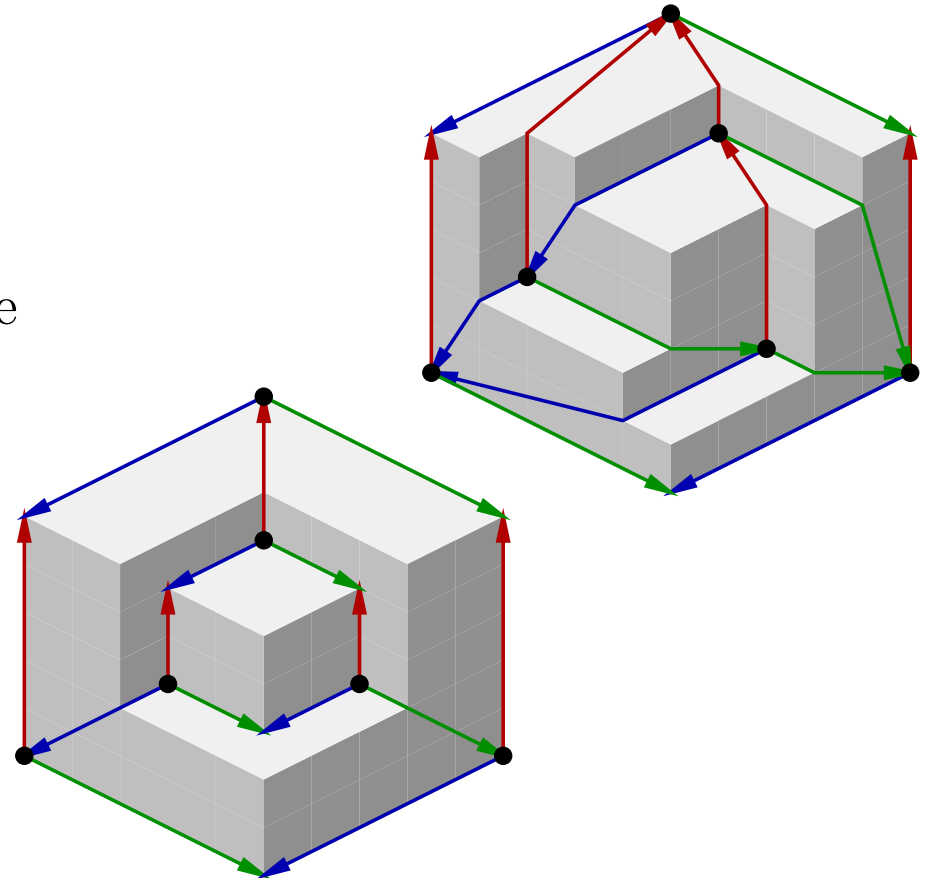


# 3-Connected Planar Graphs Orthogonal Surfaces and Flips

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# Content

## Plane Triangulations

- (1) Schnyder Structures
- (2) Applications: Dimension, Drawing and Surfaces

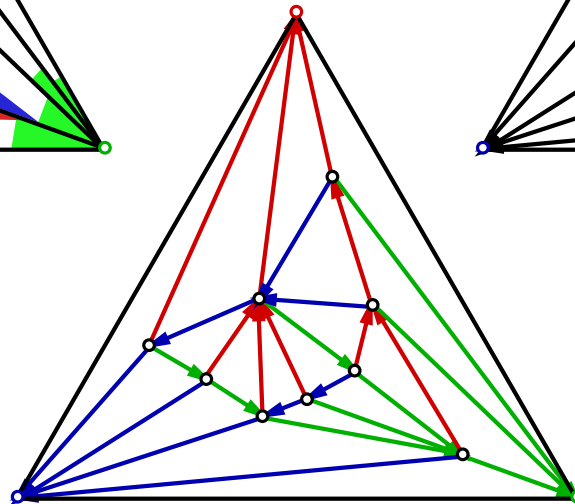
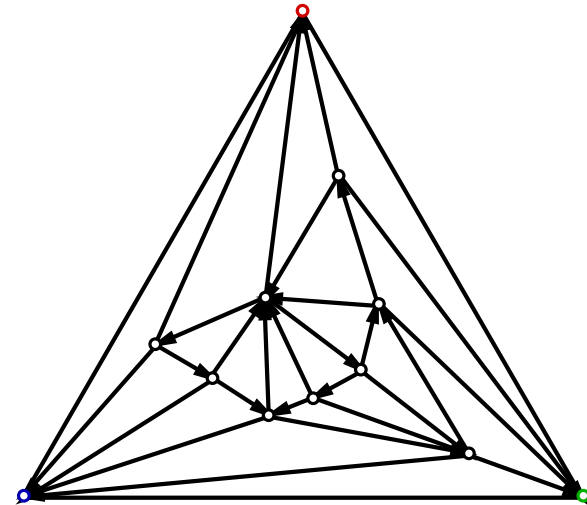
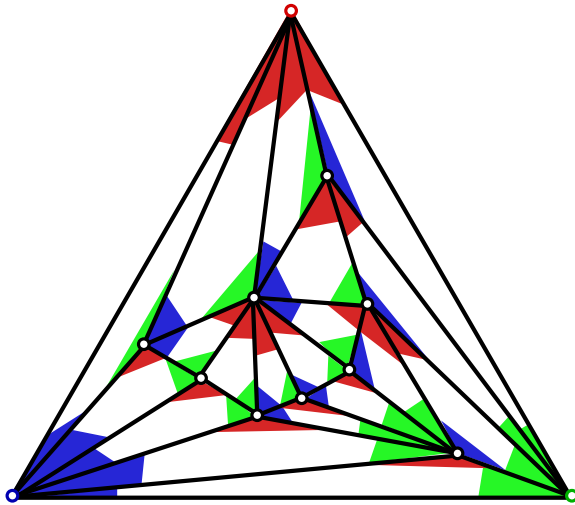
## 3-Connected Plane Graphs

- (3) Schnyder Structures Revisited
- (4) Applications and more on Orthogonal Surfaces
- (5) Flips and Lattices

Today's talk covers (3) and (4).

