Further Evidence Towards the Multiplicative 1-2-3 Conjecture

Julien Bensmail[1], Hervé Hocquard[2], Dimitri Lajou[2], Éric Sopena[2]

[1]Université Côte d'Azur, Nice.

[2]LaBRI, Bordeaux.

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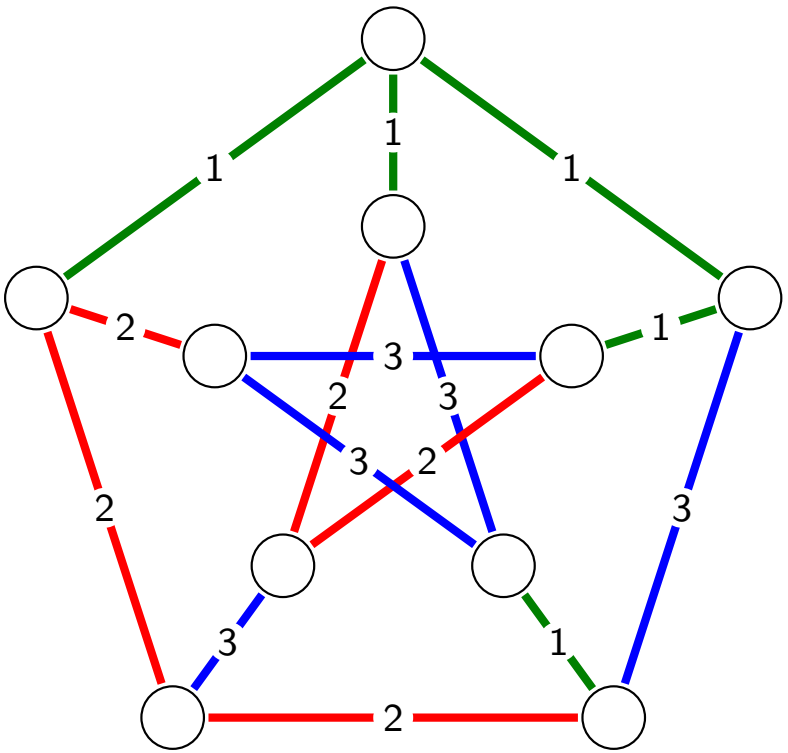
Overview

Overview

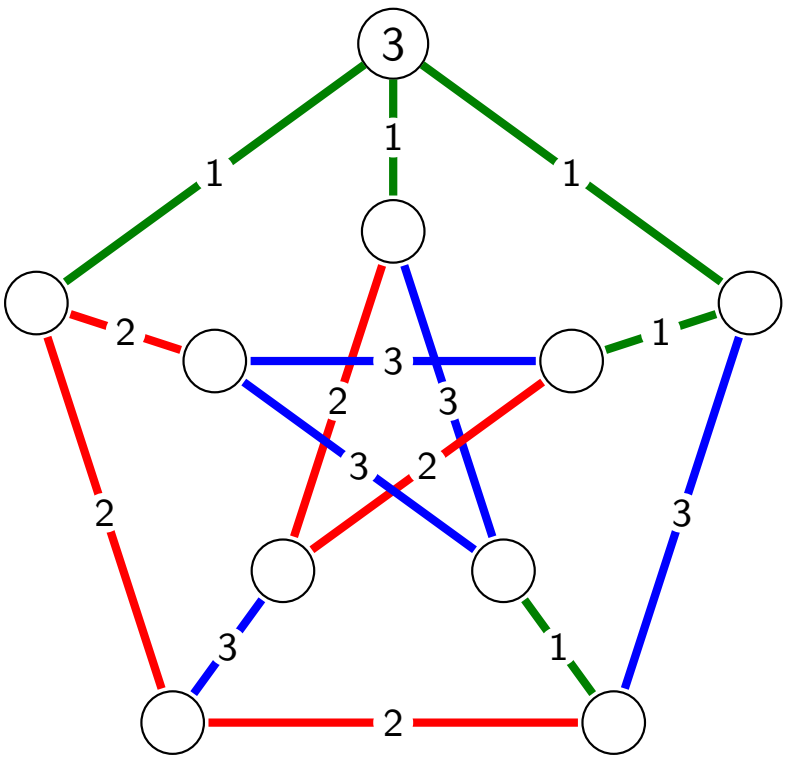
3-Edge-labelling and coloring



Definition

A k-edge-labelling is a function $\ell : E(G) \to \{1, 2, \ldots, k\}$.

3-Edge-labelling and coloring



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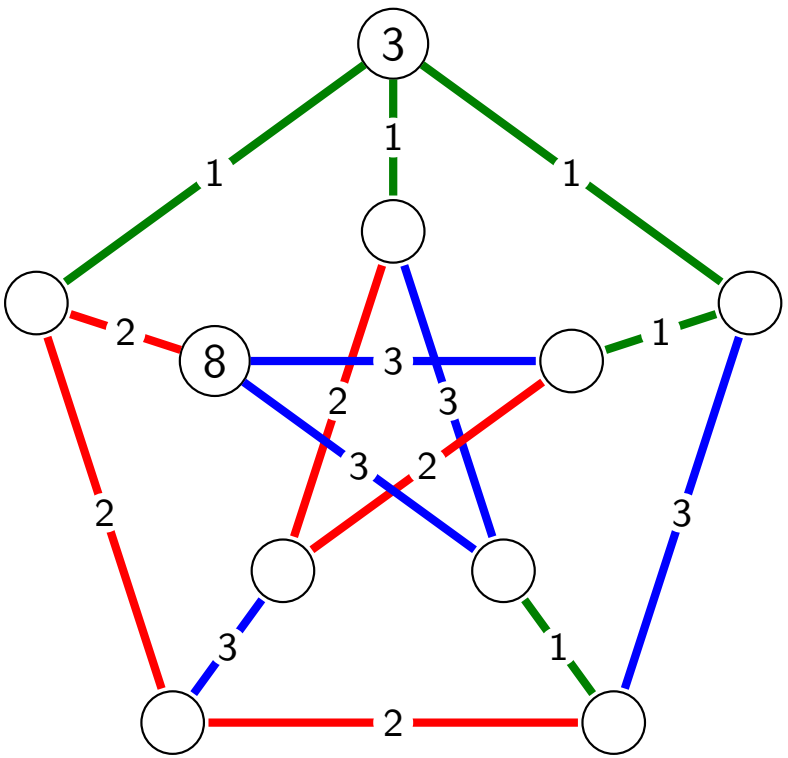

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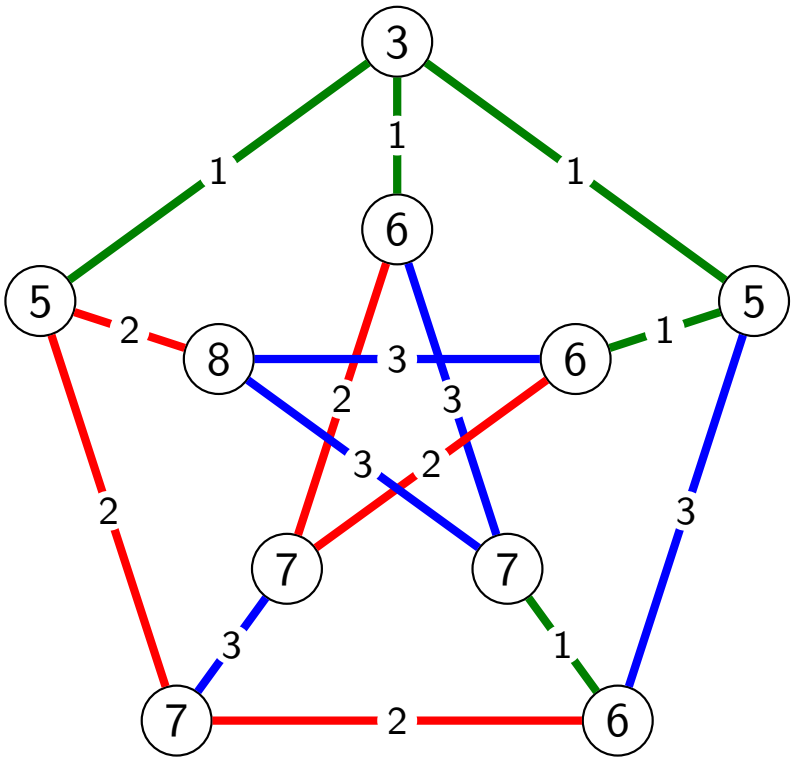

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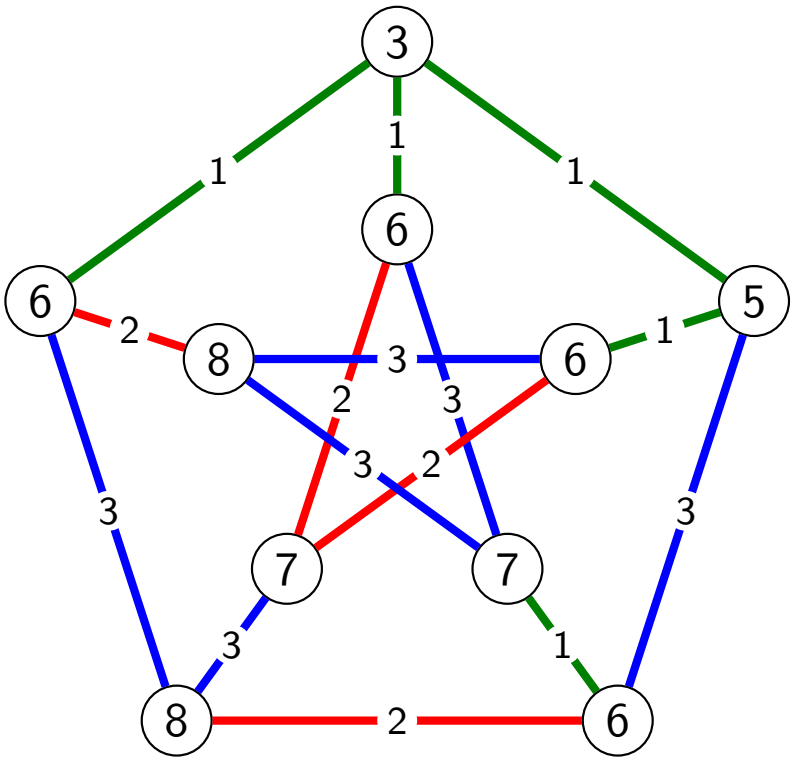

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The function σ_ℓ is a proper coloring of G.

