(A_1, \dots, A_k)$ admits a l.u.b. in N and

$$\sup_N \varphi(A_1, \dots, A_k) = \varphi(\sup_M A_1, \dots, \sup_M A_k).$$

With each F-magma M is naturally associated
 $\Xi(M)$ = the set of non-empty subsets of D_M ,
 $\Phi(M)$ = the set of non-empty finite subsets of D_M ,
 $\Delta(M)$ = the set of non-empty directed subsets of D_M
 $(A \in \Delta(M) \iff \forall d, d' \in A, \exists d'' \in A, d \leq d'' \text{ and } d' \leq d'')$,
 $\Delta_\alpha(M)$ = the set of non-empty directed subsets of cardinality at most α ,
 $\Lambda(M)$ = the set of non-empty directed subsets of D_M which admit a l.u.b. in D_M ,
 $\Theta_\alpha(M)$ = the set of monotone morphisms of an ordinal α into M (the α -chains), etc.
Hence $-\Lambda \subseteq \Delta \subseteq \Xi$.

Thus, a partial order is a complete lattice (cf. Birkhoff [1]) if and only if it is Ξ -complete, a join-semi-lattice if and only if it is Φ -complete, a lattice if and only if it is Φ' -complete with

$$\Phi'(M) = \{\{d' \in D_M; d' \leq d_1 \ \& \ \dots \ \& \ d' \leq d_k\}; k \in \mathbb{N}, d_1, \dots, d_k \in D_M\}.$$

A F-magma is *pre-complete* if and only if it is Λ -complete. Notice that this condition only affects the f_m 's. If F only contains symbols of arity 0, then every F-magma is precomplete (cf. corollaries 1 and 2, and theorem 3).

To each of these families Γ is attached a category the objects of which are the Γ -complete F-magmas, and the arrows of which are the monotone F-morphisms $f : M \rightarrow N$ that send a subset $A \in \Gamma(M)$ on a subset $f(A)$ in $\Gamma(N)$ and furthermore such that $f(\sup A) = \sup f(A)$: the Γ -*continuous morphisms*. These restrictions can be automatically satisfied by the monotone morphisms for some families, but it is not always the case, as noticed for Λ . In the sequel, we shall restrict ourselves to those “functorial” families and related morphisms.

The main theorem of this paper reads as follows.

THÉORÈME 1. — *Let $\Gamma \subseteq \Gamma'$ two families of subsets, and M a Γ -complete F-magma. There exists a Γ' -complete F-magma $M_\Gamma^{\Gamma'}$ and a Γ -continuous injective F-morphism $i : M \rightarrow M_\Gamma^{\Gamma'}$ such that for all Γ -continuous morphisms $j : M \rightarrow N$ where n is Γ' -complete, there exists a unique Γ' -continuous morphism $h : M_\Gamma^{\Gamma'} \rightarrow N$ such that $j = hi$.*

$$\begin{array}{ccc} M & \xrightarrow{i} & M_\Gamma^{\Gamma'} \\ & j \searrow & \downarrow h \\ & & N \end{array}$$

Proof: The construction of $M_\Gamma^{\Gamma'}$ will be carried out in the particular case when $\Gamma' = \Xi$.

A non-empty subset A of D is Γ -*closed* when

- (1) $d \leq d'$ and $d' \in A \Rightarrow d \in A$ (hence $\Omega_M \in A$),
- (2) $B \subseteq A$ and $B \in \Gamma \Rightarrow \sup B \in A$.

