1 Introduction and notations

The construction of one, and if possible many edge-disjoint Hamilton cycles in a network can provide advantage for algorithms that make use of a ring structure. As example, the existence of many edge-disjoint Hamilton cycles allows the message traffic to be evenly distributed across the network. Furthermore, a partition of the edges into Hamilton cycles can be used in various distributed algorithms (termination, garbage collector, ...). So, many authors have considered the problem of finding how many edge-disjoint Hamilton cycles can be found in a given network. The most significant results have been obtained for the class of Cayley graphs on abelian groups, and for (underlying) line digraphs. Here we solve this problem for the *Butterfly networks*. These networks have been proposed as suitable topologies for parallel computers, due to their interesting structure (see [8, 9]) because they are, when properly defined, both Cayley digraphs (on a non-abelian group) and iterated line digraphs.

1.1 Definitions

First, we have to warn the reader that under the name *Butterfly* and with the same notation, different networks are described. Indeed, if some authors consider the *Butterfly network* as a multistage network used to route permutations, others consider it as a point-to-point network. In what follows, we will study the point-to-point version and use Leigthon's terminology [8], namely, *wrapped Butterfly*. Also, when we use the terms edge-disjoint or arc-disjoint, it obviously means *pairwise* edge-disjoint or arc-disjoint. In this article, we will use the next definitions and notations. For definitions not given here, see [9].

$\mathbb{Z}_q$ will denote the set of integers modulo q; addition of elements in $\mathbb{Z}_q$ will always occur in $\mathbb{Z}_q$.

Definition 1.1 *The* **wrapped Butterfly digraph** *of degree d and dimension n, denoted $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$, has as vertices the couples (x,l) where x is an element of $\mathbb{Z}_d^n$, that is, a word $x_{n-1}x_{n-2}\cdots x_1x_0$ where the letters belong to $\mathbb{Z}_d$, and $l \in \mathbb{Z}_n$ (l is called the level). For any l, a vertex labelled $(x_{n-1}x_{n-2}\cdots x_l \cdots x_1x_0, l)$ is joined by an arc to d vertices labelled $(x_{n-1}\cdots x_{l+1}, x_l+\alpha, x_{l-1}\cdots x_0,\ l+1)$ where α is any element of $\mathbb{Z}_d$. Each one of these arcs is said to have the* **slope** *α.*

$\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$ is a d-regular digraph with nd^n vertices; its diameter is $2n-1$. This network is sometimes considered as undirected, but its structure being indeed directed, we will always consider the digraph.

For convenience, we repeat the level 0 when drawing the wrapped Butterfly digraph. Hence, the reader has to remember that the two occurrences of level 0 have to be identified. Figure (**1**) displays $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(3,2)$ with the arcs directed from left to right. Note that $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$ is often represented (for example in [8, 9]) in an opposite way to our drawing as the authors denote the nodes $(x_0x_1\cdots x_{n-1}, l)$.

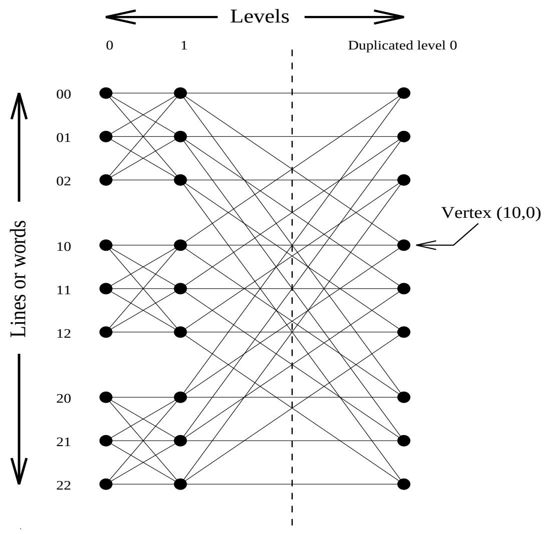

Figure 1: *The digraph $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(3,2)$, the arcs beings directed from left to right.*

Now, we give two digraph definitions we use in the following.

- $\mathcal{K}_d^+$ will denote the complete symmetric digraph with a loop on each vertex,
- $\vec{\mathcal{K}}_{d,d}$ will denote the complete bipartite digraph where each set of the bipartition has size d and with all the arcs directed from the left part to the right part.

Note that $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,1)$ is nothing else than $\mathcal{K}_d^+$.

In digraphs, the concept of dipaths and circuits (directed cycles) is well-known. Here, we need to use more general concepts valid for digraphs of paths and cycles (which are also called oriented elementary paths and oriented elementary cycles).