1-2-3-Conjecture

Definition

An ***s*-proper** labelling ℓ of a graph G is a labelling such that $\sigma_\ell = \sum_{uv \in E(G)} \ell(uv)$ is a proper coloring.

A *nice* graph is a graph with no connected component isomorphic to K_2.

Conjecture (1-2-3-Conjecture, Karonski, Luczak, and Thomason 2004)

*Every nice graph G admits an **s-proper** 3-labelling.*

Overview of some results

Theorem (Kalkowski, Karoński and Pfender 2010)

Every nice graph G admits an ***s*-proper 5-labelling**.

Theorem

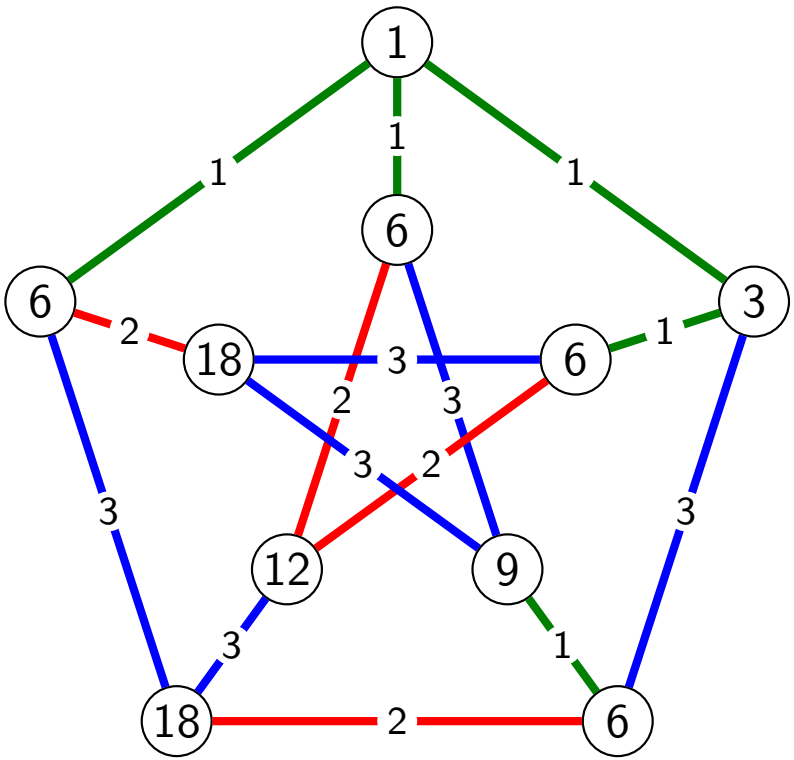
Every nice graph G such that

- *G has* **chromatic number at most** 3 *(Karoński, Łuczak, Thomason 2004), or*
- *G has* **chromatic number at most** 4 *and is* 4-**edge-connected** *(Wu, Zhang, Zhu 2017),*

admits an ***s*-proper** 3*-labelling.*

Overview

Variations



$$\pi_\ell(u) = \prod_{uv \in E(G)} \ell(uv)$$

The function π_ℓ is a proper coloring of G.

Others 1-2-3-Conjecture

Definition

An *p*-**proper** labelling ℓ of a graph G is a labelling such that $\pi_\ell = \prod_{uv \in E(G)} \ell(uv)$ is a proper coloring.

Conjecture (Multiplicative 1-2-3-Conjecture, J. Skowronek-Kaziów 2012)

Every nice graph G admits a *p*-**proper** *3-labelling .*

Overview of some results

Theorem (Skowronek-Kaziów 2012)

Every nice graph G admits a ***p-proper 4-labelling****.*

Theorem (Skowronek-Kaziów 2012)

Every nice graph G of ***chromatic number at most 3*** *admits a* ***p-proper*** *3-labelling.*

Others 1-2-3-Conjecture

Definition

For a labelling ℓ of a graph G, let μ_ℓ be the function which associates with a vertex u the multiset $\mu_\ell(u)$ containing the labels of the edges incident with u. If μ_ℓ is a proper coloring of G, we say that ℓ is *m*-**proper**.

Observation

If ℓ is s-proper or p-proper then it is m-proper.

Theorem (Vučković 2018)

Every nice graph G admits an *m*-**proper** *3-labelling.*

Overview

Main results

Theorem

Every nice graph G of **chromatic number at most** 4 *admits a* p-**proper** 3*-labelling.*

Theorem

Every graph G admits a 3*-labelling such that:*

- *if u and v are adjacent and $\pi_\ell(u) = \pi_\ell(v)$ then $\pi_\ell(u) = \pi_\ell(v) = 1$,*
- *the subgraph of G induced by the vertices u with $\pi_\ell(u) = 1$ contains only connected components with at most two vertices.*

We say that ℓ is **almost** p-**proper**.

Regular graphs

Theorem

Every nice **regular** *graph G admits a* p-**proper** 3*-labelling.*

Proof. Let ℓ be an m-proper 3-labelling of G. Suppose

$$\pi_\ell(u) = \pi_\ell(v)$$
$$1^a 2^b 3^c = 1^d 2^e 3^f$$

As 2 and 3 are coprime, $b = e$ and $c = f$.
As G is k-regular, $k = a + b + c = d + e + f$.
Hence $a = d$, a contradiction.

□

Notation

For a 3-labelling ℓ:

- $d_2(u)$, the 2-degree of a vertex u, is the number of 2-labelled edges incident with u,
- $d_3(u)$, the 3-degree of a vertex u, is the number of 3-labelled edges incident with u.

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We say that

- u is 1-monochromatic if $\pi_\ell(u) = 1$,
- u is 2-monochromatic if $d_2(u) > 0$ and $d_3(u) = 0$,
- u is 3-monochromatic if $d_2(u) = 0$ and $d_3(u) > 0$, and
- u is bichromatic if $d_2(u) > 0$ and $d_3(u) > 0$.

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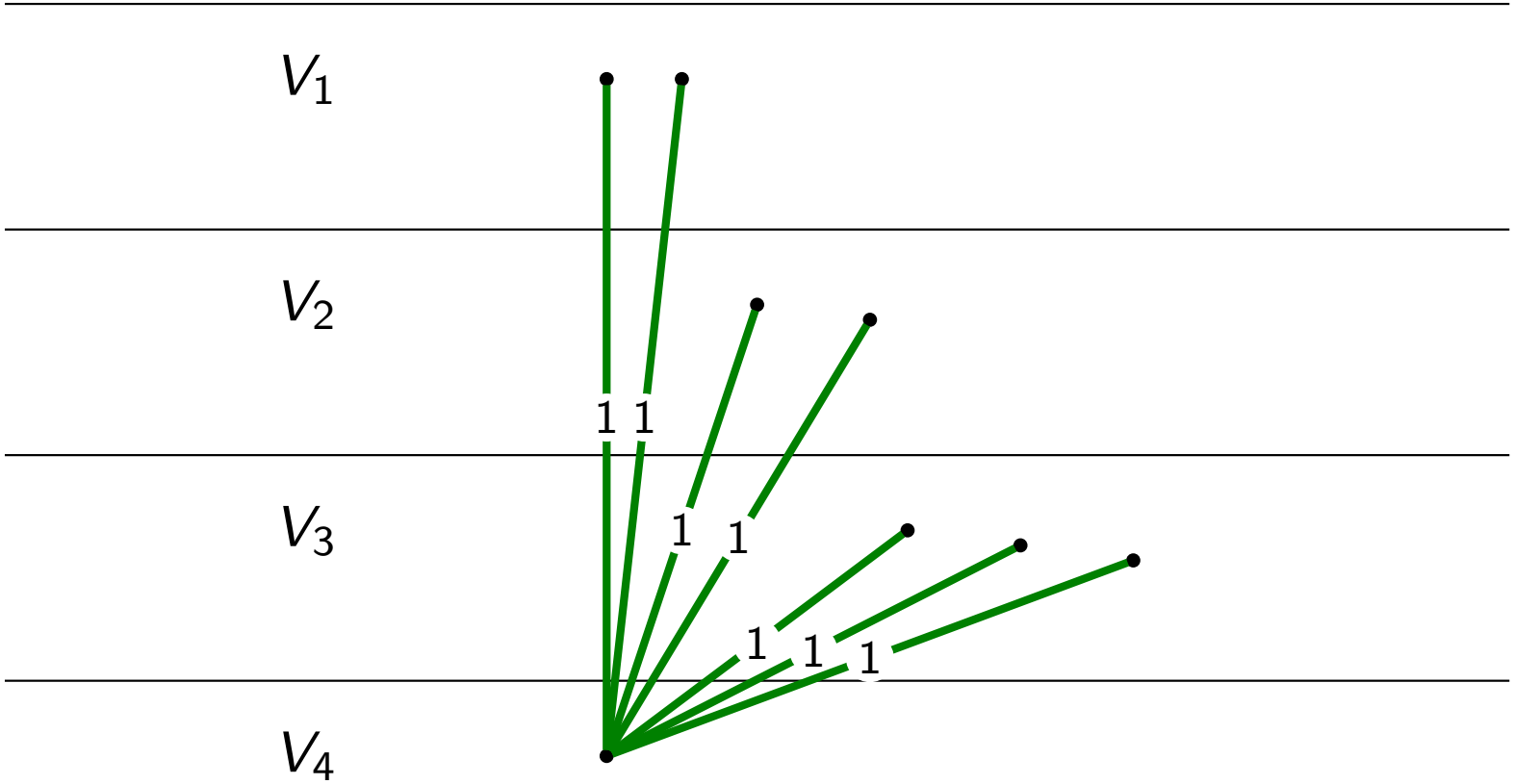

