

On Cartesian products of signed graphs

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Abstract. In this paper, we study the Cartesian product of signed graphs as defined by Germina, Hameed and Zaslavsky (2011). Here we focus on its algebraic properties and look at the chromatic number of some products. One of our main result is the unicity of the prime factor decomposition of signed graphs. This leads us to present an algorithm to compute this decomposition in quasi-linear time. Both these results use their counterparts for ordinary graphs as building blocks. We also study the signed chromatic number of graphs with underlying graph of the form $P_n \square P_m$, of products of signed paths, of products of signed complete graphs and of products of signed cycles, that is the minimum order of a signed graph to which they admit a homomorphism.

1 Introduction

Signed graphs were introduced by Harary in [8]. Later, the notion of homomorphism of signed graphs was introduced by Guenin [7], and latter studied by Naserasr, Rollová and Sopena [14] and gave rise to a notion of signed chromatic number $\chi_s(G, \sigma)$ of a signed graph (G, σ) defined as the smallest order of a signed graph (H, π) to which (G, σ) admits a homomorphism.

In this paper, we are interested in the study of the Cartesian product of signed graphs, defined by Germina, Hameed and Zaslavsky in [6]. They mainly study the spectral properties of the Cartesian product.

The Cartesian product of two ordinary graphs G and H , noted $G \square H$, has been extensively studied. In 1957, Sabidussi [15] showed that $\chi(G \square H) = \max(\chi(G), \chi(H))$ where $\chi(G)$ is the chromatic number of the graph G . Another notable article on the subject by Sabidussi [16] shows that every connected graph G admits a unique prime decomposition, *i.e.*, there is a unique way to write a graph G as a product of some graphs up to isomorphism of the factors. This result was also independently discovered by Vizing in [17]. Another algebraic property, the cancellation property, which states that if $A \square B = A \square C$, then $B = C$, was proved by Imrich and Klavžar [10] using a technique of Fernández, Leighton and López-Presa [5]. On the complexity side, the main question associated with the Cartesian product is to be able to decompose a graph with the best possible complexity. The complexity of this problem has been improved successively in [4,18,3,1] to finally reach an optimal complexity of $O(m)$ in [11] where m is the number of edges of the graph.

Our study of the Cartesian product of signed graphs is divided in several sections. First in section 2, we present general definitions of graph theory and

set our notation. In section 3, we present some useful results on signed graphs and on the Cartesian product of graphs. In section 4, we present the definition of the Cartesian product of signed graphs and give some first properties and easy consequences of the definition. We also prove the prime decomposition theorem for signed graphs and give an algorithm to decompose a signed product into its factors. We study the signed chromatic number of products of complete graphs in section 5 and products of cycles in section 6. Finally we present some concluding remarks in section 7.

2 Definitions and notation

All graphs we consider are undirected, simple and loopless. For classical graph definitions, we refer the reader to [2].

A *homomorphism* from G to H is a function φ from $V(G)$ to $V(H)$ such that for all $x, y \in V(G)$, $xy \in E(G)$ implies $\varphi(x)\varphi(y) \in E(H)$. In this case, we note $G \rightarrow H$. Note that $\chi(G)$, the *chromatic number* of G (see [2]), can also be defined as the smallest order of a graph H such that $G \rightarrow H$. An *isomorphism* from G to H is a bijection φ from $V(G)$ to $V(H)$ such that for all $x, y \in V(G)$, $xy \in E(G)$ if and only if $\varphi(x)\varphi(y) \in E(H)$. In this case, we note $G = H$.

A *walk* in a graph G is a sequence s_0, \dots, s_n of vertices of G such that $s_i s_{i+1} \in E(G)$. Its *starting vertex* is s_0 and its *end vertex* is s_n . A *closed walk* is a walk where its starting vertex s_0 and its end vertex s_n are identified (*i.e.* $s_0 = s_n$). If all elements of a walk are pairwise distinct, then the walk is a *path*. A closed walk where all elements are pairwise distinct (except s_0 and s_n) is a *cycle*. Suppose W is a walk s_0, \dots, s_n , we define the *length* of W by its number of edges n (taken with multiplicity) and the *order* of W by its number of vertices (again taken with multiplicity). Note that a walk has order $n + 1$ while a closed walk has order n since we consider that s_0 and s_n to count for only one vertex.

A graph is *connected* if for all pairs of vertices $u, v \in V(G)$, there is a path between u and v . If $X \subseteq V(G)$, then the graph $G[X]$ is the subgraph of G induced by X . We say that $G[X]$ is an *induced subgraph* of G . The *complete graph* K_p is the graph of order p such that for all pair of distinct vertices of G , u and v , uv is an edge of K_p .

A *signed graph* (G, σ) is a graph G along with a function $\sigma : E(G) \rightarrow \{+1, -1\}$ called its *signature* and $\sigma(e)$ is the *sign* of the edge $e \in E(G)$. The edges in $\sigma^{-1}(+1)$ are the *positive* edges and the edges in $\sigma^{-1}(-1)$ are the *negative* edges of (G, σ) . We often write a signed graph (G, σ) as (G, Σ) in place of (G, σ) where Σ is the set of negative edges $\sigma^{-1}(-1)$. These two ways to represent a signed graph are equivalent and will be used interchangeably. We note K_p^+ (resp. K_p^-) the complete graph (K_p, \emptyset) (resp. $(K_p, E(K_p))$) of order p with only positive (resp. negative) edges.