Definition 1.2 *A* **path of a digraph** *is a sequence $\mu = (v_0, e_0, v_1, e_1, \cdots, v_k, e_k, v_{k+1})$ where the v_i's are vertices and the e_i's are arcs such that the end vertices of e_i are v_i and v_{i+1} and where the sequence μ does not meet twice the same vertex except maybe v_0 and v_{k+1}.*

Definition 1.3 *A path such that $v_{k+1} = v_0$ in the sequence μ is called a* **cycle***.*

Note that the arc e_i can be either directed from v_i to v_{i+1} or from v_{i+1} to v_i. If all the arcs of the path (*resp. cycle*) are directed from v_i to v_{i+1} we have a dipath (*resp. circuit also called dicycle*).

Definition 1.4 *A vertex v_i of a cycle is said to be* **of type +** (resp. **of type −**) *for the cycle, if v_i is the terminal vertex of e_{i-1}* (resp. e_i) *and the initial vertex of e_i* (resp. e_{i-1}).

Note that in a circuit, all vertices are of type +, but the type is not necessarily defined for all the vertices of a cycle.

Definition 1.5 *A vertex v is said* **to be crossed by a cycle**, *or* **a cycle crosses the vertex** *v, if v is of type + or of type − for the cycle. When a vertex v is crossed by a cycle, we will define its* **sign function** *ϵ by $\epsilon(v) = +1$* (resp. $\epsilon(v) = -1$) *if v is of type +* (resp. of type −).

Remark 1 We can also define the predecessor $p(v)$ and the successor $s(v)$ of the vertex v in the order induced by the cycle. Then, the vertex v is of type + (or has sign $\epsilon(v) = +1$) if $(p(v), v)$ and $(v, s(v))$ are both arcs of the digraph, and is of type − (or has sign $\epsilon(v) = -1$) if both $(s(v), v)$ and $(v, p(v))$ are arcs of the digraph.

Definition 1.6 *A* **Hamilton cycle** (resp. **circuit**) *of a digraph is a cycle* (resp. circuit) *which contains every vertex exactly once.*

Definition 1.7 *We will say that a digraph is* **decomposable into Hamilton cycles** (resp. **circuits**) *if its arcs can be partitioned into Hamilton cycles* (resp. circuits).

Definition 1.8 *A Hamilton cycle of $\vec{\mathcal{WBF}}(d, n)$ will be said to be* **l-crossing** *if the cycle crosses all the vertices of level l and furthermore $\sum_{v=(x,l),\, x \in \mathbb{Z}_d^n} \epsilon(v) \equiv 0 \pmod{d}$.*

Figure (**3**) shows examples of 1-crossing Hamilton cycles in $\vec{\mathcal{WBF}}(3, 2)$ and $\vec{\mathcal{WBF}}(3, 3)$.

1.2 Results

Various results have been obtained on the existence of Hamilton cycles in classical networks (see for example the surveys [2, 7]). For example, it is well-known that any Cayley graph on an abelian group is hamiltonian. Furthermore, it has been conjectured by Alspach [1] that:

Conjecture 1 (Alspach) *Every connected Cayley graph on an abelian group has a Hamilton decomposition.*

This conjecture has been verified for all connected 4-regular graphs on abelian groups in [6]. It includes in particular the toroidal meshes (grids). For the hypercube, it is also known that $\mathcal{H}(2d)$ is decomposable into d Hamilton cycles (see [2, 3]).

The wrapped Butterfly digraph is actually a Cayley graph (on a non-abelian group) and a line digraph. So, the decomposition into Hamilton cycles (*resp. circuits*) of this digraph has received some attention. It is well-known that $\vec{\mathcal{WBF}}(d, n)$ has one Hamilton circuit (see [8, page 465] for a proof in the case $d = 2$ or [12]). In [4], Barth and Raspaud proved that the underlying multigraph associated to $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(2,n)$ contains two arc-disjoint Hamilton cycles answering a conjecture of J. Rowley and D. Sotteau [10]. In our terminology, their result can be stated as:

Theorem 1.1 (Barth, Raspaud) *$\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(2,n)$ is decomposable into Hamilton cycles.*

They conjectured that this result can be generalized for any degree:

Conjecture 2 (Barth, Raspaud) *For $n \geq 2$, $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$ is decomposable into Hamilton cycles.*

In this paper, we prove the conjecture (**2**). To do so, we use some techniques introduced in [5] where we studied the decomposition of $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$ into Hamilton circuits. In fact, we prove that $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$ is decomposable into d l-crossing Hamilton cycles. Indeed, the l-crossing property, combined with the recursive structure of $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$, enables us to prove that the number of l-crossing arc-disjoint Hamilton cycles that $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$ contains can only increase when n increases. Then, we prove mainly that $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,2)$ contains d arc-disjoint l-crossing Hamilton cycles, by constructing two arc-disjoint l-crossing Hamilton cycles using only arcs of slopes 0 and 1 and $d-2$ arc-disjoint Hamilton circuits using arcs of other slopes. These results are summarized in the following theorem:

