Circuit-switched gossiping in the 3-dimensional torus networks

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Abstract: in this paper we describe, in the case of short messages, an efficient gossiping algorithm for 3-dimensional torus networks (wrap-around or toroidal meshes) that uses synchronous circuit-switched routing. The algorithm is based on a recursive decomposition of a torus. The protocol requires an optimal number of rounds and a quasi-optimal number of intermediate switch settings to gossip in an $7^i \times 7^i \times 7^i$ torus.

Key-words: circuit-switching, gossiping, torus network, toroidal mesh, linear coding theory.

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Echange total en mode commutation de circuits dans les tores de dimension 3

Résumé : dans cet article nous décrivons, pour des messages courts, un algorithme d'échange total efficace dans les réseaux toriques (grilles toriques) de dimension 3 qui utilise un routage par commutation de circuits synchone. L'algorithme est basé sur une décomposition récursive du tore. Le protocole requiert un nombre d'étapes optimal et un nombre de commutateurs intermédiaires quasi-optimal pour diffuser dans le tore $7^i \times 7^i \times 7^i$.

Mots-clé : commutation de circuits, échange total, tore, grille torique, théorie des codes linéaires.

1 Introduction

We consider here algorithms on parallel or distributed memory multicomputer systems in which the processors communicate by exchanging messages over an interconnection network. We are interested in global (or structured) communication protocols as these protocols appear as basic routines in many applications for example in linear algebra [3, 4, 17] or in image processing [28]. Two basic protocols are broadcasting (a processor sends its message to all the other) and gossiping (in which a distinct message originated at each processor must be distributed to all the other processors). These algorithms strongly depend of the routing mechanism used and of the topology of the network (see [31]). We consider the "circuit-switched routing mode"; under this name we include the currently used mechanisms of routing like wormhole [32], virtual cut-through [22] or direct connect [29] routing. For example, the wormhole routing is used in the IBM SP2, the Intel Paragon or in the two last generations of Cray machines (T3D and T3E). With this kind of routing the efficiency of the protocol is less depending of the distance between processors than in the classical store and forward routing model. So simple and regular topologies like torus networks (or meshes) well adapted to numerical applications can be used. For example, the Cray T3D and T3E use a 3−dimensional torus structure. Our aim in this paper is to give an efficient gossiping algorithm under the circuit-switched mode in the 3−dimensional torus networks. Note that the same problem for the 2−dimensional torus networks was solved in [1] and other results on circuit-switched structured communications in torus can be found in [33].

This paper is organized as follows. In sections 1.1 to 1.4 we describe the different hypothesis used in this paper. Then in section 1.5 we give some definitions and notations before to establish in section 2 lower bounds for gossiping protocols in circuit-switched routing. In section 3 we will study the gossiping protocol in the 3−dimensional torus network. We give the protocol for the $7^i \times 7^i \times 7^i$ torus network. Note that we study the case of power of 7 because in our model this is the extremal case in terms of numbers of processors which can receive a message originating at one processor in a given number of rounds. The proof uses recursion and is based on a detailed study of the $7 \times 7 \times 7$ torus network. The number of rounds of the protocols matches the lower bound established in section 2. In the last part we discuss the case of large messages.

1.1 The circuit-switched model

In the circuit-switched mode, the principle is to establish between ordered pairs of nodes which must communicate, a "circuit" which will correspond to a dipath in the network between the source and the destination. As said before, this model regroup different kinds of routings like wormhole, virtual cut-through or direct connect routing. Although there exist important differences between these mechanisms (like acknowledgment, use of small or large intermediate buffer size, ...), these differences are too small and too difficult to analyze to be taken into account in a simple general modelization and to evaluate the performance of protocols. When any type of circuit-switched routing is used, and communication patterns are arbitrary, deadlock is possible and many papers on wormhole routing are more concerned with deadlock avoidance than efficiency [5, 9, 10, 25, 26]. The most common deadlock avoidance method is the use of virtual channels which uses multiplexing to share physical links.

Here, in our "circuit-switched model", we will design a global protocol like a succession of rounds. During each round, there are communications between ordered pairs of nodes. In order to avoid blocking and loss of messages, we will suppose that during a given round the dipaths realizing the communications are pairwise arc-disjoint (note that some authors consider a stronger constraint as they ask for vertex-disjoint dipaths [11]). Moreover, we will suppose that our protocol is somewhat "synchronous", that is a round can start only when the communications of the precedings one are finished.

1.2 Communication time

In the "circuit-switched model" we will consider that the transmission time for a message of length L to be sent from a processor x to a processor y along a dipath of length l is $\alpha+l\delta+L\tau$ (linear cost model), where α is the time to initiate a new message transmission, δ is the time to switch an intermediate node, and $1/\tau$ is the bandwidth of the communications links. In most current machines, message transmissions are initiated in software and switching is done in hardware, so δ is usually much smaller than α. Furthermore, α is usually much larger than τ. In this paper we will suppose that the initial message length L is small. For long messages, one can use the pipelining techniques developed in the store and forward mode (for example see the survey [13]).

1.3 Network constraint

We will modelize our interconnection network by a symmetric digraph $G = (V(G), A(G))$, where the set $V(G)$ represent the processors and the set of arcs $A(G)$ the links between the processors (more exactly between the output and input buffers of the switches). We will use the link-bounded [13] (or shouting [15] or all-port or $\Delta-$port) model of communication in which a processor can use all of its communication links simultaneously. We also assume that the communication links are full-duplex so that messages can travel in both directions simultaneously. Furthermore, we assume that each node has an initial distinct message, but all these messages have the same small length L. Finally, we allow messages to be concatenated with negligible cost.

