

# Cartesian product of signed graphs

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# Overview

## Signed Graphs

- Signed graphs

- Signed Homomorphisms

## Cartesian Product of signed graphs

- Definitions and Properties

- Algebraic properties

## Cartesian products and signed chromatic number

- Firsts results

- Products of cycles

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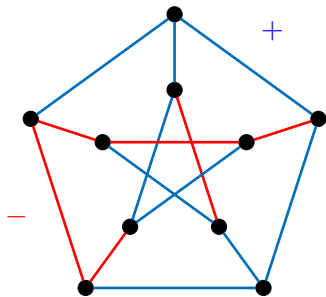
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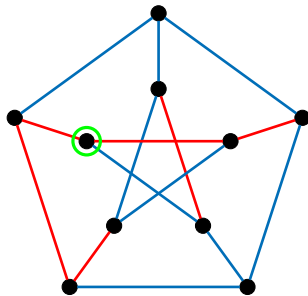
## Definition

A **signed graph**  $(G, \sigma)$  is a graph where each edge can be either positive or negative. The function  $\sigma : E(G) \rightarrow \{-, +\}$  is the **signature** of  $(G, \sigma)$ . Moreover we can switch at any vertex  $v$ . Switching at  $v$  consists in inverting the signs of all edges incident with  $v$ .



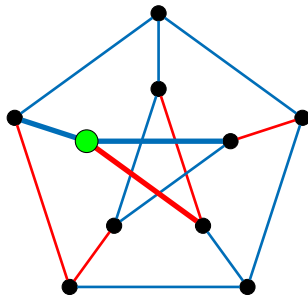
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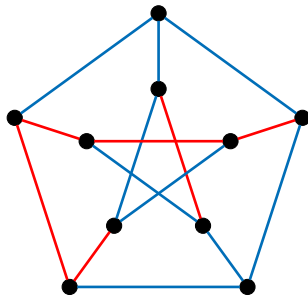
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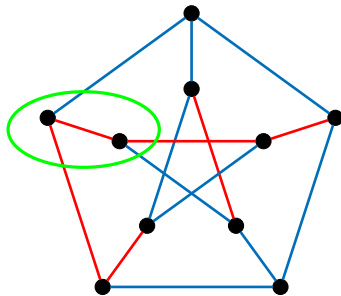
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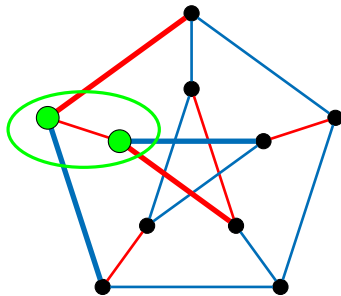
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A **signed graph**  $(G, \sigma)$  is a graph where each edge can be either positive or negative. The function  $\sigma : E(G) \rightarrow \{-1, 1\}$  is the **signature** of  $(G, \sigma)$ . Moreover we can switch at any vertex  $v$ . Switching at  $v$  consists in inverting the signs of all edges incident with  $v$ .

## Definition (Equivalent signed graphs)

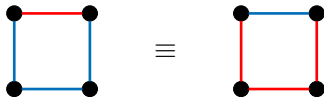
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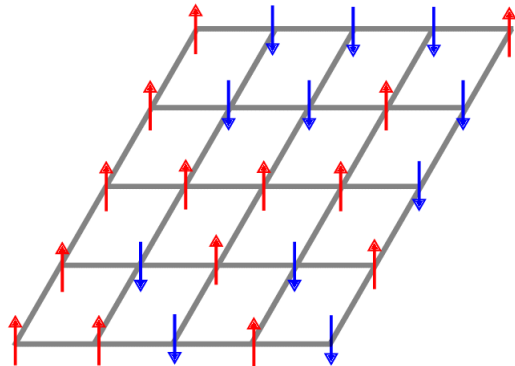
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Related to Ising Model in physics.



Picture uploaded on [researchgate.com](https://www.researchgate.com) by Sascha Wald.

## Definition

A cycle is **balanced** if it has an **even** number of negative edges.

A cycle is **unbalanced** if it has an **odd** number of negative edges.

$BC_k$  = a balanced cycle of length  $k$ .

$UC_k$  = an unbalanced cycle of length  $k$ .