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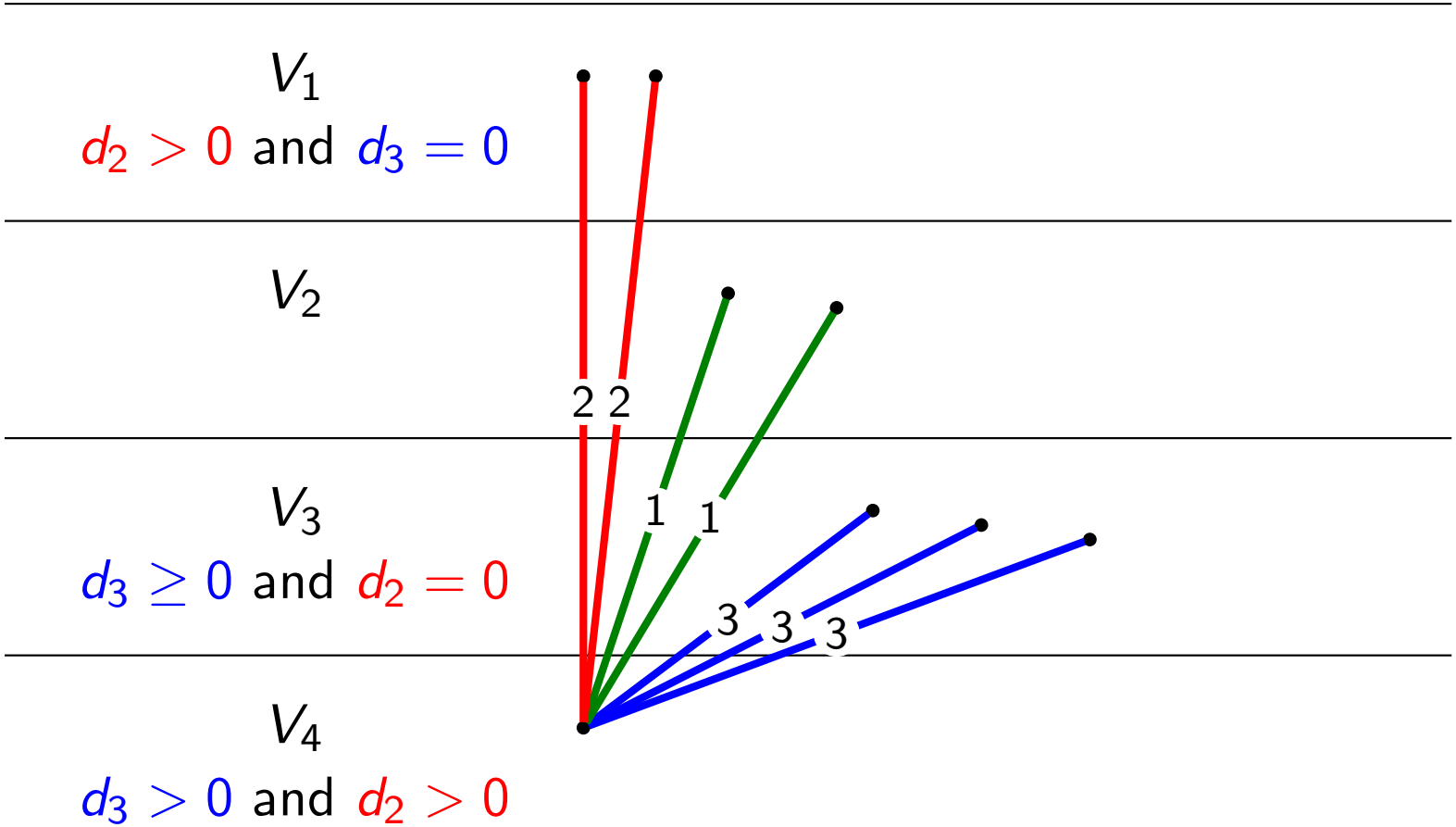

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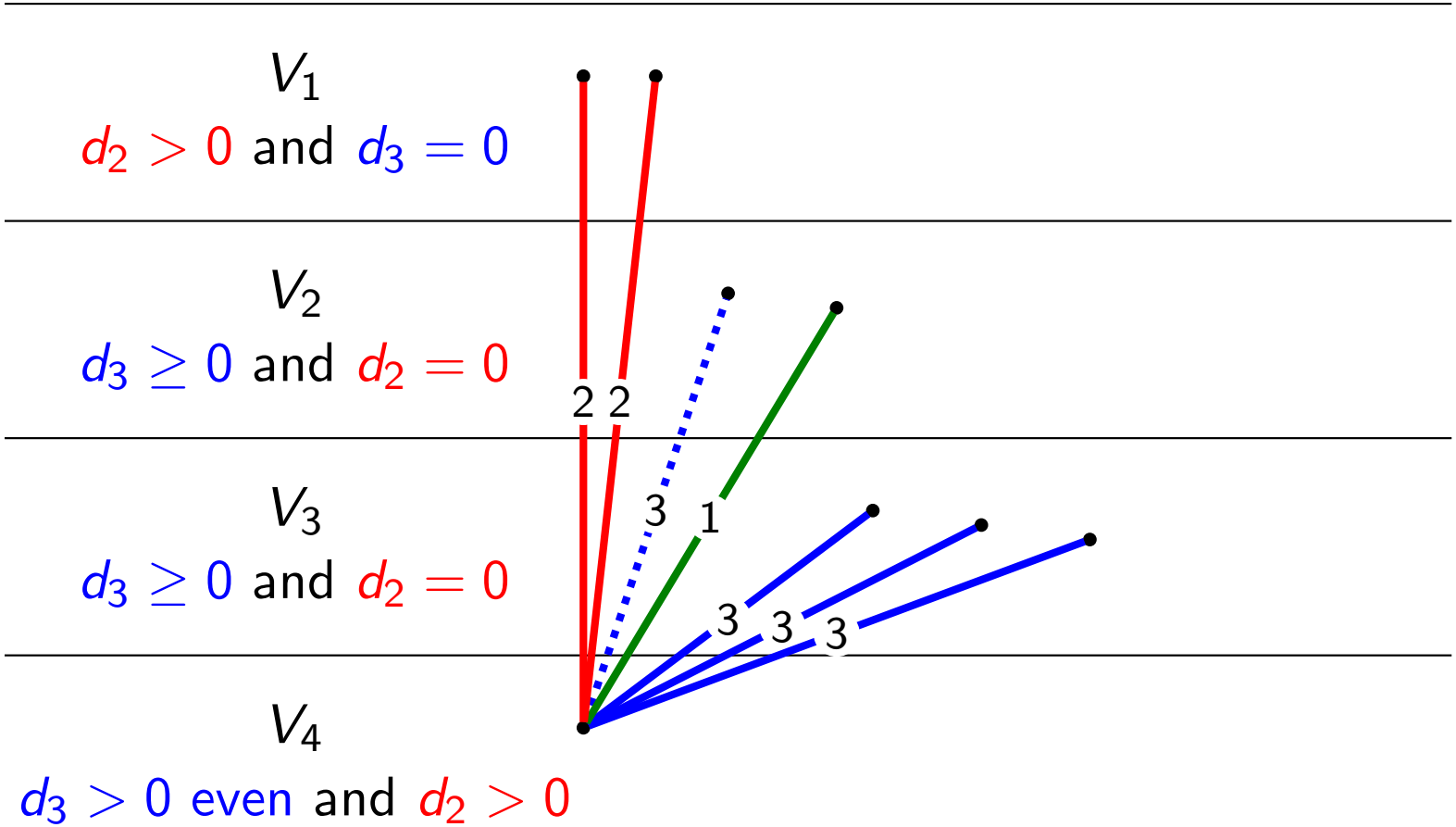

