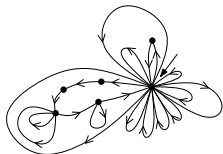


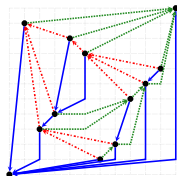
Planar graphs: non-aligned drawings, power domination and enumeration of Eulerian orientations



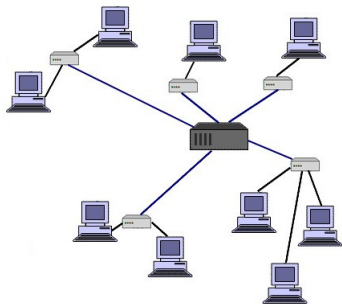
Claire Pennarun

Advisors: Nicolas Bonichon and Paul Dorbec

Jury: Mireille Bousquet-Mélou (CNRS, LaBRI)
Nadia Brauner (Univ. Joseph Fourier, G-SCOP)
Eric Fusy (CNRS, LIX, Ecole Polytechnique)
Michael Henning (Univ. of Johannesburg)

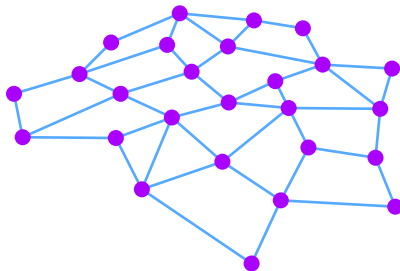
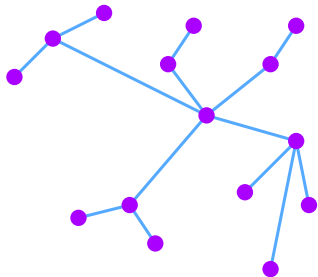


GRAPHS



Entities with some pairwise relations

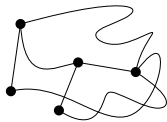
GRAPHS



Entities with some pairwise relations \rightarrow Graph: Vertices + Edges

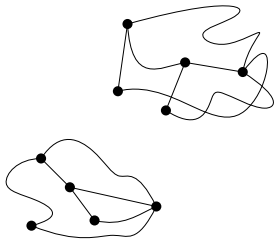
PLANAR GRAPHS

A **planar** graph can be embedded in the plane **without edges crossing**.



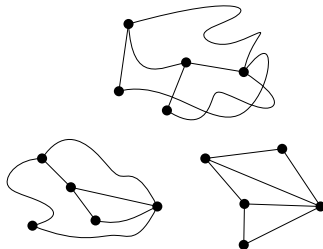
PLANAR GRAPHS

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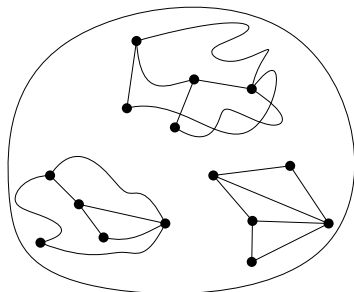
PLANAR GRAPHS

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PLANAR GRAPHS

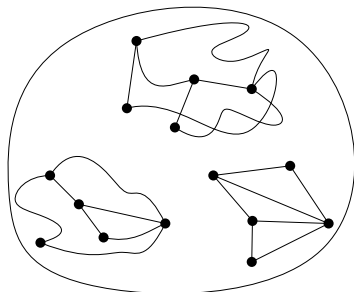
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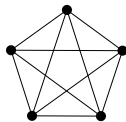
\Rightarrow planar graph

PLANAR GRAPHS

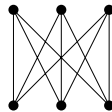
A **planar** graph can be embedded in the plane **without edges crossing**.



\Rightarrow planar graph



K_5

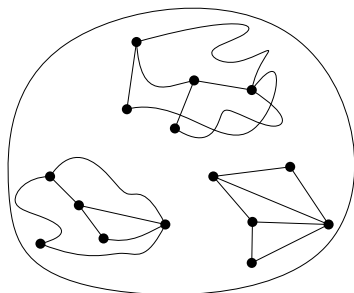


$K_{3,3}$

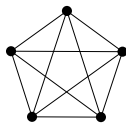
not planar

PLANAR GRAPHS

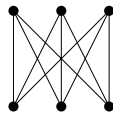
A **planar** graph can be embedded in the plane **without edges crossing**.



⇒ planar graph



K_5



$K_{3,3}$

not planar

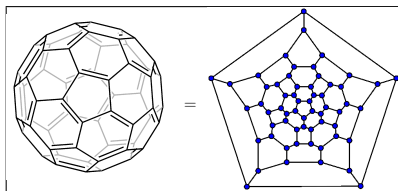
Appearing in **other**

contexts:

chemistry,

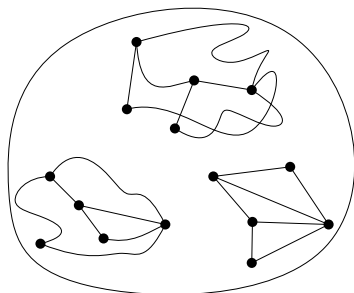
electronics,

image analysis...

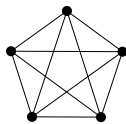


PLANAR GRAPHS

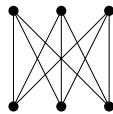
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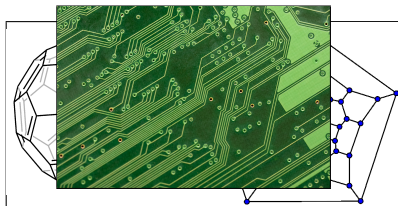


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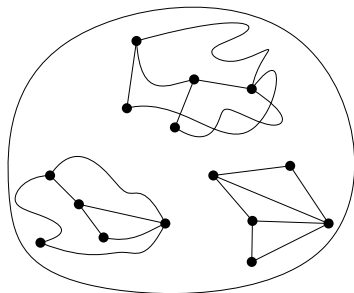
Appearing in **other contexts**:

chemistry,
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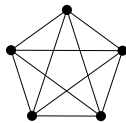


PLANAR GRAPHS

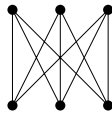
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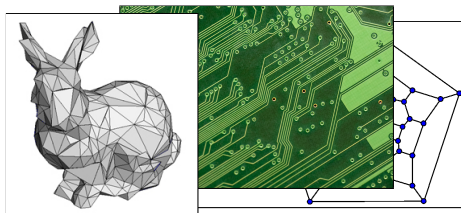


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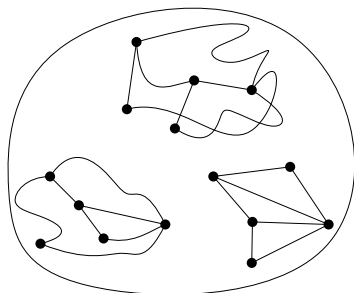
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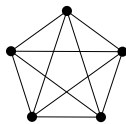


PLANAR GRAPHS

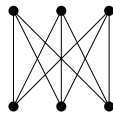
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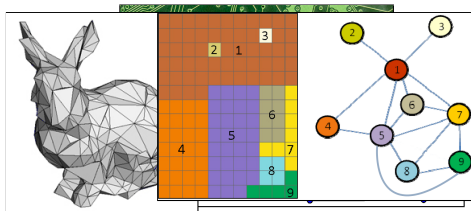


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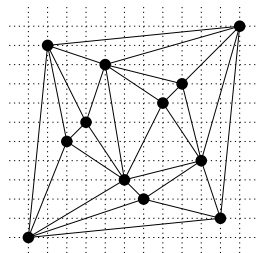
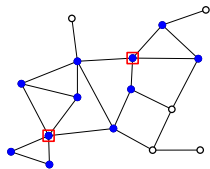
Appearing in **other contexts**:

chemistry,
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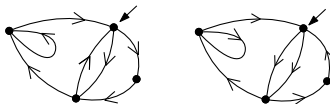
GLOBAL OVERVIEW

- Non-aligned drawings ← Graph drawing

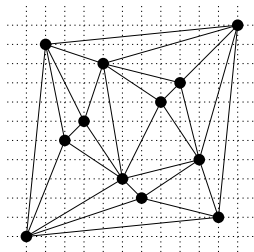
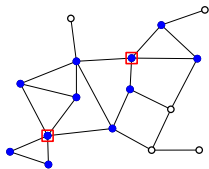


- Power domination ← Optimization problems

- Planar Eulerian orientations ← Enumerative combinatorics

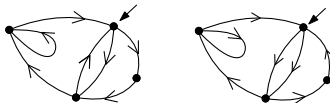


- Non-aligned drawings ← Graph drawing



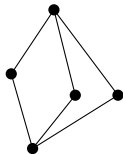
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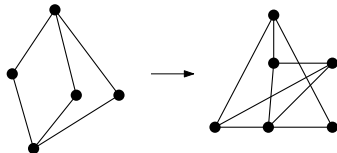
GRAPH VISUALIZATION

Dynamic graphs (addition/deletion of vertices/edges)



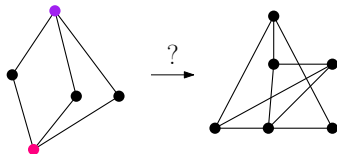
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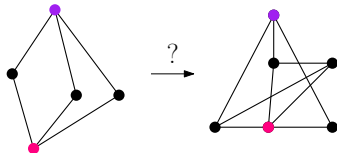
GRAPH VISUALIZATION

