

VIZING'S CONJECTURE: A SURVEY AND RECENT RESULTS

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ABSTRACT. Vizing's conjecture from 1968 asserts that the domination number of the Cartesian product of two graphs is at least as large as the product of their domination numbers. In this paper we survey the approaches to this central conjecture from domination theory and give some new results along the way. For instance, several new properties of a minimal counterexample to the conjecture are obtained and a lower bound for the domination number is proved for products of claw-free graphs with arbitrary graphs. Open problems, questions and related conjectures are discussed throughout the paper.

Keywords: *Cartesian product; domination; Vizing's conjecture*

1. INTRODUCTION

Vertex connectivity, matching number, chromatic number, crossing number, genus, and independence number are but a few examples of graph invariants. An important problem to be solved in understanding a graph invariant is “how it behaves” on graph products. Because of how the product relates to the two factors, it seems reasonable to think that the value of the invariant on the product of two graphs G and H will, in some consistent way, relate to its value—and perhaps that of other invariants—on G and H . In 1996 Nowakowski and Rall [36] explored this relationship for twelve independence, coloring and domination invariants on the ten associative graph products whose edge structure depends on that of both factors.

For some invariants and products, this relationship is known and easy to verify. An example of this situation is that the chromatic number of the Cartesian product of two graphs is the maximum of their chromatic numbers. In some cases, for example the independence number of the direct product, there are proven bounds, but in general no exact formula is known in terms of the independence numbers of the two factor graphs. For still others the invariant has a conjectured behavior, but the issue is far from being settled. This is the situation for the domination number on a Cartesian product. The following conjecture was made by V. G. Vizing in 1968.

Conjecture 1.1. ([39]) *For every pair of finite graphs G and H ,*

$$(1) \quad \gamma(G \square H) \geq \gamma(G) \gamma(H).$$

As usual, γ stands for the domination number, and $G \square H$ is the standard notation for the Cartesian product of graphs G and H .

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