Suppose that (G, σ) is a signed graph and W is a walk s_0, \dots, s_n in G . We say that W is a *balanced walk* if $\sigma(W) = \sigma(s_0 s_1) \sigma(s_1 s_2) \dots \sigma(s_i s_{i+1}) \dots \sigma(s_{n-1} s_n) = 1$ and an *unbalanced walk* otherwise. Similarly, this notion can be extended to

closed walks, paths and cycles. We note an unbalanced path (resp. balanced path) of order k by UP_k (resp. BP_k) and an unbalanced cycle (resp. balanced cycle) of order k by UC_k (resp. BC_k). Similarly, we can define an unbalanced (resp. balanced) path UP_k (resp. BP_k) of order k . A signed graph where all closed walks are balanced (resp. unbalanced) is said to be *balanced* (resp. *antibalanced*). Generally, for the same ordinary graph G , there are several signatures σ such that (G, σ) is balanced. In particular it is the case for all signatures of a forest. These notions of balanced and antibalanced were introduced by Harary in [8].

This led Zaslavsky in [19] to define the notion of equivalent signed graphs. Two signed graphs (G, σ_1) and (G, σ_2) on the same underlying graph are *equivalent* if they have the same set of balanced closed walks. Note that this is equivalent to having the same set of balanced cycles. In this case we note $(G, \sigma_1) \equiv (G, \sigma_2)$. We also say that the two signatures σ_1 and σ_2 (resp. Σ_1 and Σ_2) are equivalent and we note $\sigma_1 \equiv \sigma_2$ (resp. $\Sigma_1 \equiv \Sigma_2$).

Let (G, σ) be a signed graph and v be a vertex of G . *Switching* (G, σ) at v creates the signed graph (G, σ') where $\sigma'(e) = -\sigma(e)$ if e incident to v and $\sigma'(e) = \sigma(e)$ otherwise. One can check that the switch operation does not modify the set of balanced closed walks as switching at a vertex of a closed walk does not change the sign of this walk. Moreover, two signed graphs are equivalent if and only if one can be obtained from the other by a sequence of switches [19]. This means that we can work with the balance of closed walks or with the switches depending on which notion is the easiest to use.

A *homomorphism* of a signed graph (G, σ) to a signed graph (H, π) is a homomorphism φ from G to H which maps balanced (resp. unbalanced) closed walks of (G, σ) to balanced (resp. unbalanced) closed walks of (H, π) . Alternatively, a homomorphism from (G, σ) to (H, π) is a homomorphism from G to H such that there exists a signature σ' of G with $\sigma' \equiv \sigma$, such that if e is an edge of G , then $\pi(\varphi(e)) = \sigma'(e)$. When there is a homomorphism from (G, σ) to (H, π) , we note $(G, \sigma) \rightarrow_s (H, \pi)$ and say that (G, σ) *maps to* (H, π) . Here (H, π) is the *target graph* of the homomorphism. When constructing a homomorphism, we can always fix a given signature of the target graph as proven in [14].

The *signed chromatic number* $\chi_s(G, \sigma)$ of a signed graph (G, σ) is the smallest k for which (G, σ) admits a homomorphism to a signed graph (H, π) of order k . Alternatively, a signed graph (G, σ) admits a k -*(vertex)-colouring* if there exists $\sigma' \equiv \sigma$ such that (G, σ') admits a proper vertex colouring $\theta : V(G) \rightarrow \llbracket k \rrbracket$ verifying that for every $i, j \in \llbracket k \rrbracket$, all edges uv with $\theta(u) = i$ and $\theta(v) = j$ have the same sign in (G, σ') . Then $\chi_s(G, \sigma)$ is the smallest k such that (G, σ) admits a k -vertex-colouring. The two definitions are equivalent, as with any colouring of a signed graph, we can associate a signed homomorphism which identifies the vertices with the same colour. The homomorphism is well defined as long as the target graph is simple, which is the case here by definition of a k -vertex-colouring.

Suppose that G and H are two ordinary graphs. The *Cartesian product* of G and H is the graph $G \square H$ whose vertex set is $V(G) \times V(H)$ and where (x, y)

and (x', y') are adjacent if and only if $x = x'$ and y is adjacent to y' in H or $y = y'$ and x is adjacent to x' in G .

A graph G is *prime* if there are no graphs A and B on at least two vertices for which $G = A \square B$. A decomposition D of a graph G is a multi-set $\{G_1, \dots, G_k\}$ such that the G_i 's are graphs containing at least one edge and $G = G_1 \square \dots \square G_k$. A decomposition is *prime* if all the G_i 's are prime. The G_i 's are called *factors* of G . A decomposition D is *finer* than another decomposition D' if $D' = \{G_1, \dots, G_k\}$ and for each i there exists a decomposition D_i of G_i such that $D = \bigcup_i D_i$. Note that by definition, every decomposition is finer than itself.

Suppose that G is a graph and $D = \{G_1, \dots, G_k\}$ is a decomposition of G such that $G = G_1 \square \dots \square G_k$. A *coordinate system* for G under decomposition D is a bijection $\theta : V(G) \rightarrow \prod_{i=1}^k V(G_i)$ verifying that for each vertex v of G , the set of vertices which differ from v by the i th coordinate induces a graph noted G_i^v called a G_i -*layer* which is isomorphic to G_i by the projection on the i th coordinate. We say that a subgraph X of G is a copy of G_i if X is a G_i -layer of G . An edge uv of G is a copy of an edge ab of G_i if $\theta(u)$ and $\theta(v)$ differ only in their i th coordinate with $u_i = a$ and $v_i = b$.

3 Preliminary results

The goal of this section is to present useful results on signed graphs and on the Cartesian product.

In [19], Zaslavsky gave a way to determine if two signed graphs are equivalent in linear time. In particular, all signed forests with the same underlying graph are equivalent. This theorem comes from the following observation.