The intersection of a family of closed subsets is closed; therefore

- for all non-empty subset A of D , there exists a smallest closed subset containing A , its *closure*, denoted by $C_\Gamma(A)$ or $C(A)$ when Γ is clear from the context;
- the set \hat{D} of non-empty subsets of D ordered by set inclusion is a complete lattice: for all family $(A_i)_{i \in I}$ (which contains Ω_D)

$$\bigwedge_{i \in I} A_i = \bigcap_{i \in I} A_i$$

$$\bigvee_{i \in I} A_i = C\left(\bigcup_{i \in I} A_i\right).$$

The lattice \hat{D} is given a structure of an F-magma by setting for all closed subsets $C_i \in \hat{D}$

$$f_{\hat{D}}(C_1, \dots, C_k) = C(\{f_M(d_1, \dots, d_k; d_i \in C_i)\}).$$

One checks that, for all subsets A_1, \dots, A_k of D

$$(*) \quad C(f_M(C(A_1), \dots, C(A_k))) = C(f_M(A_1, \dots, A_k)).$$

As a result, \hat{D} is a complete F-magma; let us check it for the first argument :

$$\begin{aligned} f_{\hat{D}}(\bigvee_i A_i, B_2, \dots, B_k) &= f_{\hat{D}}(C(\bigcup_i A_i), B_2, \dots, B_k) \\ &= C(f_M(\bigcup_i A_i, B_2, \dots, B_k)) \quad \text{from } (*) \\ &= C(\bigcup_i f_M(A_i, B_2, \dots, B_k)) \\ &= \bigvee_i f_{\hat{D}}(A_i, B_2, \dots, B_k) \end{aligned}$$

because for all family of subsets P_i , $C(\cup P_i) = C(\cup C(P_i))$ holds since C is a closure operation (Birkhoff [1], p. 111). The corresponding F-magma is denoted by $M_{\Gamma}^{\bar{E}}$.

DEFINITION OF i . Let $i : D \rightarrow \hat{D}$ be defined by $i(d) = C(\{d\}) = \{x \in D; x \leq d\}$.

Clearly $d \leq d' \iff i(d) \leq i(d')$; furthermore, i is Γ -continuous : if $d = \sup A$ with $A \in \Gamma$, one should have

$$i(d) = \{x \in D; x \leq d\} = \bigvee_{x \in A} i(x).$$

Now that last element is $C(A)$ and since $A \in \Gamma$, $\sup A = d \in C(A)$.

Finally, $(*)$ shows that i is an F-morphism.

CONSTRUCTION OF $M_{\Gamma}^{\bar{E}}$. Set $E_0 = \{i(d); d \in D\}$ and let E denote the smallest subset of \hat{D} containing E_0 which is Γ' -complete (i.e. such that $\bigvee_i A_i \in E$ for all family $(A_i)_{i \in I}$ in $\Gamma'(E)$). It can be described more explicitly by

$$\begin{aligned} E &= \bigcup_{\alpha} E_{\alpha} && (\alpha \text{ is an ordinal}), \\ E_{\alpha+1} &= E_{\alpha} \cup \{\sup_{\hat{D}} X; X \in \Gamma'(E_{\alpha})\}, \\ E_{\beta} &= \bigcup_{\alpha < \beta} E_{\alpha} && \text{if } \beta \text{ is a limit ordinal.} \end{aligned}$$

In fact, $E = E_{\gamma}$ for some ordinal γ because \hat{D} is a set. Notice that

$$E_1 \supseteq \{C(A); A \in \Gamma(D)\}.$$

Since the subsets of Γ' are sent by f onto subsets of Γ , E is invariant under $f_{\hat{D}}$. Hence $(E, \subseteq, (f_E)_{f \in F})$ is a sub-F-magma of $M_{\Gamma}^{\bar{E}}$ (f_E is the restriction of $f_{\hat{D}}$ to E) which is Γ' -complete for the induced order. We shall denote it by $M_{\Gamma}^{\bar{E}}$.

UNIVERSAL PROPERTY OF $(i, M_{\Gamma}^{\bar{E}})$. Let $j : M \rightarrow N$ be a Γ -continuous morphism into a Γ' -complete F-magma. We shall define $h : M_{\Gamma}^{\bar{E}} \rightarrow N$ by transfinite induction : h is defined over $i(D) = E_0$ by the condition $hi = j$: $h(C(\{d\})) = j(d)$.

