

1 Algorithms and complexity for geodetic sets on 2 planar and chordal graphs

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15 — Abstract —

16 We study the complexity of finding the *geodetic number* on subclasses of planar graphs and chordal
17 graphs. A set S of vertices of a graph G is a *geodetic set* if every vertex of G lies in a shortest
18 path between some pair of vertices of S . The MINIMUM GEODETIC SET (MGS) problem is to
19 find a geodetic set with minimum cardinality of a given graph. The problem is known to remain
20 NP-hard on bipartite graphs, chordal graphs, planar graphs and subcubic graphs. We first study
21 MGS on restricted classes of planar graphs: we design a linear-time algorithm for MGS on solid
22 grids, improving on a 3-approximation algorithm by Chakraborty et al. (CALDAM, 2020) and
23 show that MGS remains NP-hard even for subcubic partial grids of arbitrary girth. This unifies
24 some results in the literature. We then turn our attention to chordal graphs, showing that MGS
25 is fixed parameter tractable for inputs of this class when parameterized by their *treewidth* (which
26 equals the clique number minus one). This implies a linear-time algorithm for k -trees, for fixed k .
27 Then, we show that MGS is NP-hard on interval graphs, thereby answering a question of Ekim et
28 al. (LATIN, 2012). As interval graphs are very constrained, to prove the latter result we design a
29 rather sophisticated reduction technique to work around their inherent linear structure.

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36 **1** Introduction

37 A simple undirected graph G has vertex set $V(G)$ and edge set $E(G)$. For two vertices
38 $u, v \in V(G)$, let $I(u, v)$ denote the set of all vertices in G that lie in some shortest path
39 between u and v . For a subset S of vertices of a graph G , let $I(S) = \bigcup_{u, v \in S} I(u, v)$. We
40 say that $T \subseteq V(G)$ is *covered* by S if $T \subseteq I(S)$. A set of vertices S is a *geodetic set* if
41 $V(G)$ is covered by S . The *geodetic number*, denoted $gn(G)$, is the minimum integer k such
42 that G has a geodetic set of cardinality k . Given a graph G , the MINIMUM GEODETIC
43 SET (MGS) problem, introduced in [17], is to compute a geodetic set of G with minimum



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44 cardinality. In this paper, we study the computational complexity of MGS in subclasses of
 45 planar and chordal graphs. MGS is a natural graph covering problem that falls in the class
 46 of problems dealing with the important geometric notion of *convexity*: see [11, 22] for some
 47 general discussion of graph convexities. The setting of MGS is quite natural, and it can be
 48 applied to facility location problems such as the optimal determination of bus routes in a
 49 public transport network [6]. See also [10] for further applications.

50 The algorithmic complexity of MGS has been studied actively. In 1993, Harari, Loukakis
 51 and Tsouros, in [17], proved that MGS is NP-hard. Later, Dourado et al. [8, 9] strengthened
 52 the above result to *bipartite* graphs, *chordal* graphs (*i.e.* graphs with no induced cycle of
 53 order greater than 3) and *chordal bipartite* graphs (*i.e.* bipartite graphs with no induced
 54 cycle of order greater than 4). Recently, Bueno et al. [4] proved that MGS remains NP-hard
 55 for *subcubic* graphs, and Chakraborty et al. [6] proved that MGS is NP-hard for planar
 56 graphs. Kellerhals and Koana [19] studied the parameterized complexity of MGS, proving
 57 that it is unlikely to be FPT for the parameters solution size, feedback vertex set number
 58 and pathwidth, combined.

59 On the positive side, polynomial-time algorithms to solve MGS are known for *cographs* [8],
 60 *split* graphs [8], *ptolemaic* graphs [11], *block cactus* graphs [10], *outerplanar* graphs [21] and
 61 *proper interval* graphs [10], and the problem is FPT for parameters tree-depth and feedback
 62 edge set number [19].

63 A *grid embedding* of a graph is a set of points in two dimensions with integer coordinates
 64 such that each point in the set represents a vertex of the graph and, for each edge, the points
 65 corresponding to its endpoints are at Euclidean distance 1. A graph is a *partial grid* if it has
 66 a grid embedding. A graph is a *solid grid* if it has a grid embedding such that all interior
 67 faces have unit area. Chakraborty et al. [6] gave a 3-approximation algorithm for MGS on
 68 solid grids. We improve this as follows.

69 ► **Theorem 1.** *There is a linear-time algorithm for MGS on solid grids.*

70 We note that researchers have designed polynomial-time algorithms for various problems
 71 on solid grids [12, 20, 24]. Our algorithm on solid grids does not require the grid embedding
 72 to be part of input. This is interesting since deciding whether an input graph is a solid grid
 73 is an NP-complete problem [16]. To complement Theorem 1, we prove the following.

74 ► **Theorem 2.** *MGS is NP-hard for subcubic partial grids of girth at least g , for any fixed*
 75 *integer $g \geq 4$.*

76 We note that this result jointly strengthens three existing hardness results: for bipartite
 77 graphs [8], subcubic graphs [4] and planar graphs [6]. Moreover, partial grids are subclasses
 78 of many other important graph classes such as *disk* graphs, *rectangle intersection* graphs,
 79 etc [7, 23]. Hence, our result implies that MGS remains NP-hard on the aforementioned
 80 graph classes.

81 An *interval representation* of a graph G is a collection of intervals on the real line such
 82 that two intervals intersect if and only if the corresponding vertices are adjacent in G . A
 83 graph is an *interval graph* if it has an interval representation. Ekim et al. [10] asked if there
 84 is a polynomial-time algorithm for MGS on *interval graphs*. We give a negative answer to
 85 their question (note that proper interval graphs are those interval graphs with no induced
 86 $K_{1,3}$).

87 ► **Theorem 3.** *MGS is NP-hard for interval graphs (even with no induced $K_{1,5}$).*

88 This result is somewhat surprising, as most covering problems can be solved in polynomial
 89 time on interval graphs (but other distance-based problems, like METRIC DIMENSION, are

NP-complete for interval graphs [13]). Our reduction (from 3-SAT) uses a quite involved novel technique, that we hope can be used to prove similar results for other distance-related problems on interval graphs. The main challenge here is to overcome the linear structure of the graph to transmit information across the graph. To this end, we use a sophisticated construction of many parallel *tracks*, *i.e.* shortest paths with intervals of (mostly) the same length spanning roughly the whole graph, and such that each track is shifted with respect to the previous one. Each track represents shortest paths that will be used by solution vertices from our variable and clause gadgets. In between the tracks, we are able to build our gadgets.

We remark that MGS admits a polynomial-time algorithm on proper interval graphs by a nontrivial dynamic programming scheme [10]. Problems known to be NP-complete on interval graphs but polynomial-time solvable on proper interval graphs are very rare; two examples known to us are EQUITABLE COLORING [15] and INDUCED SUBGRAPH ISOMORPHISM [18].

To complement Theorem 3, we design an FPT algorithm for MGS on interval graphs when parameterized by its *treewidth* which equals its *clique number* ω minus one. Observe that interval graphs are also *chordal graphs*, *i.e.* graphs without induced cycles of order greater than 3. We use dynamic programming on tree decompositions to prove the following.

► **Theorem 4.** *MGS can be solved in time $2^{2^{O(\omega)}}n$ for chordal graphs and in time $2^{O(\omega)}n$ for interval graphs, where n is the order of the input graph.*

This result applies to the following setting. A k -tree is a graph formed by starting with a complete graph on $(k + 1)$ vertices and then repeatedly adding vertices by making each added vertex adjacent to exactly k neighbors forming a $(k + 1)$ -clique. Allgeier [1] gave a polynomial-time algorithm to solve MGS on *maximal outerplanar* graphs, which is a subclass of 2-trees, and thus our algorithm generalizes this result (note that 2-trees are both chordal and planar). Since all k -trees are chordal graphs, MGS can be solved in time $2^{2^{O(k)}}n$ for k -trees of order n . Recall that this is unlikely to hold for *partial k -trees* (which are exactly the graphs of treewidth at most k) since MGS is $W[1]$ -hard for parameter treewidth [19].