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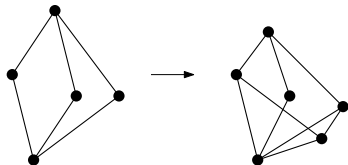
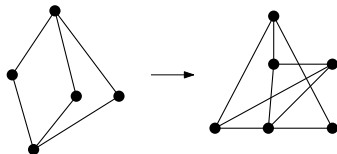
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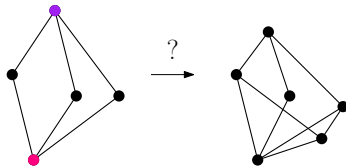
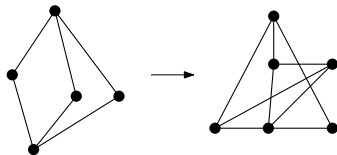
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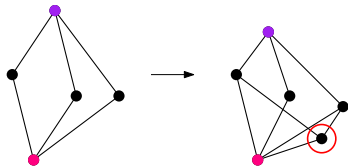
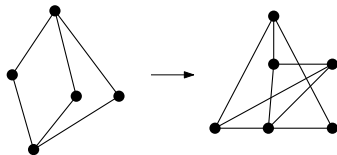
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GRAPH VISUALIZATION

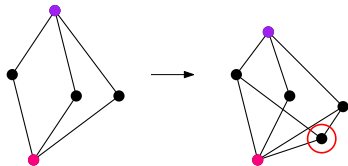
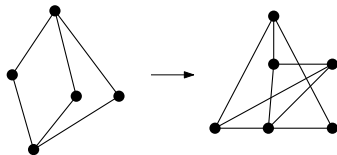
Dynamic graphs (addition/deletion of vertices/edges)



GRAPH VISUALIZATION

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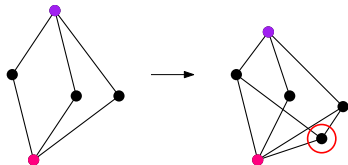
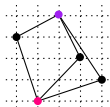
Preserving the relative positions of vertices (part of the **mental map**)



GRAPH VISUALIZATION

Dynamic graphs (addition/deletion of vertices/edges)

Preserving the relative positions of vertices (part of the **mental map**)



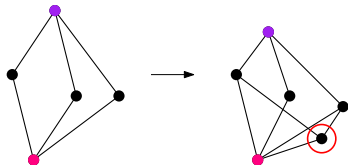
one vertex per line and column on a grid

[Kornaropoulos & Tollis'13, Didimo et al.'14, Angelini et al.'16]

GRAPH VISUALIZATION

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Preserving the relative positions of vertices (part of the **mental map**)



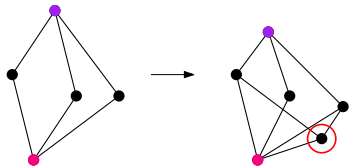
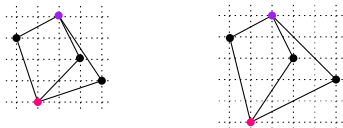
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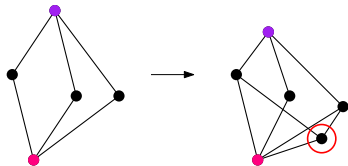
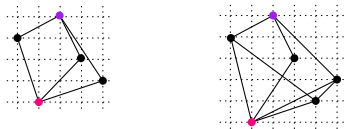
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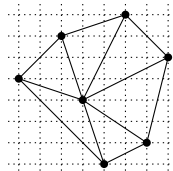
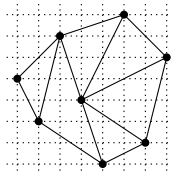
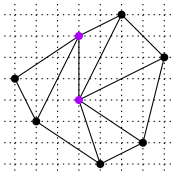
[Kornaropoulos & Tollis'13, Didimo et al.'14, Angelini et al.'16]

NON-ALIGNED DRAWINGS

Graph G , n vertices

Non-aligned drawing of a graph:

- vertices at the intersection of a regular grid
- no two vertices on the same row/column

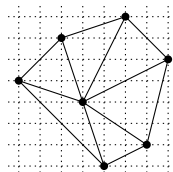
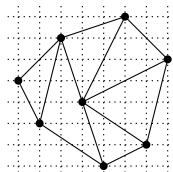
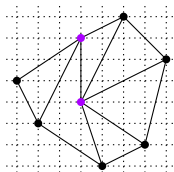


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Graph G , n vertices

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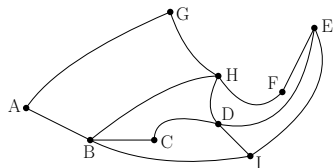


Non-aligned drawings **always** exist (two permutations of the vertices)

What if the graph is **planar**? → **planar non-aligned drawing**

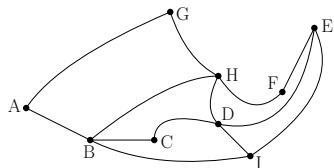
DRAWING PLANAR GRAPHS

Definition: a planar graph has a **planar drawing** with edges as **curves**



DRAWING PLANAR GRAPHS

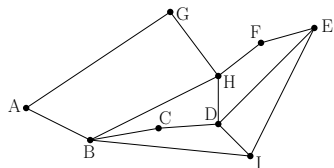
Definition: a planar graph has a **planar drawing** with edges as **curves**



[Fáry'48] Drawing planar graphs
with only **straight-lines**

DRAWING PLANAR GRAPHS

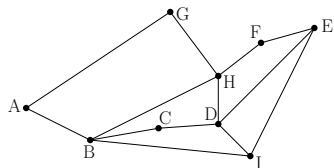
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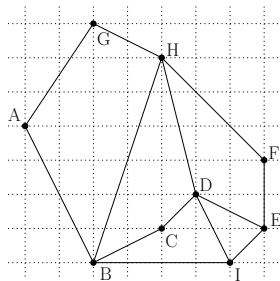
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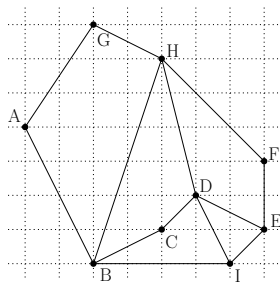
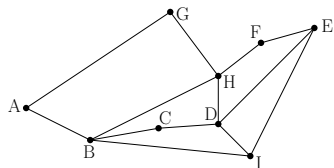


Grid drawing: vertices have **integer coordinates**

[Schnyder'90, de Fraysseix et al.'89]:
planar graphs on a grid with $n - 1$
lines/columns

DRAWING PLANAR GRAPHS

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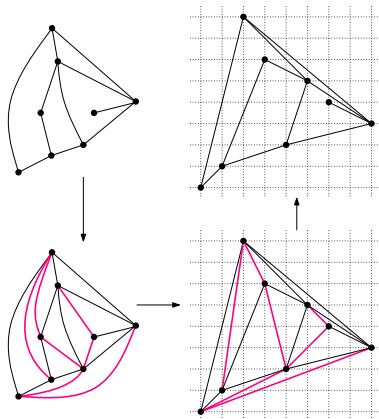
Grid drawing: vertices have **integer coordinates**

[Schnyder'90, de Fraysseix et al.'89]: planar graphs on a grid with $n - 1$ lines/columns

1 more line/column: **non-aligned?**

NON-ALIGNED DRAWINGS

Sufficient to look only at **maximal planar graphs**



Now: only planar drawings

NON-ALIGNED DRAWINGS

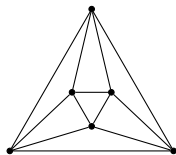
On the **minimal grid** = n lines and columns

[Auber, Bonichon, Dorbec, P. '15] There are planar graphs with **no** **straight-line** non-aligned drawing on the **minimal grid**.

NON-ALIGNED DRAWINGS

On the **minimal grid** = n lines and columns

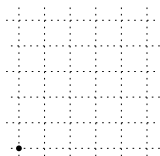
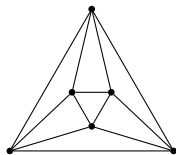
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NON-ALIGNED DRAWINGS

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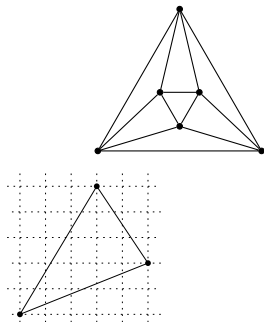
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NON-ALIGNED DRAWINGS

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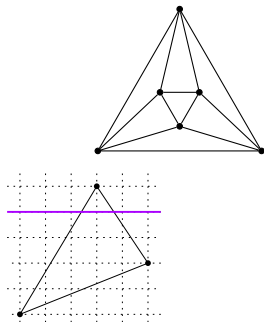
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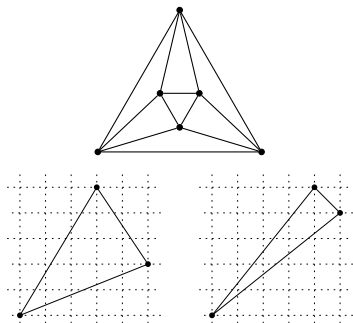
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NON-ALIGNED DRAWINGS

On the **minimal grid** = n lines and columns

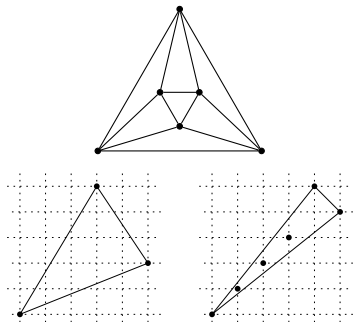
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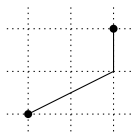
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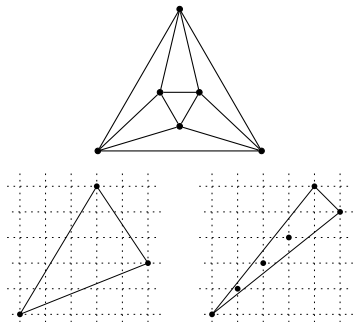
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Two possibilities:

