

# Between Proper and Strong Edge-Coloring of (sub)cubic graphs

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Joint work with

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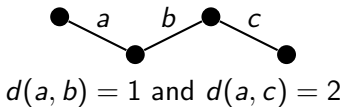
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Thanks to Borut for the slides!

## Warm-up definitions

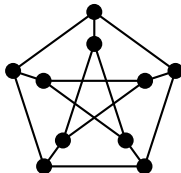
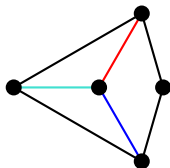
- **Subcubic graphs** → graphs with maximum degree 3;
- **Simple** graphs (although parallel edges are not problematic);
- **Distance between edges** → distance between the corresponding vertices in the line graph (adjacent edges are at distance 1);



# Proper edge-coloring

- **Adjacent edges** receive distinct colors;
- Edges of every color form a **matching**;
- The smallest  $k$  for which a graph  $G$  admits an edge-coloring with  $k$  colors is the **chromatic index** of  $G$ ,  $\chi'(G)$ ;
- By Vizing's theorem, for every subcubic graph  $G$  it holds

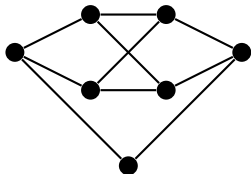
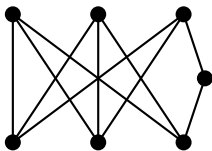
$$3 \leq \chi'(G) \leq 4$$



## Strong edge-coloring

- **Edges at distance at most 2** receive distinct colors;
- Edges of every color form an **induced matching**, i.e. the graph induced on the endvertices is a matching;
- The smallest  $k$  for which a graph  $G$  admits an edge-coloring with  $k$  colors is the **strong chromatic index** of  $G$ ,  $\chi'_s(G)$ ;
- Andersen (1992), and Horák, Qing, and Trotter (1993) proved that **for every subcubic graph  $G$  it holds**

$$\chi'_s(G) \leq 10$$

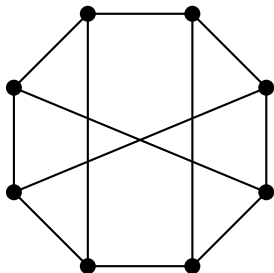


# Packings

- A set of edges is a  **$k$ -packing** if every pair of edges is at distance at least  $k + 1$ ;
- Hence, every matching is 1-packing and every induced matching is a 2-packing;
- For a non-decreasing sequence of positive integers,  $S = (s_1, \dots, s_\ell)$ , Gastineau and Togni (2019), defined an  **$S$ -packing edge-coloring** of  $G$  as a partition of the edge set of  $G$  into  $\ell$  subsets  $\{X_1, \dots, X_\ell\}$  such that each  $X_i$  is an  $s_i$ -packing;
- The notion of packing colorings is derived from its vertex analogues introduced by W. Goddard, S. M. Hedetniemi, S. T. Hedetniemi, J. M. Harris and D. F. Rall (2008), and Goddard and Xu (2012);

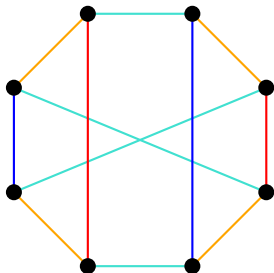
## Example

- Consider the following cubic graph with  $\chi'_s = 10$ :



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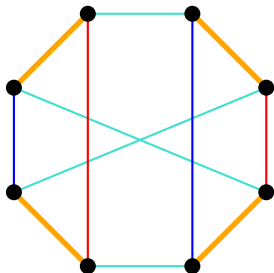
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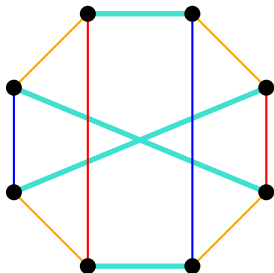


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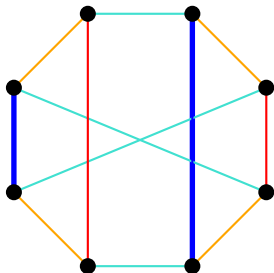
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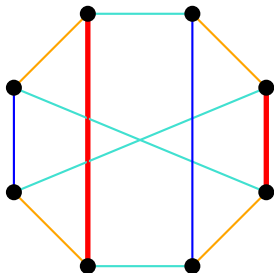
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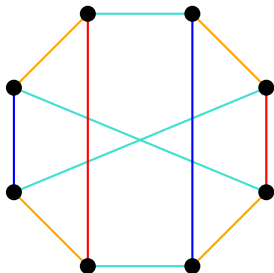
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## Today's focus

- For  $S = (s_1, \dots, s_\ell)$ , we are interested in  $S$ -packing edge-colorings with  $s_i \in \{1, 2\}$  for a given number of 1's;
- We abbreviate  $(\underbrace{1, \dots, 1}_p, \underbrace{2, \dots, 2}_q)$  as  $(1^p, 2^q)$ ;
- By Vizing (also Brooks):  
Every subcubic graph admits a  $(1, 1, 1, 1)$ -packing edge-coloring.
- By Andersen and Horák, Qing and Trotter:  
Every subcubic graph admits a  $(2^{10})$ -packing edge-coloring.
- **What is in between** and **what is open**?