Structure of the paper. In Section 2, we describe the algorithm for solid grids. In Section 3, we present the algorithm for chordal graphs. In Section 4, we prove hardness for partial grids. In Section 5, we prove hardness for interval graphs. We conclude in Section 6. Due to space restrictions, some of the proofs are only sketched. The complete proof details can be found in the full version of this paper: see [5].

2 A linear-time algorithm for solid grids

We here give a linear-time algorithm for MGS on solid grids and prove Theorem 1. In the remainder of the section, G denotes a solid grid and \mathcal{R} its grid embedding. A path P in G is a *corner path* if (i) no vertex of P is a cut-vertex, (ii) both end-vertices of P have degree 2 in G , and (iii) all other vertices of P have degree 3 in G . Chakraborty et al. [6] proved:

► **Lemma 5** ([6]). *Any geodetic set of G contains at least one vertex from each corner path.*

Any geodetic set of G contains all vertices of degree 0 or 1. We say that a vertex v of G is a *corner vertex* if v is an end-vertex of some corner path. All corner vertices can be found in linear time even if the grid embedding is not provided as an input [6].

► **Definition 6.** *We say that u_1, u_2, \dots, u_k forms a corner sequence if for each $1 \leq i \leq k - 1$,*

1. *there is a corner path with u_i and u_{i+1} as endpoints, and*
2. *there is no corner vertex in the clockwise traversal of the boundary of \mathcal{R} from u_i to u_{i+1} .*

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133 A corner sequence is *maximal* if it is not a subsequence of any other corner sequence. For
134 a corner sequence S , let $|S|$ denote the length of S .

135 ► **Lemma 7.** *Let \mathcal{S} be the set of all maximal corner sequences of G , and let t be the number*
136 *of vertices of G with degree 1. Then, $gn(G) \geq t + \sum_{S \in \mathcal{S}} \lfloor |S|/2 \rfloor$.*

137 **Proof.** Any geodetic set of G contains all vertices of degree 1 and therefore $gn(G) \geq t$.
138 Now, let X be any geodetic set of G and $S \in \mathcal{S}$ be an arbitrary maximal corner sequence.
139 Assume that $u_1, u_2, \dots, u_{|S|}$ forms the maximal subsequence S . Lemma 5 implies that for
140 each $1 \leq j < |S|$, at least one vertex of the corner path between u_j and u_{j+1} must belong to
141 X . Observe that two corner paths may have at most one corner vertex in common. Moreover,
142 a corner vertex cannot be in three corner paths. Therefore, X must contain at least $\lfloor \frac{|S|}{2} \rfloor$
143 vertices. Now, let P be a corner path with endpoints a, b and P' be a corner path with
144 endpoints a', b' . If a, b and a', b' are in two different maximal corner subsequences, then P
145 and P' have no vertex in common. ◀

146 Due to space constraints, we only sketch the proof. Let \mathcal{S} be the set of all maximal corner
147 sequences of G . For a maximal corner sequence $S = u_1, u_2, \dots, u_k$ let $f(S)$ denote the set
148 $\{u_2, u_4, \dots, u_{k-k'}\}$ where $k' = 0$ if k is even and $k' = 1$, otherwise. Observe that $|f(S)| = \lfloor \frac{k}{2} \rfloor$.
149 Let V_1 be the set of all vertices of degree 1. Now consider the sets $V_2 = \cup_{S \in \mathcal{S}} f(S)$ and
150 $D = V_1 \cup V_2$. Indeed, one can prove that D is a geodetic set of G and by Lemma 7, we are
151 done.

3 An FPT algorithm for chordal graphs parameterized by clique number

152 We now sketch the main ideas for proving Theorem 4.
153

154 We give an FPT algorithm for chordal graphs parameterized by the clique number (which
155 is also the treewidth plus 1). We explain how to improve the complexity in the case of interval
156 graphs after the proof of the chordal case. Our algorithm performs dynamic programming on
157 a *nice tree decomposition* of the input chordal graph. The main idea behind the algorithm
158 is that the internal bags of the tree-decomposition (*i.e.* those who disconnect the tree into
159 non-empty graphs) induce clique cutsets (cliques whose removal disconnects the graph).
160 Then, for two vertices u, v all whose shortest paths go through some clique cutset X , their
161 behaviour (with respect to MGS and X) can be described in terms of X only. This key
162 observation will be enough to design our algorithm.

163 Nice tree decompositions are a well-known tool for designing dynamic programming
164 algorithms for graphs of bounded treewidth. In our notation, the set of vertices of the graph
165 associated to a node v of the tree, its bag, is denoted X_v . A nice tree decomposition of a
166 chordal graph (see [2]) is a rooted tree decomposition where the bag of every node induces a
167 clique. Each node belong to one of the following types. A node is a *join node* if it has exactly
168 two children, on the same bags as the node. An *introduce node* has a unique child whose
169 bag contains exactly one vertex less. A *forget node* has a unique child whose bag contains
170 exactly one more vertex. A *leaf node* is a leaf of the tree whose bag is empty. The root node
171 is a leaf node.

172 For a nice tree decomposition and a node v , we define $G_{\leq v}$ as the subgraph of G induced
173 by the vertices of the subtree of the decomposition rooted at v . If $u \in V(G)$ and X is a clique,
174 we say that u is *close* to a nonempty set of vertices $A \subseteq X$ with respect to X , if $d(u, x) = d_u$
175 when $x \in A$ and $d(u, x) = d_u + 1$ when $x \in X \setminus A$ (for some integer d_u). Intuitively, if X
176

177 is clique cutset which creates two connected components G_1 and G_2 when removed and if
 178 $u \in V(G_1)$ then the shortest paths from u to some vertices of the other components “tends
 179 to” go through vertices of A .

180 Each maximal clique of a chordal graph will be associated to some node of the tree
 181 decomposition. Intuitively speaking, as in most tree-width based dynamic programming
 182 schemes, we need to show how the number of local solutions (*i.e.* in a bag of the tree) is
 183 bounded by a function of the maximal size of a bag, and how it can be computed from the
 184 information already computed for the node’s children. To do this, we will need the following
 185 lemma, which deals with how shortest paths interact with clique cutsets.

186 **► Lemma 8.** *Let X be a clique cutset of a chordal graph G . Let u, v be two vertices of G
 187 such that all paths from u to v intersect X . Let A (resp. B) be a nonempty set of vertices of
 188 X such that u (resp. v) is close to A (resp. B) with respect to X . Then, a vertex x belongs
 189 to $I(u, v) \cap X$ (where $I(u, v)$ is the set of vertices of G covered by a shortest path from u to
 190 v) if and only if $x \in A \cap B$ or, $A \cap B = \emptyset$ and $x \in A \cup B$.*

191 Lemma 8 implies that to compute an optimal partial solution (*i.e.* a subset of vertices of
 192 $G_{\leq v}$) for a given bag X_v , it is sufficient to “guess” for which subsets A of X_v , there will exist
 193 (in the future solution that will be computed for ancestors of v) a vertex y which is close to
 194 A with respect to X_v . Thus, roughly speaking, it will be sufficient to index our solutions by
 195 types depending on what subsets A of X_v are required to satisfy this property.

