Cartesian product of signed graphs

D. Lajou\textsuperscript{1}

\textsuperscript{1}LaBRI, Université de Bordeaux, France.

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- Signed Homomorphisms

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- Algebraic properties

Cartesian products and signed chromatic number
- First results
- Products of cycles
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A **signed graph** \((G, \sigma)\) is a graph where each edge can be either positive or negative. The function \(\sigma : E(G) \rightarrow \{-, +\}\) is the **signature** of \((G, \sigma)\). Moreover we can switch at any vertex \(v\). Switching at \(v\) consists in inverting the signs of all edges incident with \(v\).
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A signed graph \((G, \sigma)\) is a graph where each edge can be either positive or negative. The function \(\sigma : E(G) \rightarrow \{-1, 1\}\) is the signature of \((G, \sigma)\). Moreover we can switch at any vertex \(v\). Switching at \(v\) consists in inverting the signs of all edges incident with \(v\).

Two signed graphs \((G, \sigma)\) and \((G, \sigma')\) are equivalent if we can obtain one from the other by a sequence of switches. In this case we note \((G, \sigma) \equiv (G, \sigma')\).
Signed graphs

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**Definition (Equivalent signed graphs)**

Two signed graphs \((G, \sigma)\) and \((G, \sigma')\) are **equivalent** if we can obtain one from the other by a sequence of switches. In this case we note \((G, \sigma) \equiv (G, \sigma')\).
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Related to Ising Model in physics.

Picture uploaded on researchgate.com by Sascha Wald.
Definition

A cycle is **balanced** if it has an **even** number of negative edges.
A cycle is **unbalanced** if it has an **odd** number of negative edges.

$BC_k = a$ balanced cycle of length $k$.
$UC_k = a$ unbalanced cycle of length $k$. 
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- $BC_k = \text{a balanced cycle of length } k.$
- $UC_k = \text{an unbalanced cycle of length } k.$

There are four classes of cycles:
Theorem (Zaslavsky 1982)

Two signed graphs $(G, \sigma)$ and $(G, \sigma')$ on the same underlying graph $G$ are equivalent if and only if they span the same set of balanced cycles.

Corollary (Zaslavsky 1982)

There is a $O(m)$ algorithm to recognise if two signed graphs $(G, \sigma)$ and $(G, \sigma')$ are equivalent.
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Definition

A **homomorphism** $\varphi$ from $G$ to $H$ is a mapping from the vertices of $G$ to the vertices of $H$ such that $\varphi(u)\varphi(v) \in E(H)$ for every edge $uv$ of $G$. 

![Diagram showing examples of homomorphisms from a signed graph to another signed graph.](image-url)
**Definition**

A **signed homomorphism** $\varphi$ from $(G, \sigma)$ to $(H, \pi)$ is a mapping from the vertices of $G$ to the vertices of $H$ such that there exists $(G, \sigma') \equiv (G, \sigma)$ verifying that for every edge $uv$ of $G$, $\varphi(uv) \in E(H)$ and $\sigma'(uv) = \varphi(\pi(uv))$ for every edge $uv$ of $G$. 
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\begin{align*}
(G,\sigma) & \quad \quad \quad \quad \quad \quad (H,\pi) \\
\begin{tikzpicture}
  \node[fill=red,circle,inner sep=2pt] (A) at (0,0) {};
  \node[fill=green,circle,inner sep=2pt] (B) at (1,1) {};
  \node[fill=violet,circle,inner sep=2pt] (C) at (2,-1) {};
  \node[fill=black,circle,inner sep=2pt] (D) at (3,0) {};
  \draw[red,thick] (A) -- (B);
  \draw[blue,thick] (A) -- (C);
  \draw[red,thick] (B) -- (C);
  \node at (1.5,-1.5) {$u$};
  \node[fill=red,circle,inner sep=2pt] (E) at (4,-1) {};
  \node[fill=green,circle,inner sep=2pt] (F) at (5,0) {};
  \node[fill=violet,circle,inner sep=2pt] (G) at (6,1) {};
  \node[fill=red,circle,inner sep=2pt] (H) at (7,0) {};
  \draw[red,thick] (E) -- (F);
  \draw[blue,thick] (E) -- (G);
  \draw[red,thick] (F) -- (H);
\end{tikzpicture}
\end{align*}
A signed homomorphism $\varphi$ from $(G, \sigma)$ to $(H, \pi)$ is a mapping from the vertices of $G$ to the vertices of $H$ such that there exists $(G, \sigma') \equiv (G, \sigma)$ verifying that for every edge $uv$ of $G$, $\varphi(uv) \in E(H)$ and $\sigma'(uv) = \varphi(\pi(uv))$ for every edge $uv$ of $G$. 