Theorem 1.2 *For $n \geq 2$,*

- *for $d \notin \{3,4,6\}$, $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$ is decomposable into $d-2$ Hamilton circuits and 2 Hamilton cycles,*
- *for $d \in \{4,6\}$, $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$ is decomposable into Hamilton circuits,*
- *$\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(3,n)$ is decomposable into 1 Hamilton circuit and 2 Hamilton cycles.*

2 The general construction

We give below some additional definitions and properties enabling us to establish the lemma (**2.2**) which is indeed a strengthened version of the inductive lemma of [5]. This lemma is then applied in section (**3**) to construct inductively the decomposition.

2.1 Families of perfect matchings

We will denote by M a permutation of $\mathbb{Z}_d$ mapping a to $M(a)$ or equivalently the associated perfect matching of $\vec{\mathcal{K}}_{d,d}$ which contains all the arcs $(a, M(a))$.

Definition 2.1 *Let S be a set of slopes (that is a subset of $\mathbb{Z}_d$). Then, a* **perfect matching** *M of $\vec{\mathcal{K}}_{d,d}$* **uses the slopes in S** *if, for any $a \in \mathbb{Z}_d$ $M(a) \in \{a+s,\ s \in S\}$. A* **family of perfect matchings** *$\mathcal{M} = \{M_x,\ x \in \mathbb{Z}_d^n\}$ of $\vec{\mathcal{K}}_{d,d}$* **uses the slopes in S** *if, for any perfect matching M_x of the family, M_x uses the slopes in S.*

Definition 2.2 *For $1 \leq j \leq p$, let $\mathcal{M}_j = \{M_{x,j} \,|\, x \in \mathbb{Z}_d^n\}$ be p families of perfect matchings. The* **families** *$\mathcal{M}_j$ are said to be* **compatible** *if, for each x in $\mathbb{Z}_d^n$, the perfect matchings $M_{x,j}$ are arc-disjoint (i.e. $\forall a \;\; M_{x,j}(a) \neq M_{x,j'}(a)$, for $j \neq j'$).*

Definition 2.3 *A family $\mathcal{M} = \{M_x,\ x \in \mathbb{Z}_d^n\}$ of $\vec{\mathcal{K}}_{d,d}$ perfect matchings satisfies the* **cyclic-potent property** *if, for any order of composition of the M_x and any set of sign $\{\epsilon_x \mid x \in \mathbb{Z}_d^n,\ \epsilon_x \in \{-1, 1\}\}$ such that $\sum_x \epsilon_x \equiv 0 \mod d$, the permutation $\Pi_x M_x^{\epsilon_x}$ is cyclic.*

Definition 2.4 *A* **family of perfect matchings** *$\mathcal{M} = \{M_x,\ x \in \mathbb{Z}_d^n\}$ is* **of type** *(i, j) if:*

- *for $x \neq 0$, $M_x(a) = a + i$;*
- *for $x = 0$, $M_0(a) = a + j$.*

Lemma 2.1 *A family of perfect matchings of type (i, j) $\mathcal{M} = \{M_x, x \in \mathbb{Z}_d^n\}$ is cyclic-potent if and only if $j - i$ is prime with d.*

Proof. As the permutations of the family commute, the permutation $\Pi_x M_x^{\epsilon_x}$ of definition (**2.3**) can be simply expressed as $x \to x + \delta$. So, this permutation will be cyclic if and only if δ is prime with d. Here $\delta = (\sum_{x \neq 0} \epsilon_x)i + \epsilon_0 j$. As $\sum_x \epsilon_x = 0$, we have $\delta = (\sum_x \epsilon_x)i + \epsilon_0(j - i) = \epsilon_0(j - i)$. So, δ is clearly prime with d if and only if $j - i$ is prime with d. □

In section (**3**), we will need some very simple cyclic-potent families of perfect matchings that we give as examples. We will represent a set of p families of perfect matchings of type (i, j):

$$\{(i_0, j_0), (i_1, j_1), \ldots, (i_{p-1}, j_{p-1})\}$$

by the array:

$$\begin{array}{ccccccc} i_0 & i_1 & i_2 & i_3 & \ldots & i_{p-2} & i_{p-1} \\ j_0 & j_1 & j_2 & j_3 & \ldots & j_{p-2} & j_{p-1} \end{array}$$

Families 1 There exist d compatible cyclic-potent families of perfect matchings:

$$\begin{array}{cccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & \ldots & d-2 & d-1 \\ 1 & 2 & 3 & 4 & 5 & 6 & \ldots & d-1 & 0 \end{array}$$

These families are cyclic-potent as, applying lemma (**2.1**), we obtain:

$$1 - 0 = 2 - 1 = \cdots = d - 1 - (d - 2) = 0 - (d - 1) = 1$$

which is prime with d. These families use all the slopes.