## Definition





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There are four classes of cycles:

	even length	odd length
balanced	$BC_{\text{even}}$ 	$BC_{\text{odd}}$ 
unbalanced	$UC_{\text{even}}$ 	$UC_{\text{odd}}$ 

## Theorem (Zaslavsky 1982)

*Two signed graphs  $(G, \sigma)$  and  $(G, \sigma')$  on the same underlying graph  $G$  are equivalent if and only if they span the same set of balanced cycles.*

## Corollary (Zaslavsky 1982)

*There is a  **$O(m)$  algorithm** to recognise if two signed graphs  $(G, \sigma)$  and  $(G, \sigma')$  are equivalent.*



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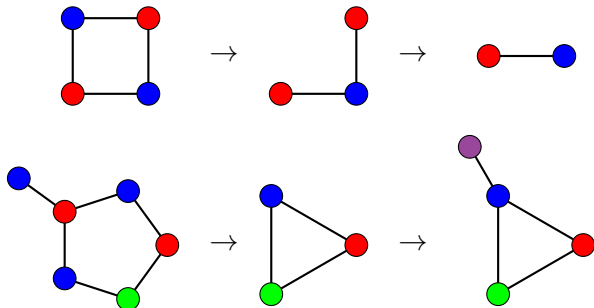
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A **homomorphism**  $\varphi$  from  $G$  to  $H$  is a mapping from the vertices of  $G$  to the vertices of  $H$  such that  $\varphi(u)\varphi(v) \in E(H)$  for every edge  $uv$  of  $G$ .

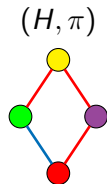
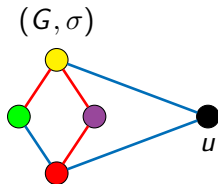


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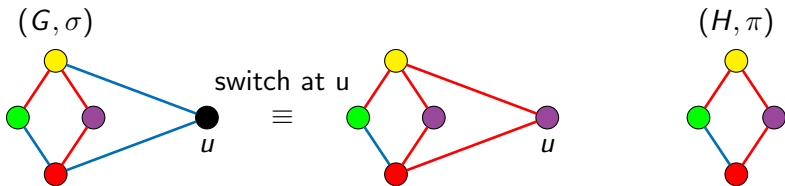
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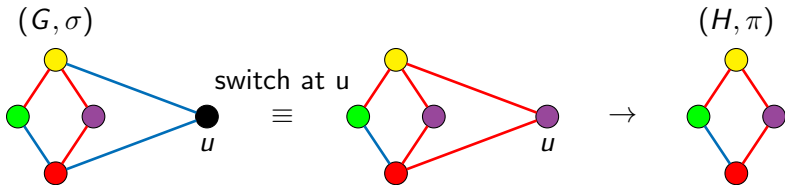
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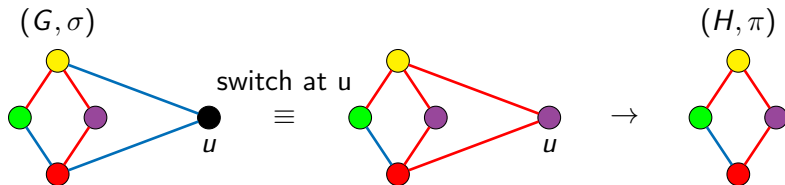
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We note  $(G, \sigma) \rightarrow_s (H, \pi)$ .

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## Definition

The **signed chromatic number**  $\chi_s(G, \sigma)$  is the order of the smallest simple signed graph  $(H, \pi)$  such that  $(G, \sigma) \rightarrow_s (H, \pi)$ .

- ▶  $\chi_s(G, \mathbb{1}) = \chi(G)$ ,
- ▶  $\chi_s(T) = 2$  where  $T$  is a signed tree,



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$$\chi_s = 2$$

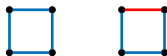
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$$\chi_s = 2 \quad \chi_s = 4$$

## Signed chromatic number of cycles

	even length	odd length
balanced	$\chi_s(BC_{\text{even}}) = 2$ 	$\chi_s(BC_{\text{odd}}) = 3$ 
unbalanced	$\chi_s(UC_{\text{even}}) = 4$ 	$\chi_s(UC_{\text{odd}}) = 3$ 

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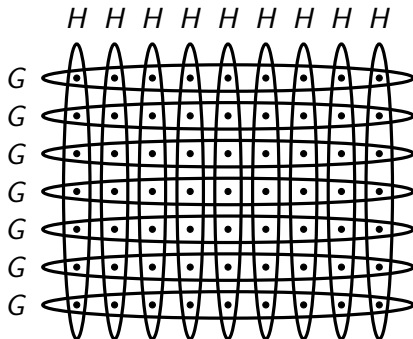
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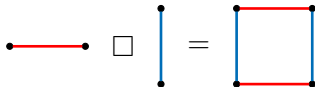
## Definition

The **Cartesian product** of  $G$  and  $H$  is the graph  $G \square H$  whose vertex set is  $V(G) \times V(H)$  and where  $(x, y)$  and  $(x', y')$  are adjacent if and only if  $x = x'$  and  $y$  is adjacent to  $y'$  in  $H$  or  $y = y'$  and  $x$  is adjacent to  $x'$  in  $G$ .