\[ (G, \sigma) \quad \equiv \quad (H, \pi) \]
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![Diagram](image-url)
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We note $(G, \sigma) \rightarrow_s (H, \pi)$. 
Signed Homomorphisms

Definition

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Definition

The signed chromatic number \( \chi_s(G, \sigma) \) is the order of the smallest simple signed graph \((H, \pi)\) such that \((G, \sigma) \to_s (H, \pi)\).

- \( \chi_s(G, 1) = \chi(G) \),
- \( \chi_s(T) = 2 \) where \( T \) is a signed tree,
**Definition**

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**Definition**

The signed chromatic number \( \chi_s(G, \sigma) \) is the order of the smallest simple signed graph \( (H, \pi) \) such that \( (G, \sigma) \to_s (H, \pi) \).

- \( \chi_s(G, \mathbb{1}) = \chi(G) \),
- \( \chi_s(T) = 2 \) where \( T \) is a signed tree,

\[ \chi_s = 2 \quad \chi_s = 4 \]
## Signed chromatic number of cycles

<table>
<thead>
<tr>
<th></th>
<th>even length</th>
<th>odd length</th>
</tr>
</thead>
<tbody>
<tr>
<td>balanced</td>
<td>$\chi_s(BC_{even}) = 2$</td>
<td>$\chi_s(BC_{odd}) = 3$</td>
</tr>
<tr>
<td></td>
<td><img src="image1.png" alt="Diagram" /></td>
<td><img src="image2.png" alt="Diagram" /></td>
</tr>
<tr>
<td>unbalanced</td>
<td>$\chi_s(UC_{even}) = 4$</td>
<td>$\chi_s(UC_{odd}) = 3$</td>
</tr>
<tr>
<td></td>
<td><img src="image3.png" alt="Diagram" /></td>
<td><img src="image4.png" alt="Diagram" /></td>
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Definition

The **Cartesian product** of $G$ and $H$ is the graph $G \Box H$ whose vertex set is $V(G) \times V(H)$ and where $(x, y)$ and $(x', y')$ are adjacent if and only if $x = x'$ and $y$ is adjacent to $y'$ in $H$ or $y = y'$ and $x$ is adjacent to $x'$ in $G$. 

![Cartesian Product Diagram]
Definition (Germina, Hameed K, Zaslavsky)

The **signed Cartesian product** of \((G, \sigma)\) and \((H, \pi)\), and note \((G, \sigma) \Box (H, \pi)\), the signed graph with vertex set \(V(G) \times V(H)\). It has for positive (resp. negative) edges the pairs \(\{(u_1, v_1), (u_2, v_2)\}\) such that \(u_1 = u_2\) and \(v_1 v_2\) is a positive (resp. negative) edge of \((H, \pi)\) or such that \(v_1 = v_2\) and \(u_1 u_2\) is a positive (resp. negative) edge of \((G, \sigma)\).
Ex: \((P, \rho) = \begin{array}{c}
0 & \square & 3 & \square & 1 \\
1 & \bullet & 2 & \triangle & 0 \\
\end{array}\)
Theorem

If \((G, \sigma) \rightarrow_s (G', \sigma')\) then

\[(G, \sigma) \square (H, \pi) \rightarrow_s (G', \sigma') \square (H, \pi)\]
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A signed graph \((G, \sigma)\) is *s-prime* if it cannot be written as 
\[(G, \sigma) \equiv (A, \pi_A) \Box (B, \pi_B).\]

**Theorem**

*If* \((G, \sigma)\) *is a connected signed graph then*

\[(G, \sigma) \equiv (G_1, \sigma_1) \Box \ldots \Box (G_k, \sigma_k)\]

*where each* \((G_i, \sigma_i)\) *is a s-prime graph and this decomposition is unique up to isomorphism and the order of the factor. Moreover the prime factor decomposition of* \(G\) *is finer than the one of* \((G, \sigma)\).*

By using replicating a proof of Imrich, Klavžar and Rall:

**Theorem**

*If* \((A, \pi_A), (B, \pi_B)\) *and* \((C, \pi_C)\) *are three signed graphs with*

\[(A, \pi_A) \Box (B, \pi_B) \equiv (A, \pi_A) \Box (C, \pi_C)\] *then* \((B, \pi_B) \equiv (C, \pi_C).\)
Claim

If \((G, \sigma)\) is a signed graph and \(C\) is a UC\(_4\) in \((G, \sigma)\) then \(C\) is induced in one of the prime factor of \((G, \sigma)\).
Claim

If \((G, \sigma)\) is a signed graph and \(C\) is a UC4 in \((G, \sigma)\) then \(C\) is induced in one of the prime factor of \((G, \sigma)\).