Families 2 There exist 2 compatible families which use the slopes $\{0,1\}$:

$$\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}$$

According to lemma (**2.1**), they are two compatible cyclic-potent families and they use the slopes $\{0,1\}$.

Families 3 When $d \neq 3$, there exist $d-2$ compatible cyclic-potent families of perfect matchings using the slopes $\{2,\ldots,d-1\}$. One possible solution is given below:

- When d is odd and $d \neq 3$, the following families can be used:

$$\begin{array}{cccccccc} 2 & 3 & 4 & 5 & \ldots & d-3 & d-2 & d-1 \\ 4 & 5 & 6 & 7 & \ldots & d-1 & 2 & 3 \end{array}$$

- When d is even, we use the following families:

$$\begin{array}{cccccc} 2 & 3 & 4 & 5 & \ldots & d-2 & d-1 \\ 3 & 2 & 5 & 4 & \ldots & d-1 & d-2 \end{array}$$

These families are cyclic-potent as, applying lemma (**2.1**), we get:

- for d odd:

$$\begin{array}{ccccccccc} 4-2 & = & 5-3 & = & \cdots & = & d-1-(d-3) & = & 2 \\ & & \text{and} & & 2-(d-2) & = & 3-(d-1) & = & 4 \end{array}$$

  as 2 and 4 are prime with d;

- for d even:

$$\begin{array}{cccccccccc} & 3-2 & = & 5-4 & = & \cdots & = & (d-1)-(d-2) & = & 1 \\ \text{and} & 2-3 & = & 4-5 & = & \cdots & = & (d-2)-(d-1) & = & -1 \end{array}$$

  which are prime with d.

In both cases, the slopes used are in $\{2,\ldots,d-1\}$.

2.2 Inductive construction

Lemma 2.2 *If $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$ admits p arc-disjoint l-crossing Hamilton cycles and if there exist p cyclic-potent families of perfect matchings in $\vec{\mathcal{K}}_{d,d}$, then $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n+1)$ admits p arc-disjoint l-crossing Hamilton cycles.*

Proof. Let H be an l-crossing Hamilton cycle of $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$. As all the levels are equivalent, we can suppose without loss of generality and for simplicity in the notations that $l=0$. Let $\mathcal{M}=\{M_x,\ x\in \mathbb{Z}_d^n\}$ be a cyclic-potent family of perfect matchings of $\vec{\mathcal{K}}_{d,d}$. The vertices of $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n+1)$ can be labeled (ax,l) with $a\in\mathbb{Z}_d$, $x\in\mathbb{Z}_d^n$ and $l\in\mathbb{Z}_{n+1}$. Now, we associate to H and $\mathcal{M}$ a partial digraph H' in $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n+1)$ as follows (for an example of such a construction see figure (**3**)):

- for $0\le l\le n-1$ and for each a, if the arc $(x,l)(x',l+1)$ belongs to H, we put in H' the arc $(ax,l)(ax',l+1)$ where the indices are taken modulo $n+1$, which means that to the arc $(x,n-1)(x',0)$ of H is associated the arc $(ax,n-1)(ax',n)$ in H';
- between levels n and 0 of $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n+1)$ we put the arcs joining (ax,n) to $(M_x(a)x,0)$.

With such a definition, each vertex of $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n+1)$ is incident to two arcs of H'. Hence, we can define for each vertex a predecessor and a successor on H' that enables us to prove that we can order H' in a cycle.

For $1\le l\le n-1$, let (x',l') (*resp.* (x'',l'')) be the predecessor (*resp. successor*) of (x,l) in H; then, the predecessor (*resp. successor*) of (ax,l) in H' will be (ax',l') (*resp.* (ax'',l'')).

For $l=0$ and n, as H is a 0-crossing Hamilton cycle, vertices $(x,0)$ are either of type $+$ or $-$ on H.

When $(x,0)$ is of type $+$, its predecessor (*resp. successor*) in the cycle H is $(x',n-1)$ (*resp.* $(x'',1)$). Then, in H' the predecessor (*resp. successor*) of (ax,n) will be $(ax',n-1)$ (*resp.* $(M_x(a)x,0)$); the predecessor (*resp. successor*) of $(ax,0)$ will be $(M_x^{-1}(a)x,n)$ (*resp.* $(ax'',1)$).

When $(x,0)$ is of type $-$, its predecessor (*resp. successor*) in H is $(x',1)$ (*resp.* $(x'',n-1)$). Then, in H' the predecessor (*resp. successor*) of $(ax,0)$ will be $(ax',1)$ (*resp.* $(M_x^{-1}(a)x,n)$); the predecessor (*resp. successor*) of (ax,n) will be $(M_x(a)x,0)$ (*resp.* $(ax'',n-1)$) in H'.