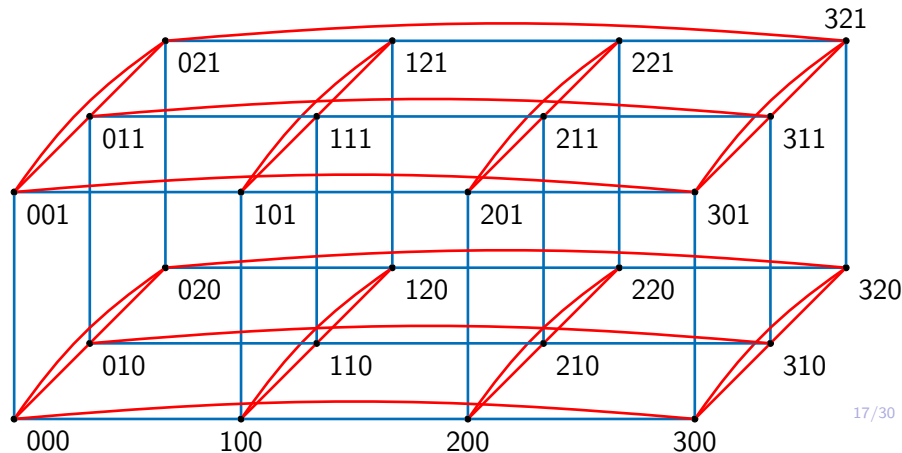
## Definition (Germina, Hameed K, Zaslavsky)

The **signed Cartesian product** of  $(G, \sigma)$  and  $(H, \pi)$ , and note  $(G, \sigma) \square (H, \pi)$ , the signed graph with vertex set  $V(G) \times V(H)$ . It has for positive (resp. negative) edges the pairs  $\{(u_1, v_1), (u_2, v_2)\}$  such that  $u_1 = u_2$  and  $v_1 v_2$  is a positive (resp. negative) edge of  $(H, \pi)$  or such that  $v_1 = v_2$  and  $u_1 u_2$  is a positive (resp. negative) edge of  $(G, \sigma)$ .



## Definitions and Properties

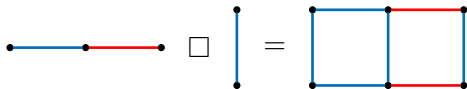
$$\text{Ex: } (P, \rho) = \begin{array}{c} 0 \\ \text{---} \\ 1 \end{array} \square \begin{array}{c} 1 \\ \text{---} \\ 2 \end{array} \square \begin{array}{c} 0 \\ \text{---} \\ 1 \end{array}$$



## Theorem

If  $(G, \sigma) \rightarrow_s (G', \sigma')$  then

$$(G, \sigma) \square (H, \pi) \rightarrow_s (G', \sigma') \square (H, \pi)$$

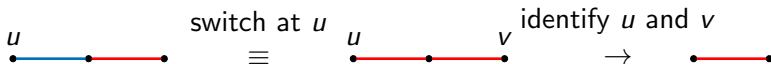
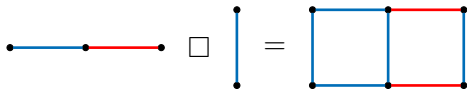