Suppose that h is defined over E_{α} for all $\alpha < \beta$. Then, if β is a limit ordinal, then h is defined over $E_{\beta} = \bigcup_{\alpha < \beta} E_{\alpha}$. Else h is defined by the Γ' -continuity : if $e \in E_{\beta} - E_{\alpha}$ then

$$e = \sup X \quad \text{where } X \in \Gamma'(E_{\alpha})$$

and $\{h(x); x \in X\} \in \Gamma'(N)$ admits a l.u.b. n in N . We set

$$h(e) = n,$$

so that h is now defined over E_β . By induction, it is defined over E , and is clearly Γ' -continuous by construction. Finally, one can check that h is indeed a morphism, i.e. that it is compatible with the operators.

This construction generalizes the completion of Markowski-Rosen [7] and Markowski [6] with the introduction of operators (cf. theorem 3 and corollary 4) and that of Bleicher-Schneider [2] by using a parameterized notion of limits and continuity. That last possibility is also considered in [10].

Since the magma $M_\Gamma^{\Gamma'}$ is completely specified, it is the object function of a functor from the category of Γ -complete magmas to that of Γ' -complete magmas, and is a left adjoint to the obvious “forgetful” functor. From this fact, it can be deduced easily that if $\Gamma \subseteq \Gamma' \subseteq \Gamma''$, then

$$(M_\Gamma^{\Gamma'})_{\Gamma''}^{\Gamma''} = M_\Gamma^{\Gamma''}.$$

II. Consequences of theorem 1. An important point in the proof of the theorem is that all morphisms should belong to the appropriate category, including the functions f_M . It goes without saying when Γ is Ξ of Φ or even Δ , that any monotone function will send a subset of $\Gamma(M)$ onto another subset of $\Gamma(M)$ (or $\Gamma(N)$ if it is a morphism $M \rightarrow N$). It is not so obvious when Γ is Λ : this was the purpose of the remark following the definition of a precomplete magma.

If M is a lattice, i.e. a Φ' -complete ordered set, $M_{\Phi'}^{\Xi}$ is exactly the ideal completion as defined by Birkhoff ([1] theorem 5 p. 113).

If M is also a F-magma, the corresponding object in the category of all Δ -complete F-magmas is M_{Δ}^{Δ} . One checks easily that its domain is the set of non-empty ideals. In particular, if $M = M(F)$ is the *free ordered F-magma* (identified with the set of finite terms over alphabet F), the corresponding object is the *free Δ -complete F-magma* (from theorem 1), which can also be identified with the set of *infinite terms* or *trees*, as defined in [3], [4] (and [6] with a somewhat different terminology).

Completion by cuts. Let M be a *precomplete F-magma*, i.e. a Λ -complete, magma where Λ is the set of directed subsets of M which admit a l.u.b. In this case, M_{Λ}^{Δ} is a completion of M which preserves the already existing l.u.b.'s of directed sets, and corresponds to the completion by cuts in lattices as defined by Birkhoff ([1], theorem 22, p. 126). This construction can also be found in Markowski [6]. These results are regrouped in the following corollary.

COROLLAIRE 2. — *Let M be an ordered (resp. precomplete) F-magma. There exists a Δ -complete F-magma M_{Δ}^{Δ} (resp. M_{Λ}^{Δ}) such that M is a full sub-magma (and furthermore the inclusion $i : M \rightarrow M_{\Lambda}^{\Delta}$ is Λ -continuous), and for all Δ -complete F-magma N , and all morphisms $j : M \rightarrow N$ (resp. Λ -continuous morphisms), there exists a unique Δ -continuous morphism $k : M_{\Delta}^{\Delta} \rightarrow N$ (resp. $k : M_{\Lambda}^{\Delta} \rightarrow N$) extending j .*

Remark : It is necessary for M to be precomplete if it is to be embedded in a complete ordered magma M_{Λ}^{Δ} , so that

COROLLAIRE 3. — *A magma is precomplete if and only if it is a full Δ -sub-magma of a Δ -complete magma.*

Notice that if M is not precomplete, M_{Λ}^{Δ} is an F-magma with domain Δ -complete but the $f_{\overline{M}}$ are not Δ -continuous. This point will be dealt with in corollary 4. We shall use \overline{M} as an other notation for M_{Λ}^{Δ} .