Therefore, when $(x,0)$ is of type $+$ (*resp.* $-$), (ax,n) and $(ax,0)$ are vertices of type $+$ (*resp.* $-$) in H'. Hence, all the vertices of levels 0 and n are crossed by H'; furthermore, the sum of the signs of the vertices of H' of levels 0 or n will be d times the sum of the signs of the vertices of H of level 0, that is, by hypothesis, 0. Hence, H' is 0-crossing (and also n-crossing).

Now, we have to prove that H' is effectively a Hamiltoncycle. For this, it suffices to prove that if we start at some vertex $(ax,0)$ and follow H', we meet successively all the vertices of level 0 and n before coming back to $(ax,0)$. Indeed, suppose that (y,l) was on the portion of cycle H between $(x_1,0)$ and $(x_2,0)$. Then, (ay,l) will be on the portion of H' between (ax_1,α) and (ax_2,β), where $\alpha=0$ (*resp.* $\alpha=n$) if $(x_1,0)$ is of type $+$ (*resp.* $-$), and $\beta=0$ (*resp.* $\beta=n$) if $(x_2,0)$ is of type $-$ (*resp.* $+$). These cases are described on figure (**2**).

Now, let $(x_0,0),(x_1,0),\ldots,(x_{d^n}=x_0,0)$ be the sequence of vertices of H at level 0 in the order we meet them on H. Starting from $(a_0x_0,0)$, we will meet successively $(a_1x_1,0)$, $(a_2x_2,0)$, … and $(a_{d^n}x_{d^n}=a_{d^n}x_0,0)$ on H'. Following such a path, we can meet either x_i of type $+$ by going from level n to level 0, in which case we will apply the perfect matching M_{x_i} to some a, or x_i of type $-$ by going from level 0 to n, in which case we will apply $M_{x_i}^{-1}$ to a. So, $a_{d^n}=\Pi M_{x_i}^{\epsilon_{x_i}}(a)$ where the product is taken in an order depending on x_0. As all the x_i differ, we can meet again $(a_0x_0,0)$ only at some $a_{qd^n}x_0$, but M being cyclic-potent, the values $a_{d^n},a_{2d^n},\ldots,a_{qd^n},\ldots,a_{(d)d^n}$ are all distinct. So, we meet again $(a_0x_0,0)$ only after having encountered the d^{n+1} vertices of level 0.

Now, note that we can perform this construction with p arc-disjoint 0-crossing cycles and p compatible cyclic-potent families. From construction, the p 0-crossing cycles that we will obtain will be arc-disjoint. □

Remark 2 When the 0-crossing Hamilton cycles used in the lemma above are circuits of $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$, all the vertices are of type $+$, and the construction leads to circuits of $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n+1)$, giving another proof of the inductive lemma of [5].

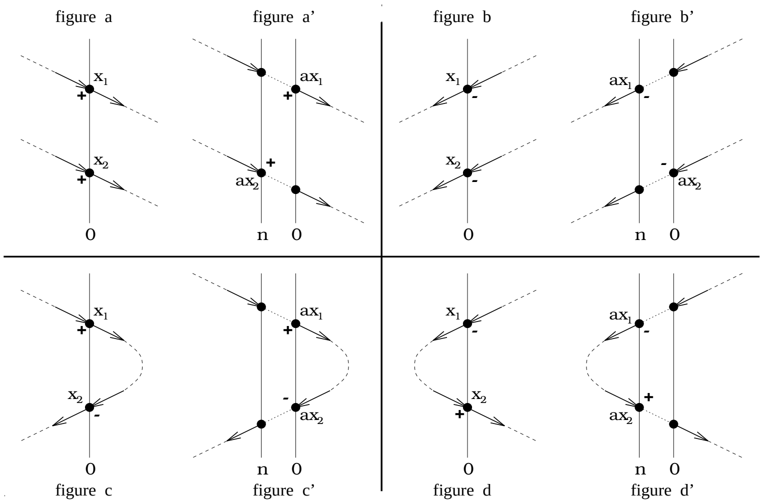

Figure 2: *This figure shows the four possible cases when we perform the inductive construction of $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n+1)$ from $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$. In figure a and a′* (resp. *b* and *b′) the vertices x_1 and x_2 are of type $+$* (resp. *$-$). Figure c and c′* (resp. *d* and *d′) displays the case where the vertex x_1 is of type $+$* (resp. *$-$) and the vertex x_2 is of type $-$* (resp. *$+$).*

3 Decomposition of $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$

We will use a decomposition of $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$ into two partial digraphs.