Lemma 1 (Zaslavsky [19]). *If C is a cycle of a graph G , then the parity of the number of negative edges of C in (G, σ) is the same in all (G, σ') with $\sigma' \equiv \sigma$.*

This implies that we can separate the set of all cycles into four families BC_{even} , BC_{odd} , UC_{even} and UC_{odd} , depending on the parity of the number of negative edges (even for BC_{even} and BC_{odd} and odd for UC_{even} and UC_{odd}) and the parity of the length of the cycle (even for BC_{even} and UC_{even} and odd for BC_{odd} and UC_{odd}).

Theorem 1. *If (C, σ) is a signed cycle with $(C, \sigma) \in BC_{even}$, then $\chi_s(C, \sigma) = 2$. If $(C, \sigma) \in BC_{odd} \cup UC_{odd}$, then $\chi_s(C, \sigma) = 3$. Finally, if $(C, \sigma) \in UC_{even}$, then $\chi_s(C, \sigma) = 4$.*

Proof. By [13], we already have the upper bounds. A homomorphism of signed graphs is also a homomorphism of graphs thus $\chi(C) \leq \chi_s(C, \sigma)$. If $(C, \sigma) \equiv UC_{2q}$ suppose its signed chromatic number is less than 3. Then $(C, \sigma) \rightarrow_s (H, \pi)$ where (H, π) is a triangle or a path. In each case, (H, π) can be switched to be all positive or all negative. This means that (C, σ) can be switched to be all positive or all negative, which is not the case, a contradiction. We get the desired lower bounds in each case. \square

One of the first results on Cartesian products is a result from Sabidussi on the chromatic number of the product of two graphs.

Theorem 2 (Sabidussi [15]). *If G and H are two graphs, then $\chi(G \square H) = \max(\chi(G), \chi(H))$.*

Following this paper, Sabidussi proved one of the most important results on the Cartesian product: the unicity of the prime decomposition of connected graphs. This result was also proved by Vizing.

Theorem 3 (Sabidussi [16] and Vizing [17]). *Every connected graph G admits a unique prime decomposition up to the order and isomorphisms of the factors.*

Using some arguments of [5] and the previous theorem, Imrich and Klavžar proved the following theorem.

Theorem 4 (Imrich and Klavžar in [9] and [10]). *If A , B and C are three ordinary graphs such that $A \square B = A \square C$, then $B = C$.*

The unicity of the prime decomposition raises the question of the complexity of finding such a decomposition. The complexity of decomposing algorithms has been extensively studied. The first algorithm, by Feigenbaum *et al.* [4], had a complexity of $O(n^{4.5})$ where n is the order of the graph (its size is denoted by m). In [18], Winkler proposed a different algorithm improving the complexity to $O(n^4)$. Then Feder [3] gave an algorithm in $O(mn)$ time and $O(m)$ space. The same year, Aurenhammer *et al.* [1] gave an algorithm in $O(m \log n)$ time and $O(m)$ space. The latest result is this time optimal algorithm.

Theorem 5 (Imrich and Peterin [11]). *The prime factorization of connected graphs can be found in $O(m)$ time and space. Additionally a coordinate system can be computed in $O(m)$ time and space.*

4 Cartesian products of signed graphs

We recall the definition of the signed Cartesian product.

Definition 1 ([6]). *Let (G, σ) and (H, π) two signed graphs. We define the signed Cartesian product of (G, σ) and (H, π) , and note $(G, \sigma) \square (H, \pi)$, the signed graph with vertex set $V(G) \times V(H)$. It has for positive (resp. negative) edges the pairs $\{(u_1, v_1), (u_2, v_2)\}$ such that $u_1 = u_2$ and $v_1 v_2$ is a positive (resp. negative) edge of (H, π) or such that $v_1 = v_2$ and $u_1 u_2$ is a positive (resp. negative) edge of (G, σ) . The underlying graph of $(G, \sigma) \square (H, \pi)$ is the ordinary graph $G \square H$.*

Using this definition, we can derive that this product is associative and commutative.

The following result shows that it is compatible with the homomorphism of signed graphs and in particular with the switching operation.

Theorem 6. *If (G, σ) , (G', σ') , (H, π) , (H', π') are four signed graphs such that $(G, \sigma) \rightarrow_s (G', \sigma')$ and $(H, \pi) \rightarrow_s (H', \pi')$, then:*

$$(G, \sigma) \square (H, \pi) \rightarrow_s (G', \sigma') \square (H', \pi').$$

Proof. By commutativity of the Cartesian product and composition of signed homomorphisms, it suffices to show that $(G, \sigma) \square (H, \pi) \rightarrow_s (G', \sigma') \square (H, \pi)$. Since $(G, \sigma) \rightarrow_s (G', \sigma')$, there exists a set S of switched vertices and a homomorphism φ from G to G' such that if (G, σ_S) is the signed graph obtained from (G, σ) by switching at the vertices of S , then $\sigma'(\varphi(e)) = \sigma_S(e)$ for every edge e of G . We note $P = (G, \sigma) \square (H, \pi)$ and $X = \{(g, h) \in V(G \square H) \mid g \in S\}$. Let P' be the signed graph obtained from P by switching at the vertices in X .

If $(g, h)(g, h')$ is an edge of P , then in P' this edge was either switched twice if $g \in S$ or not switched if $g \notin S$. In both cases its sign did not change. If $(g, h)(g', h)$ is an edge of P , then in P' this edge was switched twice if $g, g' \in S$, switched once if $g \in S, g' \notin S$ or $g \notin S, g' \in S$, and not switched if $g, g' \notin S$. In each case its new sign is $\sigma_S(gg')$. Thus $P' = (G, \sigma_S) \square (H, \pi)$. Now define $\varphi_P(g, h) = (\varphi(g), h)$. It is a homomorphism from $G \square H$ to $G' \square H$ by definition. By construction, the target graph of φ_P is $(G', \sigma') \square (H, \pi)$ as the edges of H do not change and the target graph of φ is (G', σ') . \square

As mentioned, we can derive the following corollary from Theorem 6.