Within this model (when the goal is first to minimize the number of rounds), broadcasting has been studied for different networks like hypercube [18, 24], $2-$dimensional torus [30], $k-$dimensional torus [7, 8] or Butterfly network [6]. For gossiping the problem is much harder and to our knowledge it has been studied only for hypercube [14] and $2-$dimensional torus network [1].

1.4 Gossiping protocol

Let G be a symmetric digraph. Then in our circuit-switched model, we can define the cost of a gossiping protocol as:

Notation 1.1 The total time necessary to achieve a gossiping protocol *in a digraph G will be denoted by $g(G) = g_\alpha(G)\alpha + g_\delta(G)\delta + g_\tau(G)\tau$ where $g_\alpha(G)$ is the number of rounds of the protocol, $g_\delta(G)$ is the sum of the maximum communication distances of each ordered pairs of processors used in each round of the protocol and $g_\tau(G)$ is the flow of information.*

As said before, here we don't use pipeline techniques and consider only short messages (or equivalently suppose $\tau << \alpha$ and $\tau << \delta$). **So, we are mainly interested in determining the optimal $g_\alpha(G)$ and for an optimal $g_\alpha(G)$ we search to minimize the $g_\delta(G)$ parameter.**

For $g_\delta(G)$ a trivial lower-bound is the diameter $D(G)$. In section 2 we give a new no-trivial lower-bound for $g_\alpha(G)$.

1.5 Definitions

In this article, we will use the following definitions and notations.

- $\mathbb{Z}_q$ will denote the **additive group of integers modulo q**. In the following, when we use the set $\mathbb{Z}_{2k+1}$, the elements of this set will be taken in $\{-k, \ldots, -1, 0, 1, \ldots, k\}$.
- G will denote a **digraph** with vertex set $V(G)$ and arc set $A(G)$.
- N will denote the **number of vertices** of G, that is $N = |V(G)|$.
- $d_G(x, y)$ will denote the **distance** from a vertex $x \in V(G)$ to a vertex $y \in V(G)$ that is the length of the shortest dipath from x to y.
- $D(G)$ will denote the **diameter** of a digraph G, that is the maximum of the distances between every ordered pair of vertices: $D(G) = \max_{(x,y)\in V^2(G)} d_G(x, y)$.
- In a symmetric digraph G, $\Delta(G)$ (or Δ for short) will denote **maximum in-degree (or out-degree)** of G, that is the maximum over the in-degrees (or out-degrees) of all vertices $V(G)$.
- C_N will denote the symmetric directed cycle of order N.

Definition 1.2 *The* **cartesian sum** *(also called* **cartesian product** *or* **box product***) denoted by $G\Box G'$ of two digraphs $G = (V, A)$ and $G' = (V', A')$, is the digraph whose vertices are the ordered pairs (x, x') where x is a vertex of G and x' is a vertex of G'. The vertex (y, y') is a successor of the vertex (x, x') in $G\Box G'$ if and only $x = y$ and (x', y') is an arc of G', or if $x' = y'$ and (x, y) is an arc of G.*

Definition 1.3 **The $k-$dimensional torus** *is the cartesian sum of k symmetric directed cycles of orders $p_1, p_2, \ldots, p_k$ and is denoted by $TM(p_1, p_2, \ldots, p_k) = C_{p_1} \square C_{p_2} \square \cdots \square C_{p_k}$.*

Remark 1.4 If all $p_i \geq 3$, $TM(p_1, p_2, \ldots, p_k)$ is a regular digraph of degree $\Delta = 2k$. Its order is $p_1 \times p_2 \times \cdots \times p_k$, the number of arcs is $2kN$ and its diameter is $\sum_{i=1}^{k} \lfloor \frac{p_i}{2} \rfloor$.

Notation 1.5 *When $p_1 = p_2 = \cdots = p_k$, we will use the abbreviated notation $TM(p)^k$ and suppose in what follows that $p \geq 3$.*

Remark 1.6 **The $k-$dimensional torus** is a vertex-transitive digraph where each vertex x of the $TM(p_1, p_2, \ldots, p_k)$ digraph can be seen as a k-tuple vector $(x_1, x_2, \ldots, x_k) \in \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_k}$. Each vertex $(x_1, x_2, \ldots, x_k)$ (or k-tuple vector) is joined by an arc to the $2k$ vertices (or k-tuple vectors) $(x_1, x_2, \ldots, x_i \pm 1, \ldots, x_k) \in \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_i} \times \cdots \times \mathbb{Z}_{p_k}$ for $1 \leq i \leq k$.

Definition 1.7 *A* **routing function** *R for a digraph G is a set of $N(N-1)$ dipaths $R = \{R(x,y) | x, y \in V(G)\}$, where $R(x,y)$ is a dipath in G from x to y.*

Definition 1.8 *Given a routing function R for the digraph G, the* **load of an arc** *$u \in A(G)$, denoted by $\pi(G, R, u)$, is the number of dipaths of R going through u.*

Definition 1.9 *Given a routing function R for the digraph G, the* **arc-forwarding index of** *(G, R), denoted by $\pi(G, R)$, is the maximum number of dipaths of R going through any arc of G, that is $\pi(G, R) = \max_{u \in A(G)} \pi(G, R, u)$.*

Definition 1.10 *The* **arc-forwarding index of the digraph** *G, denoted by $\pi(G)$, is defined as $\pi(G) = \min_R \pi(G, R)$.*

2 Lower bounds

We first give some additional definitions and notations.