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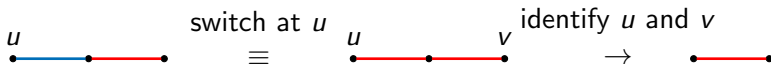
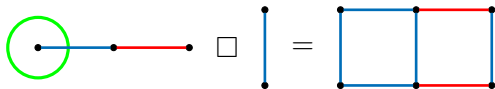
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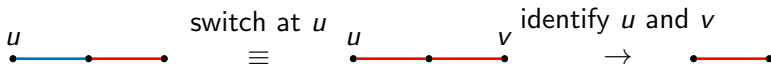
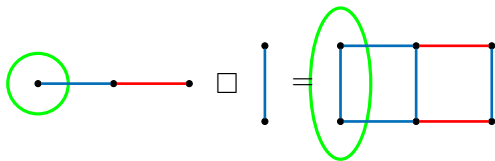
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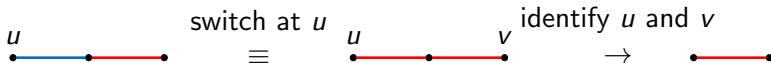
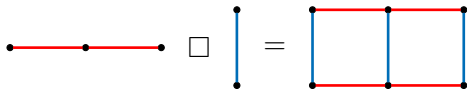
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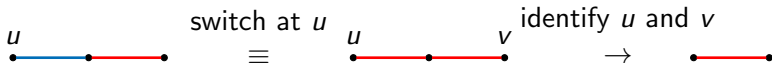
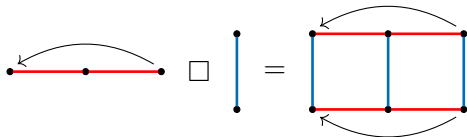
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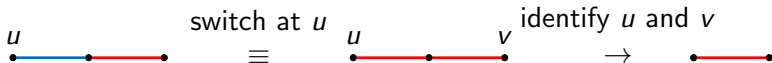
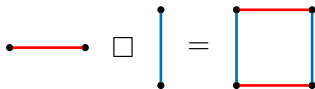
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A signed graph  $(G, \sigma)$  is *s-prime* if it cannot be written as  $(G, \sigma) \equiv (A, \pi_A) \square (B, \pi_B)$ .

## Theorem

*If  $(G, \sigma)$  is a connected signed graph then*

$$(G, \sigma) \equiv (G_1, \sigma_1) \square \dots \square (G_k, \sigma_k)$$

*where each  $(G_i, \sigma_i)$  is a s-prime graph and this decomposition is unique up to isomorphism and the order of the factor. Moreover the prime factor decomposition of  $G$  is finer than the one of  $(G, \sigma)$ .*

By using replicating a proof of Imrich, Klavžar and Rall:

## Theorem

*If  $(A, \pi_A)$ ,  $(B, \pi_B)$  and  $(C, \pi_C)$  are three signed graphs with  $(A, \pi_A) \square (B, \pi_B) \equiv (A, \pi_A) \square (C, \pi_C)$  then  $(B, \pi_B) \equiv (C, \pi_C)$ .*



## Claim

*If  $(G, \sigma)$  is a signed graph and  $C$  is a  $UC_4$  in  $(G, \sigma)$  then  $C$  is induced in one of the prime factor of  $(G, \sigma)$ .*



## Claim

*If  $(G, \sigma)$  is a signed graph and  $C$  is a  $UC_4$  in  $(G, \sigma)$  then  $C$  is induced in one of the prime factor of  $(G, \sigma)$ .*



By adapting a technique from Imrich and Peterin:

## Theorem

*We can decompose a signed graph into its prime decomposition in time  $O(m)$ .*

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If  $(F, \pi)$  is a forest then

$$\chi_s((G, \sigma) \square (F, \pi)) = \chi_s((G, \sigma) \square (K_2, \mathbb{1})).$$

## Corollary

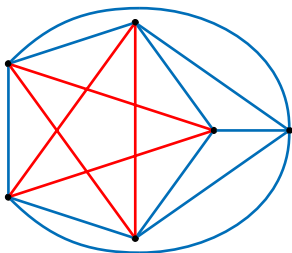
If  $(F_i, \pi_i)$  is a forest for  $i \in \{1, \dots, k\}$  then

$$\chi_s((F_1, \pi_1) \square \dots \square (F_k, \pi_k)) = 2.$$

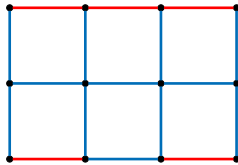
Note that,  $\chi_s((P_n, \sigma) \square (P_m, \pi)) = 2$ .

## Theorem

If  $G = P_n \square P_m$  and  $\sigma$  is any signature of  $G$  then  $(G, \sigma) \rightarrow_s SPal_5^*$  and  $\chi_s(G, \sigma) \leq 6$ . If  $\min(n, m) \leq 4$  then  $\chi_s(G, \sigma) \leq 5$ .



The signed graph  $SPal_5^*$ .



A grid with  $\chi_s(G, \sigma) = 5$ .