$(1, 1, 1, 2^k)$

## (1, 1, 1, 2)-packing edge-coloring

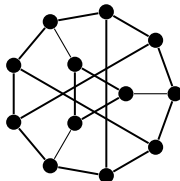
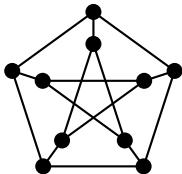
- The fourth color, call it  $\delta$ , in a proper edge-coloring of subcubic graphs is rare;
- Albertson and Haas (1996):  
If  $G$  is cubic, at most  $\frac{2}{15}$  edges are colored with  $\delta$ .
- Steffen (2004):  
The Petersen graph is the only bridgeless cubic graph such that all four colors must have at least  $\frac{2}{15}$  edges.
- Fouquet and Vanherpe (2013):  
For any subcubic graph, less than  $\frac{2}{15}$  edges are colored with  $\delta$ , except for three graphs.
- As a side product...

## $(1, 1, 1, 2)$ -packing edge-coloring

- Fouquet and Vanherpe (2013) and Payan (1977):  
Every subcubic graph admits a  $(1, 1, 1, 2)$ -packing edge-coloring.
- Here the 2-packing cannot be replaced by a 3-packing, due to the Petersen and the Tietze graphs.

### Conjecture 1 (Gastineau and Togni)

*Every cubic graph different from the Petersen and the Tietze graph is  $(1, 1, 1, 3)$ -packing edge-colorable.*

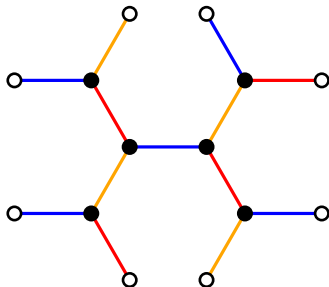




$$(1, 1, 2^k)$$

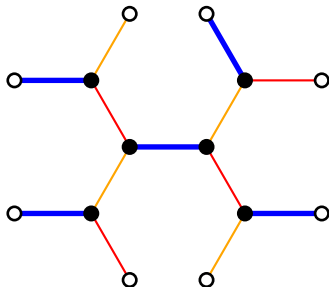
# $(1, 1, 2^k)$ -packing edge-coloring

- Trivial:  $k \geq 6$   
Take a  $(1, 1, 1, 2)$ -packing edge-coloring of  $G$  and replace one 1-packing with five 2-packings;



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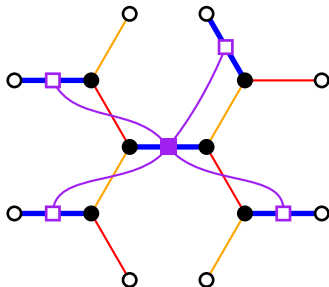
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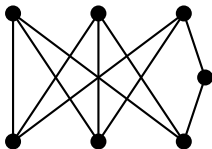


## $(1, 1, 2^k)$ -packing edge-coloring

- Showing there is a 1-packing  $A$  in a  $(1, 1, 1, 2)$ -packing edge-coloring such that no five edges of  $A$  are pairwise at distance 2, Gastineau and Togni (2019) proved:  
Every bridgeless cubic graph admits a  $(1, 1, 2^5)$ -packing edge-coloring.
- Hocquard, L and Lužar (2019<sup>+</sup>):  
Every subcubic graph admits a  $(1, 1, 2^5)$ -packing edge-coloring.

# $(1, 1, 2^k)$ -packing edge-coloring

- There are graph(s) that do not admit a  $(1, 1, 2^3)$ -packing edge-coloring;



## Conjecture 2 (Gastineau and Togni)

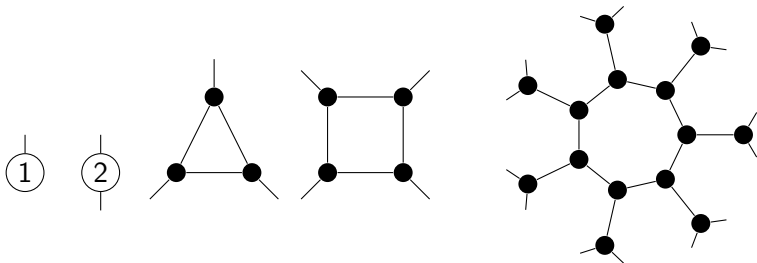
*Every subcubic graph is  $(1, 1, 2^4)$ -packing edge-colorable.*

- The conjecture is supported by checking all bridgeless subcubic graphs on at most 17 vertices;
- Is  $K_{3,3}$  with a subdivided edge the only bridgeless subcubic graph needing four 2-packings?