Definition 3.1 *The Butterfly digraph $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$ is the sum of* **two partial digraphs** *$\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{0,1}(d,n)$ and $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{2,\dots,d-1}(d,n)$ defined as follows:*

- *$\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{0,1}(d,n)$ contains the arcs which slopes belong to $\{0,1\}$,*
- *$\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{2,\dots,d-1}(d,n)$ contains the arcs which slopes belong to $\{2,\dots,d-1\}$.*

3.1 Decomposition of $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{2,\dots,d-1}(d,n)$

The proof is inductive on n. We start the induction for $n = 1$.

Lemma 3.1 *When $d \notin \{4,6\}$, $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{2,\dots,d-1}(d,1)$ is decomposable into Hamilton circuits.*

Proof. As $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,1) = K_d^+$, $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{2,\dots,d-1}(d,1)$ is obtained from K_d^+ by removing the loops and the arcs of slope 1. Following Tillson [11], we know that K_d^+ without the loops contains $d-1$ arc-disjoint Hamilton circuits when $d \neq 4,6$. So, using Tillson decomposition, we can label the vertices of K_d^+ such that one of the circuits uses all the arcs of slope 1. By removing it we get $d-2$ arc-disjoint Hamilton circuits in $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{2,\dots,d-1}(d,1)$. $\square$

Proposition 3.1 *For $d \notin \{3,4,6\}$, $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{2,\dots,d-1}(d,n)$ is decomposable into Hamilton circuits.*

Proof. As $d \notin \{4,6\}$, the proposition is proved for $n = 1$ by lemma (**3.1**). Then, as $d \neq 3$, the $d-2$ compatible cyclic-potent families (**3**) in section (**2.1**) use the slopes $\{2,\dots,d-1\}$ and satisfy the hypothesis of lemma (**2.2**). Hence, we can apply that lemma inductively, in order to construct $d-2$ arc-disjoint Hamilton circuits (see remark (**2**)) in $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{2,\dots,d-1}(d,n)$. $\square$

3.2 Decomposition of $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{0,1}(d,n)$

Lemma 3.2 *$\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{0,1}(d,2)$ is decomposable into l-crossing Hamilton cycles.*

Proof. For this proof, the vertices of $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{0,1}(d,2)$ will be denoted by the couples (xy,l) with $x \in \mathbb{Z}_d$, $y \in \mathbb{Z}_d$ and $l \in \mathbb{Z}_2$. We will show that we can build two arc-disjoint 1-crossing Hamilton cycles in $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{0,1}(d,2)$ by using two sets of arcs of $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{0,1}(d,2)$ defined by the next two rules:

1. $\boxed{\textit{Arcs of } H_0\textit{:}}$

$$\begin{cases} \text{if } x \neq y, & (x(y-1),0) \overset{+1}{\rightarrow} (xy,1) \overset{+0}{\rightarrow} (xy,0), & (1) \\ \text{if } x = y, & (xx,0) \overset{+0}{\rightarrow} (xx,1) \overset{+1}{\rightarrow} ((x+1)x,0). & (2) \end{cases}$$

2. $\boxed{\textit{Arcs of } H_1\textit{:}}$

$$\begin{cases} \text{if } x \neq y, & (xy,0) \overset{+0}{\rightarrow} (xy,1) \overset{+1}{\rightarrow} ((x+1)y,0), & (1) \\ \text{if } x = y, & (x(x-1),0) \overset{+1}{\rightarrow} (xx,1) \overset{+0}{\rightarrow} (xx,0). & (2) \end{cases}$$

It is easy to verify that H_0 and H_1 are arc-disjoint. With the arcs (1) of H_0, we can define for each $x \in \mathbb{Z}_d$ a dipath P_x as follows:

$$P_x \begin{cases} & (xx,0) & \rightarrow & (x(x+1),1) & \rightarrow & (x(x+1),0) & \rightarrow \\ \rightarrow & (x(x+2),1) & \rightarrow & \cdots & \rightarrow & (x(x+d-2),1) & \rightarrow \\ \rightarrow & (x(x+d-2),0) & \rightarrow & (x(x+d-1),1) & \rightarrow & (x(x+d-1),0) & \end{cases}$$

The d dipaths P_x, $x \in \mathbb{Z}_d$, are clearly vertex-disjoint. Only the vertices noted $(xx,1)$ are not in these d dipaths. The arcs (2) of H_0 allows us to join the end vertices of the d dipaths through the missing vertices $(xx,1)$ as follows:

$$\begin{array}{lllll} P_x & \leftarrow & ((x+d-1)(x+d-1),1) & \leftarrow & P_{x+d-1} & \leftarrow \\ & \leftarrow & ((x+d-2)(x+d-2),1) & \leftarrow & \cdots & \leftarrow \\ & \leftarrow & ((x+1)(x+1),1) & \leftarrow & P_{x+1} & \leftarrow \\ & \leftarrow & (xx,1) & \leftarrow & P_x & \end{array}$$

One can easily check that we have defined a Hamilton cycle. The d dipaths are joined through their extremal vertices in a cyclic way, using only arcs (2) of H_0.