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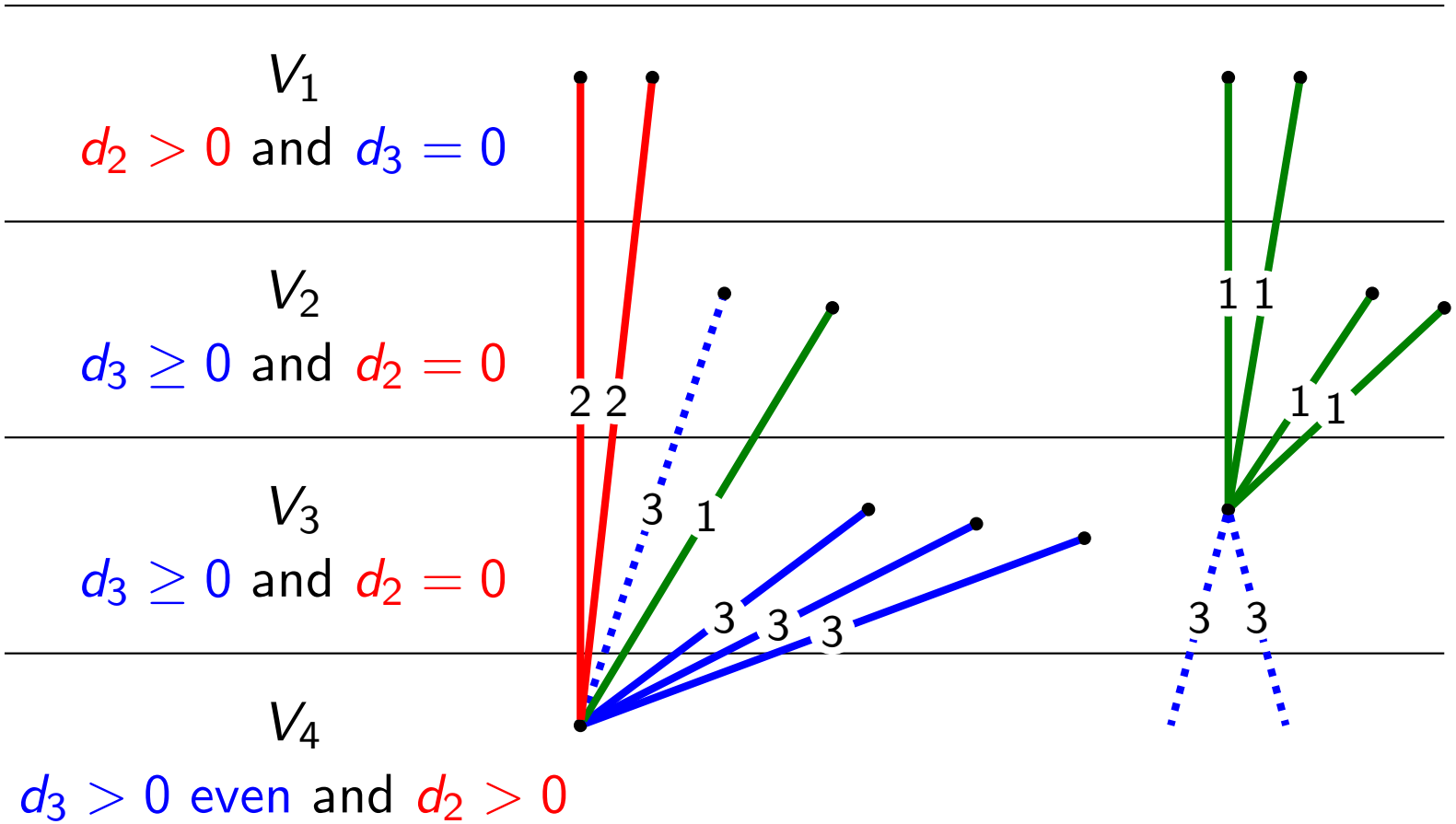

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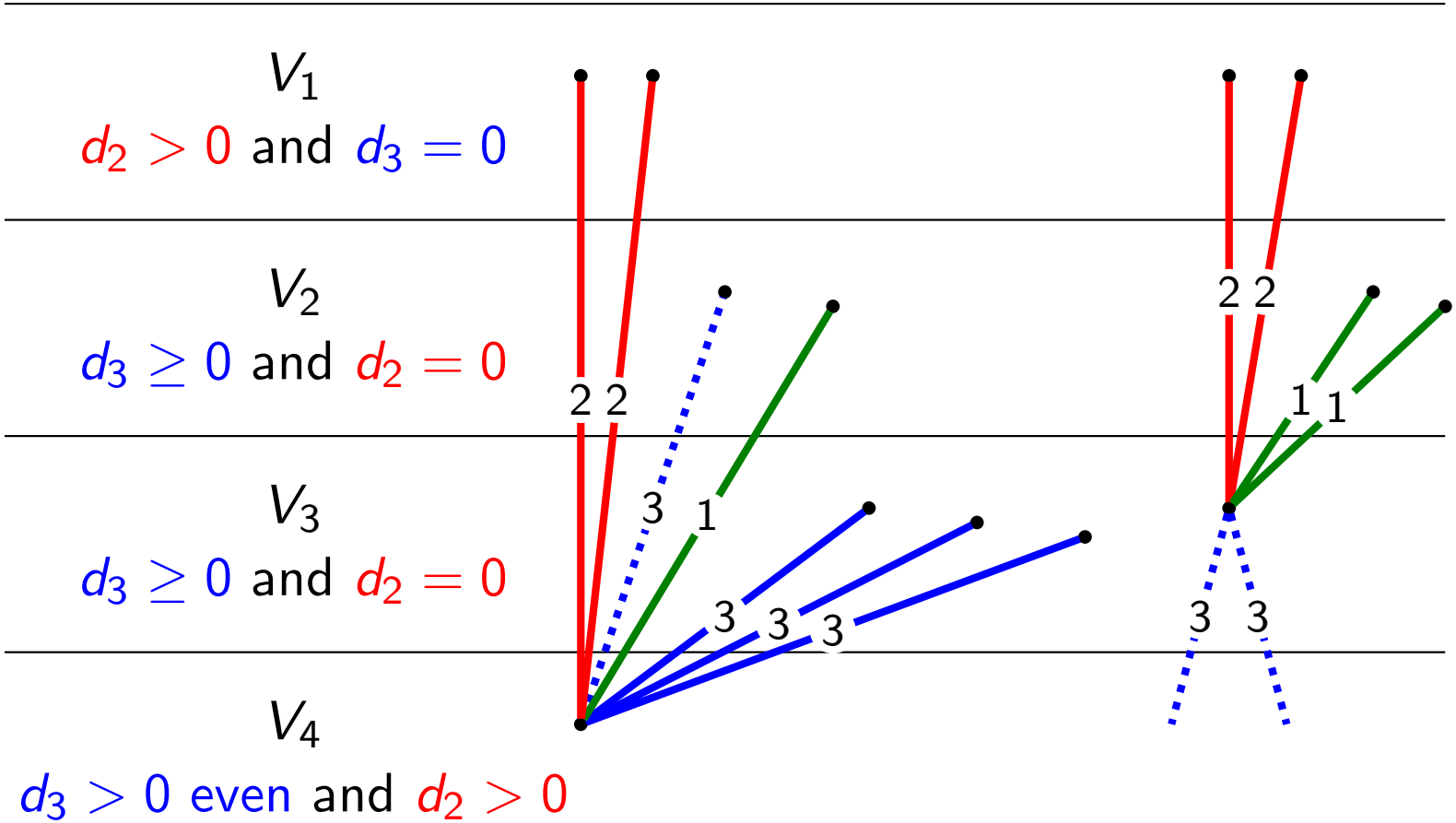