Definition 2.1 *Given a gossip protocol in a digraph G, we will say that the ordered pair $(x, y) \in V(G) \times V(G)$* **uses the arc** *$u \in A(G)$* **at round** *$t$ if a message originated from x and finally reaching y goes through the arc u at round t of the gossip protocol.*

Definition 2.2 *Given a gossip protocol in a digraph G, the* **gossip load** *of an arc $u \in A(G)$ at round t denoted $\mathcal{GL}(u, t)$ is the number of couples $(x, y) \in V^2(G)$ using the arc u at round t of the gossip protocol.*

Definition 2.3 *Given a gossip protocol in a digraph G completed at round $g_\alpha(G)$, the* **total gossip load** *of an arc $u \in A(G)$ noted $\mathcal{TGL}(u)$ is defined as $\mathcal{TGL}(u) = \sum_{t=1}^{g_\alpha(G)} \mathcal{GL}(u, t)$.*

Proposition 2.4 *For a gossip protocol in a digraph G, there exists an arc u such that $\mathcal{TGL}(u) \geq \pi(G)$.*

Proof. Any gossip algorithm constructs at least a dipath from each vertex to each other one. By choosing for each ordered pair (x, y) one of these dipaths, we can associate with a gossip algorithm a routing R for the digraph G. For any dipath created from x to y by the gossip algorithm and for any arc u of the dipath there exists a round t such that the couple (x, y) uses the arc u at round t. Hence the total gossip load of any arc u is at least the load of u for the routing R. It remains to observe that there exists an arc of G whose load for the routing R is at least $\pi(G)$. □

Let G be a digraph of maximum degree Δ, we calculate an upper bound on the total gossip load of an arc $u \in A(G)$, for any gossip protocol with $g_\alpha(G)$ rounds.

Proposition 2.5 *Let G be a digraph with maximum degree Δ and order N. Let $t_0 = \lceil \log_{\Delta+1}(N) \rceil$. If $g_\alpha(G) \leq 2t_0$, then $\forall u \in A(G)$:*

$$\mathcal{TGL}(u) \quad \leq \quad N\left[\left(\frac{2}{\Delta} + 2t_0 - g_\alpha(G)\right) \cdot (\Delta+1)^{g_\alpha(G)-t_0} - \frac{2}{\Delta}\right]$$

Proof. The total gossip load of an arc u is the sum of the gossip load of u at round t for all the rounds $t \in \{1, \ldots, g_\alpha(G)\}$. At the end of the round $t-1$ a vertex knows at most $\mathcal{I}(t-1) = \min((\Delta+1)^{t-1}, N)$ pieces of information and so at round t at most $\min((\Delta+1)^{t-1}, N)$ pieces of information can go through arc u. After round t the information which has gone through u will be able to reach at most $\mathcal{I}(g_\alpha(G)-t) = \min((\Delta+1)^{g_\alpha(G)-t}, N)$ nodes. Hence the gossip load of u at round t cannot exceed $\mathcal{I}(t-1) \cdot \mathcal{I}(g_\alpha(G)-t) = \min((\Delta+1)^{t-1}, N) \cdot \min((\Delta+1)^{g_\alpha(G)-t}, N)$. So, as $(\Delta+1)^{t_0} \geq N$ and $(\Delta+1)^{t_0-1} < N$ (and we suppose that $g_\alpha(G) \leq 2t_0$, that is $g_\alpha(G) - t_0 \leq t_0$), we have the following array:

Round t	$\mathcal{I}(t-1)$	$\mathcal{I}(g_\alpha(G)-t)$	$\mathcal{GL}(u,t)$
$\{1, \ldots, g_\alpha(G) - t_0\}$	$(\Delta+1)^{t-1} < N$	$(\Delta+1)^{g_\alpha(G)-t} \geq N$	$N(\Delta+1)^{t-1}$
$\{g_\alpha(G) - t_0 + 1, \ldots, t_0\}$	$(\Delta+1)^{t-1} < N$	$(\Delta+1)^{g_\alpha(G)-t} < N$	$(\Delta+1)^{g_\alpha(G)-1}$
$\{t_0 + 1, \ldots, g_\alpha(G)\}$	$(\Delta+1)^{t-1} \geq N$	$(\Delta+1)^{g_\alpha(G)-t} < N$	$N(\Delta+1)^{g_\alpha(G)-t}$

Hence, the number of the dipaths on the arc u is:

$$\begin{aligned}
\mathcal{TGL}(u) &\leq N \sum_{i=0}^{g_\alpha(G)-t_0-1} (\Delta+1)^i + (2t_0 - g_\alpha(G))(\Delta+1)^{g_\alpha(G)-1} + N \sum_{i=0}^{g_\alpha(G)-t_0-1} (\Delta+1)^i \\
&\leq 2N \sum_{i=0}^{g_\alpha(G)-t_0-1} (\Delta+1)^i + (2t_0 - g_\alpha(G))(\Delta+1)^{g_\alpha(G)-1} \\
&\leq 2N \frac{(\Delta+1)^{g_\alpha(G)-t_0} - 1}{\Delta} + (2t_0 - g_\alpha(G))(\Delta+1)^{g_\alpha(G)-t_0}(\Delta+1)^{t_0-1}
\end{aligned}$$

as $(\Delta+1)^{t_0-1} < N$, we obtain:

$$\mathcal{TGL}(u) \quad \leq \quad N\left((\Delta+1)^{g_\alpha(G)-t_0}\left(\frac{2}{\Delta}+2t_0-g_\alpha(G)\right)-\frac{2}{\Delta}\right)$$

□

Consequently for any digraph we have the following theorem.