Corollary 1. *If (G, σ) , (G, σ') , (H, π) , (H, π') are four signed graph such that $\sigma \equiv \sigma'$ and $\pi \equiv \pi'$, then:*

$$(G, \sigma) \square (H, \pi) \equiv (G, \sigma') \square (H, \pi').$$

One first observation is that we can apply Theorem 6 to the case of forests.

Corollary 2. *If (G, σ) is a signed graph and (F, π) is a signed forest with at least one edge, then:*

$$\chi_s((G, \sigma) \square (F, \pi)) = \chi_s((G, \sigma) \square K_2).$$

In particular, for $n, m \geq 2$, $\chi_s((P_n, \sigma_1) \square (P_m, \sigma_2)) = 2$.

Note that there is a difference between considering the chromatic number of a signed product and the chromatic number of a signed graph with the product graph as underlying graph. For example, $BC_4 = K_2 \square K_2$ but $\chi_s(UC_4) \neq \chi_s(BC_4)$. Another example: $\chi_s(P_n \square P_m) = 2$, for any $n, m \in \mathbb{N}$ but the following theorem shows that not all signed grids have chromatic number 2.

Theorem 7. *If n, m are two integers and (G, σ) is a signed grid with $G = P_n \square P_m$, then $\chi_s(G) \leq 6$. If n or m is less than 4, then $\chi_s(G) \leq 5$. Moreover there exist grids with signed chromatic number 5.*

Due to page constraints, we do not include the proof of this result. Nonetheless, the proof is available in the full version of this paper which you can find on the author's web page. Note that we are not aware of a signed grid with signed chromatic number six.

Question 1. What is the maximal value of $\chi_s(G, \sigma)$ when (G, σ) is a signed grid? Is it 5 or 6?

Our goal is now to prove that each connected signed graph has a unique prime s -decomposition. Let us start with some definitions.

Definition 2. A signed graph (G, σ) is said to be s -prime if and only if there do not exist two signed graphs (A, π_A) and (B, π_B) such that $(G, \sigma) \equiv (A, \pi_A) \square (B, \pi_B)$. An s -decomposition D of a signed connected graph (G, σ) is a multi-set $\{(G_1, \pi_1), \dots, (G_k, \pi_k)\}$ such that:

1. the (G_i, π_i) 's are signed graphs containing at least one edge and
2. $(G, \pi) \equiv (G_1, \pi_1) \square \dots \square (G_k, \pi_k)$.

An s -decomposition D is prime if all the (G_i, π_i) 's are s -prime. The (G_i, π_i) 's are called factors of D . An s -decomposition D is finer than another s -decomposition D' if $D' = \{(G_1, \pi_1), \dots, (G_k, \pi_k)\}$ and for each $i \in \llbracket 1, k \rrbracket$ there exists an s -decomposition D_i of (G_i, π_i) such that $D = \bigcup_i D_i$. Recall that, if D is finer than D' , then we may have $D = D'$.

Suppose $D = \{G_1, \dots, G_k\}$ is a decomposition of a graph G . We say that two copies X_1 and X_2 of G_i are adjacent by G_j if and only if there exists an edge ab of G_j such that for all $u \in V(G_i)$ and u_1, u_2 its corresponding vertices in X_1 and X_2 , $u_1 u_2$ is a copy of ab . In other words, the subgraph induced by the vertices of X_1 and X_2 is isomorphic to $G_i \square K_2$ where the edge K_2 corresponds to a copy of an edge of G_j .

Note that if $G = A \square B$, then it is not always true that (G, σ) is the product of two signed graphs. For example, for $(G, \sigma) \equiv UC_4$, it is s -prime but C_4 is not a prime graph as $C_4 = K_2 \square K_2$.

We now present a useful tool to show that a signed graph is a product.

Lemma 2. If (G, σ) , (A, π_A) and (B, π_B) are three connected signed graphs with $G = A \square B$, then we have $(G, \sigma) \equiv (A, \pi_A) \square (B, \pi_B)$ if and only if:

1. all copies of A are equivalent to (A, π_A) ,
2. one copy of B is equivalent to (B, π_B) and
3. for each edge e of A , for each pair of distinct copies e_1, e_2 of e , if these two edges belong to the same square, then it is a BC_4 .

Proof. (\Rightarrow) This follows from the definition of the Cartesian product.

(\Leftarrow) We will do the following switches: switch all copies of A to have the same signature π_A .

Now we claim that all copies of B have the same signature π'_B equivalent to π_B . Indeed take one edge xy of B and two copies of this edge $x_1 y_1$ and $x_2 y_2$ in G . Take a shortest path from x_1 to x_2 in the first copy of A . Now if u_1, u_2 are two consecutive vertices along the path and v_1 and v_2 are their copy in the second copy of A , then $u_1 u_2 v_2 v_1$ is a BC_4 by the third hypothesis.

As u_1u_2 and v_1v_2 have the same sign by the previous switches, it must be that u_1v_1 and u_2v_2 have the same sign. Thus all copies of an edge of B have the same sign.

Thus $(G, \sigma) \equiv (A, \pi_A) \square (B, \pi'_B) \equiv (A, \pi_A) \square (B, \pi_B)$ by Theorem 6. \square

One of our main results is the prime decompositions theorem.

Theorem 8. *If (G, σ) is a connected signed graph and D is the prime decomposition of G , then (G, σ) admits a unique (up to isomorphism of the factors) prime s -decomposition D_s . Moreover if we see D_s as a decomposition, then D is finer than D_s .*

For its proof, we need the following lemma.