Remarks : McNeille's construction given in Birkhoff [1] consists in taking the set of all subsets of the lattice to be completed, then their upper bounds, and finally the sets of lower bounds of these upper bounds, as elements of the completed domain. But it does not give the desired result when applied to a precomplete magma.

Anyway, McNeille's completion does not satisfy the expected universal property.

An example is shown on figure 1.

Let M^+ denote McNeille's completion of lattice M . The inclusion $i : M \rightarrow N$ is a lattice-morphism which preserves all l.u.b.'s (and all g.l.b.'s) existing in M . It does not extend into a morphism of complete lattices $M^+ \rightarrow N$.

III. Quotients of complete F-magmas. Let M be a Γ -complete F-magma, for some $\Gamma \in \Delta$, and π an F-preorder over M , i.e. a preorder over D_M such that

Proof: Let (D, \leq) be a preorder. An element d of D is a l.u.b. of $A \subseteq D$ when

1) $d \leq d'$ for all $d' \in A$,

2) d is smaller than all upper bounds of A , i.e. $d' \leq d''$ for all $d' \in A$ implies that $d \leq d''$.

Notice that A may well have several l.u.b.'s. Let M be an ordered F-magma.

There exists a least F-preorder $\pi \subset M \times M$ such that

(**) if A_1, \dots, A_k are π -directed subsets of D_M with l.u.b.'s $\delta_1, \dots, \delta_k$ respectively, and if the π -directed subset $f_M(A_1, \dots, A_k)$ admits an upper bound δ for π , then $f_M(\delta_1, \dots, \delta_k) \pi \delta$.

We claim that M/π is precomplete and satisfies the following universal property : for all Δ -continuous F-morphism $j : M \rightarrow N$ in a Δ -continuous F-magma N , there exists a unique Δ -continuous F-morphism $k : M/\pi \rightarrow N$ such that $j = kh_\pi$ (h_π being the natural epimorphism $M \rightarrow M/\pi$). The theorem will be deduced from corollary 1 with $\overline{M} = \overline{M/\pi}$.

Let $A \subseteq D_M$, $A' = h_\pi(A) \in D_{M/\pi}$, and $h_\pi(d)$ the l.u.b. of A' . It is easy to see that A is directed for π and D is one of the l.u.b.'s of A for π . If $A'_1, \dots, A'_k \in \Delta_{M/\pi}$ admit l.u.b.'s $h_\pi(d_1), \dots, h_\pi(d_k)$ and $d = f_M(d_1, \dots, d_k)$, we must show that $h_\pi(d)$ is the l.u.b. of $f_{M/\pi}(A'_1, \dots, A'_k)$. Now, since π is an F-preorder,

$$f_{M/\pi}(h_\pi(a_1), \dots, h_\pi(a_k)) = h_\pi(f_M(a_1, \dots, a_k)) \leq h_\pi(f_M(d_1, \dots, d_k)) = h_\pi(d).$$

On the other hand, if $h_\pi(d') \geq f_{M/\pi}(A'_1, \dots, A'_k)$, then d' is an upper bound of $f_M(A_1, \dots, A_k)$ with respect to π ; hence $f_M(d_1, \dots, d_k) \pi d'$, i.e. $h_\pi(d) \leq h_\pi(d')$, QED.

Therefore the following diagram in which \overline{M} exists even if M is not precomplete is commutative :

$$\begin{array}{ccc} & & \overline{M} \\ & i_1 \nearrow & \\ M & & \downarrow h \\ & i_2 \searrow & \\ & & \overline{M} \end{array}$$

COROLLARY 4. — *The morphism h is onto. It is one-to-one if and only if for all $A_1, \dots, A_k \in \Delta_M$ such that $f(A_1, \dots, A_k) \in \Delta_M$ then*

$$f(\sup A_1, \dots, \sup A_k) = \sup f_M(A_1, \dots, A_k).$$

Proof: On one hand, if the condition holds, then \leq_M itself satisfies (**), hence $\pi = \leq_M$.