Theorem 2.6 *Let G be a digraph with maximum degree Δ and order N, and let $t_0 = \lceil \log_{\Delta+1}(N) \rceil$. If $g_\alpha(G) \leq 2t_0$ then*

$$g_\alpha(G) \quad \geq \quad t_0 + \log_{\Delta+1}\left(\frac{\pi(G)}{N}\right) - O\left(\log_{\Delta+1}\log_{\Delta+1} N\right)$$

Proof. By the proposition 2.4 we have,

$$\begin{aligned} N\left((\Delta+1)^{g_\alpha(G)-t_0}\left(\frac{2}{\Delta}+2t_0-g_\alpha(G)\right)-\frac{2}{\Delta}\right) &\geq \pi(G) \\ (\Delta+1)^{g_\alpha(G)-t_0}\left(\frac{2}{\Delta}+2t_0-g_\alpha(G)\right) &\geq \frac{\pi(G)}{N} \end{aligned}$$

When we take the logarithm in base $\Delta+1$ we obtain,

$$g_\alpha(G) - t_0 + \log_{\Delta+1}\left(\frac{2}{\Delta}+2t_0-g_\alpha(G)\right) \geq \log_{\Delta+1}\left(\frac{\pi(G)}{N}\right)$$

$$\begin{aligned} g_\alpha(G) &\geq t_0 + \log_{\Delta+1}\left(\frac{\pi(G)}{N}\right) - \log_{\Delta+1}\left(\frac{2}{\Delta}+2t_0-g_\alpha(G)\right) \\ &\geq t_0 + \log_{\Delta+1}\left(\frac{\pi(G)}{N}\right) - O\left(\log_{\Delta+1} t_0\right) \end{aligned}$$

□

In order to exploit the bound above we can use the lower bound on the arc-forwarding index.

Proposition 2.7 (M.C. Heydemann, J.C. Meyer, D. Sotteau [16]) *The arc-forwarding index for the $TM(n)^k$ symmetric digraph is $\pi(TM(n)^k) = \frac{n^{k-1}}{2}\lfloor\frac{n^2}{4}\rfloor$.*

Now, we are able to state the following corollary.

Corollary 2.8 *Given a gossip protocol in the $TM(n)^k$ digraph of degree $\Delta = 2k$, the number of rounds necessary to achieve this protocol is:*

$$g_\alpha(G) \geq (k+1)\log_{2k+1}(n) - O\left(\log_{2k+1}\log_{2k+1} n\right)$$

Proof. This corollary is correct as in [2, 7, 8] it has been shown that the total number of rounds to achieve a broadcasting (and the inverse as gathering) protocol in the $TM(n)^k$ digraph is t_0. Then a trivial gossiping protocol will be the concatenation of a gathering and a broadcasting protocol, so $g_\alpha(G) \leq 2t_0$. □

Remark 2.9 There exists a relationship between the arc-forwarding index and the vertex bisection, since in [23] R. Klasing established a lower bound of the same kind (in a model named two-way vertex-disjoint paths mode) by using vertex bisection width.

3 Gossiping in the 3-dimensional torus TM$(7^i)^3$

3.1 Case of TM$(7)^3$

Gossiping can be derived from the broadcast protocol given in [7, 8] and the forthcoming paper [2]. We will recall briefly the principle of this broadcast protocol. In the following, we will make use of the following notation and definitions.

- Here G denotes the $TM(7)^3$ symmetric digraph.
- According to remark 1.6 we will consider vertices of G as elements of the 3-dimensional vector-space $\mathbb{Z}_7^3$, with canonical base $\{e_1, e_2, e_3\}$. Vertex $(x_1e_1 + x_2e_2 + x_3e_3)$ will be denoted $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.
- M will be a 3-dimensional matrix. For a given set U of vectors, MU will denote the image of U by $M : \{Mx \mid x \in U\}$.
- The sum of two sets of vectors U_1 and U_2 will be $U_1 + U_2 = \{x \mid x = u_1 + u_2, u_1 \in U_1, u_2 \in U_2\}$.

Definition 3.1 *$\mathcal{E}$ is the set of the 3-dimensional vector-space $\mathbb{Z}_7^3$ associated with all the vertices of G that is $\forall x \in V(G)$.*

Definition 3.2 *We will denote by $\mathcal{B}_1$ the set $\{e_1, e_2, e_3, 0, -e_1, -e_2, -e_3\}$.*

Remark 3.3 Note that $\mathcal{B}_1$ is the sphere of radius 1 centered on zero of $\mathbb{Z}_7^3$ for the Lee distance (see [27]). As $x + \mathcal{B}_1$ is the sphere of radius 1 centered on x, it also contains the neighbors of x in G union x (see figure 1).

Definition 3.4 *The* **code** *$\mathcal{C}$ is the set of vertices such that $\mathcal{C} = \{(x_1, x_2, x_3) \in \mathbb{Z}_7^3 | x_1 + 2x_2 + 3x_3 = 0\}$*

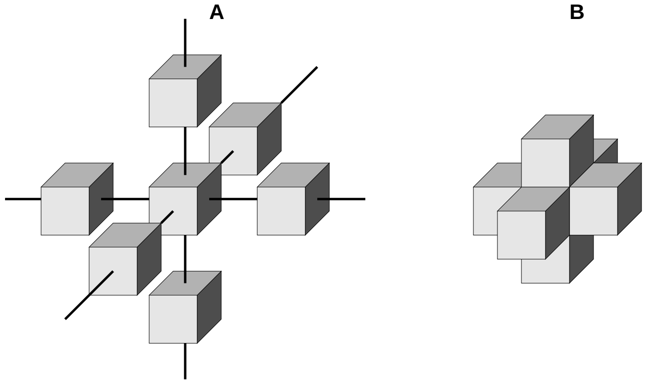

Figure 1: *A Lee sphere $\mathcal{B}_1$. Cubes represent vertices and the central cube is a vertex of the code $\mathcal{C}$ (see definition 3.4). On figure 1-A we represent the links (a link represents symmetric arcs) and on figure 1-B the representation without links. It's this pattern that we must pack to cover the entire torus (see remark 3.5).*

Remark 3.5 In coding theory [27], it is well known that $\mathcal{C}$ is a linear code of length 3 defined over $\mathbb{Z}_7$. Note that $\mathcal{C}$ has 49 elements like $(0,0,0), (-2,1,0), (2,-3,-1)$, and so on.