196 More precisely, for each node v of our tree decomposition, we compute a table of partial
 197 solutions, indexed by *types*. For a node v , a type $\tau = (\tau^{int}, \tau^{ext}, \tau^{bag}, \tau^{cov})$ is an element of
 198 $\{0, 1\}^{2^{X_v}} \times \{0, 1\}^{2^{X_v}} \times 2^{X_v} \times 2^{X_v}$ where 2^{X_v} is the power set of X_v . We see τ^{int} and τ^{ext} as
 199 Boolean vectors indexed by the elements of 2^{X_v} , and τ^{bag} and τ^{cov} , as subsets of X_v . The
 200 table of partial solutions of the node v , denoted as $sol[v, \tau]$, will contain an optimal partial
 201 solution for X_v of the given type τ , computed using the partial solutions of the children of
 202 v . Our goal is to compute all such partial solutions $sol[v, \tau]$. The partial solution $sol[v, \tau]$,
 203 must follow some additional constraints that we detail below. Note that it is possible that
 204 no partial solution verify those constraints. In this case, $sol[v, \tau]$ is left empty.

205 For a node v of type τ and for a set $A \subseteq X_v$ of vertices, the Boolean $\tau^{int}[A]$ represents
 206 whether there is some $y \in sol[v, \tau]$ such that y is close to A with respect to X_v (“int” stands
 207 for “interior”). For $A \subseteq X_v$, the Boolean $\tau^{ext}[A]$ represents whether we need to add, at a
 208 later step of our algorithm, some vertex y such that y is close to A with respect to X_v . Here,
 209 y is a vertex that needs to be added later to the solution, in the upper part of the tree (“ext”
 210 stands for “exterior”). By Lemma 8, it is not necessary to keep track of all such vertices, as
 211 it is sufficient to record which subsets of vertices of X_v they are close to. This is a crucial
 212 property used to construct our solution: if there exists such a y , then we can cover some
 213 vertices of the subtree using this y , and Lemma 8 tells us exactly how.

214 The set τ^{cov} represents the vertices of X_v that we require to cover with $sol[v, \tau]$. Due to
 215 the existence of join nodes, we might want to cover the other vertices of X_v at a later step of
 216 the algorithm. The set τ^{bag} represents the vertices of X_v in $sol[v, \tau]$, and is essentially used
 217 to know which solutions of the children nodes of a join node can be merged.

218 To formalise the notion of types associated with a node we introduce the following
 219 definition which essentially asserts what our solution table for a given node must satisfy
 220 to be correct. To this end, we define a helper graph H_v^τ that simulates the vertices (using
 221 simplicial vertices) whose types are required to belong to the (future) solution.

222 For a node v , let \mathcal{T}_v denote the set of all types of v . For a fixed τ let H_v^τ denote the graph
 223 obtained by adding a vertex S to $G_{\leq v}$ whenever there is a set $S \subseteq X_v$ with $\tau^{ext}[S] = 1$,

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224 and making S adjacent to each $x \in S$. Let $S_v^\tau = \{S: S \subseteq X_v, \tau^{ext}[S] = 1\}$ denote the newly
225 added vertices.

226 ► **Definition 9.** Let v be a node of T . A 4-tuple $\tau = (\tau^{int}, \tau^{ext}, \tau^{cov}, \tau^{bag})$ of \mathcal{T}_v is a “type
227 associated with v ” if there exists a set $D \subseteq V(H_v^\tau)$ such that the following hold.

- 228 (i) $S_v^\tau \subseteq D$ and $\tau^{bag} = D \cap X_v$.
229 (ii) For a vertex $w \in (V(G_{\leq v}) \setminus X_v) \cup \tau^{cov}$ there exists a pair $w_1, w_2 \in D$ such that $w \in$
230 $I(w_1, w_2)$ and $w_1 \in D \setminus S_v^\tau$.
231 (iii) For a subset $A \subseteq X_v$, we have $\tau^{int}[A] = 1$ if and only if $D \cap V(G_{\leq v})$ contains a vertex
232 which is close to A with respect to X_v .

233 Moreover, we shall say that the set $D \setminus S_v^\tau$ is a “certificate” for (v, τ) .

234 The proof of Theorem 4 boils down to showing, by induction, that it is possible to construct
235 certificates of minimum cardinality for each (valid) pair (v, τ) in a total of $2^{2^{O(\omega(G))}} n$ time.
236 This is possible as the tree decomposition contains a $O(n)$ nodes and for each of them,
237 computing our table for one particular node can be achieved in $2^{2^{O(\omega(G))}}$ time. For the root
238 r , there is only one type τ_0 and therefore the minimum cardinality certificate for (r, τ_0) is an
239 optimal geodetic set of G .

240 For interval graphs, the tree decomposition is a path decomposition. τ^{int} and τ^{ext} can be
241 chosen in $\{0, 1\}^{\mathcal{A}}$, where \mathcal{A} is a set of size $O(\omega)$, reducing the time complexity to $2^{O(\omega)} n$.

242 4 Hardness for partial grids

243 We now prove Theorem 2. Let $\mathcal{PG}(3, g)$ denote the class of subcubic partial grids of girth at
244 least g . Given a graph G , a subset $S \subseteq V(G)$ is a *vertex cover* of G if every edge in $E(G)$
245 has at least one end-vertex in S . The problem MINIMUM VERTEX COVER is to compute
246 a vertex cover of an input graph G with minimum cardinality. To prove Theorem 2, we
247 reduce the NP-hard MINIMUM VERTEX COVER problem on cubic planar graphs [14] to MGS
248 on graphs in $\mathcal{PG}(3, g)$. We use a result of Valiant [25] which says that any planar graph
249 G with maximum degree at most 4 has a drawing using $O(|V(G)|)$ area where vertices are
250 represented as points on the integer grid, and edges are drawn as rectilinear curves on the
251 integer grid.

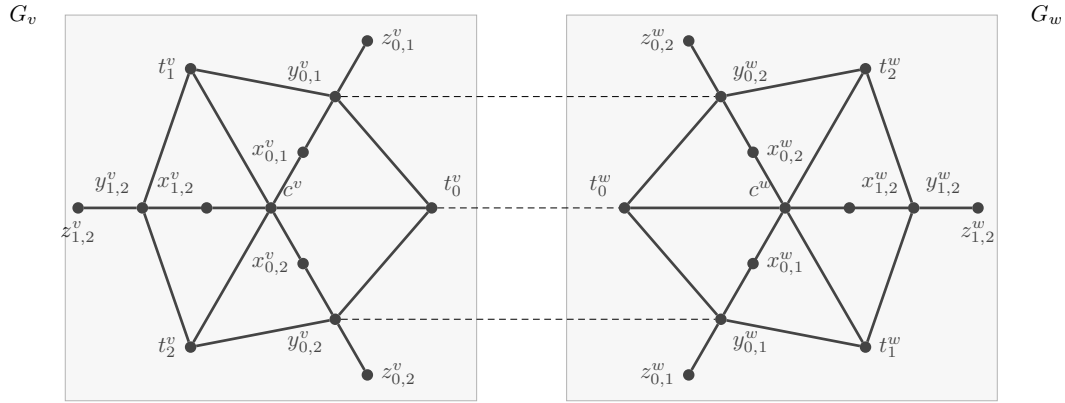
252 Let \mathcal{R} be an embedding of a cubic planar graph G as described above. One can ensure
253 that the distance between two vertices is at least 100, and two parallel lines are at distance
254 at least 100. (Any large constant would be sufficient). We call such an embedding a *good*
255 *embedding* of G . A set S of vertices of a graph is an *edge geodetic set* if every edge lies in
256 some shortest path between a pair of vertices in S . Note that an edge geodetic set is also a
257 geodetic set (if there are no isolated vertices). We need the following lemma.

258 ► **Lemma 10.** Let H be a graph having a geodetic set S which is also an edge geodetic set.
259 If H' denotes the graph obtained by replacing each edge of H with a path having $k \geq 0$ edges,
260 then S is a geodetic set of H' .