Lemma 3. *If (G, σ) is a connected signed graph that admits two prime s -decomposition D_1 and D_2 , then there is a signed graph (X, π_X) such that $(G, \sigma) \equiv (X, \pi_X) \square (Y, \pi_Y)$ with $D_1 = \{(X, \pi_X)\} \cup D'_1$ and $D_2 = \{(X, \pi_X)\} \cup D'_2$ where D'_1 and D'_2 are two decompositions of (Y, π_Y) .*

Proof. Suppose there exists a signed graph (G, σ) that admits two s -decompositions D_1 and D_2 . Fix an edge e , e belongs to some factor Z of the prime decomposition of G . The edge e belongs to some copy of some signed graph (A, π_A) in D_1 and to some copy of (B, π_B) in D_2 . The graph Z is a factor of A and B by unicity of the prime factor decomposition of G . Let X be the greatest common divisor of A and B , we have $e \in E(X)$ as $e \in E(Z)$. Now $G = X \square Y$ for some graph Y . Let us show that $(G, \sigma) \equiv (X, \pi_X) \square (Y, \pi_Y)$ for some signature π_X and π_Y of X and Y . We can suppose that $Y \neq K_1$, as otherwise the result is trivial.

First we want to show that all copies of X have equivalent signatures. Take two adjacent copies of X , if they are in different copies of A , then they are equivalent since they represent the same part of (A, π_A) . If they are in the same copy of A , then they are in different copies of B since X is the greatest common divisor of A and B . The same argument works in this case. Thus two adjacent copies of X are two equivalent copies of some signed graph (X, π_X) , and since there is only one connected component in Y , all copies of X have equivalent signatures.

Let π_Y be the signature of one copy of Y . Fix e an edge of X and X_1 and X_2 two copies of X . Now consider the square containing the two copies of this edge (if exists), if X_1 and X_2 are in different copies of A , then this is a BC_4 by Lemma 2 in G , otherwise it is a BC_4 as they are in different copies of B .

By Lemma 2, we can conclude that $(G, \sigma) \equiv (X, \pi_X) \square (Y, \pi_Y)$.

Now suppose that $A = X \square W$, then we can use Lemma 2 to show $(A, \pi_A) \equiv (X, \pi_X) \square (W, \pi_W)$ as all copies of X have equivalent signatures by $(G, \sigma) \equiv (X, \pi_X) \square (Y, \pi_Y)$ and all C_4 between two copies of an edge are BC_4 by the same argument. As (A, π_A) is s -prime this is absurd, so $(X, \pi_X) \equiv (A, \pi_A)$. Thus $(X, \pi_X) \equiv (A, \pi_A) \equiv (B, \pi_B)$ and this proves the lemma. \square

Proof of Theorem 8. Any signed graph has a prime s -decomposition by taking an s -decomposition of (G, σ) that cannot be refined. If we take a prime

s -decomposition D_s of (G, σ) , then it is a decomposition of G thus the prime decomposition of G is finer than D_s .

It is left to show that the prime s -decomposition of (G, σ) is unique. Suppose, to the contrary, that (G, σ) is a minimal counterexample to the unicity. Thus (G, σ) has two prime s -decompositions D_1 and D_2 and by Lemma 3, $(G, \sigma) \equiv (X, \pi_X) \sqcup (Y, \pi_Y)$ with $D_1 = \{(X, \pi_X)\} \cup D'_1$ and $D_2 = \{(X, \pi_X)\} \cup D'_2$ where D'_1 and D'_2 are two decompositions of (Y, π_Y) . By minimality of (G, σ) , $D'_1 = D'_2$ in Y . Thus $D_1 = D_2$, a contradiction. \square

Note that Theorem 8 implies the following result.

Theorem 9. *If (A, π_A) , (B, π_B) and (C, π_C) are three signed graphs with $(A, \pi_A) \sqcup (B, \pi_B) \equiv (A, \pi_A) \sqcup (C, \pi_C)$, then $(B, \pi_B) \equiv (C, \pi_C)$.*

The proof of this result is exactly the same as the proof for ordinary graphs presented in [10]. Indeed, we have all the necessary tools for the proof. The first one is Theorem 8, the other one is the semi-ring structure of signed graphs (quotiented by the equivalence relation) with the disjoint union and the Cartesian product which follows from the definition. See [10] for more details.

In the last part of this section, we propose an algorithm to decompose connected signed graphs. Decomposing a graph can be interpreted in multiple ways: finding a decomposition, identifying which edge of G belongs to which factor, or even better getting a coordinate system that is compatible with the decomposition. In [11], Imrich and Peterin gave an $O(m)$ time and space (m is the number of edges of G) algorithm for these three questions for ordinary graphs.

Our goal is to give an algorithm that does as much for signed graphs based on their algorithm. Due to size constraints, we do not give all the details of the algorithm but only the key ideas. Nonetheless, the proof is available in the full version of this paper which you can find on the author's web page.

A coordinate system for an s -decomposition D of (G, σ) is a coordinate system for D seen as an ordinary decomposition such that there exists a fixed equivalent signed graph (G, σ') of (G, σ) in which all layers have the same signature. This definition is very similar to the definition of a coordinate system for ordinary graphs but differ by the fact that we can switch our graph in order to have a fixed signature for which all isomorphisms are tested.

The idea of the algorithm is to start by partitioning (G, σ) with the prime decomposition of G and to merge some factors in order to get the prime s -decomposition of (G, σ) . If at one point of the algorithm we get an s -decomposition, then we can stop as all factors must be prime by the merging rules. As we remarked before, UC_4 's are s -prime graphs thus in an s -decomposition of a signed graph, the UC_4 are included in the prime factors, this means that finding a UC_4 in (G, σ) between two distinct factors of our current decomposition implies that these two factors (of the ordinary graph) belong to the same s -prime factor of the signed graph.