Definition 3.6 *Let* $M_0 = \begin{pmatrix} -2 & 0 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 0 \end{pmatrix}$ *and note that* $M_0^2 = \begin{pmatrix} -3 & 1 & -2 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \end{pmatrix}$.

The broadcast algorithm relies on the following properties:

Property 3.7 $\mathcal{C} = M_0^2\mathcal{B}_1 + M_0\mathcal{B}_1$, and $\mathcal{E} = \mathcal{C} + \mathcal{B}_1$.

Proof. Just check the first equality. The second equation is well known in coding theory, because $\mathcal{C}$ is also a perfect Lee code [27]. $\square$

Proposition 3.8 *There exists a gossiping protocol on the symmetric digraph $G = TM(7)^3$ with time $g(G) = 4\alpha + 12\delta + (1 + 7 + 7^2 + 7^3)L\tau$.*

To describe the gossiping protocol of proposition 3.8 we introduce some additional notation.

Notation 3.9 *Let x be a vertex of a digraph G and let $\mathcal{A} \subset V(G)$ be a subset of the set of vertices of G. The notation $x \rightarrow \mathcal{A}$* (resp. $x \leftarrow \mathcal{A}$) *is used when the vertex x sends his message towards all the vertices of $\mathcal{A}$ (*resp. all the vertices of $\mathcal{A}$ send their own message toward the vertex x*).*

3.1.1 Description of the gossiping algorithm in $TM(7)^3$

The algorithm that we present here is similar to a method for designing so-called *"three-phase algorithms"* [19, 20, 21] (this method uses an accumulation, a gossip and a broadcast phase).

Begin ———— *Gossiping Algorithm* ———— *in* $TM(7)^3$.

All the information of the digraph G is concentrated (more exactly equally distributed) among the set of vertices of the code $\mathcal{C}$. This round uses dipaths of length 1 and messages of length L.

▷ **Round 1** ◁ **Concentration**	
Scheme of communication	*Cost*
$\forall x \in \mathcal{C},\ x \leftarrow \{x + \mathcal{B}_1\}$	$\alpha + \delta + L\tau$

Here the scheme of communications is decomposed into two rounds. We will describe later the arc-disjoint dipaths used to perform these two rounds, and prove the cost of these rounds.

▷ **Step 2** ◁ **Gossiping among the vertices of $\mathcal{C}$**	
Round 2-a	
Scheme of communication	*Cost*
$\forall x \in \mathcal{C},\ x \rightarrow \{x + M_0\mathcal{B}_1\}$	$\alpha + 5\delta + 7L\tau$
Round 2-b	
Scheme of communication	*Cost*
$\forall x \in \mathcal{C},\ x \rightarrow \{x + M_0^2\mathcal{B}_1\}$	$\alpha + 5\delta + 7^2L\tau$

Each vertex of the code $\mathcal{C}$ send its information to its 6 direct neighbors. This round uses dipaths of length 1 and messages of length 7^3L.

▷ **Round 3** ◁ **Final broadcasting**	
Scheme of communication	*Cost*
$\forall x \in \mathcal{C},\ x \rightarrow \{x + \mathcal{B}_1\}$	$\alpha + \delta + 7^3L\tau$

End ———— *Gossiping Algorithm* ———— *in* $TM(7)^3$.

3.1.2 Analysis of the algorithm

- **Round 1:** As $\mathcal{C}$ is a perfect code, the first round equally distributes the entire information of G on the set of the vertices of $\mathcal{C}$.

- **Step 2:** During second step, each vertex $x \in \mathcal{C}$ has sent his information to $x + M_0\mathcal{B}_1 + M_0^2\mathcal{B}_1 = x + \mathcal{C} = \mathcal{C}$, thus step 2 performs a gossiping among the vertices of $\mathcal{C}$. Note that at the end of the round 1, vertices of $\mathcal{C}$ have a message of length $L_0 = 7L$. We will show in section 3.1.3 that rounds 2-a and 2-b use dipaths of length at most 5, hence step 2 is completed in time $(\alpha + 5\delta + L_0\tau) + (\alpha + 5\delta + 7L_0\tau)$.

  After step 2 each vertex x of $\mathcal{C}$ has received the whole information initially distributed on G.

- **Round 3:** As $\mathcal{E} = \mathcal{C} + \mathcal{B}_1$, this last round enables us to achieve the gossiping of the digraph.

3.1.3 Dipaths used by the algorithm

Now the problem is to exhibit a set of dipaths in G realizing each round of communication of the algorithm. As rounds 1 and 3 raise no problem, we give now the dipaths associated to step 2.

Round 2-a (resp. 2-b) uses communications of the kind $\forall x \in \mathcal{C}$, $x \to \{x + A\}$ (resp. $x \to \{x + A'\}$) with $A = M_0\mathcal{B}_1$ (resp. $A' = M_0^2\mathcal{B}_1$). In both cases, A and A' are symmetric, that is of the kind: $u_1, u_2, u_3, 0, -u_1, -u_2, -u_3$. We will describe for each u_i the dipath associated, the dipath associated to $-u_i$ being the opposite one. As there are many possible dipaths from x to $x + u_i$, we will describe them by decomposing the associated dipaths as $u_i = a_1 + a_2 + \ldots + a_l$, where a_i is proportional to some vector of the base, meaning that from x we first go to $x + a_1$ along vector a_1 then to $x + a_1 + a_2$ following a_2 and so on.