261 **Proof.** Let $w \in V(H')$ be a new vertex that was introduced while replacing an edge e of H
262 with a path. Let $u_e, v_e \in S$ such that e belongs to a shortest path P between u_e and v_e . Let
263 P' be the path obtained by replacing each edge of P by a path having k edges. Observe that
264 P' is a shortest path between u_e and v_e in H' . Hence w belongs to a shortest path between
265 u_e and v_e in H' . Thus S is a geodetic set of H' . ◀

266 **Overview of the reduction.** From a cubic planar graph G with a given good embedding,
 267 first we construct a planar graph $f_1(G)$ having maximum degree at most 6 and girth 4.
 268 We show that G has a vertex cover of size k if and only if $f_1(G)$ has a geodetic set of size
 269 $3|V(G)| + k$. Then, we construct a graph $f_2(G) \in \mathcal{PG}(3, 42)$ such that the geodetic numbers
 270 of $f_2(G)$ and $f_1(G)$ are the same. When $g > 42$, we construct a graph $f_3(G) \in \mathcal{PG}(3, g)$ such
 271 that the geodetic numbers of $f_3(G)$ and $f_2(G)$ are the same.

272 **Construction of $f_1(G)$.** From a cubic planar graph G with a given good embedding \mathcal{R} , we
 273 construct a graph $f_1(G)$ as follows. Each vertex v of G will be replaced by a *vertex-gadget*
 274 G_v which is shown in Figure 1. The edges outside of the vertex-gadgets will depend on \mathcal{R} .
 275 We assume that the edges incident with any vertex v are labeled e_i^v with $0 \leq i < 3$, in such a
 276 way that the numbering increases counterclockwise around v with respect to the embedding
 277 (thus the edge vw will have two labels: e_i^v and e_j^w , for some $i, j \in \{1, 2, 3\}$). Consider two
 278 vertices v and w that are adjacent in G , and let e_i^v and e_j^w be the two labels of edge vw
 279 in G . Add the edges (t_i^v, t_j^w) , $(y_{i,i+1}^v, y_{j-1,j}^w)$ and $(y_{i-1,i}^v, y_{j+1,j}^w)$ (See Figure 1). All indices
 280 are taken modulo 3. There are no other edges in $f_1(G)$. Observe that $f_1(G)$ is planar, and
 281 has maximum degree at most 6 and girth 4. We have the following lemma whose proof is
 282 omitted due to space restrictions.

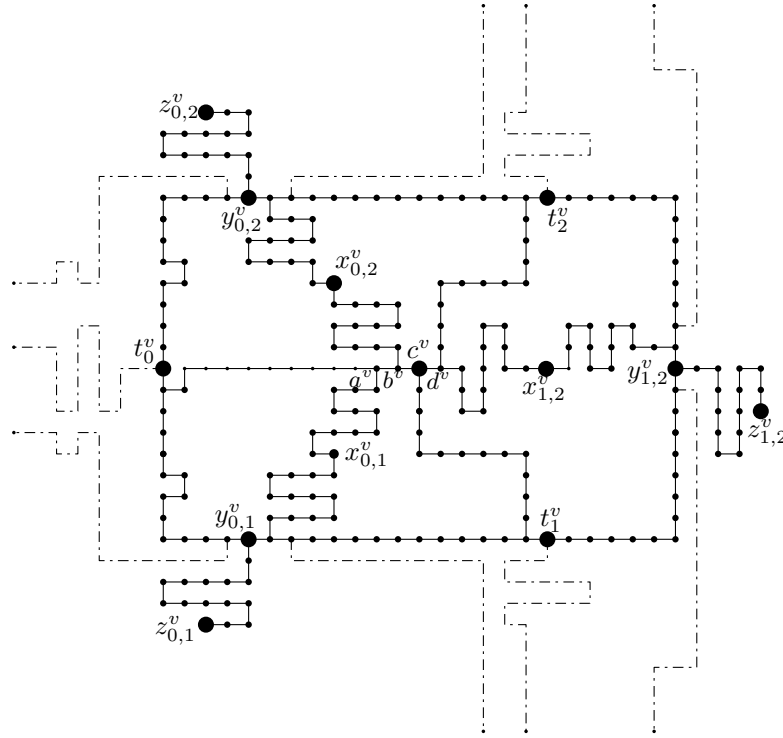


■ **Figure 1** Illustration of vertex-gadgets in the reduction in the proof of Theorem 2. For an edge vw , we have the vertex gadgets G_v and G_w and the dashed lines indicate edges between two vertex-gadgets.

283 ► **Lemma 11.** *The graph G has a vertex cover C of size k if and only if $f_1(G)$ has a geodetic*
 284 *set of size $3|V(G)| + k$.*

285 **Proof.** We construct a geodetic set S of $f_1(G)$ of size $3|V(G)| + k$ as follows. For each vertex
 286 v in G , we add the three vertices $z_{i,j}^v$ ($0 \leq i < j \leq 2$) of G_v to S . If v is in C , we also add
 287 vertex c^v to S .

288 Let us show that S is indeed a geodetic set. First, we observe that in any vertex gadget
 289 G_v that is part of $f_1(G)$, the unique shortest path between two distinct vertices $z_{i,j}^v, z_{i',j'}^v$
 290 has length 4 and goes through vertices $y_{i,j}^v, t_k^v$ and $y_{i',j'}^v$ (where $\{k\} = \{i, j\} \cap \{i', j'\}$). Thus,
 291 it only remains to show that the vertices $\{c^v, x_{i,j}^v\}$ ($0 \leq i < j \leq 2$) belong to some shortest
 292 path of vertices of S . Assume that v is a vertex of G in C . The shortest paths between c^v
 293 and $z_{i,j}^v$ have length 3 and one of them goes through vertex $x_{i,j}^v$. Thus, all vertices of G_v
 294 belong to some shortest path between vertices of S . Now, consider a vertex $w \notin C$ of G .
 295 Since G is a cubic planar graph, all three neighbours of w , say, w_1, w_2, w_3 must lie in C . Let



■ **Figure 2** Construction of $f_2(G)$.

296 $A = \{c^{w_1}, c^{w_2}, c^{w_3}\}$ and $Z = \{z_{0,1}^w, z_{1,2}^w, z_{0,2}^w\}$. Observe that all vertices of G_w lie in the set
 297 $I(A \cup Z)$. Therefore, S is a geodetic set.

298 For the converse, assume we have a geodetic set S' of $f_1(G)$ of size $3|V(G)| + k$. We
 299 will show that G has a vertex cover of size k . First of all, observe that all the $3|V(G)|$
 300 vertices of type $z_{i,j}^v$ are necessarily in S' , since they have degree 1. As observed earlier, the
 301 shortest paths between those vertices already go through all vertices of type t_i^v and $y_{i,j}^v$.
 302 However, no other vertex lies on a shortest path between two such vertices: these shortest
 303 paths always go through the boundary 6-cycle of the vertex-gadgets. Let S'_0 be the set of
 304 the remaining k vertices of S' . These vertices are there to cover the vertices of type c^v and
 305 $x_{i,j}^v$. We construct a subset C' of $V(G)$ as follows: C' contains those vertices v of G whose
 306 vertex-gadget G_v contains a vertex of S'_0 . We claim that C' is a vertex cover of G . Suppose
 307 by contradiction that there is an edge vw of G such that neither G_v nor G_w contains any
 308 vertex of S'_0 . Without loss of generality assume that e_0^v and e_0^w are the two labels of the edge
 309 vw . Here also, the shortest paths between vertices of S always go through the boundary
 310 6-cycle of G_v and thus, they never include vertex $x_{1,2}^v$. Let a and b be the neighbours of v
 311 different from w . Observe that no shortest path between a vertex of G_a and a vertex of G_b
 312 contains the vertex $x_{1,2}^v$, a contradiction. Thus S' is a vertex cover of G . ◀

313 **Construction of $f_2(G)$.** An edge uv of $f_1(G)$ is an *internal* edge if both u and v belong to
 314 G_w for some $w \in V(G)$. The other edges of $f_1(G)$ are *external* edges. We construct $f_2(G)$ in
 315 three steps described below.