By construction, all the vertices at level 1 are crossed. In order to compute the sign of the vertices at level 1, we can choose to walk along the cycle in the direction $(xx,0) \rightarrow (x(x+1),1)$. Therefore, all the vertices $(xy,1)$ with $x \neq y$ are of type $+$ and have $+1$ as sign, while the vertices $(xx,1)$ are of type $-$ and have -1 as sign. So, the sum of the signs is $(d^2-d)-(d) \equiv 0 \pmod{d}$.

To prove that the second set of rules builds a second 1-crossing Hamilton cycle, it suffices to notice that we can rewrite this rule up to a permutation of the letters x and y as being:

- $\boxed{\textit{Arcs of } H_1 \textit{ (with permutation of } x \textit{ and } y\textit{):}}$

$$\begin{cases} \text{if } y \neq x, & (y(x+1),0) \overset{+1}{\leftarrow} (yx,1) \overset{+0}{\leftarrow} (yx,0), \\ \text{if } y = x, & (yy,0) \overset{+0}{\leftarrow} (yy,1) \overset{+1}{\leftarrow} ((y-1)y,0). \end{cases}$$

Construction (2) is then clearly similar to construction (1); to be convinced, just exchange x and y, and replace 1 by -1 in the proof for construction (1).

Hence, H_0 and H_1 are two arc-disjoint 1-crossing Hamilton cycles. As the levels are equivalent, the result holds also for level 0. $\square$

Figure (**3**) gives a decomposition of $\vec{\mathcal{WBF}}_{0,1}(3,2)$ into two 1-crossing Hamilton cycles.

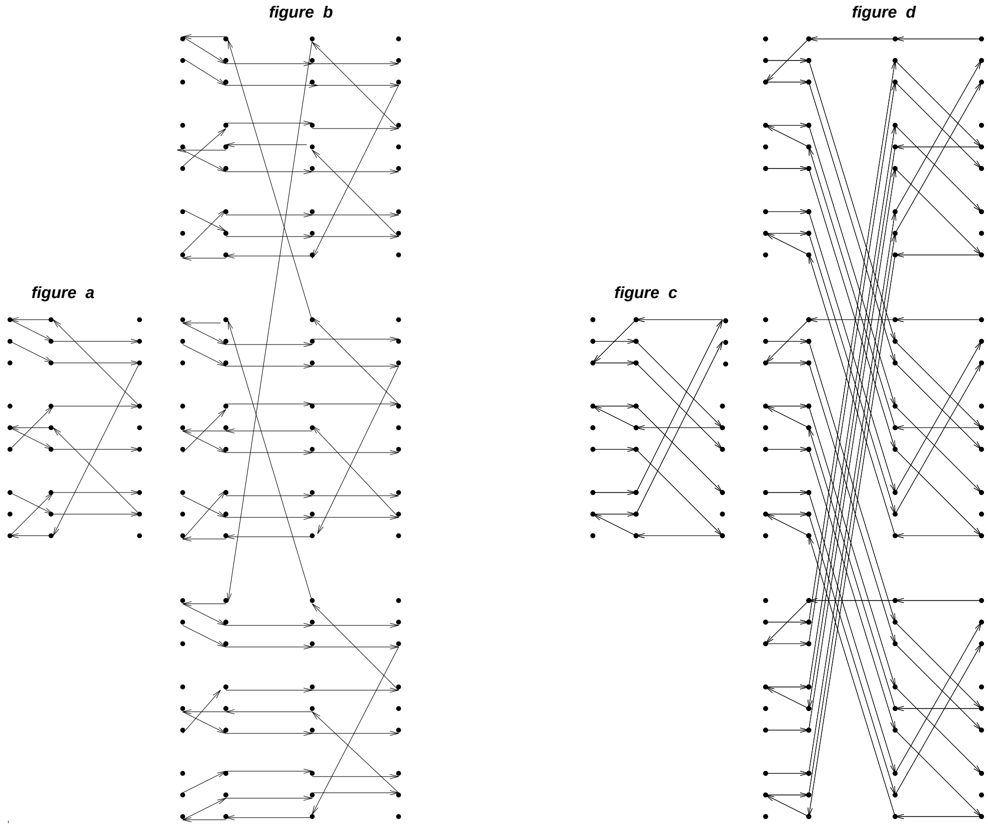

Figure 3: *Figures a and c show the two 1-crossing arc-disjoint Hamilton cycles of* $\vec{\mathcal{WBF}}_{0,1}(3,2)$*. We display on figures b and d, two 1-crossing arc-disjoint Hamilton cycles in* $\vec{\mathcal{WBF}}_{0,1}(3,3)$ *obtained by applying lemma (***2.2***) with the families (***2***).*