Dipaths used in the round 2-a In this case the vectors u_1, u_2, u_3 are

$$\begin{aligned}
u_1 &= \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} \\
u_2 &= \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \\
u_3 &= \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}
\end{aligned}$$

The maximum length of these dipaths is $\max(|+1|+|-2|; |-1|+|+2|+|+2|; |+1|+|+3|) = 5$ and the length of the messages is $7L$. Indeed after the first round each vertex $x \in \mathcal{C}$ has accumulated 7 pieces of information. The cost of this round is $\alpha + 5\delta + 7L\tau$. *Figure* 2-(2a) shows clearly that the scheme of communications described above is arc-contention free.

Dipaths used in the round 2-b In that case the vectors u_1, u_2, u_3 are

$$u_1 = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

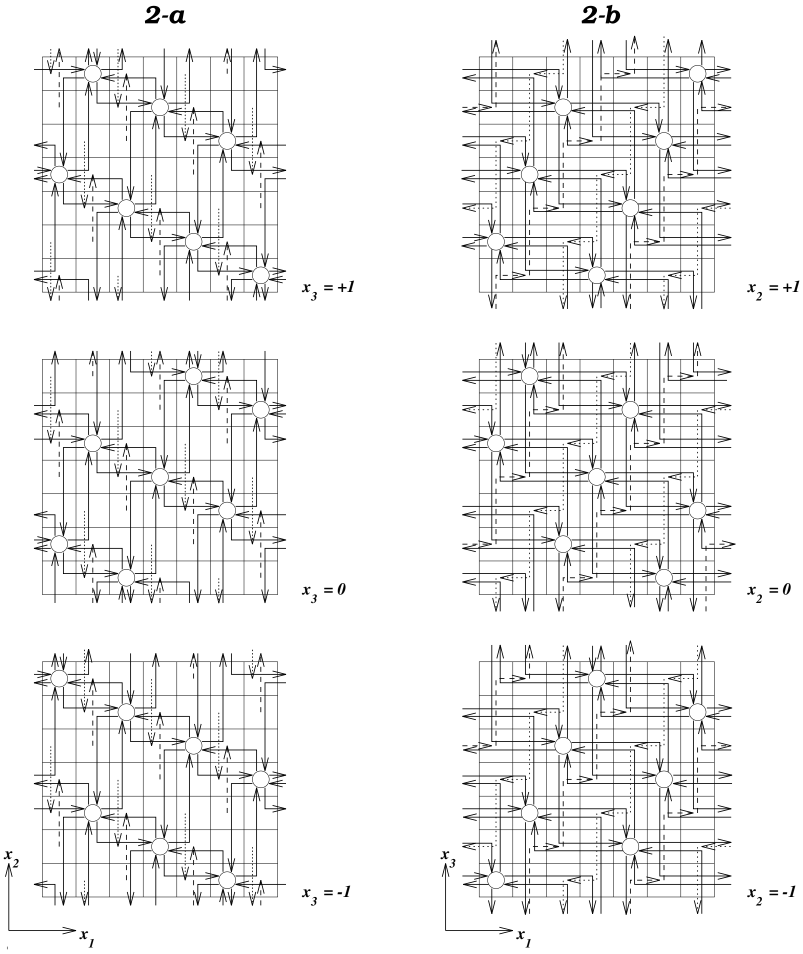

Figure 2: *Communication patterns for step 2 of the gossiping algorithm in the* $TM(7)^3$ *digraph. This step is decomposed into two rounds 2-a and 2-b. In the left* (resp. right) *drawing the torus is displayed with layers for a fixed* x_3 (resp. x_2). *Here, only three layers are displayed. In these figures the solid arcs correspond to the vectors* $u_1, -u_1, u_3, -u_3$, *the dashed arcs to vectors* u_2 *and the dotted arcs to vectors* $-u_2$ *for each round 2-a and 2-b.*

$$u_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$u_3 = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}$$

The maximum length of the dipaths is $\max(|-3|+|+1|;|-1|+|+2|+|+1|+|+1|;|+3|+|-2|) = 5$ and the length of the messages is 7^2L. Indeed, after round 1 and 2-a each vertex of the code has 7×7 pieces of information. The cost of this round is $\alpha+5\delta+7^2L\tau$. *Figure* 2-(2b) shows clearly that the scheme of communications used in this round creates no-arc-contention.

3.2 Generalization for torus $TM(7^i)^3$

The main idea of this section is to use recursively the gossiping protocol designed for the torus $TM(7)^3$ in the torus $TM(7^i)^3$.

Notation 3.10 *Here G_i denotes the symmetric digraph $TM(7^i)^3$.*

Definition 3.11 *The* **code** *$\mathcal{C}_i$ is the set of the vertices of G_i defined as $\mathcal{C}_i = \{(x_1,x_2,x_3)\in \mathbb{Z}^3_{7^i} | x_1+2x_2+3x_3 \equiv 0 \mod 7\}$.*

Remark 3.12 This code is once again a perfect code for the Lee distance [27]. Indeed, spheres of radius 1 centered on each vertex of the code $\mathcal{C}_i$ cover completely the digraph G_i. That is $V(G_i)=\mathbb{Z}^3_{7^i}=\mathcal{B}_1+\mathcal{C}_i$. The code $\mathcal{C}_i$ has 7^{3i-1} elements.

Notation 3.13 *Let U_0 be a sub-group of $\mathbb{Z}^3_{7^i}$ defined as $U_0\equiv 0 \mod 7$.*

Remark 3.14 U_0 is clearly isomorphic to $\mathbb{Z}^3_{7^{i-1}}$. The sub-group U_0 which contains the vectors $x=(x_1,x_2,x_3)$ such that $x_1\equiv 0 \mod 7$, $x_2\equiv 0 \mod 7$ and $x_3\equiv 0 \mod 7$, has $7^{i-1}\times 7^{i-1}\times 7^{i-1}$ vertices (or vectors).