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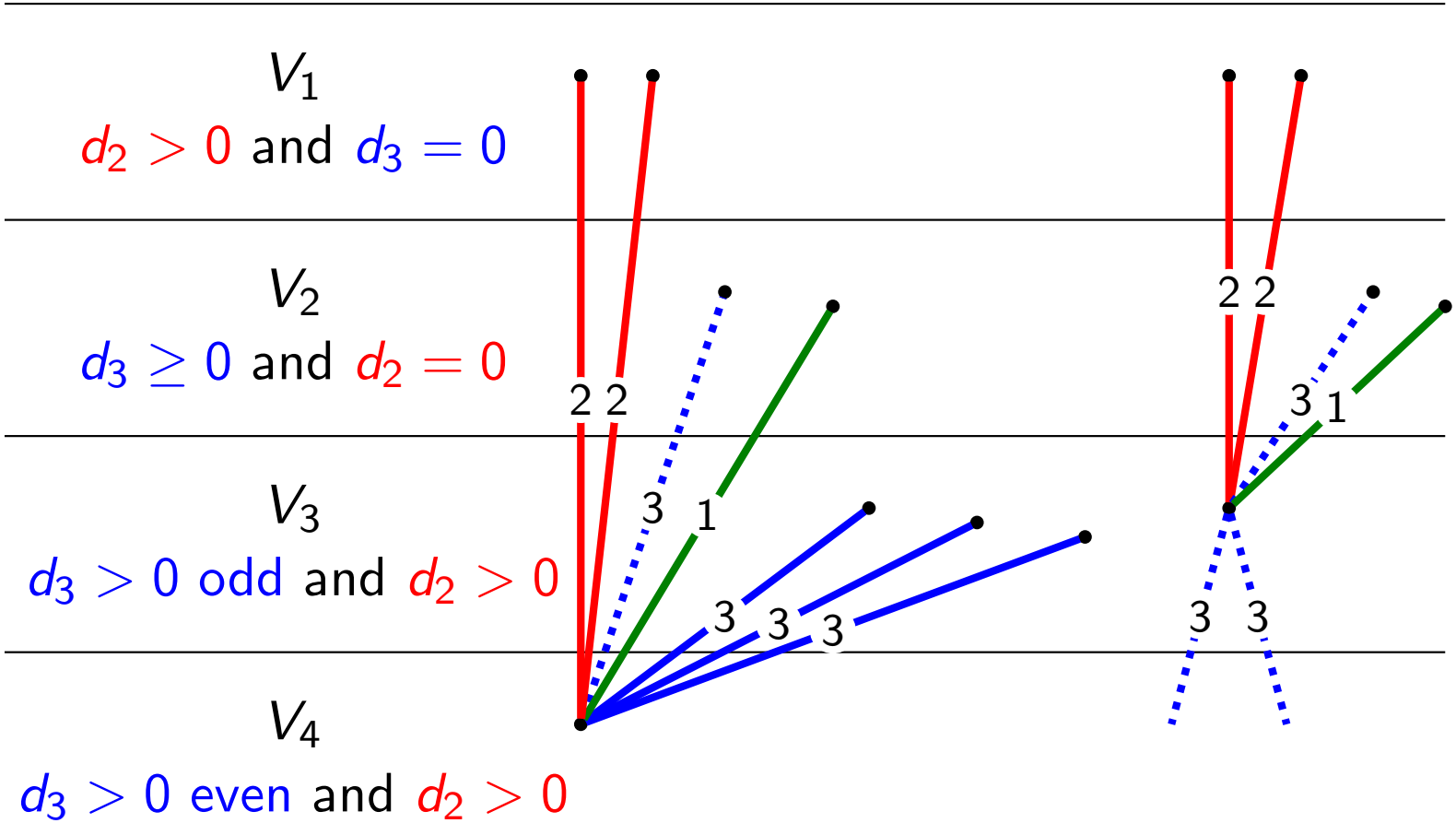

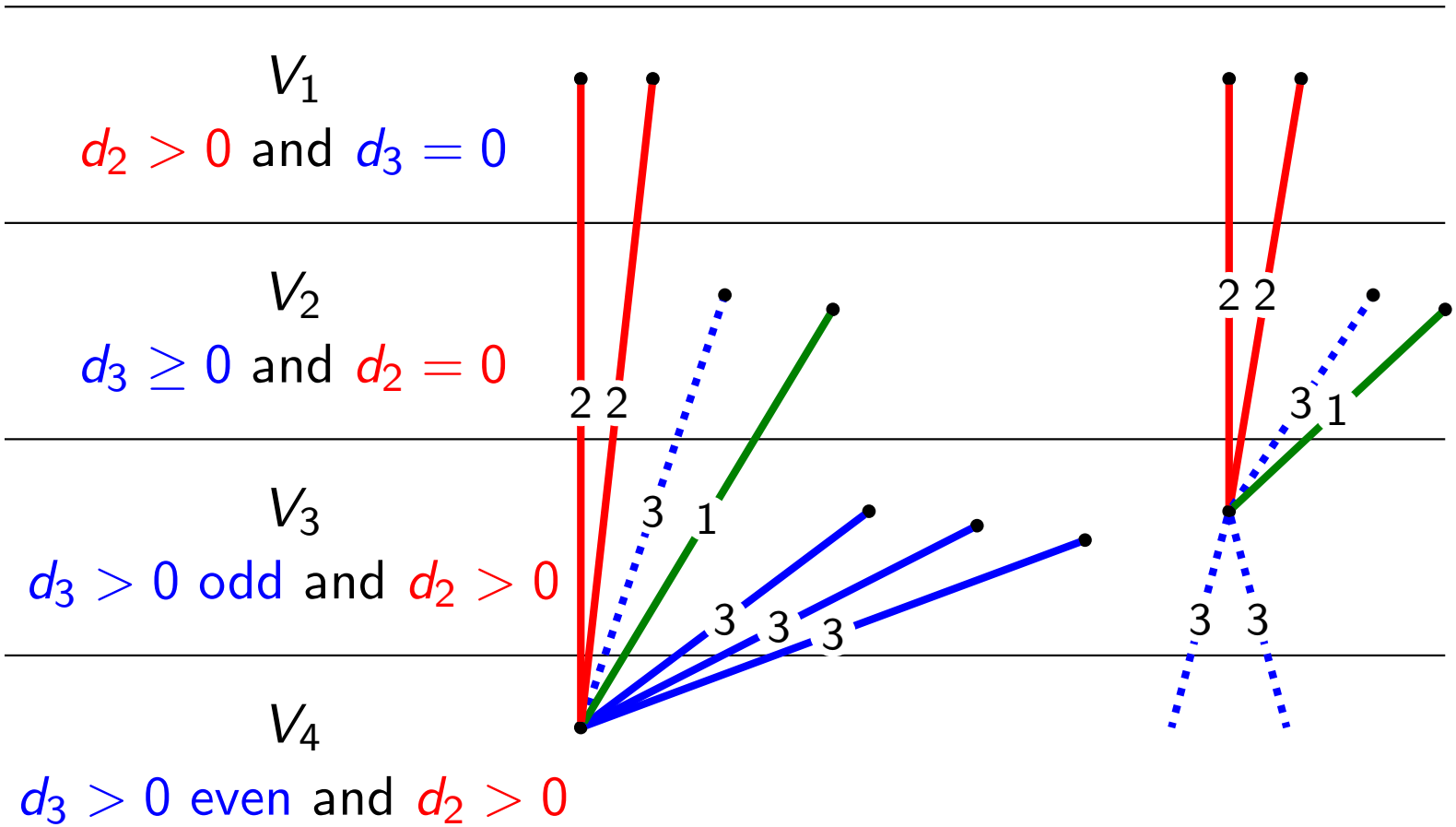

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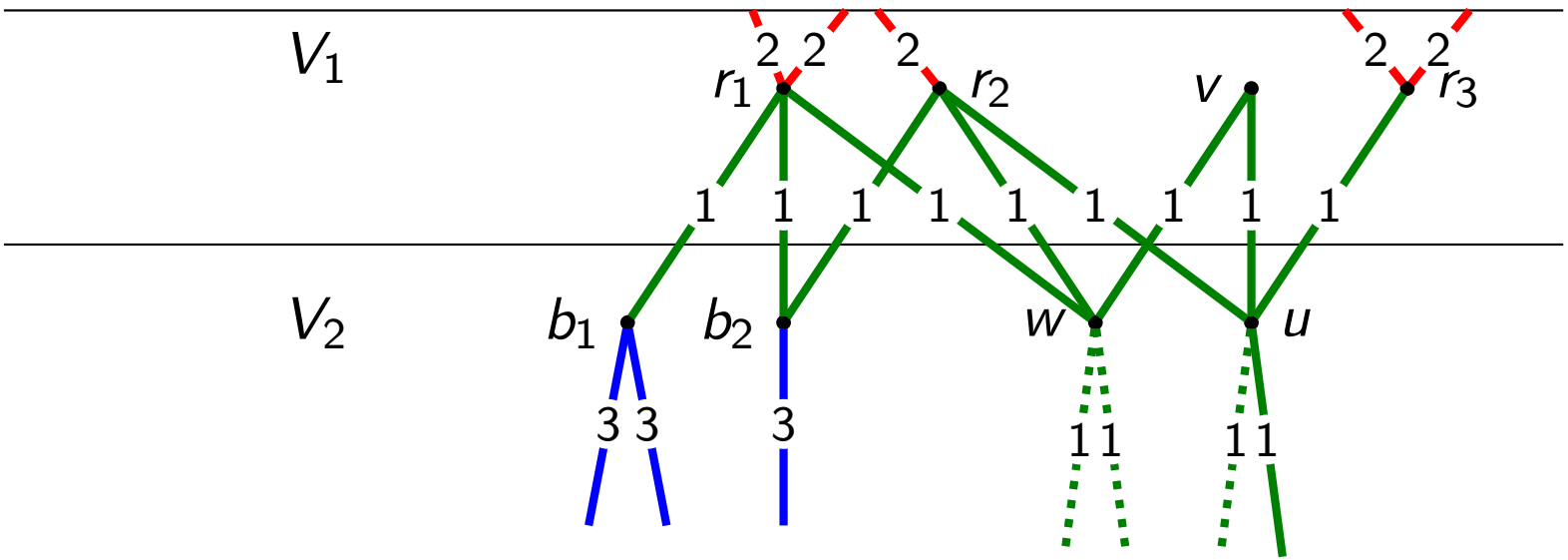