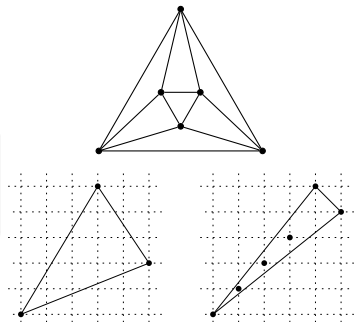
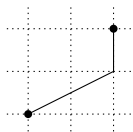
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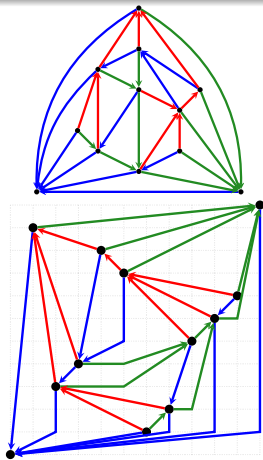
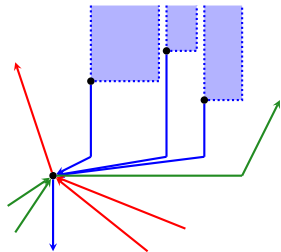
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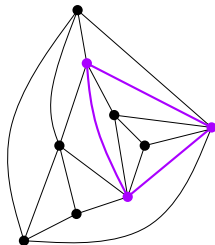
Use **Schnyder woods**:

- coloring the edges (local constraints) \rightarrow 3 trees
- coordinates given by the trees

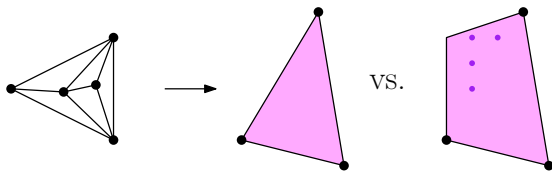


NON-ALIGNED DRAWINGS

Filled triangle = vertices inside



- Constrain **drawing areas**
- Draw them in **rectangles** not triangles

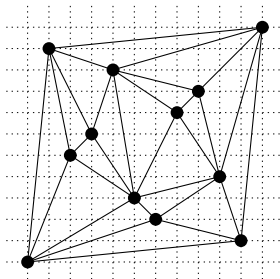


- Add bends only for filled triangles

NON-ALIGNED DRAWINGS

4-connected graph: no filled triangle (except outerface)

[Biedl et al. '99] Every **4-connected** graph with outerface of size ≥ 4 has a (non-aligned) **straight-line drawing on the minimal grid**.

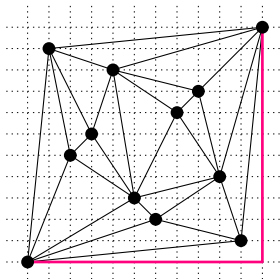


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Corollary Every **4-connected** graph has a non-aligned drawing on the minimal grid with **one bend**.



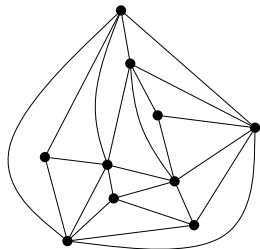
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f_G = number of filled triangles in G

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Set \mathbf{E} s.t.:

- every filled triangle has exactly one edge in \mathbf{E}
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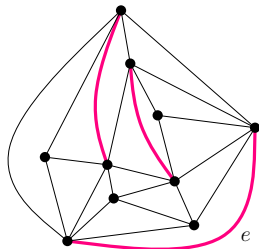
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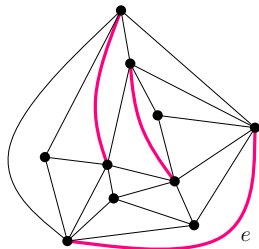
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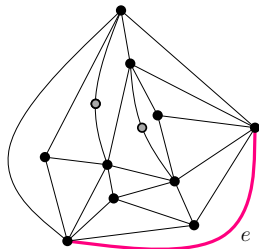
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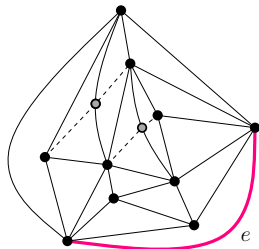
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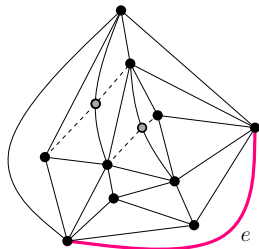
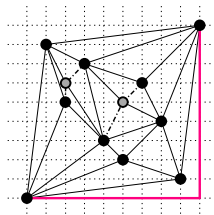
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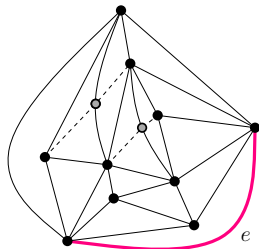
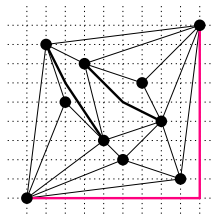
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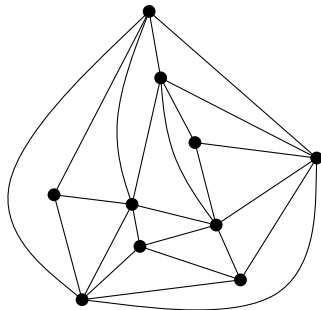
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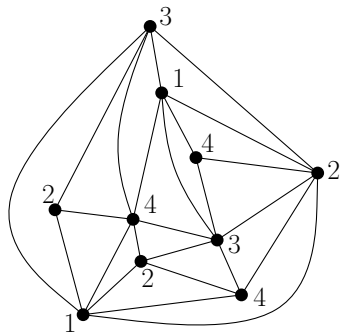
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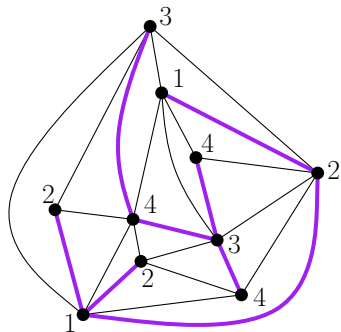
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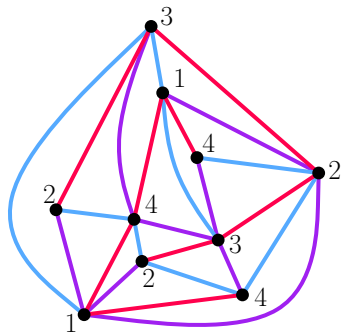
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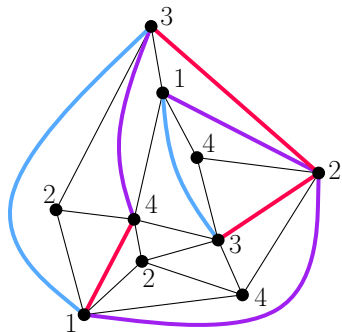
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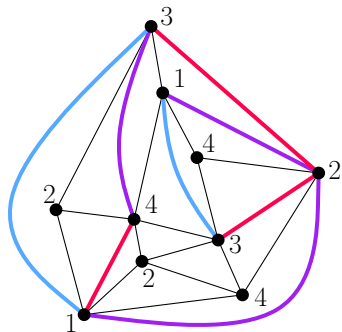
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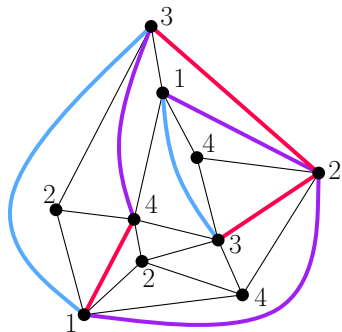
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Lemma: At most $2n - 5$ edges belong to a filled triangle.

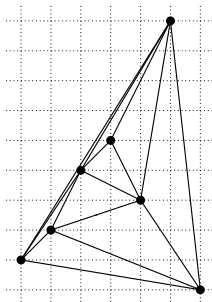
$$\rightarrow |M_i| \leq \frac{2n - 5}{3}$$

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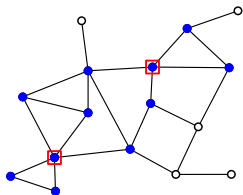
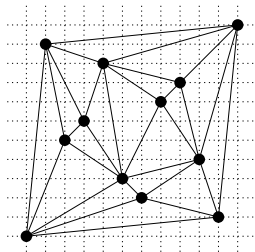
Other possibility: increasing the grid-size

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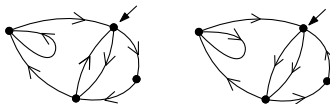


- Non-aligned drawings ← Graph drawing



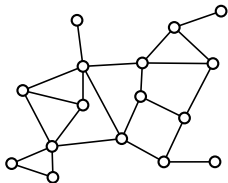
- **Power domination ← Optimization problems**

- Planar Eulerian orientations ← Enumerative combinatorics



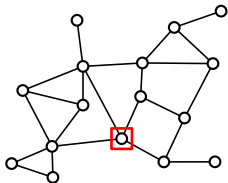
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Find a set of vertices S s.t. $N[S] = M = V(G)$



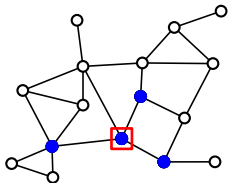
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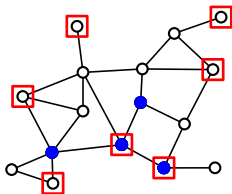
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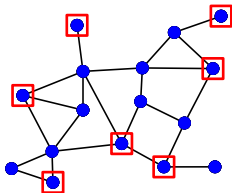
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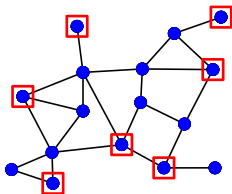
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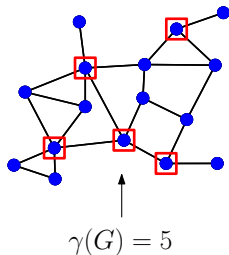
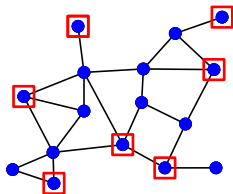
Optimization problem

The **domination number** $\gamma(G)$ is the minimum size of S s.t. $M = V(G)$.