Definition 3.15 *The* **family** *associated with a vector (or vertex) x is the set of vectors (or vertices) defined as $x+U_0$.*

Lemma 3.16 *In the $TM(7^i)^3$ symmetric digraph the vertices of the code $\mathcal{C}_i$ form 7^2 disjoint dilated sub-torus of the $TM(7^{i-1})^3$ symmetric digraph with a dilated factor of 7.*

Proof. The distance between two vertices of U_0 is a multiple of 7. If we join by a dipath of length 7 any ordered pair of vertices at distance exactly 7, we obtain a sub-graph H_0 of G_i which is a dilated sub-torus (obtained from the torus G_{i-1} by dilating each arc in a dipath of length 7). Now we can partition the vertices of the code $\mathcal{C}_i$ into 7^2 disjoint families. Indeed, any vertex (x_1,x_2,x_3) of the code $\mathcal{C}_i$ belongs to the family $(a,b,c)+U_0$ where $x_1\equiv a \mod 7$, $x_2\equiv b \mod 7$ and $x_3\equiv c \mod 7$ with $-3\le a\le 3$, $-3\le b\le 3$ and $-3\le c\le 3$. So, we have 7^2 possible choices for (a,b,c). Indeed, c is determined as soon as we fix a and b. If we consider the subgraph $H_{a,b}$ generated by the vertices of the family associated with (a,b,c), two vertices at distance 7 being joined by a dipath of length 7, $H_{a,b}$ is isomorphic to H_0, the dilated sub-torus of $TM(7^{i-1})^3$. Note that by definition any two different $H_{a,b}$ have no arc in common. So, we can in a given round, do concurrent communications on each $H_{a,b}$ (or families). □

Proposition 3.17 *There exists a gossiping protocol on the symmetric digraph $G_i = TM(7^i)^3$ with time $g(G_i) = 4i\alpha + \frac{12}{9}D(G_i)\delta + [\frac{57}{49}(7^{i-1}-1) + \frac{7^3}{7^{2i}} - \frac{1}{7^{3i}}]\frac{NL}{6}\tau$.*

We describe now the gossiping protocol of this proposition.

3.2.1 Description of the gossiping algorithm in $TM(7^i)^3$

Here the gossiping algorithm in $TM(7^i)^3$ is similar to gossiping algorithm in $TM(7)^3$.

Begin ——— *Gossiping Algorithm* ——— *in $TM(7^i)^3$.*

All the information of the digraph G_i is concentrated (more exactly equitably distributed) on the set of vertices of the code $\mathcal{C}_i$. This round uses dipaths of length 1 and messages of length L.

▷ **First Round** ◁ **Concentration**	
Scheme of communication	*Cost*
$\forall x \in \mathcal{C}_i,\, x \leftarrow \{x + \mathcal{B}_1\}$	$\alpha + \delta + L\tau$

Vertices of the code $\mathcal{C}_i$ are partitioned into families. The vertices of the same family perform the gossiping protocol on the dilated subtorus $TM(7^{i-1})^3$ recursively. We will describe later the cost of this phase.

▷ **Recursive phase** ◁
Scheme of communication
Recursive gossiping protocol on G_{i-1}
Cost
$g_\alpha(G_{i-1})\alpha + 7g_\delta(G_{i-1})\delta + 7g_\tau(G_{i-1})L\tau$

For this step, the communications are similar to communications used in the step 2 of the gossiping protocol for the vertices of the code on $\mathbb{Z}_7^3$. At the beginning of this step the length of messages is $\frac{N}{7^2}L$.

▷ **Local Gossiping Step** ◁ **Gossiping between the vertices of $\mathcal{C}_i$**	
Round a	
Scheme of communication	*Cost*
$\forall x \in \mathcal{C}_i,\, x \rightarrow \{x + M_0\mathcal{B}_1\}$	$\alpha + 5\delta + \frac{N}{7^2}L\tau$
Round b	
Scheme of communication	*Cost*
$\forall x \in \mathcal{C}_i,\, x \rightarrow \{x + M_0^2\mathcal{B}_1\}$	$\alpha + 5\delta + \frac{N}{7}L\tau$

Each vertex of the code $\mathcal{C}_i$ sends its information to its 6 direct neighbors. This round uses dipaths of length 1 and messages of length NL.

▷ **Last Round** ◁ **Final broadcasting**	
Scheme of communication	*Cost*
$\forall x \in \mathcal{C}_i,\, x \rightarrow \{x + \mathcal{B}_1\}$	$\alpha + \delta + NL\tau$

End ——— *Gossiping Algorithm* ——— *in $TM(7^i)^3$.*

3.2.2 Analysis of the algorithm and recursion

- The **first round** raises no problem.
- At the beginning of the **recursive phase** the length of messages is $7L$. As this phase performs a gossiping protocol on the dilated sub-torus G_{i-1}, then the distances of the communications have a dilation factor of 7 and this recursive phase can be applied on each family in a same partition in the same time because these families are disjoint.
- As this **local gossiping step** used the same scheme of communication as step 2 on $TM(7)^3$, then each vertex of a family receives information from a vertex of each other family. So, the cost of this step is $(\alpha + 5\delta + \frac{N}{7^2}L\tau) + (\alpha + 5\delta + 7\frac{N}{7^2}L\tau)$.
- This **last step** raises no problem. Just note that at this step each vertex of the code $\mathcal{C}_i$ knows all the information of G_i, so the length of the messages is $7^{3i}L$.