- 316 1. Replace each vertex of type t_i^w (for $w \in V(G)$) with a new edge $T_i^w = (t_i^w t_i'^w)$. Replace
 317 each vertex of type $y_{i,j}^w$ with a path $Y_{i,j}^w = a_{i,j}^w y_{i,j}^w b_{i,j}^w d_{i,j}^w$. Replace each vertex of type
 318 c^w with a path $C^w = a^w b^w c^w d^w$. (See Figure 2).
- 319 2. Replace each internal edge between vertices having labels p, q with a new path such that
 320 the shortest path in the new graph between the vertices with label p, q has length 14 (this
 321 constant is required to get a valid good embedding).
- 322 3. For an edge $uv \in E(G)$, let E_{uv} denote the set of three external edges between G_u and
 323 G_v in $f_1(G)$. Recall that \mathcal{R} is a good embedding of G . Let l_{uv} denote the length of the
 324 edge uv in \mathcal{R} . Replace all three external edges $p_i q_i \in E_{uv}$ ($1 \leq i \leq 3$) with three new
 325 paths P_i ($1 \leq i \leq 3$) such that lengths of all three paths are equal and in $O(l_{uv})$.

326 Clearly, $f_2(G_v)$ is a partial grid for each $v \in V(G)$ (Figure 2). It is not difficult to verify
 327 that $f_2(G)$ has maximum degree 3 and girth at least 42. Let C be a vertex cover of G with
 328 cardinality k . We construct a geodetic set S of $f_2(G)$ of cardinality $3|V(G)| + k$ as follows.
 329 For each vertex v in G , we add the three vertices with labels $z_{i,j}^v$ ($0 \leq i < j \leq 2$) to S . If v
 330 is in C , we also add vertex c^v to S . From the construction of $f_2(G)$ and using arguments
 331 similar to that of Lemma 11, G has a vertex cover of size k if and only if $f_2(G)$ has a geodetic
 332 set of size $3|V(G)| + k$. Moreover, we can prove the following.

333 ► **Lemma 12.** *The set S is both a geodetic set of minimum cardinality and an edge geodetic*
 334 *set of minimum cardinality of $f_2(G)$.*

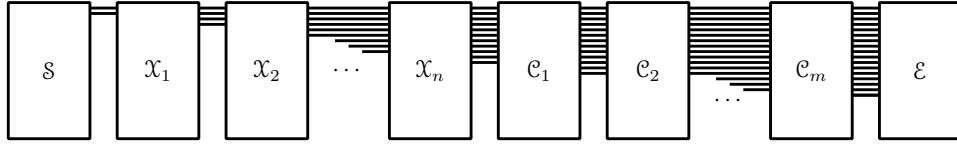
335 **Completion of the proof of Theorem 2.** If $g \leq 42$, then observe that $f_2(G) \in \mathcal{PG}(3, g)$
 336 and from the previous discussions, we have that MGS is NP-hard for graphs in $\mathcal{PG}(3, g)$.
 337 Otherwise, we replace every edge of $f_2(G)$ with a path of length g . Call this modified
 338 graph $f_3(G)$, and observe that $f_3(G) \in \mathcal{PG}(3, g)$. By Lemma 12, S is both a geodetic set of
 339 minimum cardinality and an edge geodetic set of minimum cardinality in $f_2(G)$. Now, due
 340 to Lemma 10, S is a geodetic set of $f_3(G)$ of cardinality $3|V(G)| + k$.

341 5 Hardness for interval graphs

342 We now give a sketch of the proof of Theorem 3. Let F be an instance of 3-SAT with
 343 variables x_1, \dots, x_n and clauses C_1, \dots, C_m . We construct a set D of intervals in polynomial
 344 time such that the geodetic number of the intersection graph of D (denoted as $\mathcal{I}(D)$) is at
 345 most $4 + 7n + 58m$ if and only if F is a positive instance of 3-SAT.

346 The key intuition that explains why the problem is hard on interval graphs, is that
 347 considering two solution vertices x, y , the structure of the covered set $I(x, y)$ can be very
 348 complicated. Indeed, it can be that many vertices lying “in between” x and y in the interval
 349 representation, are not covered. This allows us to construct gadgets, by controlling which
 350 such vertices get covered, and which do not. Moreover, we can easily force some vertices
 351 to be part of the solution by representing them by an interval of length 0 (then, they are
 352 simplicial vertices), which is very useful to design our reduction. Nevertheless, implementing
 353 this idea turns out to be far from trivial, and to this end we need the crucial idea of *tracks*,
 354 which are shortest paths spanning a large part of the construction. Each track starts at a
 355 key interval called its *root* (representing a literal, for example) and serves as a shortest path
 356 from the root to the rightmost end of the construction. In a way, each track “carries the
 357 effect of the root” being chosen in a solution to the rest of the graph. The tracks are shifted
 358 in a way that no shortcut can be used going from one track to another. We are then able to
 359 locally modify the tracks and place our gadgets so that the track of, say, a literal, enables

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■ **Figure 3** Overview of the construction. Boxes represent gadgets, and lines represent tracks.

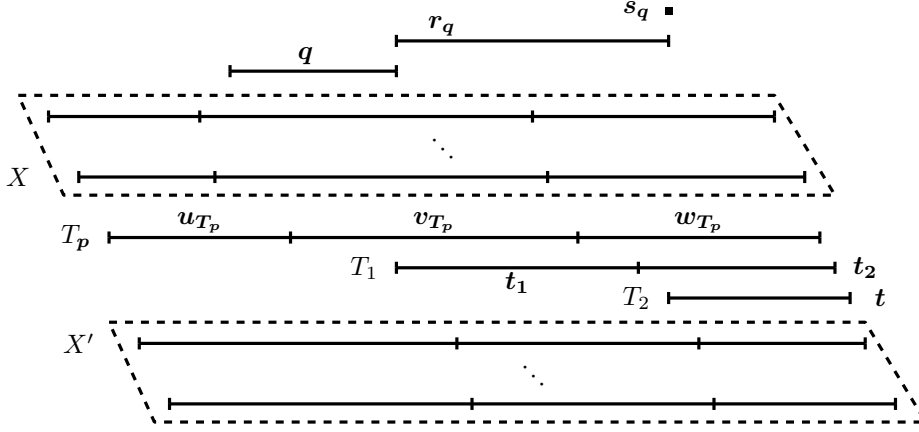
360 the interval of that literal to cover an interval of a specific clause gadget (while the other
 361 tracks are of no use for this purpose).

362 **Overview of the reduction:** There are four stages of our reduction. We initialise it by
 363 constructing a set of intervals which we call the *start gadget* (denoted as S). After creating
 364 the start gadget, we create the variable gadgets, which are placed consecutively, after the start
 365 gadget. For each variable x_i with $1 \leq i \leq n$, we create the variable gadget X_i . Each variable
 366 gadget is composed of several *implication gadgets*. An implication gadget $IMP[\neg p \rightarrow q]$
 367 ensures (under some extra hypotheses) that if p is not chosen in a geodetic set of our
 368 constructed intervals, then q must be chosen. These are used to encode the behaviour of
 369 the variables of the 3-SAT instance: there will be two possible solutions, corresponding to
 370 both truth values of x_i . After creating all the variable gadgets, we create the clause gadgets,
 371 also placed consecutively, after the variable gadgets. For each clause C_j with $1 \leq j \leq m$,
 372 we construct the clause gadget C_j . Each clause gadget is composed of a *covering gadget*,
 373 several implication gadgets and several *AND gadgets*. The covering gadget of a clause C_i is
 374 denoted by $COV[i]$. For two intervals p and q , the corresponding AND gadget is denoted
 375 by $AND[p, q]$. Together, these gadgets will ensure that all intervals of the clause gadget
 376 corresponding to the clause C_i are covered by six intervals if and only if one of the intervals
 377 corresponding to the literals of C_i is chosen in a geodetic set. This encodes the behaviour of
 378 the clauses of the 3-SAT instance.