Given G , what is $\gamma(G)$?

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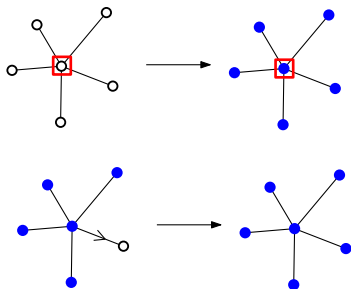
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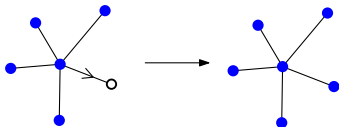
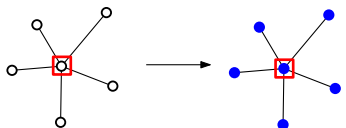
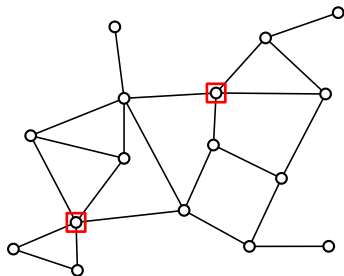
POWER DOMINATION [HAYNES ET AL. '02]

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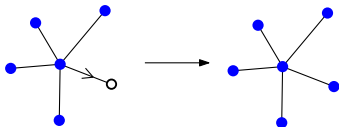
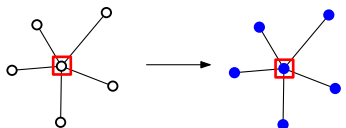
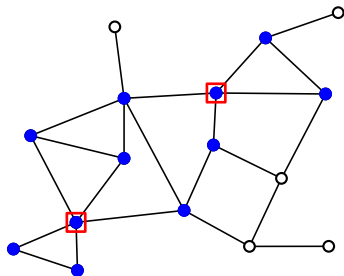
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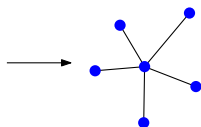
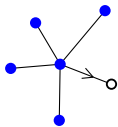
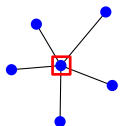
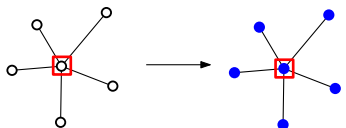
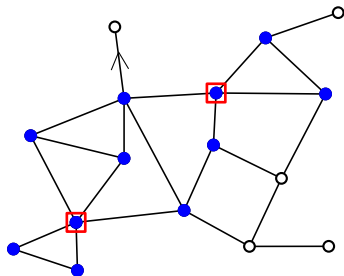
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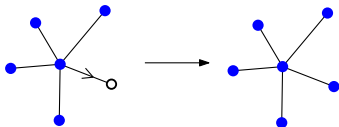
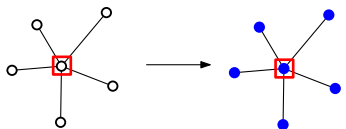
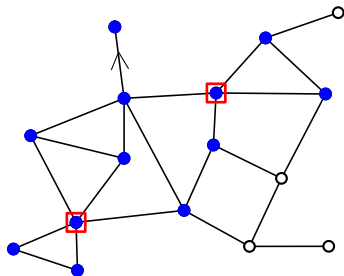
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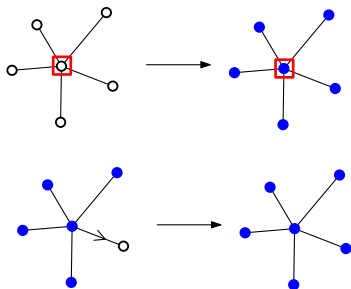
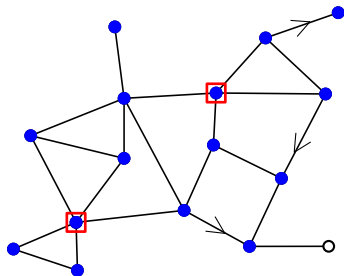
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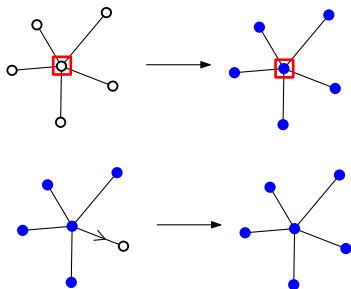
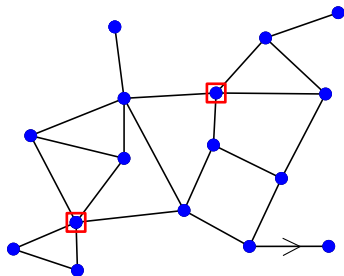
POWER DOMINATION [HAYNES ET AL. '02]

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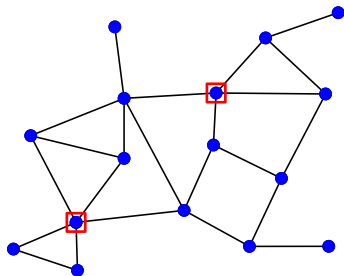
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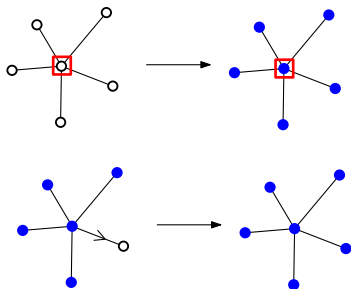


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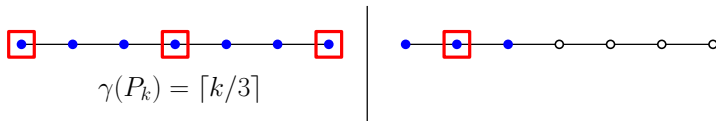
$$\gamma_P(G) = 2$$



The **power domination number** $\gamma_P(G)$ of G is the minimum size of S s.t. $M = V(G)$ at the end.

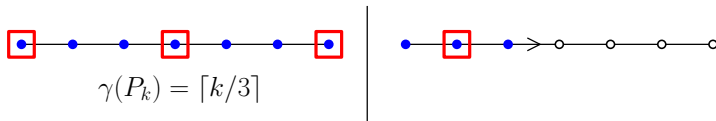
POWER DOMINATION VS DOMINATION

By definition, $\gamma_P(G) \leq \gamma(G)$, but difference arbitrarily large



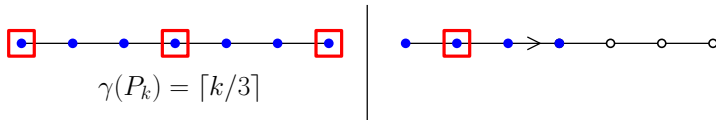
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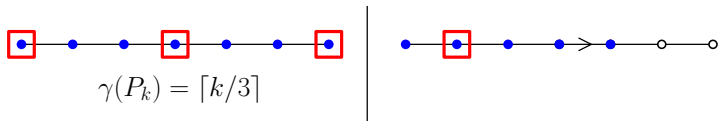
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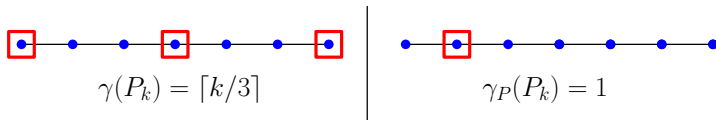
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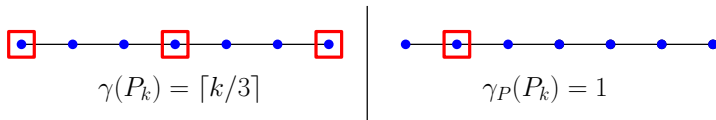
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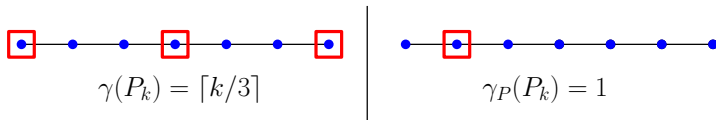


In **maximal planar graphs**: finding tight bounds is an **open problem**

[Matheson & Tarjan '96] For n sufficiently large, every **maximal planar graph** has $\gamma(G) \leq \alpha$, with $\frac{n}{4} \leq \alpha \leq \frac{n}{3}$.

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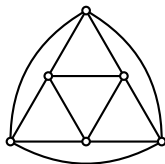
[Matheson & Tarjan '96] For n sufficiently large, every **maximal planar graph** has $\gamma(G) \leq \alpha$, with $\frac{n}{4} \leq \alpha \leq \frac{n}{3}$.