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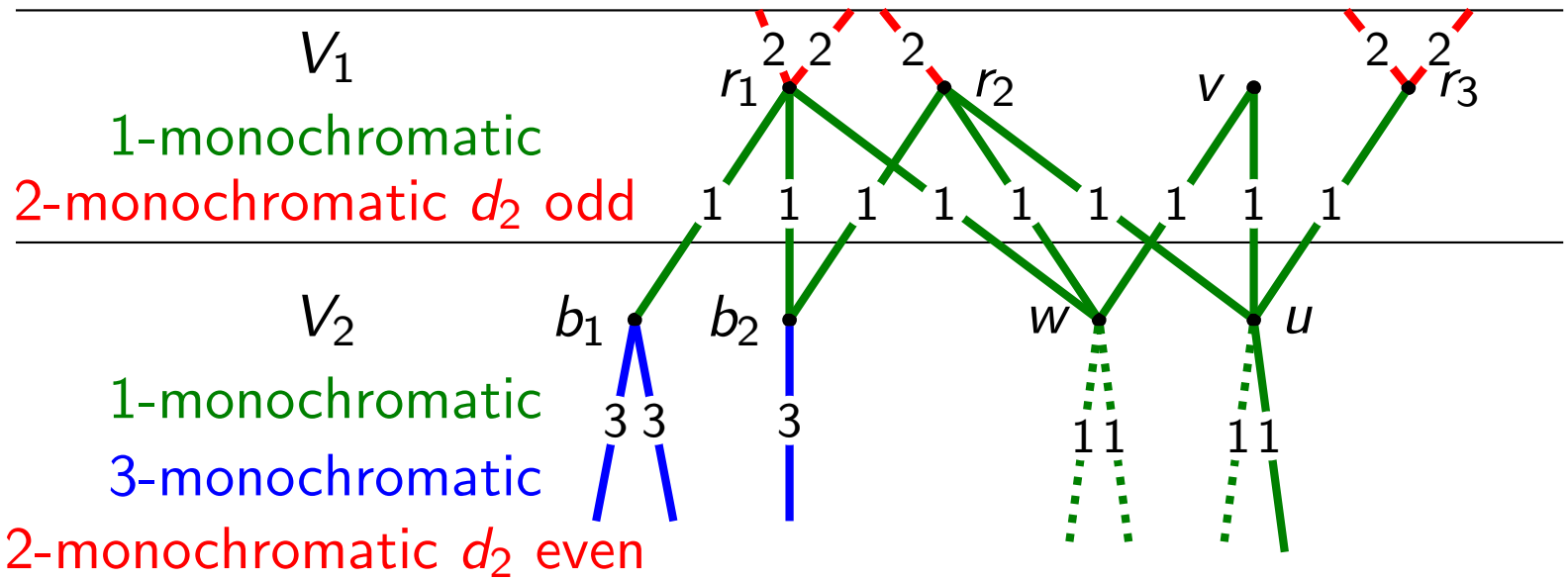

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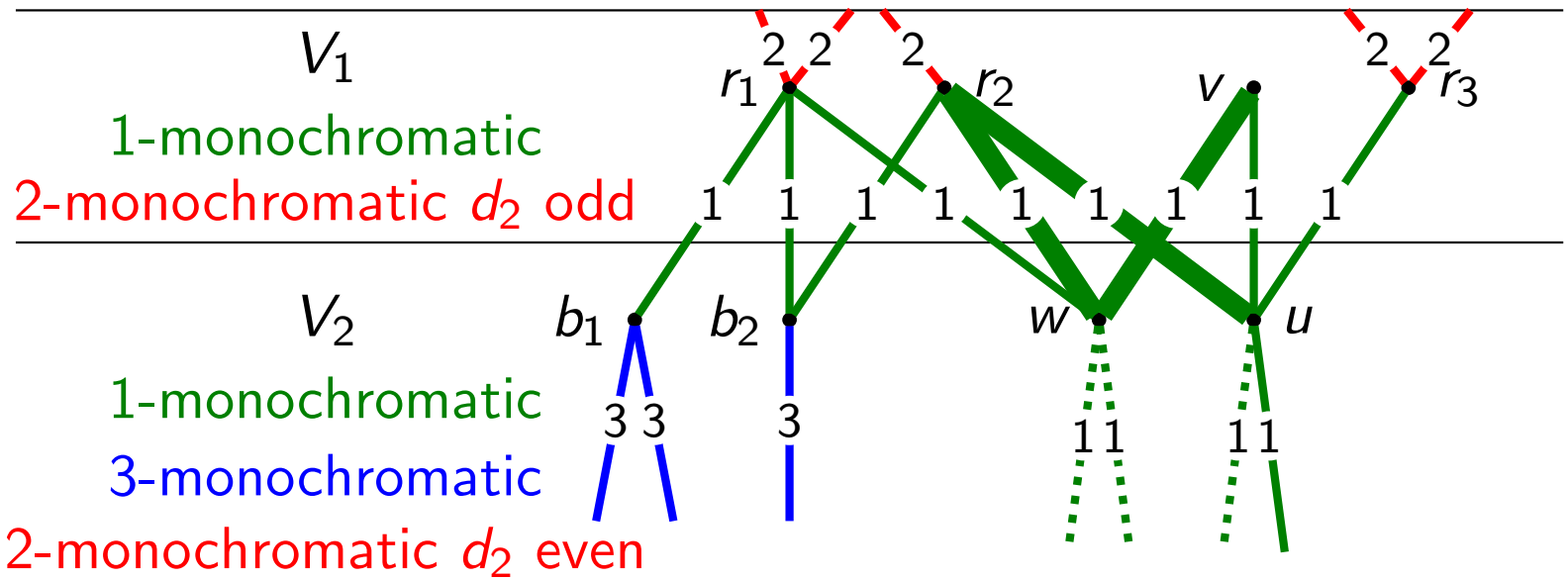

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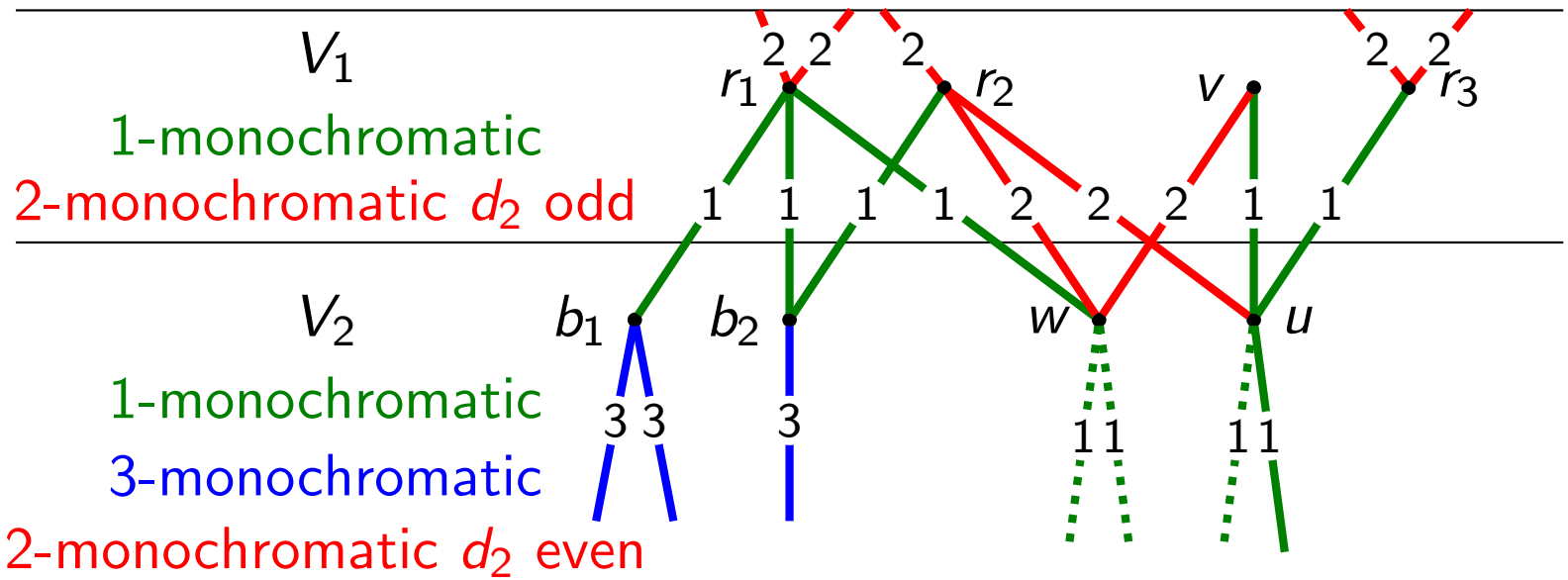