379 After creating all the clause gadgets, we conclude our construction by creating the *end*
 380 *gadget* E , placed after all clause gadgets. See Figure 3 for a schematic diagram.

381 **Notations:** Let S be a set of intervals with no isolated interval. For a vertex $v \in V(\mathcal{I}(S))$, let
 382 $\mathbf{v} = [min(v), max(v)]$ denote the interval corresponding to v in S , where $min(\mathbf{v})$ and $max(\mathbf{v})$
 383 refer the left boundary and right boundary of \mathbf{v} , respectively. From now on, we only work with
 384 intervals. The *rightmost neighbour* of \mathbf{v} is the interval intersecting \mathbf{v} that has the maximum
 385 right boundary. For a nonempty set S of intervals, let $min(S) = \min\{min(\mathbf{v}) : \mathbf{v} \in S\}$,
 386 $max(S) = \max\{max(\mathbf{v}) : \mathbf{v} \in S\}$. For two intervals \mathbf{u}, \mathbf{v} we have $\mathbf{u} < \mathbf{v}$ if $max(\mathbf{u}) < min(\mathbf{v})$.
 387 Let S be a set of intervals and $\mathbf{u}, \mathbf{v} \in S$. A shortest path between \mathbf{u}, \mathbf{v} is a shortest path
 388 between \mathbf{u}, \mathbf{v} in $\mathcal{I}(S)$. The set $I(\mathbf{u}, \mathbf{v})$ is the set of intervals that belongs to some shortest
 389 path between \mathbf{u}, \mathbf{v} . The geodetic set of S is analogously defined. For a subset S' of S the
 390 phrase “ S is covered by S' ” means that S' is a geodetic set of S . A *point interval* is an
 391 interval of the form $[a, a]$. A *unit interval* is an interval of the form $[a, a + 1]$. A set of
 392 intervals is *proper* if no interval contains another. A set $T = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_t\}$ of intervals is a
 393 *track* if $max(\mathbf{u}_i) = min(\mathbf{u}_{i+1})$ for all $1 \leq i < t$ and no \mathbf{u}_i is a point interval for all $1 \leq i \leq t$.
 394 Observe that if T is a track, then $\mathcal{I}(T)$ is a path. In our construction, each track T will be
 395 associated with a number of intervals called its *roots*. The set of roots of T is denoted by
 396 $R(T)$. For an intuition of how the tracks and roots are used, the track for which \mathbf{v} is the root
 397 will almost entirely be a shortest path from \mathbf{v} to any interval \mathbf{w} to the right of \mathbf{v} (except for
 398 some local shortcuts in the gadgets involving \mathbf{w} , that can be controlled).

399 ► **Definition 13.** Let T and T' be two tracks such that $T \cup T'$ is a proper set of intervals.
 400 Then $T < T'$ if $max(T) < max(T')$.



■ **Figure 4** The implication gadget $IMP[\neg p \rightarrow q]$.

401 Let \mathcal{T} be a set of tracks and $T \in \mathcal{T}$. By construction each $T \cup T'$ will be a proper set
 402 of intervals for $T, T' \in \mathcal{T}$. The phrase “the track just preceding T ” refers to the track T'
 403 such that $T' < T$ and there is no T'' such that $T' < T'' < T$. The phrases “the track just
 404 following T ”, “maximal track of \mathcal{T} ” and “minimal track of \mathcal{T} ” are analogously defined.

405 **Construction of \mathcal{S} .** Let $\epsilon = \frac{1}{(n+m)^4}$ where n is the number of variables and m is the number
 406 of clauses. The start gadget \mathcal{S} consists of four intervals which are defined as follows: the *start*
 407 *interval* $\mathbf{o} = [1, 1]$, $\mathbf{u}_{\mathbf{o}} = [1, 2]$, the *true interval* $\top = [1 + \epsilon, 1 + \epsilon]$ and $\mathbf{u}_{\top} = [1 + \epsilon, 2 + \epsilon]$.
 408 Let $T_1 = \{\mathbf{u}_{\mathbf{o}}\}$ and $T_2 = \{\mathbf{u}_{\top}\}$. Observe that T_1, T_2 are tracks and $T_1 < T_2$.

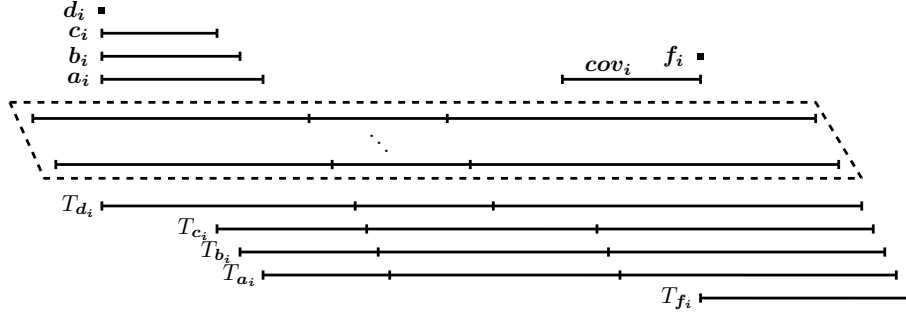
409 We initialize two more sets, the set $\mathcal{T} = \{T_1, T_2\}$ of all tracks, and the set $D = \mathcal{S}$ of
 410 all intervals. As we proceed with the construction, we shall insert more intervals in T_1, T_2
 411 while maintaining that both of them are tracks. We shall also add more tracks in \mathcal{T} . Let
 412 $R(T_1) = \{\mathbf{o}\}$ and $R(T_2) = \{\top\}$.

413 **Implication gadget of a root p .** To construct the variable gadgets and the clause gadgets,
 414 we need to define the implication gadget. On Figure 4, we present the implication gadget of
 415 a root p which is different from \mathbf{o} of \mathcal{S} . The track $T_p \in \mathcal{T}$ is the track such that $p \in R(T_p)$.
 416 Since $p \neq \mathbf{o}$, T_p is not the minimal element in \mathcal{T} . The interval q is a new interval constructed
 417 in the gadget for which we create a track T_1 . The goal of this gadget is to ensure that q is
 418 part of our solution when p is not.

419 **Construction of variable gadgets.** We construct the variable gadgets sequentially and
 420 connect each of them to the previous one (\mathcal{X}_1 is connected to the start gadget \mathcal{S}). Assuming
 421 that we have placed $\mathcal{S}, \mathcal{X}_1, \dots, \mathcal{X}_{i-1}$, we construct \mathcal{X}_i as follows. For variable x_i , the gadget
 422 \mathcal{X}_i consists of two implication gadgets. Let D and \mathcal{T} be the set of intervals and tracks created
 423 so far. First, we construct $IMP[\neg \top \rightarrow \mathbf{x}_i]$. Observe that the sets D and \mathcal{T} have been
 424 updated after the last operation. There is an interval \mathbf{x}_i in D and there is a track $T \in \mathcal{T}$
 425 whose root is \mathbf{x}_i . Now we construct $IMP[\neg \mathbf{x}_i \rightarrow \bar{\mathbf{x}}_i]$. Observe that after constructing all
 426 the variable gadgets, for each literal ℓ , there is an interval named ℓ in D . Also note that
 427 nothing prevent us, at this point, from taking both \mathbf{x}_i and $\bar{\mathbf{x}}_i$ in our solution. This property
 428 will follow from the cardinality constraints on the size of the solution.

