

Limits of near-coloring of sparse graphs

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Abstract

Let a, b, d be non-negative integers. A graph G is $(d, a, b)^*$ -colorable if its vertex set can be partitioned into $a + b$ sets $I_1, \dots, I_a, O_1, \dots, O_b$ such that the graph $G[I_i]$ induced by I_i has maximum degree at most d for $1 \leq i \leq a$, while the graph $G[O_j]$ induced by O_j is an edgeless graph for $1 \leq j \leq b$. In this paper, we give two real-valued functions f and g such that any graph with maximum average degree at most $f(d, a, b)$ is $(d, a, b)^*$ -colorable, and there exist non- $(d, a, b)^*$ -colorable graphs with average degree at most $g(d, a, b)$. Both these functions converge (from below) to $2a + b$ when d tends to infinity. Counterintuitively, this implies that allowing a color to be d -improper (i.e. of type I_i) even for a large degree d increases the maximum average degree that guarantees the existence of a valid coloring only by 1. Using a color of type I_i (even with a very large degree d) is somehow less powerful than using two colors of type O_j (two stable sets).

1 Introduction

A graph G is (d_1, \dots, d_k) -colorable if the vertex set of G can be partitioned into subsets V_1, \dots, V_k such that the graph $G[V_i]$ induced by the vertices of V_i has maximum degree at most d_i for all $1 \leq i \leq k$. This notion generalizes those of proper k -coloring (when $d_1 = \dots = d_k = 0$) and of d -improper k -coloring (when $d_1 = \dots = d_k = d \geq 1$).

Planar graphs are known to be $(0, 0, 0, 0)$ -colorable (Appel and Haken [1, 2]) and $(2, 2, 2)$ -colorable (Cowen *et al.* [8]; for a list version, see Eaton and Hull [9] or Škrekovski [13]). This last result was improved for planar graphs with large girth [14]. We recall that the girth of a graph G is the length of a shortest cycle in G (by convention, the girth of G is ∞ when G is a forest). Havet and Sereni [12] gave new bounds for graphs with low maximum average degree. The maximum average degree of a graph G , denoted by $\text{mad}(G)$, is the maximum of the average degrees of all subgraphs of G , i.e. $\text{mad}(G) = \max \{2|E(H)|/|V(H)|, H \subseteq G\}$.

Theorem 1 (Havet and Sereni [12]) *Every graph G with $\text{mad}(G) < k + \frac{kd}{k+d}$ is d -improper k -colorable (in fact, d -improper k -choosable), i.e. (d, \dots, d) -colorable (where the tuple is of size k).*

Moreover their upper bound on the maximum average degree is asymptotically sharp:

Theorem 2 (Havet and Sereni [12]) *There exists a non- d -improper k -colorable graph whose maximum average degree tends to $2k$ when d goes to infinity.*

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