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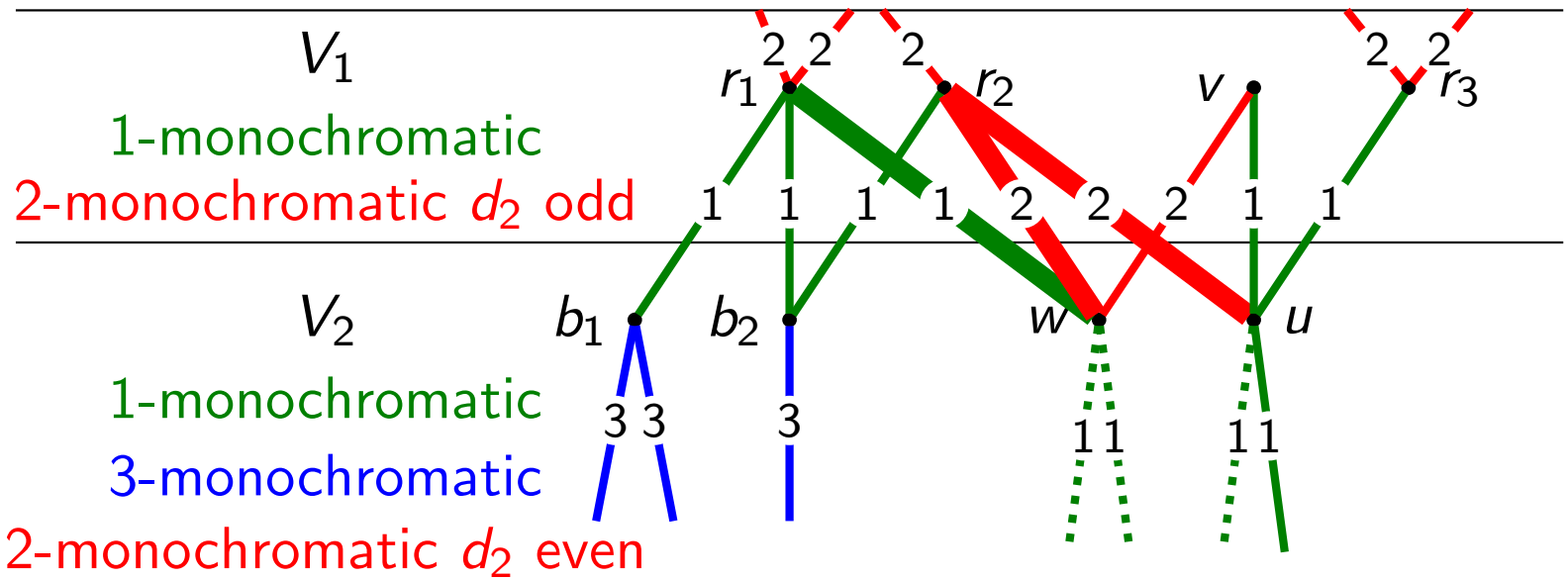

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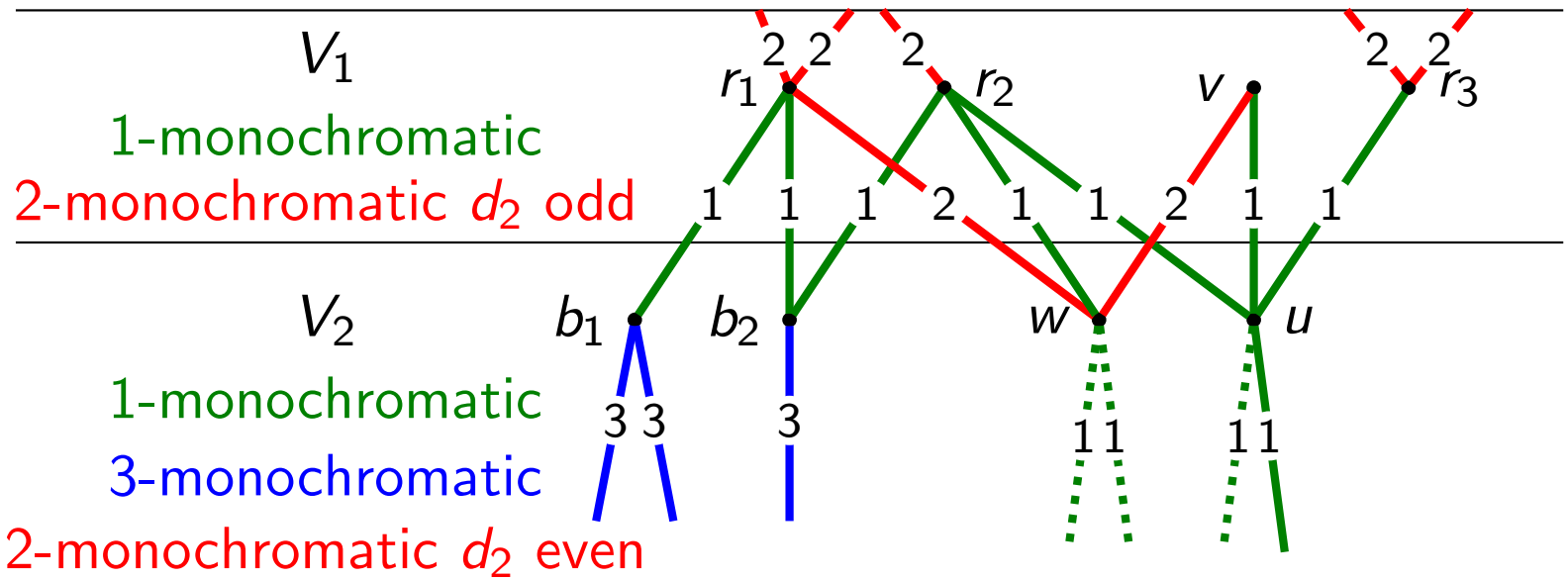

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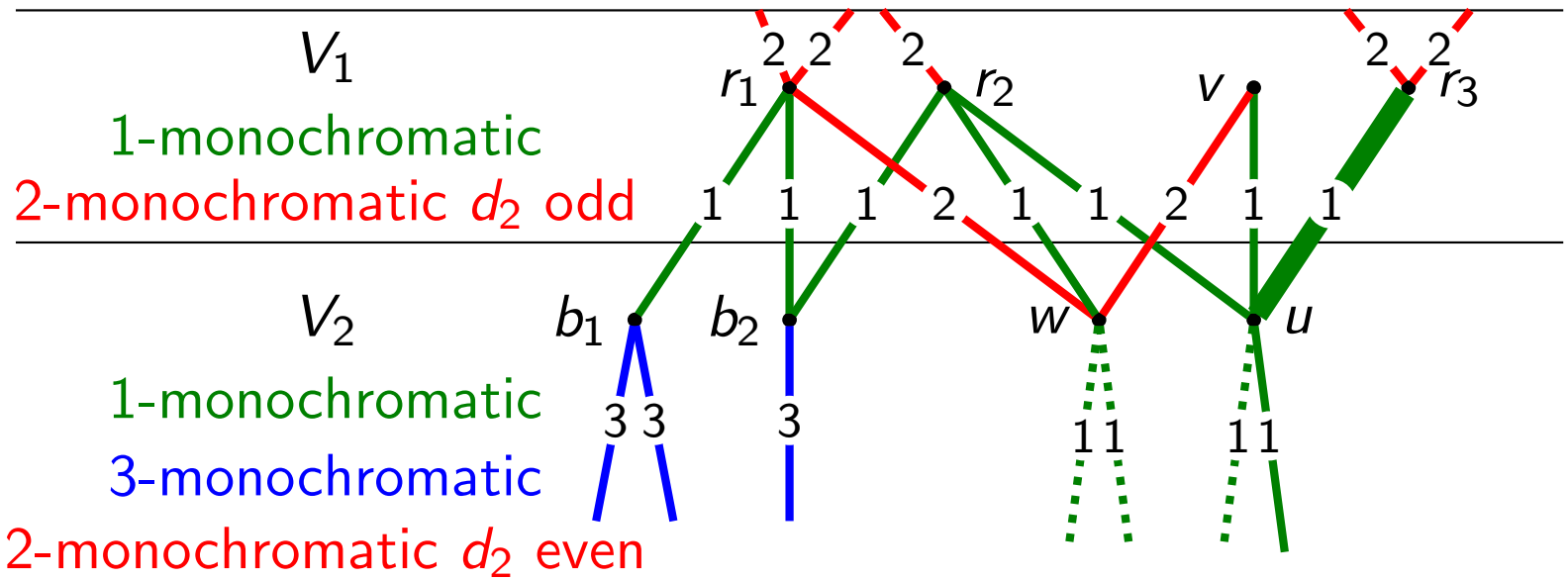