[Dorbec, González, P.'17+] For n sufficiently large, every **maximal planar graph** has $\gamma_P(G) \leq \beta$, with $\frac{n}{6} \leq \beta \leq \frac{n-2}{4} < \alpha$.

POWER DOMINATION IN MAXIMAL PLANAR GRAPHS

[Dorbec, González, P.'17+] For n sufficiently large, every maximal planar graph has $\gamma_P(G) \leq \beta$, with $\frac{n}{6} \leq \beta \leq \frac{n-2}{4} < \alpha$.

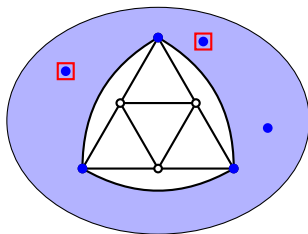
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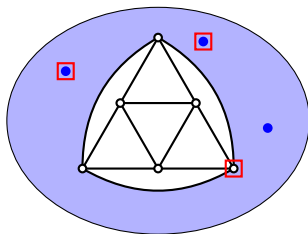
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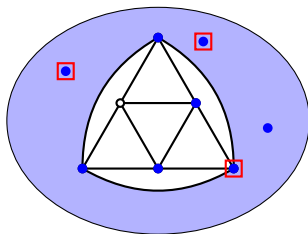
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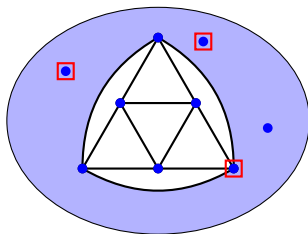
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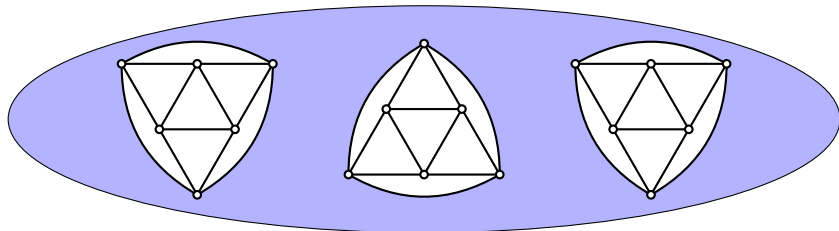
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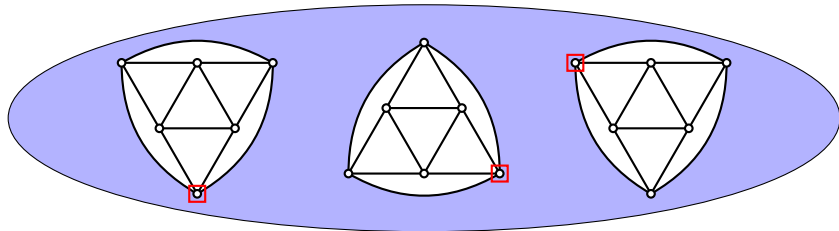
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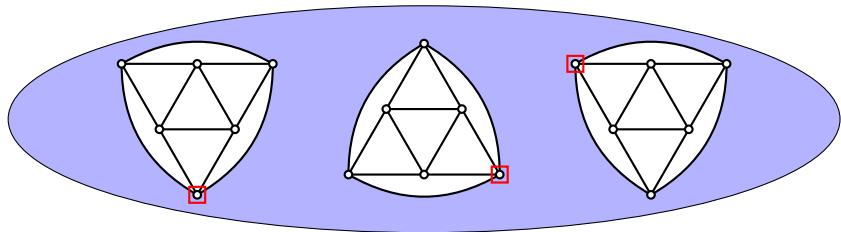
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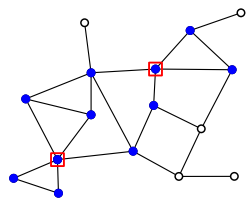
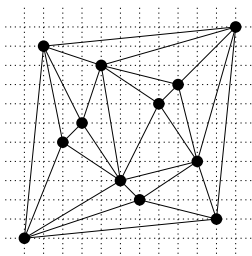
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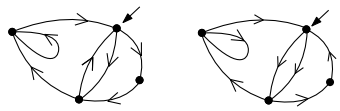
- Upper bound: Constructive process

- Non-aligned drawings ← Graph drawing



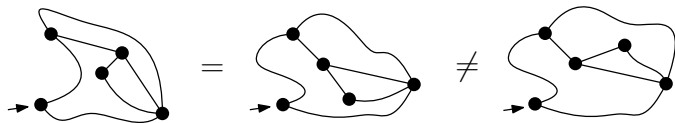
- Power domination ← Optimization problems

- Planar Eulerian orientations ← Enumerative combinatorics



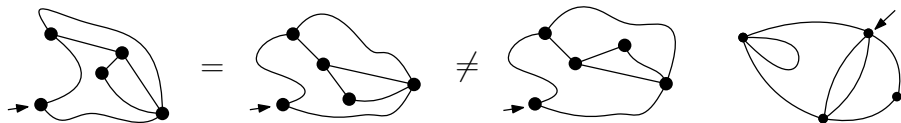
EULERIAN ORIENTATIONS

Planar maps, rooted in a corner



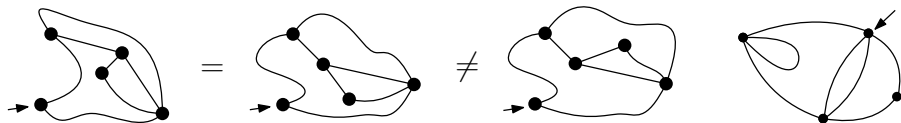
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Planar maps, rooted in a corner (loops and multiple edges allowed)



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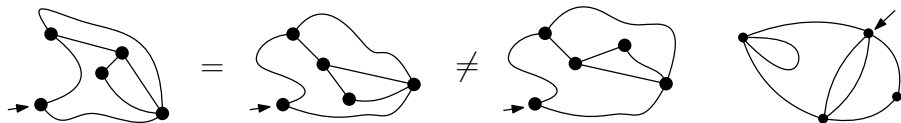
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A planar map is **Eulerian** if every vertex has **even degree**.

EULERIAN ORIENTATIONS

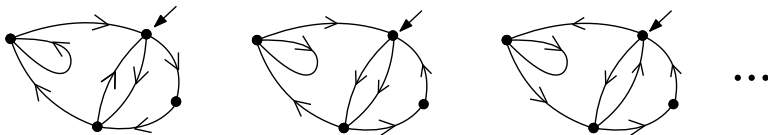
Planar maps, rooted in a corner (loops and multiple edges allowed)



A planar map is **Eulerian** if every vertex has **even degree**.

Maps + **structure**

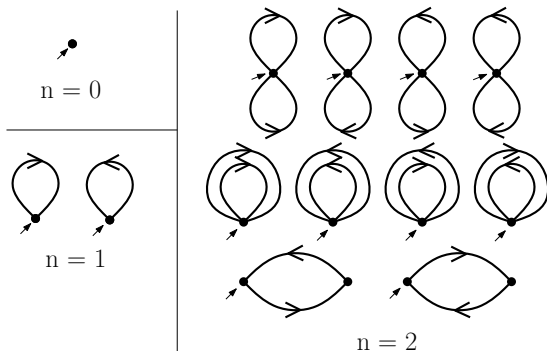
An oriented planar map is a **planar Eulerian orientation** (PEO) if every vertex has **in-degree and out-degree equal**.



EULERIAN ORIENTATIONS

Enumeration: "How many planar Eulerian orientations with n edges?"

→ determining the sequence $(a_n)_{n \geq 0}$

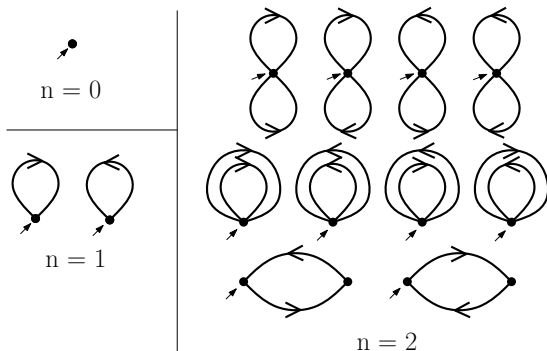


$$a_0 = 1, a_1 = 2, a_2 = 10, \dots$$

EULERIAN ORIENTATIONS

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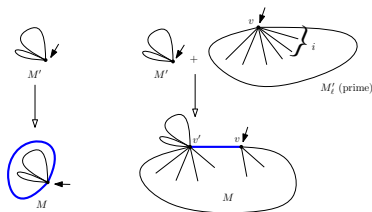
→ determining the sequence $(a_n)_{n \geq 0}$



$$a_0 = 1, a_1 = 2, a_2 = 10, \dots a_{37} = ?$$

EULERIAN ORIENTATIONS

New **recursive decomposition** of planar maps



Knowing the maps with **up to i edges** \rightarrow compute maps with **$i+1$ edges**

Computing the **first terms** until **a_{15}** \rightarrow OEIS A277493

n	a_n	n	a_n	n	a_n
0	1	6	37 548	12	37 003 723 200
1	2	7	350 090	13	393 856 445 664
2	10	8	3 380 520	14	4 240 313 009 272
3	66	9	33 558 024	15	46 109 094 112 170
4	504	10	340 670 720		
5	4 216	11	3 522 993 656		

EULERIAN ORIENTATIONS

Study the **growth rate** $\mu = \lim_{n \rightarrow \infty} a_n^{1/n}$

Eulerian maps \subset PEO \subset arbitrarily oriented Eulerian maps

$$\Rightarrow 8 \leq \mu \leq 16$$

EULERIAN ORIENTATIONS

Study the **growth rate** $\mu = \lim_{n \rightarrow \infty} a_n^{1/n}$

Eulerian maps \subset PEO \subset arbitrarily oriented Eulerian maps
 $\Rightarrow 8 \leq \mu \leq 16$

[Bonichon, Bousquet-Mélou, Dorbec, P. '17]

The growth rate μ of the planar Eulerian orientations satisfies
 $11.68 \leq \mu \leq 13.005$.