429 A clause gadget consists of a covering gadget, several implication and AND gadgets.
 430 **Construction of covering gadgets.** The covering gadget $COV[i]$ of a clause C_i is
 431 presented on Figure 5. In particular, this gadget is used to say that, under some extra

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■ **Figure 5** The covering gadget $\text{COV}[i]$.

432 assumptions, one of a_i , b_i and c_i is in our solution and covers the interval cov_i with a
 433 shortest path to f_i .

434 **Construction of the AND gadget.** For two previously constructed interval p and q , we
 435 can construct a gadget named $\text{AND}[p, q]$. We do not present the construction of this gadget
 436 as it is slightly more complicated than the previous ones. This gadget contains a number
 437 of new intervals. Among them is a particular interval denoted by $\gamma(p, q)$. The role of this
 438 gadget is as follows. If p and q are in our solution then every interval of the gadget is covered.
 439 Otherwise we need to add an interval of the gadget to cover it, namely the interval $\gamma(p, q)$
 440 is sufficient to cover every interval of $\text{AND}[p, q]$.

441 **Construction of \mathcal{C}_i .** We shall complete our construction of clause gadget \mathcal{C}_i corresponding
 442 to the clause $C_i = (\ell_i^1, \ell_i^2, \ell_i^3)$. First, we create the covering gadget $\text{COV}[i]$ and update D, \mathcal{T} .
 443 Recall from the construction of $\text{COV}[i]$ that the intervals named a_i, b_i, c_i exist. Also recall
 444 from the construction of variable gadgets that the intervals $\ell_i^1, \ell_i^2, \ell_i^3$ and $\bar{\ell}_i^1, \bar{\ell}_i^2, \bar{\ell}_i^3$ exist. Now
 445 we create, in this order, $\text{IMP}[-a_i \rightarrow a'_i]$, $\text{AND}[a_i, \ell_i^1]$, $\text{AND}[a'_i, \bar{\ell}_i^1]$, $\text{IMP}[-b_i \rightarrow b'_i]$,
 446 $\text{AND}[b_i, \ell_i^2]$, $\text{AND}[b'_i, \bar{\ell}_i^2]$, $\text{IMP}[-c_i \rightarrow c'_i]$, $\text{AND}[c_i, \ell_i^3]$, $\text{AND}[c'_i, \bar{\ell}_i^3]$ where a'_i, b'_i
 447 and c'_i are three new intervals constructed in the corresponding implication gadgets. The
 448 role of this last part is to ensure that a_i and ℓ_i^1 are either both in the solution or both not in
 449 the solution. The reality is slightly more complex but one should think of a_i as a copy of ℓ_i^1
 450 and a'_i as a copy of $\bar{\ell}_i^1$. The same holds for b_i and c_i . This completes the construction of \mathcal{C}_i .

451 **Construction of end gadget.** For each $T \in \mathcal{T}$, we introduce two new intervals $u_T =$
 452 $[\max(T), \max(T) + 1]$, $e_T = [\max(u_T), \max(u_T)]$ and define $T = T \cup \{u_T\}$, $D = D \cup$
 453 $\{u_T, e_T\}_{T \in \mathcal{T}}$. For each $T \in \mathcal{T}$, let e_T be the *tail* of T . The end gadget \mathcal{E} consists of all the
 454 new intervals created above. The role of this gadget is to ensure that every interval belonging
 455 to a track is covered.

456 **5.1 Analysis**

457 First, note that there are $2 + 4n + 35m$ tracks in \mathcal{T} and $n_{\text{point}} = 4 + 6n + 52m$ point intervals
 458 in D . The total number of intervals in D is $O((n + m)^2)$. Remark that the point intervals
 459 are exactly the simplicial vertices of D , hence they belong to every geodetic set of D . Let
 460 T be a track in \mathcal{T} such that $T = \{u_1, \dots, u_k\}$ with $\max(u_i) = \min(u_{i+1})$ ($1 \leq i \leq k - 1$).
 461 Observe that for each i with $1 \leq i \leq k - 1$, u_{i+1} is the rightmost neighbour of u_i .

462 ► **Proposition 14.** *Let u and v be two intervals of D such that $\min(u) < \min(v)$. The
 463 path u_0, u_1, \dots, u_k, v is a shortest path from u to v (where $u = u_0$, u_{i+1} is the rightmost
 464 neighbour of u_i for $i \in 1 \leq i \leq k - 1$, and $u_{k-1} \notin N(v)$, while $u_k \in N(v)$). We say that*

465 *such a path is a good shortest path.*

466 An interval \mathbf{u} is a *track interval* if $\mathbf{u} \in T$ for some $T \in \mathcal{T}$. From now on, U shall denote
467 the set of track intervals. Let S_p be the set of all point intervals and recall that S_p is a
468 subset of every geodetic set of D . For an interval \mathbf{z} , let $T(\mathbf{z})$ denote the track T such that
469 $\mathbf{z} \in R(T)$. From our construction, one can observe that all track intervals are covered by
470 pair of vertices in S_p .

471 ► **Proposition 15.** *For an implication gadget $IMP[-\mathbf{p} \rightarrow \mathbf{q}]$, let T be the track with root \mathbf{q} .
472 Then $\mathbf{q} \in I(\mathbf{p}, \mathbf{s}_q)$, $\mathbf{r}_q \in I(\mathbf{o}, \mathbf{s}_q)$.*

473 ► **Proposition 16.** *Consider the cover gadget $COV[i]$ and let $\mathbf{z} \in \{\mathbf{a}_i, \mathbf{d}_i, \mathbf{c}_i\}$. Then $\mathbf{z} \in$
474 $I(\mathbf{d}_i, \mathbf{e}_{T(\mathbf{z})})$ and $\mathbf{cov}_i \in I(\mathbf{z}, \mathbf{f}_i)$.*

475 ► **Proposition 17.** *Consider an AND gadget $AND[\mathbf{p}, \mathbf{q}]$. The set $\{\mathbf{p}, \mathbf{q}\} \cup S_p$ covers all
476 vertices in $AND[\mathbf{p}, \mathbf{q}]$. The set $\{\gamma(\mathbf{p}, \mathbf{q})\} \cup S_p$ covers all vertices in $AND[\mathbf{p}, \mathbf{q}]$ where
477 $\gamma(\mathbf{p}, \mathbf{q})$ is a vertex of $AND[\mathbf{p}, \mathbf{q}]$.*

478 We shall show that if F is satisfiable, then D has a geodetic set of cardinality $4+7n+58m =$
479 $n_{point} + n + 6m$. Let $\phi: \{x_1, x_2, \dots, x_n\} \rightarrow \{0, 1\}$ be a satisfying assignment of F (we
480 also define $\phi(\bar{x}_i) = 1 - \phi(x_i)$). Now, define the following sets. Let $S_1 = \{\mathbf{x}_i: \phi(x_i) =$
481 $1\} \cup \{\bar{\mathbf{x}}_i: \phi(\bar{x}_i) = 1\}$. Let $S_2 = \emptyset$. Now, for each clause $C_i = (\ell_i^1, \ell_i^2, \ell_i^3)$, and for each
482 $(\mathbf{v}, \mathbf{v}', \ell, \bar{\ell}) \in \{(\mathbf{a}_i, \mathbf{a}'_i, \ell_i^1, \bar{\ell}_i^1), (\mathbf{b}_i, \mathbf{b}'_i, \ell_i^2, \bar{\ell}_i^2), (\mathbf{c}_i, \mathbf{c}'_i, \ell_i^3, \bar{\ell}_i^3)\}$, if $\phi(\ell) = 1$, then put $S_2 =$
483 $S_2 \cup \{\mathbf{v}, \gamma(\mathbf{v}', \bar{\ell})\}$ otherwise put $S_2 = S_2 \cup \{\mathbf{v}', \gamma(\mathbf{v}, \ell)\}$. We have that $|S_1 \cup S_2 \cup S_p| =$
484 $4 + 7n + 58m$.