Therefore $\overline{M} = \overline{M/\pi} = \overline{M}$.

On the other hand, suppose that h is one-to-one, and that $A_1, \dots, A_k, f_M(A_1, \dots, A_k) \in \Delta_M$ with respective l.u.b.'s $\delta_1, \dots, \delta_k, \delta$, and $\delta' = f_M(\delta_1, \dots, \delta_k)$. Obviously $\delta \leq \delta'$. But $\delta' \pi \delta$ entails that $i_2(\delta) = i_2(\delta')$. Since h is one-to-one, and the diagram commutes, $i_1(\delta) = i_1(\delta')$. Since i_1 is the inclusion $M \subset \overline{M}$, the same holds in M : $\delta = \delta'$, QED.

EXAMPLE. Let M denote the F-magma $N \cup \{\alpha, \beta\}$ ordered by $i \leq_M j \leq_M \alpha \leq_M \beta$ for $i \leq j$ in N . The only element of N is of arity one defined by

$$\begin{cases} f(i) = i & \text{if } i \in N, \\ f(\alpha) = f(\beta) = \beta, \end{cases}$$

which is not Δ -continuous. Since the domain of M is Δ -complete, as an ordered set, $\overline{M} = M$. But in order to have a continuous f , one must identify α and β , thus obtaining \overline{M} with domain $N \cup \gamma$ and $h : \overline{M} \rightarrow \overline{M}$, as shown by figure 2.

$$\begin{array}{ccc} \beta & & \\ \downarrow & \searrow & \\ \alpha & \longrightarrow & \gamma \\ \downarrow & & \downarrow \\ \vdots & & \vdots \\ i & \longrightarrow & i \\ \downarrow & & \downarrow \\ \vdots & & \vdots \\ 0 & \longrightarrow & 0 \\ M=\overline{M} & & \overline{M} \end{array}$$

Fig. 2

References

- [1] Birkhoff, G., *Lattice theory*, Amer. Math. Soc., Providence 1967
- [2] Bleicher, M.N., et al., *Permanence of identities on algebras*, Algebra Universalis 3 (1973), 72–93.
- [3] Courcelle, B., Nivat, M., *Algebraic families of interpretations*, 17th Symposium on FOCS, Houston 1976.
- [4] Goguen, J., et al., *Initial algebra semantics and continuous algebras*, J. Assoc Comput. Mach. 24 (1977), 68–95.
- [5] Kleene, S.C., *Introduction to metamathematics*, Van Nostran, Princeton 1952.
- [6] Markowski G., *Chain-complete posets and directed sets with applications*, Algebra Univ. 6 (1976), 53–68.
- [7] Markowski G., Rosen, B., *Bases for chain-complete posets*, IBM J. Res. Develop. 20 (1976), 138–147.
- [8] Milner, R., *Fully abstract models of types lambda-calculi*, Theoretical Computer Science 4 (1977), 1–22.
- [9] Plotkin, G., T^ω as a universal domain, JCSS 17 (1978), 209–236.
- [10] Thatcher, J., Wagner, E., Wright, J., *A uniform approach to inductive posets and inductible closure*, Theoretical Computer Science 7 (1978), 57–77.
- [11] Vuillemin, J. *Syntaxe, sémantique et axiomatique d'un langage de programmation simple*, Birkhäuser, Basel 1975.

References added in proof (June 1980) and not cited in the text :

- [12] Bishop, A., *A universal characterization of the completion by cuts*, Algebra Universalis 8 (1978), 349–354.
- [13] Courcelle, B., Nivat, M., *The algebraic semantics of recursive program schemes*, Mathematical Foundations of Computer Science '78, Lec. Notes Comput. Sci. 64 (1978), 16–30.
- [14] Meseguer, J., *Completions, factorizations and colimits for ω -posets*, Proc. of Coll. “Logic in Programming”, Salgotarjan, Hungary (1978); also *Semantics and computation report 13*, U. C. L. A. (revised version, June 1979).