## Theorem

Let  $(G_i, \sigma_i)$  for  $i \in \{1, \dots, k\}$  be  $k$  signed graphs then

$$\chi_s((G_1, \sigma_1) \square \dots \square (G_k, \sigma_k)) \leq \prod_{i=1}^k \chi_s(G_i, \sigma_i).$$

## Problem

What is the smallest  $\lambda$  such that for all non trivial signed graphs:

$$\chi_s((G_1, \sigma_1) \square \dots \square (G_k, \sigma_k)) \leq \lambda \prod_{i=1}^k \chi_s(G_i, \sigma_i)?$$

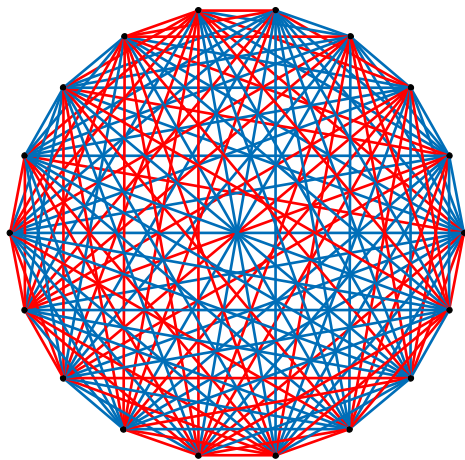


Figure: A signed graph  $K$  of order 18 such that  $\chi_s(K \square K_2) = 25$ .

Thus:

$$0.69444 \simeq \frac{25}{36} \leq \lambda \leq 1$$

# Overview

## Signed Graphs

Signed graphs

Signed Homomorphisms

## Cartesian Product of signed graphs

Definitions and Properties

Algebraic properties

## Cartesian products and signed chromatic number

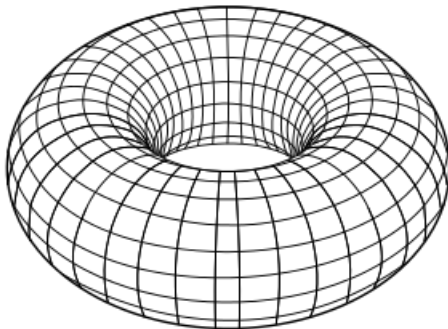
Firsts results

Products of cycles



## Theorem

*If  $(G, \sigma)$  and  $(H, \pi)$  are two signed cycles then the signed chromatic number of  $(P, \rho) = (G, \sigma) \square (H, \pi)$  is given by the table below.*



## Theorem

If  $(G, \sigma)$  and  $(H, \pi)$  are two signed cycles then the signed chromatic number of  $(P, \rho) = (G, \sigma) \square (H, \pi)$  is given by the table below.

$(G, \sigma) \square (H, \pi)$	$BC_{\text{even}}$	$BC_{\text{odd}}$	$UC_{\text{even}}$	$UC_{\text{odd}}$
$BC_{\text{even}}$	2	3	4	3
$BC_{\text{odd}}$	3	3	5	5
$UC_{\text{even}}$	4	5	4	5
$UC_{\text{odd}}$	3	5	5	3

**Table:** The signed chromatic number for each type of products of two cycles.

$(G, \sigma) \square (H, \pi)$	$BC_{even}$	$BC_{odd}$	$UC_{even}$	$UC_{odd}$
$BC_{even}$	2	3	4	3
$BC_{odd}$	3	3	5	5
$UC_{even}$	4	5	4	5
$UC_{odd}$	3	5	5	3

$(G, \sigma) \square (H, \pi)$	$BC_{even}$	$BC_{odd}$	$UC_{even}$	$UC_{odd}$
$BC_{even}$	2	3	4	3
$BC_{odd}$	3	3	<b>5</b>	<b>5</b>
$UC_{even}$	4	<b>5</b>	4	<b>5</b>
$UC_{odd}$	3	<b>5</b>	<b>5</b>	3

$(G, \sigma) \square (H, \pi)$	$BC_{even}$	$BC_{odd}$	$UC_{even}$	$UC_{odd}$
$BC_{even}$	2	3	4	3
$BC_{odd}$	3	3	<b>5</b>	<b>5</b>
$UC_{even}$	4	<b>5</b>	4	<b>5</b>
$UC_{odd}$	3	<b>5</b>	<b>5</b>	3

## Lemma

For every two integers  $p, q \in \mathbb{N}$ :

$$\chi_s(UC_q \square BC_{2p+1}) > 4.$$

Thank you for your attention.