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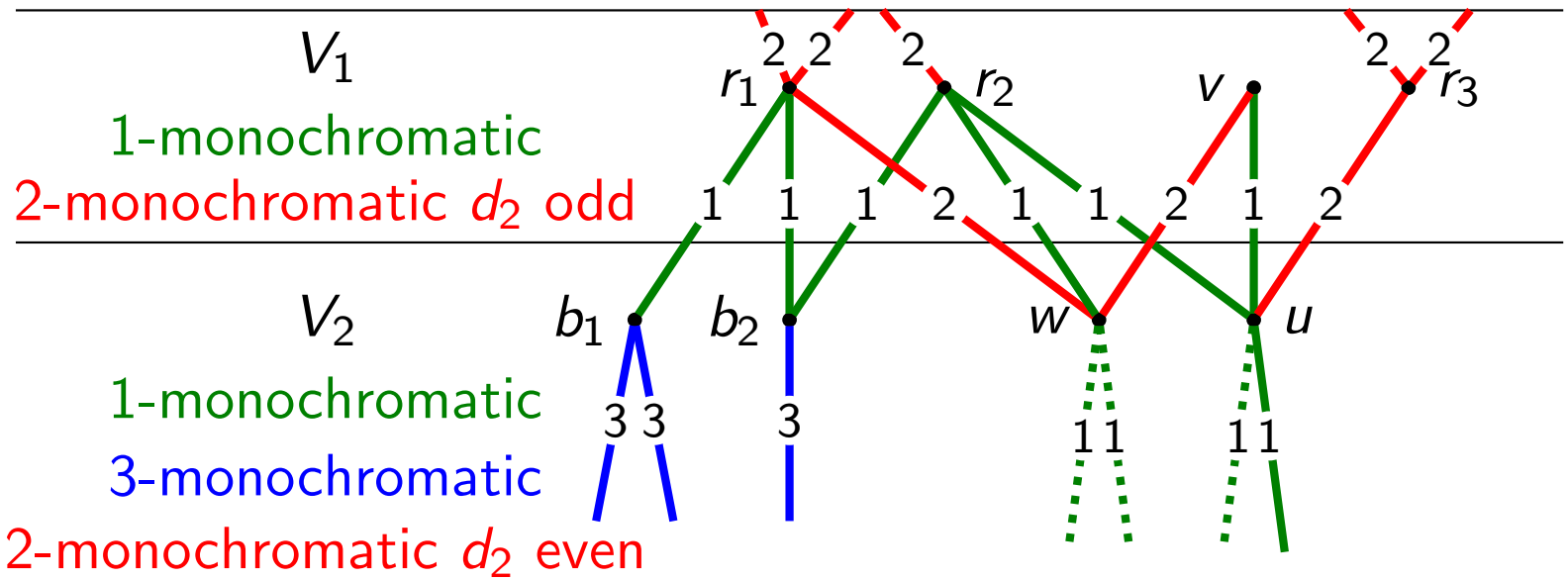

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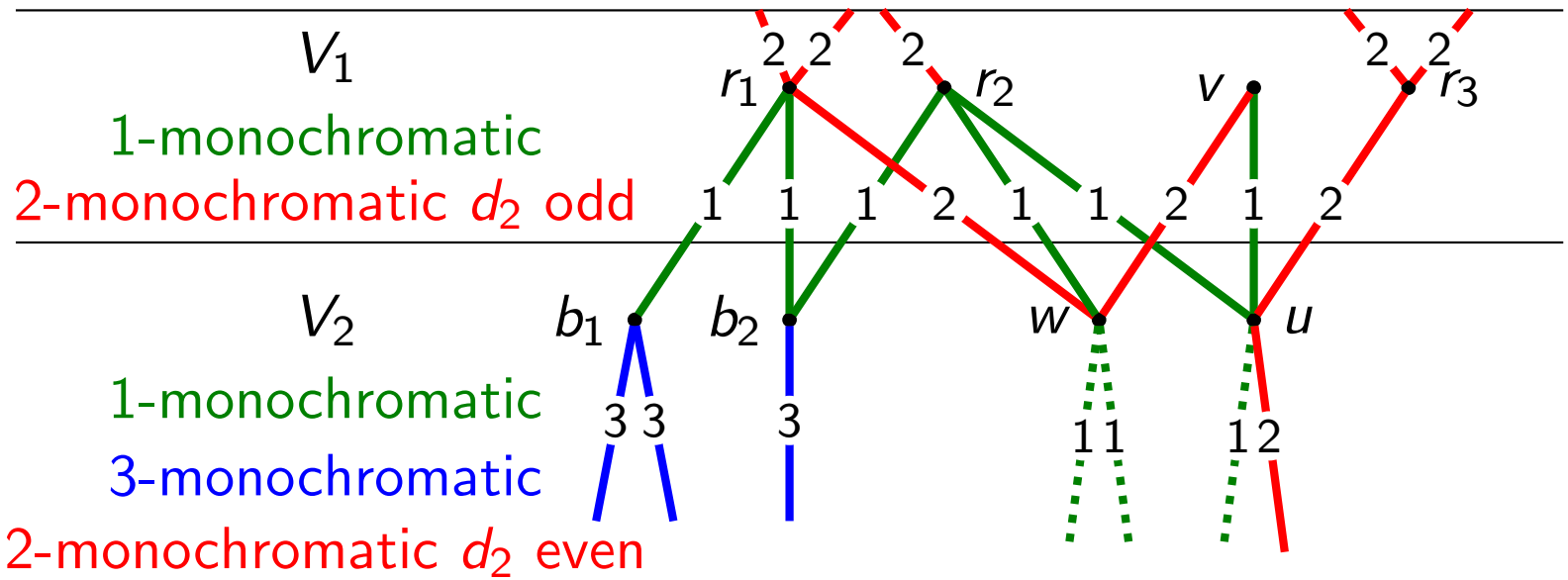

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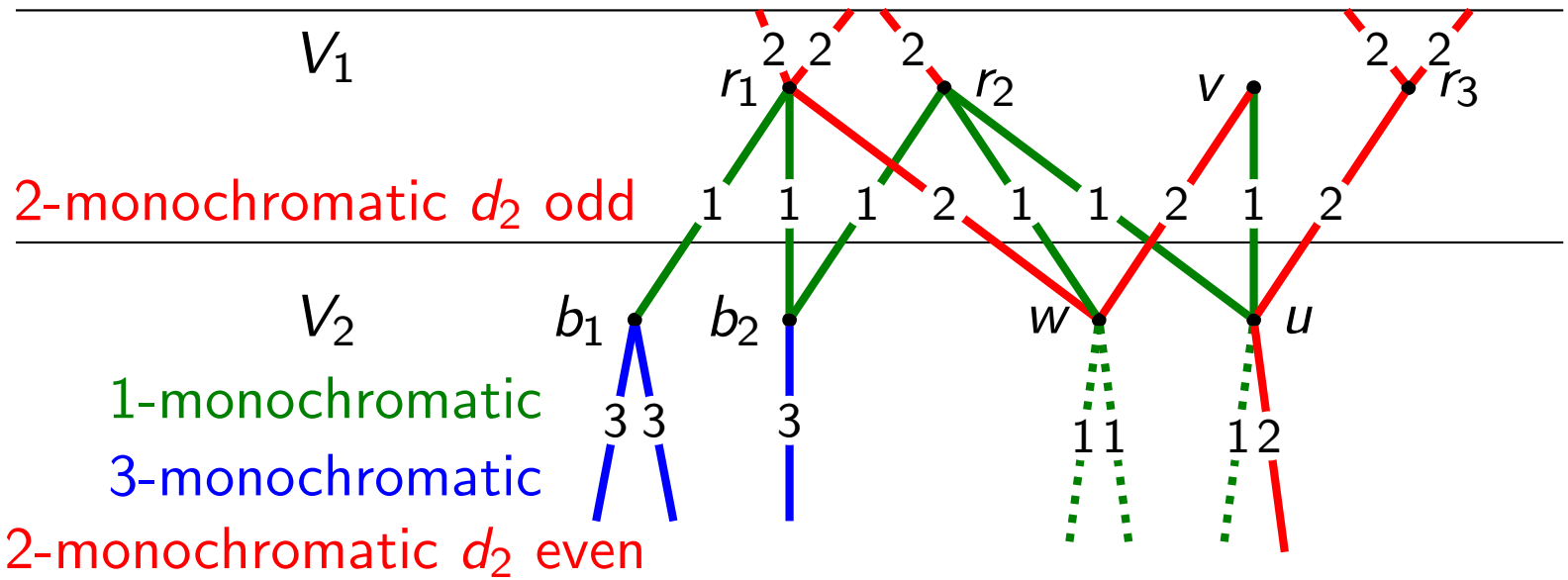

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Proof idea for "almost p-proper labellings"

Step 1: Partition $V(G)$ into k independent sets $V_1, \ldots, V_k$ such that each $u \in V_i$ has a neighbor in every V_j for $j < i$.

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- $v \in V_4$: bichromatic, $d_3(v) = 2$, and $d_2(v) + d_3(v)$ **odd**;
- $v \in V_5$: bichromatic, $d_2(v) = 2$, and $d_2(v) + d_3(v)$ **even**;
- ...
- $v \in V_{2n}$: bichromatic, $d_3(v) = n$, and $d_2(v) + d_3(v)$ **odd**;
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Step 3: Try to fix conflicts between vertices of V_1 and V_2.

Overview

Corollary

Every graph G admits a 3-**labelling** *such that, for every $x \in \mathbb{N}$, the subgraph induced by x-colored vertices is a* **forest**.

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Conjecture

*Every graph G admits a 2-**labelling** such that, for every $x \in \mathbb{N}$, the subgraph induced by x-colored vertices is a **forest**.*

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Theorem

*A graph G admits a 3-**labelling** such that, for every $x \in \mathbb{N}$, the subgraph induced by x-colored vertices is a **forest**, if*

- *G is a **complete graph**, or*
- *G is **bipartite**, or*
- *G is **subcubic**.*

Thank you for your attention.