Another reason to merge is when one factor X has two copies that do not represent the same signed graph. Then we need to find two adjacent copies X_1

and X_2 of X adjacent by factor Y , and we must merge the factor X with the factor Y . This can be tested efficiently in $O(m)$. Note that the proof of this fact amounts to finding a UC_4 between the two factors but it is faster to keep this remark as a separate rule. Note that it might be possible to use techniques of [12] to improve the time complexity to an optimal $O(m)$ by avoiding to use the second merging rule.

With these two merging rules we can prove that once no merging is needed to be done, we get the prime s -decomposition of (G, σ) .

Theorem 10. *Let (G, σ) be a connected signed graph of order n and size m . If k is the number of factors in the prime decomposition of G , then we can find in time $O(mk) = O(m \log n)$ and space $O(m)$ the prime s -decomposition of (G, σ) and a coordinate system for this decomposition.*

5 Signed chromatic number of Cartesian products of complete graphs

In this section, we show a simple upper bound on the signed chromatic number of a product and compute the signed chromatic number of some special complete graphs. We start by defining a useful tool on signed graphs.

In what follows we define the notion of an s -redundant set in a signed graph. Informally, it is a set of vertices such that removing them does not remove any distance two constraint between the pairs of vertices left in the graph.

Definition 3. *Let (G, σ) be a signed graph and $S \subseteq V(G)$. We say that the set S is s -redundant if and only if, for every $x, y \in V(G) - S$ such that $xy \notin E(G)$, for every $z \in S$ and for all signatures σ' with $\sigma' \equiv \sigma$, in the signed graph (G, σ') , $xzy = UP_3$ implies that there exists $w \in V(G) - S$ such that $xwy = UP_3$.*

The following proposition is an alternative formulation of the definition which is useful in order to prove that a set is an s -redundant set.

Proposition 1. *If (G, σ) is a signed graph and $S \subseteq V(G)$, then S is s -redundant if and only if for every $z \in S$, and every $x, y \in N(z) \setminus S$ with $xy \notin E(G)$, there exists $w \in V(G) \setminus S$ such that $xwyz$ is a BC_4 .*

Proof. Take $x, y \in V(G) - S$ such that $xy \notin E(G)$ and $z \in S$. If $xzy = UP_3$ in a signature $\sigma' \equiv \sigma$, then $x, y \in N(z)$. Now if S is a redundant set, then with the notation of the definition $xzyw$ is a BC_4 in (G, σ') and thus in (G, σ) . If $xzyw$ is a BC_4 and xzy is a UP_3 in a given signature σ' , then xwy is also a UP_3 as $xzyw$ is balanced. This proves the equivalence between the two statements. \square

The next theorem is the reason why we defined this notion. It allows us to compute an upper bound of the chromatic number of a graph given the chromatic number of one of its subgraphs. One example of utilisation of this notion is given by the proof of Theorem 12.

Theorem 11. *If (G, σ) is a signed graph and S is an s -redundant set of (G, σ) , then:*

$$\chi_s(G, \sigma) \leq |S| + \chi_s((G, \sigma) - S).$$

Proof. Let φ be a homomorphism from $(G, \sigma) - S$ to a signed graph (H, π) of order $\chi_s((G, \sigma) - S)$. We construct a homomorphism φ' of (G, σ) as follows:

$$\varphi'(v) = \begin{cases} \varphi(v) & \text{if } v \notin S, \\ v & \text{otherwise.} \end{cases}$$

This homomorphism is well defined. To show this we need to prove that the image of φ' has no loops and no UC_2 (i.e. is simple). It does not create loops as φ does not. As φ is well defined, if there was a UC_2 , it would come from identification of two vertices x and y in $(G, \sigma) - S$ and $z \in S$ such that xzy is a UP_3 . By definition of S , if so, there would be a UC_2 in $(G, \sigma) - S$, which cannot be.

Thus φ is a well defined homomorphism. Thus $\chi_s(G, \sigma) \leq |S| + \chi_s((G, \sigma) - S)$. \square

This result does not hold for any set S . For example, if $(G, \sigma) = UC_4$ and $S = \{v\}$ is a single vertex of G , then $\chi_s(G, \sigma) = 4$ but $\chi_s((G, \sigma) - v) \leq 3$.

As a direct corollary of Theorem 6, we get the following upper bound on the chromatic number of a product of signed graphs.

Corollary 3. *If $(G_1, \sigma_1), \dots, (G_k, \sigma_k)$ are k signed graphs, then:*

$$\chi_s((G_1, \sigma_1) \square \dots \square (G_k, \sigma_k)) \leq \prod_{1 \leq i \leq k} \chi_s(G_i, \sigma_i).$$

We study the product of balanced and antibalanced complete graphs in our next result. Recall that K_p^+ (resp. K_q^-) is the complete graph with only positive edges (resp. negative edges).

Theorem 12. *For every two integers p, q with $p, q \geq 2$, we have*

$$\chi_s(K_p^+ \square K_q^-) = \left\lceil \frac{pq}{2} \right\rceil.$$

Proof. Let us note $(P, \pi) = K_p^+ \square K_q^-$. First let us show that $\chi_s(P, \pi) \geq \lceil \frac{pq}{2} \rceil$.

Suppose it is not the case. By the pigeon hole principle, there exist x, y and z three vertices of the product that were identified. These vertices cannot be in the same lines or columns as these induce cliques. Consider the subgraph (H, σ) of the product composed of vertices which are in the same line as one of x, y, z and in the same column as one of x, y and z . We have $(H, \sigma) = K_3^+ \square K_3^-$ (see Figure 1).