→ study families of **subsets** and **supersets**

- all have **computable algebraic** generating functions
- subsets behave like **trees**, (conjecture) supersets behave like **maps**?

CONCLUSION

Planar Eulerian orientations

Future research:

- continue the **study** of Eulerian orientations (find new decomposition?)
- apply the **same method** for families difficult to enumerate (polyominoes, self-avoiding walks, meanders?)

CONCLUSION

Power domination

[Dorbec, González, P. '17+] For n sufficiently large, every maximal planar graph has $\gamma_P(G) \leq \beta$, with $\frac{n}{6} \leq \beta \leq \frac{n-2}{4} < \alpha$.

Future research:

- study the complexity of the decision problems on maximal planar graphs
- other difficult problems: hypercubes?
- generalization of domination/power domination: constraint on the number of parallel propagation steps

CONCLUSION

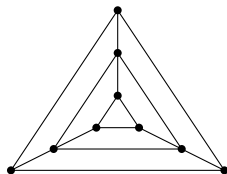
Non-aligned drawings

Not always possible on the minimal grid with straight-lines

Size of the grid	Numb. bends	Complexity	Comments
$n \times n$	$n - 3$	$O(n)$	Schnyder wood
$n \times n$	$\min \left\{ f_G, \frac{2n-5}{3} \right\}$	$O(n \log n) \dots$	filled triangles
$n \times O(n^3)$	0	$O(n^2)$	canonical ordering
$O(n^2) \times O(n^2)$	0	$O(n)$	Schnyder construction

Lower bound?

Maybe $(n + o(n)) \times (n + o(n))$ with $o(n)$ "bad" structural configurations

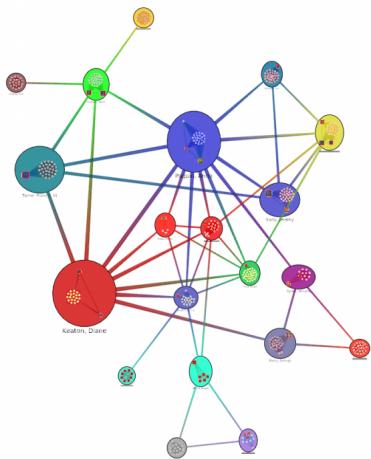


Thank you!

DRAWING GRAPHS

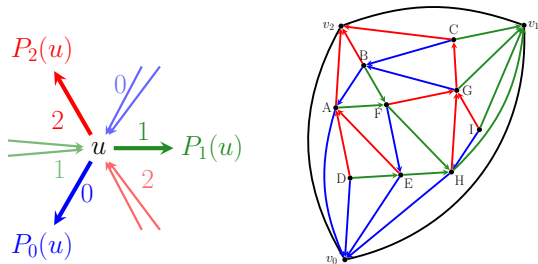
We want to draw **large** graphs with hierarchical view: a vertex in the drawing = a group of vertices in the graph

- preserve the relative positions of vertices → mental map



SCHNYDER WOODS

A **Schnyder wood** is a partition of the internal edges of a triangulation in three trees T_0 , T_1 and T_2 (directed toward the root) and with a particular configuration around each inner vertex:

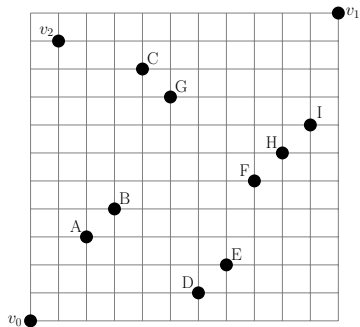
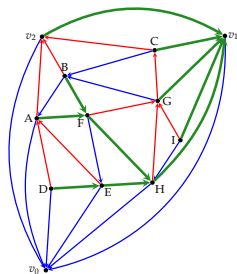


[Schnyder 1989]

Every plane triangulation admits at least one Schnyder wood, and it can be computed in linear time.

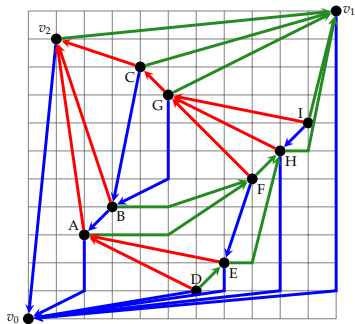
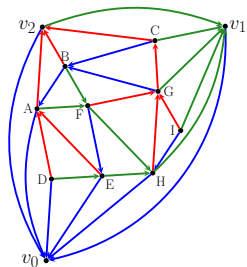
PLANAR POLYLINE ROOK-DRAWING - VERTICES

- (T_0, T_1, T_2) : Schnyder wood of G .
- $(v_1v_0), (v_2v_0), (v_2v_1)$
- x : clockwise preordering of $T_0 = \{v_0v_2ABCGDEFHIv_1\}$.
- y : clockwise postordering of $T_1 = \{DEABFHIGCv_2v_1\}$
($v_0 = 0$).



PLANAR POLYLINE ROOK-DRAWING - EDGES

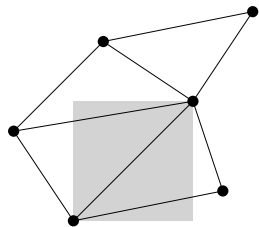
- The edges $(u, P_0(u))$ are bent at $(x(u), y(P_0(u)) + 1)$ (except for the first child in T_0)
- The edges $(u, P_1(u))$ are bent at $(x(\text{last descendant}_0(u)), y(u))$ (no bend if u is a leaf of T_0)
- Edges of T_2 : not bent



NON-ALIGNED DRAWINGS ON THE MINIMAL GRID

A **rectangle-of-influence (RI) drawing**:

- a straight-line planar drawing
- the minimum open rectangle containing u and v is empty if (u, v) is an edge

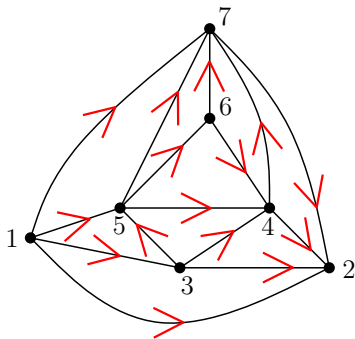


[Biedl et al. '99] If G is **4-connected**, and e is an edge of the outerface, then $G - e$ has a **planar non-aligned RI-drawing on the minimal grid**.

NON-ALIGNED DRAWING ON AN $(n - 1) \times O(n^3)$ -GRID

- 1 Canonical ordering $v_1 \cdots v_n$ of $V(G)$
- 2 (Acyclic) orientation of the edges:
 - $v_1 \rightarrow v_2$
 - $w \neq c_r$ pred. of v_k : $w \rightarrow v_k$
 - $v_k \rightarrow c_r$
- 3 Topological order $x: V \rightarrow \{1 \cdots n\}$ s.t. if $u \rightarrow v$ then $x(u) < x(v)$

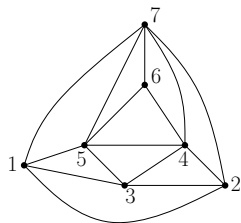
$$x(1) < x(3) < x(5) < x(6) < x(4) < x(7) < x(2)$$



NON-ALIGNED DRAWING ON AN $(n - 1) \times O(n^3)$ -GRID

$$x(1) < x(3) < x(5) < x(6) < x(4) < x(7) < x(2)$$

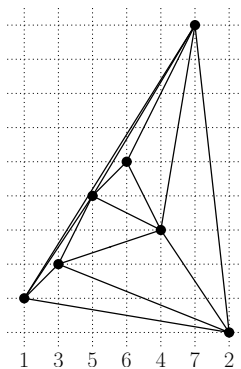
Place v_1 at $(1, 2)$, v_3 at $(x(v_3), 3)$, v_2 at $(n, 1)$
 $\rightarrow G_3$



Suppose $G_k = G[v_1 \cdots v_k]$ is drawn.

$y(v_{k+1})$ is the smallest possible such that:

- v_{k+1} can see all its predecessors
- the edge from c_ℓ has positive slope
- the row $\{y = y(v_{k+1})\}$ is empty



NON-ALIGNED DRAWING ON AN $(n-1) \times O(n^3)$ -GRID

Left-steepness of a vertex: $s(v) = \left| \frac{y(v) - y(c_\ell)}{x(v) - x(c_\ell)} \right|$

In the non-aligned drawing of G_k , $s(v_k) \leq \frac{(k-1)(k-2)}{2}$ for $k \geq 3$.

$$s(v_n) \leq \frac{1}{2}(n-1)(n-2)$$

$$c_\ell = v_1 \rightarrow x(v_n) - x(v_1) \leq n-2 \text{ and } y(v_1) = 2$$

$$y(v_n) \leq 2 + \frac{1}{2}(n-1)(n-2)^2 \leftarrow \text{Maximal height}$$

Every planar graph with n vertices has a non-aligned straight-line drawing in an $(n-1) \times \left(1 + \frac{1}{2}(n-1)(n-2)^2\right)$ grid.