From this algorithm and recursion we obtain (with $N = 7^{3i}$),

$$\begin{aligned}
g_\alpha(G_i) &= g_\alpha(G_{i-1}) + 4 &&= 4i \\
g_\delta(G_i) &= 7g_\delta(G_{i-1}) + 12 &&= \tfrac{12}{9}D(G_i) \\
g_\tau(G_i) &= 7g_\tau(G_{i-1}) + 1 + (1 + \tfrac{1}{7} + \tfrac{1}{7^2})N &&= [\tfrac{57}{49}(7^{i-1} - 1) + \tfrac{7^3}{7^{2i}} - \tfrac{1}{7^{3i}}]\tfrac{N}{6}
\end{aligned}$$

4 Case of large messages

In the previous parts we presented the case for short messages as we focused only on the parameters $\alpha(G)$ and $\delta(G)$. But our protocols are not efficient if we consider large messages. Indeed the predominant parameter becomes $\tau(G)$. Our previous algorithms imply a maximal parameter $\tau(G)$ at the last round (final broadcasting). In this part we discuss the case of large messages and will show that it is possible to decrease the parameter $\tau(G)$ using classical methods, if we allow an increase in the parameter $\alpha(G)$ (see [1]). Here we use the following definition and lemma.

Definition 4.1 *Let x be a vertex (or vector) of the $TM(7^i)^3$ symmetric digraph, and let $c(x)$ denote the color of the vertex x defined by $c(x) = (x_1 + 2x_2 + 3x_3) \mod 7$, so $c(x) \in \Omega = \{-3, -2, -1, 0, 1, 2, 3\}$.*

Lemma 4.2 *The vertices of the $TM(7^i)^3$ digraph can be partitioned with 7 different colors such that a vertex x and its direct neighbors all have different colors.*

Proof. Just use the $c(x)$ colors and note that the direct neighbors of the vertex x are the vertices with ± 1 on only one coordinate x_1 or x_2 or x_3. Thus x and its direct neighbors all have different values. $\square$

4.1 Case of $TM(7)^3$

As we saw in section 3.1, at the beginning of the last round (final broadcasting) of the gossiping protocol on the symmetric digraph $G = TM(7)^3$, all the vertices of the code $\mathcal{C}$ know the entire information of G. During this last round each vertex of the code sends the total information to its direct neighbors using its out-links (out-arcs) and no out-arc of the vertices which are not in the code are used. The idea is to split this last round into two using out-arcs of the vertices which are not in the code to decrease the data flow on the out-arcs of the vertices in the code by involving smaller messages.

Proposition 4.3 *There exists a gossiping protocol on the symmetric digraph $G = TM(7)^3$ with time $g(G) = 5\alpha + 13\delta + (1 + 7 + 3 \cdot 7^2)L\tau$.*

Proof. We split the last round (final broadcasting) into two.

Just before step 3, the message is splitted into 7 pieces and during rounds 3-a and 3-b the vertices of the code send a different piece of the total message (that is $1/7th$ of the total length of the initial message) to each neighbor. During round 3-b the vertices which are not in the code send the message they have received at round 3-a toward their direct neighbors.

▷ **Step 3** ◁ **Final broadcasting**	
Round 3-a	
Scheme of communication	*Cost*
$\forall x \in \mathcal{C},\ x \xrightarrow{\frac{1}{7}} \{x + \mathcal{B}_1\}$	$\alpha + \delta + \frac{7^3}{7} L\tau$
Round 3-b	
Scheme of communication	*Cost*
$\forall x \in \mathcal{C},\ x \xrightarrow{\frac{1}{7}} \{x + \mathcal{B}_1\}$ and $\forall x \notin \mathcal{C},\ x \rightarrow \{x + \mathcal{B}_1\}$	$\alpha + \delta + \frac{7^3}{7} L\tau$

At the beginning of round 3-a, all the vertices of the code $\mathcal{C}$ which know the entire information of G, partition their total message m of length 7^3 into 7 pieces (as for the concatenation, we forget the time necessary to partition the message). Now we give a different color from the set Ω to each piece of the message and we will denote each one by its color, that is $m = m_{-3}, m_{-2}, \cdots, m_2, m_3$ (this operation is exactly the same for all the vertices of the code). Then, during step 3-a each vertex of the code sends the piece of message m_i to y $\forall y \in \{x + \mathcal{B}_1\}$ if and only if $c(y) = i$. So, at the end of this round, the vertex x, $\forall x \notin \mathcal{C}$, knows the piece of message $m_{c(x)}$.

Now, lemma 4.2 assure that round 3-b achieves the gossiping protocol. Just note that during this last round the vertices of the code send the piece of message m_0. □

4.2 Case of $TM(7^i)^3$

Now we are able to state the generalization for the symmetric digraph $G_i = TM(7^i)^3$.

Proposition 4.4 *There exists a gossiping protocol on the symmetric digraph $G_i = TM(7^i)^3$ with time $g(G_i) = 5i\alpha + \frac{13}{9}D(G_i)\delta + [\frac{22}{49}(7^{i-1} - 1) + \frac{19 \cdot 7}{7^{2i}} - \frac{1}{7^{3i}}]\frac{NL}{6}\tau$.*

Proof. As the lemma 4.2 is true for G_i, just split the last round of each recursive phase and the last round (final broadcasting) of the gossiping protocol in $TM(7^i)^3$ by the 2 rounds described in the proof of proposition 4.3. □

Remark 4.5 If we are not concerned by the $g_\alpha(G)$ parameter, it is possible to decrease the data flow much more, as we can simulate the store-and-forward model with the wormhole model. Such technics in the store-and-forward model are described in the survey [13] and P. Fraignaud presents in [12] a protocol which decreases significantly the data flow as he obtains for the graph $TM(p)^k$ of order N, $g(TM(p)^k) \leq k\lfloor \frac{p}{2} \rfloor(\alpha + \delta) + \frac{(N-1)}{2k}\tau$.

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