By adapting a technique from Imrich and Peterin:

Theorem

We can decompose a signed graph into its prime decomposition in time \(O(m)\).
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<thead>
<tr>
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<th>Cartesian Product of signed graphs</th>
<th>Cartesian products and signed chromatic number</th>
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Firsts results

**Theorem**

If $(F, \pi)$ is a forest then

\[ \chi_s((G, \sigma) \Box (F, \pi)) = \chi_s((G, \sigma) \Box (K_2, 1)). \]

**Corollary**

If $(F_i, \pi_i)$ is a forest for $i \in \{1, \ldots, k\}$ then

\[ \chi_s((F_1, \pi_1) \Box \ldots \Box (F_k, \pi_k)) = 2. \]
Note that, $\chi_s((P_n, \sigma) \square (P_m, \pi)) = 2$.

**Theorem**

If $G = P_n \square P_m$ and $\sigma$ is any signature of $G$ then $(G, \sigma) \rightarrow_s SPal_5^*$ and $\chi_s(G, \sigma) \leq 6$. If $\min(n, m) \leq 4$ then $\chi_s(G, \sigma) \leq 5$.

The signed graph $SPal_5^*$.

A grid with $\chi_s(G, \sigma) = 5$. 
Firsts results

**Theorem**

Let \((G_i, \sigma_i)\) for \(i \in \{1, \ldots, k\}\) be \(k\) signed graphs then

\[
\chi_s((G_1, \sigma_1) \square \ldots \square (G_k, \sigma_k)) \leq \prod_{i=1}^{k} \chi_s(G_i, \sigma_i).
\]

**Problem**

What is the smallest \(\lambda\) such that for all non trivial signed graphs:

\[
\chi_s((G_1, \sigma_1) \square \ldots \square (G_k, \sigma_k)) \leq \lambda \prod_{i=1}^{k} \chi_s(G_i, \sigma_i)\
\]
Figure: A signed graph $K$ of order 18 such that $\chi_s(K \square K_2) = 25$.
Thus:

$$0.69444 \approx \frac{25}{36} \leq \lambda \leq 1$$
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Theorem

If \((G, \sigma)\) and \((H, \pi)\) are two signed cycles then the signed chromatic number of \((P, \rho) = (G, \sigma) \square (H, \pi)\) is given by the table below.

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If \((G, \sigma)\) and \((H, \pi)\) are two signed cycles then the signed chromatic number of \((P, \rho) = (G, \sigma) \square (H, \pi)\) is given by the table below.

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<tr>
<th>((G, \sigma) \square (H, \pi))</th>
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<th>(BC_{odd})</th>
<th>(UC_{even})</th>
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</tr>
</thead>
<tbody>
<tr>
<td>(BC_{even})</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>(BC_{odd})</td>
<td>3</td>
<td>3</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
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</tr>
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Table: The signed chromatic number for each type of products of two cycles.
### Signed Graphs

Cartesian Product of signed graphs

Cartesian products and signed chromatic number

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#### Products of cycles

<table>
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<tr>
<th>((G, \sigma) \Box (H, \pi))</th>
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<td>3</td>
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</tr>
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Products of cycles

\[
\begin{array}{|c|c|c|c|c|}
\hline
(G, \sigma) \boxtimes (H, \pi) & BC_{even} & BC_{odd} & UC_{even} & UC_{odd} \\
\hline
BC_{even} & 2 & 3 & 4 & 3 \\
\hline
BC_{odd} & 3 & 3 & 5 & 5 \\
\hline
UC_{even} & 4 & 5 & 4 & 5 \\
\hline
UC_{odd} & 3 & 5 & 5 & 3 \\
\hline
\end{array}
\]

**Lemma**

*For every two integers* \( p,q \in \mathbb{N} *:

\[
\chi_s(UC_q \boxtimes BC_{2p+1}) > 4.
\]
Thank you for your attention.