NON-ALIGNED DRAWING ON AN $O(n^2) \times O(n^2)$ -GRID

Mapping $v \in V(G)$ to a point $(p_1(v), p_2(v), p_3(v))$

Lexicographic order: For vertices u, v and $i = 0, 1, 2$,
 $(p_i(u), p_{i+1}(u)) <_{lex} (p_i(v), p_{i+1}(v))$ if either $p_i(u) < p_i(v)$ or $p_i(u) = p_i(v)$ and $p_{i+1}(u) < p_{i+1}(v)$.

Weak barycentric representation of G :

- $p_0(v) + p_1(v) + p_2(v) = c$ for every vertex v
- for each edge (u, v) and each vertex $w \neq \{u, v\}$, there is k s.t.
 $(p_k(u), p_{k+1}(u)), (p_k(v), p_{k+1}(v)) <_{lex} (p_k(w), p_{k+1}(w))$.

[Schnyder 90] Every maximal planar graph G has a **straight-line planar drawing** on a grid with $n - 1$ rows and columns where **coordinates are given by a weak barycentric representation** of G .

NON-ALIGNED DRAWING ON AN $O(n^2) \times O(n^2)$ -GRID

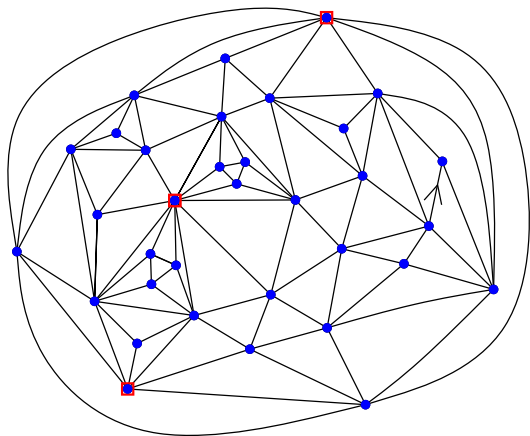
$p'_i(v) := (n-1) \times p_i(v) + p_{i+1}(v)$, for $i = 0, 1, 2$, is also a weak barycentric representation.

- [Schnyder 90] mapping vertices to $(p'_0(v), p'_1(v))$
→ **planar straight-line** drawing
- $1 \leq p_i(v) \leq n-2 \rightarrow p'_i(v) \leq (n-1)(n-2) + (n-2) = n(n-2)$
→ drawing on an $(n(n-2)-1 \times n(n-2)-1)$ -grid
- $p'_i(u) \neq p'_i(v)$ for any vertices u, v and any i → **non-aligned** drawing

Every planar graph with n vertices has a non-aligned straight-line drawing on an $(n(n-2)-1 \times n(n-2)-1)$ grid.

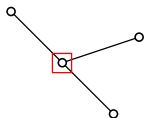
STEPS 1 AND 2

- **Step 1.** Monitor the octahedra with $l \leq n'/6$ sensors
- **Step 2.** While we can:
For every vertex v in \overline{M} in decreasing degree (in G) order: if adding v to S adds at least 4 vertices in M (with propagation): add v to S and propagate

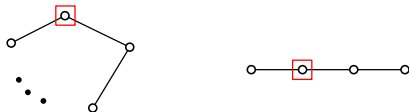


AFTER STEP 2

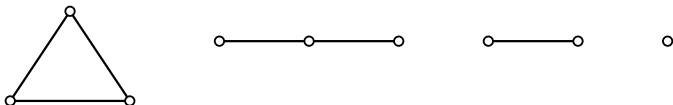
(a) $G[\overline{M}]$ has maximum degree 2 (it is a union of cycles or paths).



(b) Each connected component of $G[\overline{M}]$ has at most three vertices.

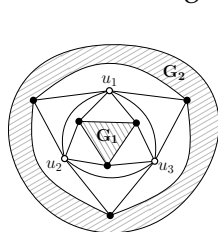


→ A connected component of $G[\overline{M}]$ is isomorphic to K_3 , P_3 , P_2 or K_1 .

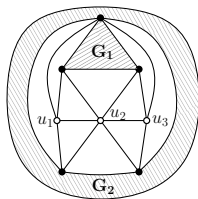


AFTER STEP 2

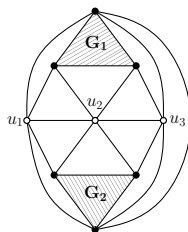
After Step 2, each non-monitored induced triangulation G' of G is isomorphic to one configuration of $\{C_1, \dots, C_7\}$:



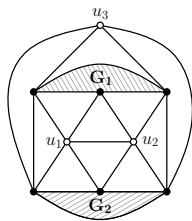
C_1



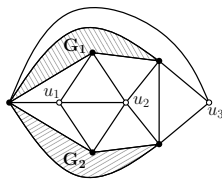
C_2



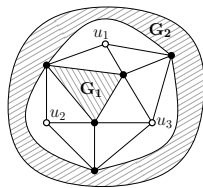
C_3



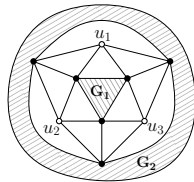
C_4



C_5



C_6



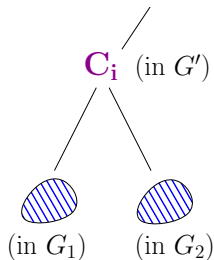
C_7

STEP 3

Bottom-up addition of vertices to S :

Induced triangulation G' with $C_i = \{u, v, w\}$

G_1, G_2 monitored with sets S_1, S_2



$$|S_1| \leq \frac{n_1 - 2}{4}, |S_2| \leq \frac{n_2 - 2}{4}$$

→ new set S' ($= S_1 \cup S_2 \cup \{v\}$)

$$|S'| \leq \frac{n_1 + n_2}{4} \text{ and } |G'| \geq n_1 + n_2 + 2 \rightarrow |S'| \leq \frac{|G'| - 2}{4}$$

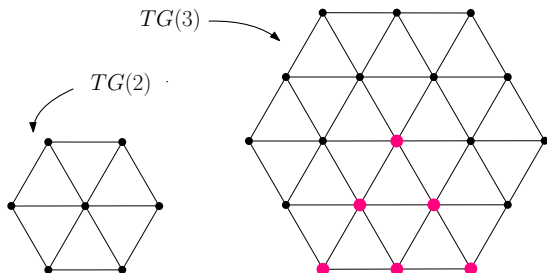
S has the good number of nodes after of Step 3.

POWER DOMINATION OF TRIANGULAR GRIDS

Another problem of power-domination: regular graphs

Hexagonal grids [Ferrero et al. '11], some products of paths [Dorfling and Henning '06, Dorbec et al. '08] ...

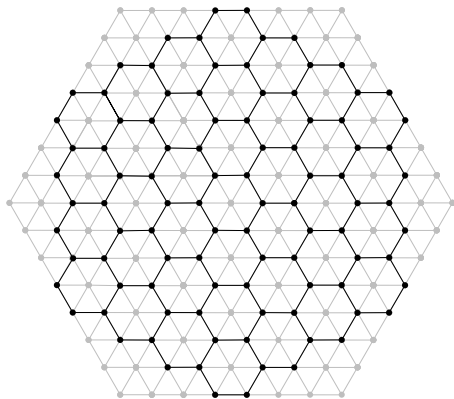
Triangular grid $TG(k)$: regular triangular grid with border of hexagonal shape (k vertices on each side)



One property: monitoring \mathcal{T} is sufficient!

RELATION WITH THE HEXAGONAL GRID

Hexagonal grid with dimension ℓ (number of hexagons per side) is an induced subgraph of $TG(2\ell)$...



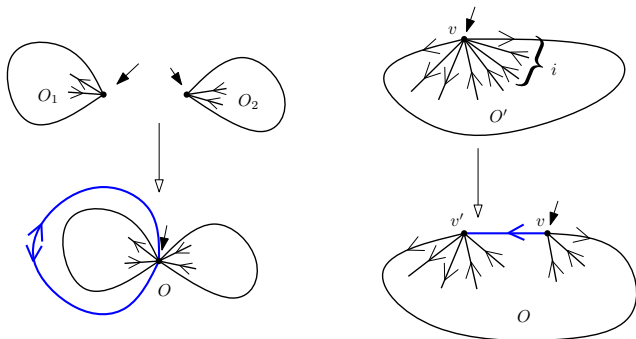
... and $\gamma_P(HG(\ell)) = \gamma_P(TG(2\ell))$ (if our conjecture is true)

When/why does this happen? May be interesting to look to induced regular grids?

DECOMPOSITION OF PEO

Two ways of creating a PEO:

- merge two PEOs O_1, O_2 and orient the new edge
- split the root-vertex at index i iff the resulting map is still a PEO



Splits at index 1 or $\Delta - 1$ are always possible; oth. we must check!
Remember the **full orientation** around the root: **no recurrence relation**
with a finite number of parameters

APPROXIMATION OF THE GROWTH RATE

μ = growth rate of PEOs = $\lim_{n \rightarrow \infty} o(n)^{1/n}$

Merging two PEOs with n and n' edges gives a PEO with $n + n'$ edges

→ $\{o(n)\}_{n \geq 0}$ is **super-multiplicative**, i.e. $o(n + n') \geq o(n) o(n')$.