By assumption x, y and z of (H, σ) are identified. By the pigeon hole principle, two of x, y and z have been switched the same way. Without loss of generality suppose they are x and y . Then if a is one of their common neighbours in H , the

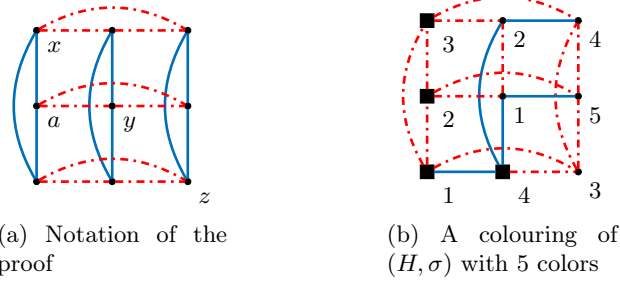


Fig. 1: The graph (H, σ) of the proof of the Theorem 12. The big squared vertices have been switched.

edges xa and ya are of different signs, thus x and y cannot be identified. This is a contradiction.

Now let us show that $\chi_s(P, \pi) \leq \lceil \frac{pq}{2} \rceil$. By symmetry suppose that $p \geq q$. We will prove this statement by induction. If $p = 2$, then $(P, \pi) \equiv BC_4$ and $\chi(P, \pi) = 2 \leq 2$. If $p = 3$ and $q = 2$, then $(P, \pi) \equiv BC_3 \square K_2$ whose chromatic number is $3 \leq 3$. If $p = 3$ and $q = 3$, then $(P, \pi) \equiv (H, \sigma)$ from the previous part of the proof. We have $\chi_s(P, \pi) = 5$, indeed Figure 1 gives a 5-colouring of (P, π) .

Now we can assume that $p \geq 4$. Suppose the vertices of P are labelled $v_{(i,j)}$ for $1 \leq i \leq p$ and $1 \leq j \leq q$ such that the labelling corresponds to the product. Now switch all vertices in $\{v_{(i,j)} \mid i = 1\}$. Now identify $v_{(1,j)}$ with $v_{(2,j+1)}$ (which are non adjacent) where the indices are taken modulo q to get the graph (P', π') . Let S be the set of identified vertices in (P', π') . We want to show that S is s -redundant to use the induction hypothesis. Take $z \in S$ and $x, y \in N(z) \setminus S$ such that $xy \notin E(P)$. If xzy is an unbalanced path of length 2, then x is some $v_{(i,j)}$ and y is some $v_{(k,j+1)}$ with $i, k \geq 3$. For $a = v_{(i,j+1)}$, $xayz$ is a BC_4 .

By Proposition 1, S is s -redundant thus $\chi_s(P, \pi) \leq \chi_s(P', \pi') \leq |S| + \chi_s((P', \pi') - S)$ by Theorem 11. By induction hypothesis, as $(P', \pi') - S = K_{p-2}^+ \square K_q^-$, we get $\chi_s((P', \pi') - S) \leq \lceil \frac{(p-2)q}{2} \rceil$. Thus $\chi_s(P, \pi) \leq q + \lceil \frac{pq}{2} \rceil - q \leq \lceil \frac{pq}{2} \rceil$. \square

For this product the upper bound of Corollary 3 is pq . We thus have an example where the chromatic number is greater than half the simple upper bound.

Question 2. What is the supremum of the set of $\lambda \in [\frac{1}{2}, 1]$ such that there exist arbitrarily big signed graphs (G_1, σ_1) and (G_2, σ_2) such that $\chi_s((G_1, \sigma_1) \square (G_2, \sigma_2)) \geq \lambda \cdot \chi_s(G_1, \sigma_1) \cdot \chi_s(G_2, \sigma_2)$?

$(G, \sigma) \square (H, \pi)$	BC_{even}	BC_{odd}	UC_{even}	UC_{odd}
BC_{even}	2	3	4	3
BC_{odd}	3	3	5	5
UC_{even}	4	5	4	5
UC_{odd}	3	5	5	3

Table 1: The signed chromatic number for each type of products of two cycles.

6 Signed chromatic number of Cartesian products of cycles

The goal of this section is to determine the chromatic number of the product of two cycles. As there are four kind of cycles (balanced/unbalanced and of even/odd length), we have a number of cases to analyse. In most cases some simple observations are sufficient to conclude. For the other cases, we need the following lemma.

Lemma 4. *For every two integers $p, q \in \mathbb{N}$:*

$$\chi_s(UC_q \square BC_{2p+1}) > 4.$$

Due to size constraints we do not include the proof of Lemma 4. Nonetheless, the proof is available in the full version of this paper which you can find on the author's web page. With this lemma, we can state this section's main result.

Theorem 13. *If (G, σ) and (H, π) are two signed cycles, then the signed chromatic number of $(P, \rho) = (G, \sigma) \square (H, \pi)$ is given by Table 1.*

Proof. If G is a cycle of type BC_{even} (resp. $BC_{odd}, UC_{even}, UC_{odd}$), then $G \rightarrow_s BC_2 = K_2$ (resp. BC_3, UC_4, UC_3). By computing the signed chromatic numbers of the products of (G, σ) and (H, π) when they belong to $\{K_2, BC_3, UC_4, UC_3\}$, we get an upper bound for each of the product type equal to the corresponding value in the table. These cases are represented in Figure 2. Note that to color some graphs, we switched at some vertices.