485 ► **Lemma 18.** *The set S is a geodetic set of D .*

486 **Proof.** As $S_p \subseteq S$, we know that all track intervals of D are covered. Moreover, every interval
487 of the form \mathbf{r}_q is covered by Proposition 15. Consider any variable gadget \mathcal{X}_i corresponding to
488 the variable x_i . Recall from construction, that $\mathcal{X}_i = IMP[-\top \rightarrow \mathbf{x}_i] \cup IMP[-\mathbf{x}_i \rightarrow \bar{\mathbf{x}}_i]$. Due
489 to Proposition 15, we have that $\mathbf{x}_i \in I(\top, \mathbf{e}_{T(\mathbf{x}_i)})$. Hence, all intervals of $IMP[-\top \rightarrow \mathbf{x}_i]$
490 are covered by S . Due to Proposition 15, either $\bar{\mathbf{x}}_i \in I(\mathbf{x}_i, \mathbf{e}_{T(\bar{\mathbf{x}}_i)})$ when $\mathbf{x}_i \in S$ or $\bar{\mathbf{x}}_i \in S$
491 otherwise. The above arguments imply that all intervals in \mathcal{X}_i are covered by S .

492 Now, consider any clause $C_i = (\ell_i^1, \ell_i^2, \ell_i^3)$ and recall the construction of \mathcal{C}_i . Observe
493 that there exists at least one interval $\mathbf{z} \in \{\mathbf{a}_i, \mathbf{b}_i, \mathbf{c}_i\} \cap S$. Using Proposition 16, we can
494 infer that all intervals in $COV[i]$ are covered by S . Now, consider the implication gadget
495 $IMP[-\mathbf{a}_i \rightarrow \mathbf{a}'_i]$, note that $\{\mathbf{a}_i, \mathbf{a}'_i\} \cap S \neq \emptyset$ and therefore using Proposition 15, we can
496 infer that all intervals in $IMP[-\mathbf{a}_i \rightarrow \mathbf{a}'_i]$ are covered by S . Repeating the above arguments
497 for $IMP[-\mathbf{b}_i \rightarrow \mathbf{b}'_i]$ and $IMP[-\mathbf{c}_i \rightarrow \mathbf{c}'_i]$, we infer that all intervals in these implication
498 gadgets are covered by S . Now, consider the AND gadget $AND[\mathbf{a}_i, \ell_i^1]$. From our definition
499 of S_2 , it follows that either $\{\mathbf{a}_i, \ell_i^1\} \subseteq S$ or $\gamma(\mathbf{p}, \mathbf{q}) \in S$. In both cases, we can use
500 Proposition 17 to show that all intervals in $AND[\mathbf{a}_i, \ell_i^1]$ are covered by S . Repeating the
501 above arguments for all the AND gadgets in \mathcal{C}_i , we can show that all intervals of \mathcal{C}_i are
502 covered by S . ◀

503 Now, we show that if the geodetic number of D is at most $4 + 7n + 58m$, then F is
504 satisfiable. The next proposition is key in showing this (recall that U contains all track
505 vertices). It is proven by considering an interval \mathbf{v} with minimum $\min(\mathbf{v})$ contradicting the
506 statement.

507 ► **Proposition 19.** *There is a minimum-size geodetic set S^* of D such that $S^* \cap U = \emptyset$.*

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508 A good geodetic set of D is a geodetic set of minimum cardinality which does not contain
509 any interval belonging to a track (*i.e.* intervals of U).

510 ► **Proposition 20.** Let S^* be a good geodetic set of D and $IMP[\neg \mathbf{p} \rightarrow \mathbf{q}]$ be an implication
511 gadget where \mathbf{p} is the only root of $T(\mathbf{p})$. Then, either $\mathbf{p} \in S^*$ or $\mathbf{q} \in S^*$.

512 ► **Proposition 21.** Let S^* be a good geodetic set of D and let $C_i = (\ell_i^1, \ell_i^2, \ell_i^3)$ be a clause.
513 Then $|S^* \cap C_i| \geq 6$. Moreover if none of $\ell_1^i, \ell_2^i, \ell_3^i$ is in S^* then $|S^* \cap C_i| \geq 7$.

514 ► **Lemma 22.** If there is a good geodetic set of D with cardinality $4 + 7n + 58m$, then F is
515 satisfiable.

516 **Proof.** Let S^* be a good geodetic set of D with cardinality $4 + 7n + 58m$. Recall that
517 a variable gadget \mathcal{X}_i is $IMP[\neg \top \rightarrow \mathbf{x}_i] \cup IMP[\neg \mathbf{x}_i \rightarrow \bar{\mathbf{x}}_i]$. Due to Proposition 20, we
518 know that at least one among $\{\mathbf{x}_i, \bar{\mathbf{x}}_i\}$ lies in S^* . Let $S_1 = (S^* \setminus S_p) \cap (\cup_{1 \leq i \leq n} \mathcal{X}_i)$, and
519 $S_2 = (S^* \setminus S_p) \cap (\cup_{1 \leq i \leq m} C_i)$. Note that $S_1 \cup S_2 \cup S_p \subseteq S^*$. We have $|S_1| \geq n$, $|S_2| \geq 6m$ by
520 Proposition 21, and $|S_p| = 4 + 6n + 52m$. Therefore, $|S_1| = n$ as $|S^*| \leq 4 + 7n + 58m$. This
521 means that for each $1 \leq i \leq n$, exactly one of $\mathbf{x}_i, \bar{\mathbf{x}}_i$ lies in S^* . Based on these, we define the
522 following truth assignment $\phi: \{x_1, \dots, x_n\} \rightarrow \{1, 0\}$ of F . Define $\phi(x_i) = 1$ if $\mathbf{x}_i \in S^*$ and
523 $\phi(x_i) = 0$, otherwise. Using Proposition 21 we can infer that for each $1 \leq i \leq m$, we have
524 that $|S^* \cap C_i| = 6$ and at least one of the intervals $\ell_1^i, \ell_2^i, \ell_3^i$ lies in S^* . Thus, for at least one
525 literal ℓ_i^j , we have that $\phi(\ell_i^j) = 1$, as needed. ◀

6 Conclusion

526 We gave a polynomial-time algorithm for MGS on solid grids and proved that MGS is
527 NP-hard on partial grids and interval graphs. We proved that MGS is FPT on chordal
528 graphs when parameterized by the clique number.

529 Are there FPT algorithms for MGS on interval graphs, chordal graphs, partial grids,
530 planar graphs when parameterized by the geodetic number?

531 Assuming the *Exponential Time Hypothesis*, our reduction implies that there cannot be
532 a $2^{o(\sqrt{n})}$ time algorithm for MGS on interval graphs of order n . Are there subexponential
533 time algorithms for MGS on interval graphs or chordal graphs, matching this lower bound?
534 (This is the case for many graph problems for geometric intersection graphs, see [3].)
535

536 We have seen that for every k , MGS is solvable in time $f(k)n$ for k -trees, but such a
537 running time is unlikely to be possible for *partial* k -trees, since MGS is known to be W[1]-hard
538 for parameter tree-width [19]. However, there could still exist an XP-time algorithm for
539 MGS, running in time $n^{g(k)}$ on partial k -trees. In fact, it is already unknown whether MGS
540 is solvable in polynomial time on partial 2-trees (also known as series-parallel graphs or
541 K_4 -minor-free graphs).

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