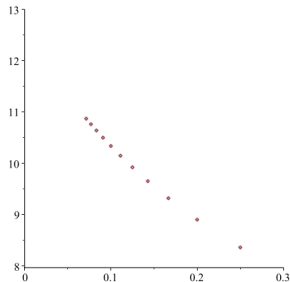
Variant of Fekete's Lemma (1923): $\mu = \sup_{n \geq 1} o(n)^{1/n} \in \mathbb{R}_+^*$

⇒ $\mu \geq (o(15))^{1/15} \sim 8.145525470$

PEO \subset arbitrary orientations of
Eulerian maps

⇒ $8.14 < \mu < 16$

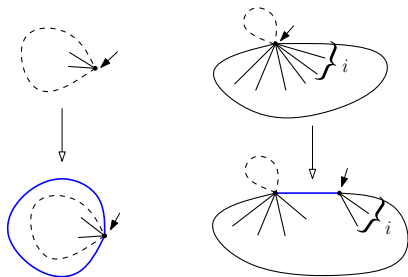
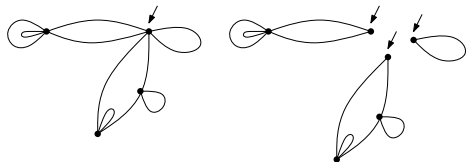
$\frac{o(n+1)}{o(n)}$ as a function of $1/n \rightarrow$



PRIME DECOMPOSITION OF MAPS

A map is **prime** if the root-vertex appears **exactly once** on the root-face.

Planar map = concatenation of prime maps



Operations to create a prime map:

- Add a loop around any map
- Split at index $i \leq \Delta(P_\ell)$ in the last prime P_ℓ of any map

SUBSETS (AND SUPERSSETS) OF \mathcal{O}

Two families of sets of orientations \mathcal{O}_k^- and \mathcal{O}_k^+ s.t.

$$\mathcal{O}_k^- \subset \mathcal{O}_{k+1}^- \subset \mathcal{O} \subset \mathcal{O}_{k+1}^+ \subset \mathcal{O}_k^+$$

Definition

A map of \mathcal{O}_k^- is obtained by either:

- a concatenation of prime maps of \mathcal{O}_k^- ,
- adding a loop around a map $O \in \mathcal{O}_k^-$ and orienting it,
- a split on the last prime component P_ℓ of a map $P_1 \dots P_\ell \in \mathcal{O}_k^-$ at index $i < 2k$ or $i = \Delta(P_\ell) - 1$.

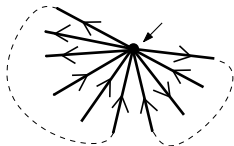
The atomic map (one vertex, no edges) is in \mathcal{O}_k^- .

Fewer splits allowed \rightarrow the number of orientations necessary to look at form now a word of finite length, which we can use as a parameter

ALGEBRAIC SYSTEM FOR $\mathcal{O}^{(k)-} \equiv \mathcal{O}^-$

The **root-word** $w(O)$ of a map O is the binary word formed as follows in counterclockwise order around the root-vertex:

- 1 if there is an out-edge,
- 0 if there is an in-edge.



1110000101

A word w is **balanced** iff $|w|_0 - |w|_1 = 0$.

$F_w(t)$: g.f. of the set $\{O \in \mathcal{O}^- \mid \mathbf{w}(O) = w\}$

$L_w(t)$: g.f. of the set $\{O \in \mathcal{O}^- \mid \mathbf{w}(O) = uw \text{ for some } u\}$

$F'_w(t)$, $L'_w(t)$: their counterparts for **prime** maps of \mathcal{O}^- .

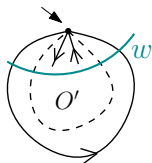
AN EXAMPLE: EQUATION FOR $F'_w(t)$

Prime oriented maps of \mathcal{O}^- with root-word w .

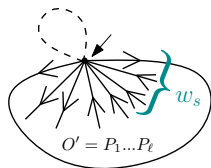
w_s : maximal proper suffix of w , w_c : central factor of w ($w = \alpha w_c \bar{\alpha}$)

For w balanced, $2 \leq |w| \leq 2k$:

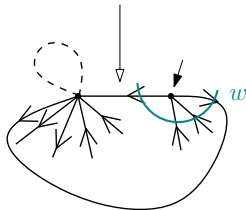
$$F'_w = tF_{w_c} + tL_\varepsilon L'_{w_s}$$



$$\mathbf{w}(O') = w_c$$



w_s is a suffix of $\mathbf{w}(P_\ell)$



ALGEBRAIC SYSTEM FOR $\mathcal{O}^{(k)}$

$$\left\{ \begin{array}{l}
 F_w = \sum_{w=uv} F_u F'_v \quad |w| \leq 2k - 2 \\
 L_w = \begin{cases} L_\varepsilon L'_w + \sum_{w=uv, u \neq \varepsilon} L_u F'_v & |w| \leq 2k - 2 \\
 1 + L_\varepsilon L'_\varepsilon & w = \varepsilon \end{cases} \\
 F'_w = tF_{w_c} + tL_\varepsilon L'_{w_s} \quad |w| \leq 2k \\
 L'_w = \begin{cases} tL_{w_p} + tF_{w_c} + tL_\varepsilon L'_w + \\
 tL_\varepsilon \sum_{\substack{u=vw \\ u \text{ balanced} \\ 0 < |u| \leq 2k}} (L'_u - F_u) + tL_\varepsilon (L'_{w'} - F'_{w'}) & |w| \leq 2k - 2 \\
 2tL_\varepsilon + tL_\varepsilon L'_\varepsilon & w = \varepsilon \end{cases}
 \end{array} \right.$$

$$w = \varepsilon \Rightarrow F_w = 1, F'_w = 0$$

$$w \text{ non-balanced} \Rightarrow F_w = F'_w = 0.$$

SMALL EXAMPLE: SUBSETS, $k = 2$

$$\left\{ \begin{array}{l}
 F_{01} = F_{10} = F'_{01}, \\
 F'_{10} = F'_{01} = t + tL_\varepsilon L'_1, \\
 F'_{1100} = tF_{10} + tL_\varepsilon L'_{100}, \\
 F'_{1010} = tF_{01} + tL_\varepsilon L'_{010}, \\
 F'_{0110} = tL_\varepsilon L'_{110}, \\
 L_\varepsilon = 1 + L_\varepsilon L'_\varepsilon, \\
 L_0 = L_1 = L_\varepsilon L'_0, \\
 L_{00} = L_{11} = L_\varepsilon L'_{00}, \\
 L_{01} = L_{10} = L_\varepsilon L'_{01}, \\
 L'_\varepsilon = 2tL_\varepsilon + tL_\varepsilon(L'_\varepsilon + 2(L'_0 - F'_{10} + L'_{100} - F'_{1100} + L'_{010} - F'_{1010} + L'_{110} - F'_{0110})), \\
 L'_0 = L'_1 = tL_\varepsilon + tL_\varepsilon(L'_0 + L'_0 - F'_{10} + L'_{100} - F'_{1100} + L'_{010} - F'_{1010} + L'_{110} - F'_{0110}), \\
 L'_{00} = tL_0 + tL_\varepsilon(L'_{00} + L'_{100} - F'_{1100}), \\
 L'_{10} = L'_{01} = tL_1 + t + tL_\varepsilon(L'_{10} + L'_1 - F'_{01} + L'_{010} - F'_{1010} + L'_{110} - F'_{0110}), \\
 L'_{100} = tL_{10} + tL_\varepsilon(L'_{100} + L'_{100} - F'_{1100}), \\
 L'_{010} = tL_{01} + tL_\varepsilon(L'_{010} + L'_{010} - F'_{1010}), \\
 L'_{110} = tL_{11} + tL_\varepsilon(L'_{110} + L'_{110} - F'_{0110}).
 \end{array} \right.$$

FINDING L_ε

Generate the systems automatically then **eliminate** the variables with Maple (keeping L_ε)

$k \geq 4$: find the first terms using the **Newton GF package**

nature	k	degree	growth rate
inf	1	3	10.60
inf	2	6	10.97
inf	3	20	11.22
inf	4	258	11.44 ^(*)
inf	5	–	11.56 ^(*)
inf	6	–	11.68 ^(*)
PEO	–	–	?

(*) not proven, use of quadratic approximants

For each $k > 0$, $o_k(n) \sim \gamma n^{-3/2} \rho^{-n}$ (ρ and γ depend on k).

Let μ_k^- be the growth rate of the set \mathcal{O}_k^- . Then $\mu_k^- \rightarrow_{k \rightarrow \infty} \mu$.

SUPERSETS OF PEO

General idea: allowing splits at indices $i \geq \Delta(P_\ell)$, creating **non Eulerian orientations**

One **catalytic variable** x (for the half-degree of the root)

Same kind of systems, but with **divided differences** !

For $k = 1$:

$$\begin{cases} L_\varepsilon(t, x) &= 1 + L_\varepsilon(t, x)L'_\varepsilon(t, x), \\ L'_\varepsilon(t, x) &= 2txL_\varepsilon(t, x) + tL_\varepsilon(t, 1) \left(2xL'_0(t, 1) + \frac{x}{x-1} (L'_\varepsilon(t, x) - xL'_\varepsilon(t, 1)) \right), \\ L'_0(t, x) &= txL_\varepsilon(t, x) + tL_\varepsilon(t, 1) \left(xL'_0(t, 1) + \frac{x}{x-1} (L'_0(t, x) - xL'_0(t, 1)) \right). \end{cases}$$

SUPERSETS OF PEO

The supersets of PEO have **algebraic** generating functions.

Conjecture

For each $k > 0$, $o_k(n) \sim \gamma n^{-5/2} \rho^{-n}$ (ρ and γ depend on k).

nature	k	degree	growth rate
PEO	–	–	?
sup	5	–	13.005 ^(*)
sup	4	–	13.017 ^(*)
sup	3	–	13.031 ^(*)
sup	2	28	13.047
sup	1	3	13.065

(*) not proven, use of quadratic approximants