For the lower bound, we can see that $\chi_s((G, \sigma) \square (H, \pi)) \geq \max(\chi_s(G, \sigma), \chi_s(H, \pi))$. Note that $\chi_s(BC_{even}) = 2$, $\chi_s(BC_{odd}) = 3$, $\chi_s(UC_{even}) = 4$ and $\chi_s(UC_{odd}) = 3$ by Theorem 1. Lemma 4 allows us to conclude for the remaining cases as $\chi_s(UC_q \square BC_{2p+1}) = \chi_s(UC_q \square UC_{2p+1})$ by symmetry between the two edge types. \square

One further question would be to compute the signed chromatic number of an arbitrary number of signed cycles. An interesting remark is that $BC_3 \square BC_3 \rightarrow_s BC_3$. This is also true for K_2, UC_3 and UC_4 . Thus if we suppose that the length of the cycles does not impact the result outside of their parity, the only interesting case for upper bounds would be to compute $\chi_s(BC_3 \square UC_3 \square UC_4)$.

We now give the idea of the proof of Lemma 4. The proof is by contradiction. If $\chi_s(UC_q \square BC_{2p+1}) > 4$, then $(P, \rho) = UC_q \square BC_{2p+1} \xrightarrow{s} (T, \theta)$ where (T, θ) is a complete graph of order 4. By looking at which (T, θ) are suitable targets, we get that (T, θ) must be the graph K_4 where only one edge is negative. (P, ρ) must be switched into another signed graph (P, ρ') which maps to (T, θ) without switches. By counting the negative edges in (P, ρ') in different ways we get our contradiction.

In particular, we use topological arguments on (P, ρ') seen as a toroidal grid. We look at connected components of the negative edges and create a set of closed walks “surrounding” them. These closed walks are bipartite, this fact gives us constraints on how they can wrap around the torus. These constraints allows us to count the negative edges one way. The other way to count is directly given by the definition of the product.

7 Conclusion

To conclude, in this paper, we showed a number of results on Cartesian products of signed graphs. We proved a number of algebraic properties: Theorem 6, Theorem 8 and Theorem 9. We also presented an algorithm to decompose a signed graph into its factors in time $O(m \log n)$. This complexity is theoretically not optimal, thus we can ask the question: can we decompose a signed graph in linear time?

Finally, we computed the chromatic number of products: products of any graphs by a signed forest, products of signed paths, signed graphs with underlying graph $P_n \square P_m$, products of some signed complete graphs and products of signed cycles. We also presented a tool called a s -redundant set that helped us compute chromatic number of signed graphs. It would be interesting to know how close to the simple upper bound can be the chromatic number of the product of two signed graphs. It is not clear if there is a sequence of signed graphs approaching this bound or if there is a constant $\lambda < 1$ such that for all signed graphs (G, Σ) and (H, Π) , $\chi_s((G, \Sigma) \square (H, \Pi)) \leq \lambda \max(\chi_s(G, \Sigma), \chi_s(H, \Pi))$. It would also be interesting to compute the chromatic number of more products.

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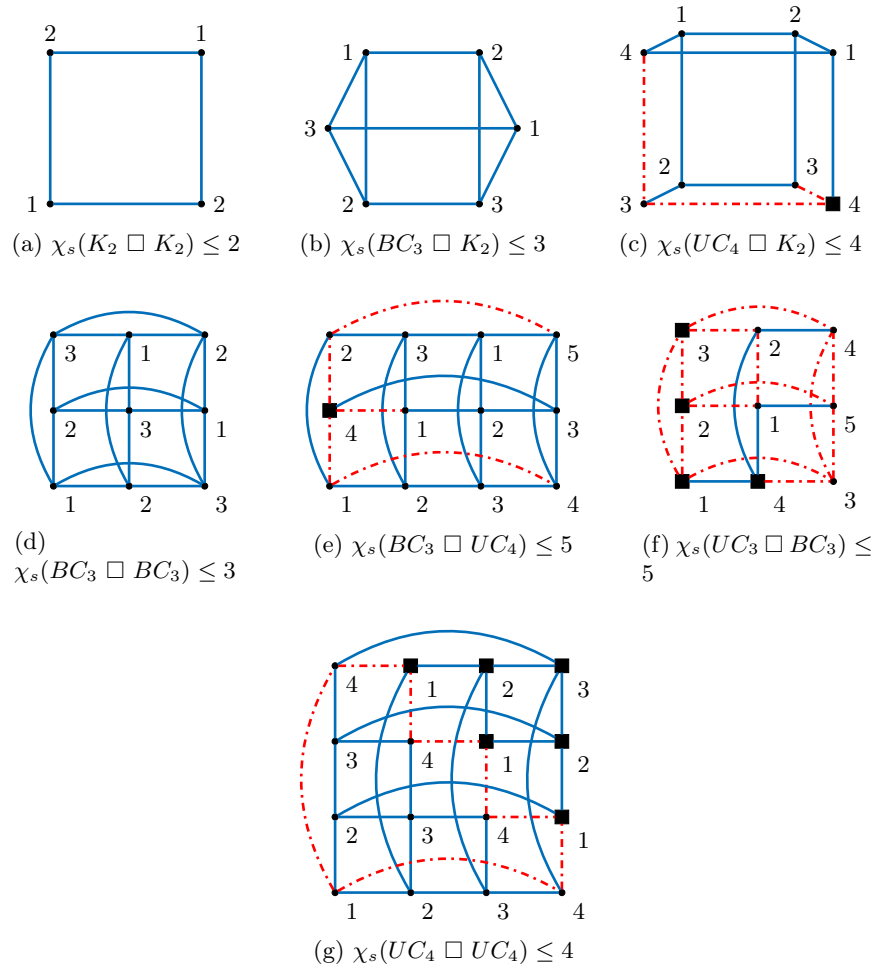


Fig. 2: All the cases for the product of cycles up to symmetry positive/negative. The large squared vertices have been switched from the product signature.

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