Proposition 3.2 *For* $n \geq 2$, $\vec{\mathcal{WBF}}_{0,1}(d,n)$ *is decomposable into l-crossing Hamilton cycles.*

Proof. The proposition is proved for $n = 2$ by the lemma (**3.2**). Then, we use lemma (**2.2**) with the two compatible cyclic-potent families (**2**) in section (**2.1**) which use the slopes $\{0,1\}$ to construct inductively two arcs-disjoint l-crossing Hamilton cycles in $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{0,1}(d,n)$. $\square$

Figure (**3**) gives the recursive construction of two 1-crossing arc-disjoint Hamilton cycles in $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{0,1}(3,3)$ from two 1-crossing arc-disjoint cycles in $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(3,2)$.

3.3 Global Decomposition

We are now ready to prove the main result:

Theorem 3.3 *For $n \geq 2$,*

- *for $d \notin \{3,4,6\}$, $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$ is decomposable into $d-2$ Hamilton circuits and 2 Hamilton cycles,*
- *for $d \in \{4,6\}$, $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$ is decomposable into Hamilton circuits,*
- *$\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(3,n)$ is decomposable into 1 Hamilton circuit and 2 Hamilton cycles.*

Proof. According to propositions (**3.1**) and (**3.2**) we have, when $d \notin \{3,4,6\}$, $d-2$ arc-disjoint circuits in $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{2,\ldots,d-1}(d,n)$ and 2 arc-disjoint cycles in $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}_{0,1}(d,n)$. So, the result holds in these cases. For $d \in \{4,6\}$ and $n=2$, an exhaustive computer search shows that $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$ is decomposable into Hamilton circuits, and so, for $n \geq 2$, $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(4,n)$ and $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(6,n)$ are decomposable into Hamilton circuits. For $d=3$, we can construct two 1-crossing arc-disjoint Hamilton cycles and one arc-disjoint Hamilton circuit in $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(3,2)$ (see figure (**4**)). Then, we can apply lemma (**2.2**) with families (**1**) and the result holds for $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(3,n)$ with $n \geq 2$. $\square$

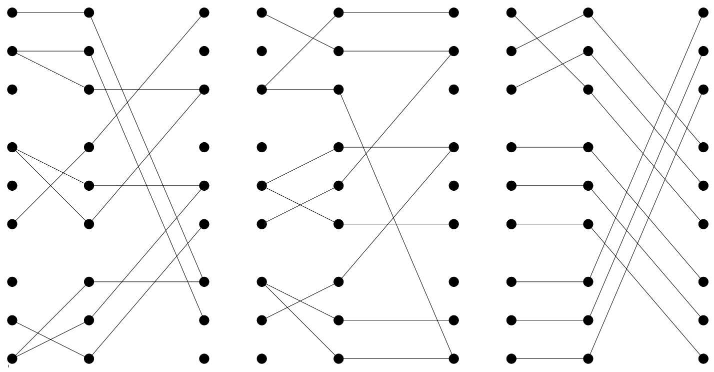

Figure 4: *The decomposition of $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(3,2)$ into two 1-crossing arc-disjoint Hamilton cycles and one arc-disjoint Hamilton circuit.*

The preceding result implies the conjecture of Barth and Raspaud:

Theorem 3.4 *For any d and $n \geq 2$, $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$ is decomposable into Hamilton cycles.*

Remark 3 We could also have derived theorem (**3.4**) by proving that, if $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$ is decomposable into l-crossing Hamilton cycles, then $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n+1)$ is also decomposable into l-crossing Hamilton cycles. This can be done by applying lemma (**2.2**) with the families (**1**) in section (**2.1**). But to start the induction we needed to split the Butterfly digraph into two partial digraphs in order to prove that $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,2)$ is decomposable into l-crossing Hamilton cycles for $n = 2$ and $d \neq 3$.

4 Conclusion

In this paper we have proved that $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$ is always decomposable into Hamilton cycles. However, the problem of decomposing $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,n)$ into Hamilton circuits remains open and is considered in [5]. The difficulty in that case is to start the induction. In fact, we conjecture that $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(d,2)$ is decomposable into Hamilton circuits for $d > 3$. Unfortunately such a decomposition is not yet known, even if in [5] we were able to reduce the problem to the case d prime and to solve it in many cases. Consequently, we propose as open problem the following conjecture:

Conjecture 3 ([5]) *For any prime number $p > 3$, $\mathcal{W}\vec{\mathcal{B}}\mathcal{F}(p,2)$ is decomposable into Hamilton circuits.*

Proving this conjecture would completely close the problem of the Hamilton decomposition of the Butterfly network.

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