$$(1, 2^k)$$

# $(1, 2^k)$ -packing edge-coloring

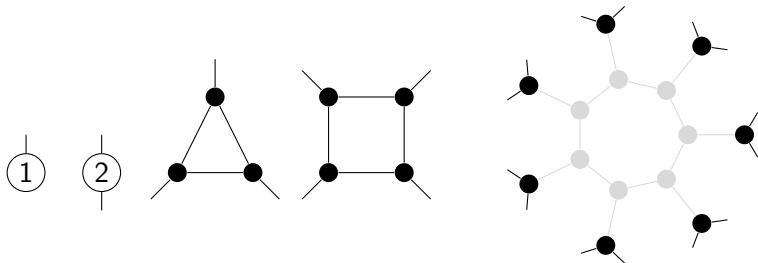
- Trivial:  $k \geq 9$   
By the strong edge-coloring result;
- Hocquard, L and Lužar (2019<sup>+</sup>):  
Every subcubic graph admits a  $(1, 2^8)$ -packing edge-coloring.
- Proof idea: a **good**- $(1, 2^8)$ -packing edge-coloring is a  $(1, 2^8)$ -packing edge-coloring where vertices of degree 2 do not receive the color  $\alpha$  (= the 1-packing).





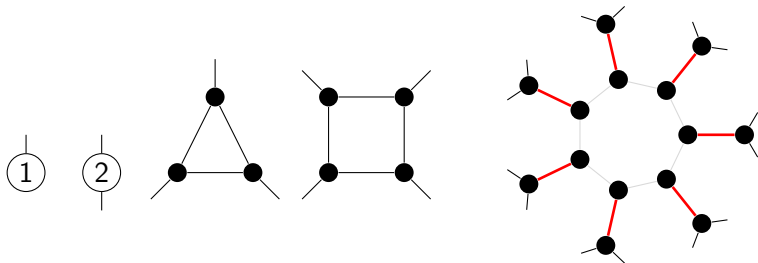
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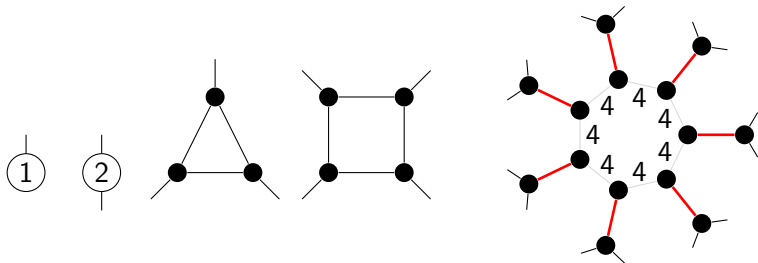
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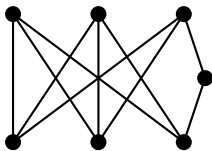
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## $(1, 2^k)$ -packing edge-coloring

- There are graph(s) that do not admit a  $(1, 2^6)$ -packing edge-coloring;



- Gastineau and Togni asked, and we conjecture the following:

Conjecture 3 (Hocquard, L and Lužar (2019<sup>+</sup>))

*Every subcubic graph is  $(1, 2^7)$ -packing edge-colorable.*

- The conjecture is supported by checking all bridgeless subcubic graphs on at most 17 vertices;
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Why is it interesting?

## How are they connected?

- The (conjectured) bounds series:

$$(1, 1, 1, 2) \quad (1, 1, 2^4) \quad (1, 2^7) \quad (2^{10})$$

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- It seems that we can always “replace” a 1-packing with three 2-packings;
- Note that the other 1-packings **cannot** stay fixed in general.



Adding constraints:  
Class I graphs

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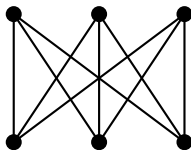
Conjecture 4 (Gastineau and Togni)

*Every subcubic class I graph is  $(1, 1, 2^3)$ -packing edge-colorable.*

Conjecture 5 (Hocquard, L and Lužar (2019<sup>+</sup>))

*Every subcubic class I graph is  $(1, 2^6)$ -packing edge-colorable.*

- Both conjectures, if true, are tight, due to  $K_{3,3}$ ;



# Proof ideas of $(1, 2^7)$ for class I

The colors are  $\alpha, 1, 2, 3, 4, 5, 6, 7$ .

- Fix a  $(1^3)$ -coloring of a minimal counter-example  $G$  and  $\alpha$  is one of those colors, the other two are blue.
- $G$  is cubic of girth at least 5.
- $G$  has no chordless blue cycles.
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Tools:

- Local modifications to reduce the graph.
- Combinatorial Nullstellensatz to color some paths and cycles.  
I.e if some well chosen polynomial  $P$  has a particular monomial with a non zero coefficient then our structure is strongly edge-choosable with lists  $L_1, \dots, L_k$ .

# Summary



## Summary

	$(1^3, 2^a)$	$(1^2, 2^b)$	$(1, 2^c)$	$(2^d)$
general	$1 \leq a \leq 1$	$4 \leq b \leq 5$	$7 \leq c \leq 8$	$10 \leq d \leq 10$
class I	$0 \leq a \leq 0$	$3 \leq b \leq 4$	$6 \leq c \leq 7$	$10 \leq d \leq 10$

- red color - only a finitely many known examples of **bridgeless** graphs attaining the bound;
- blue color - resolved completely;

Thank you for your attention!