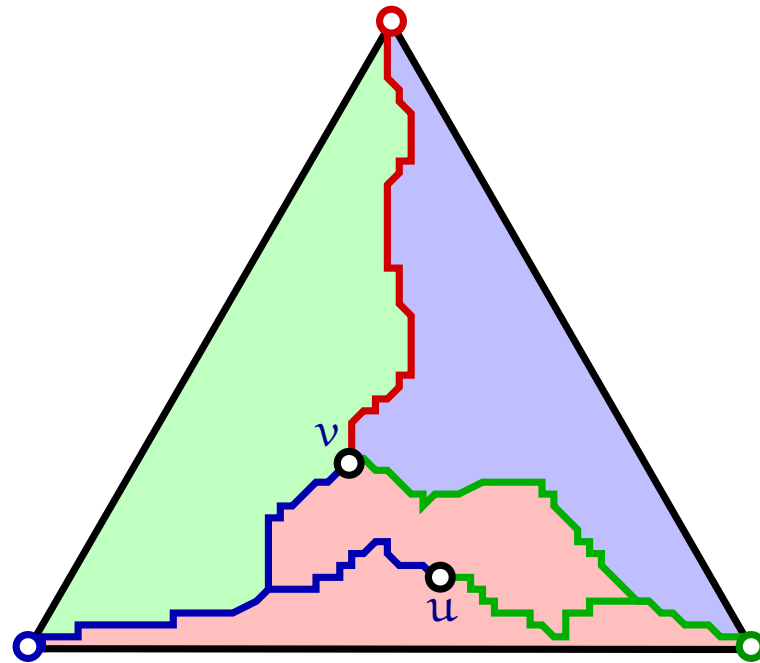
# Schnyder Structures

- Angle labelings.
- Woods.
- 3-Orientations.



# The Regions of a Vertex

- Every vertex has three distinguished regions.



- If  $u \in R_i(v)$  then  $R_i(u) \subset R_i(v)$ .

# Schnyder Region Counts

$\phi : \text{regions} \rightarrow \mathbb{R}^+$  such that

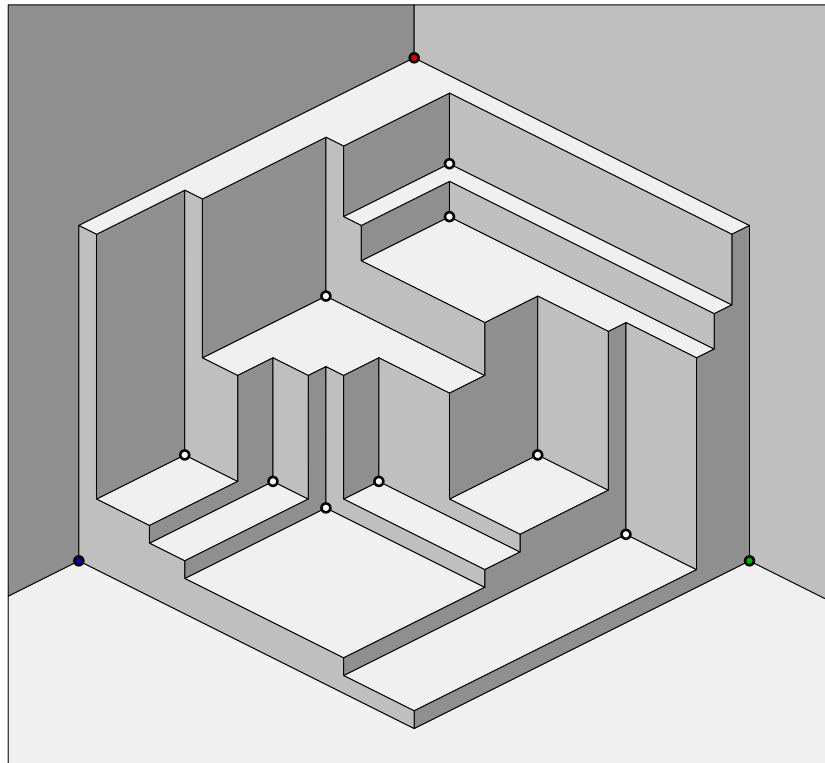
- $R_i(u) \subset R_i(v) \implies \phi(R_i(u)) < \phi(R_i(v))$
- $\phi(R_1(v)) + \phi(R_2(v)) + \phi(R_3(v)) = C$

Applications:

- 3-realizer for planar graphs.  
 $\implies \dim(G) \leq 3$  for  $G$  planar.
- Straight line drawings on small integer grids.

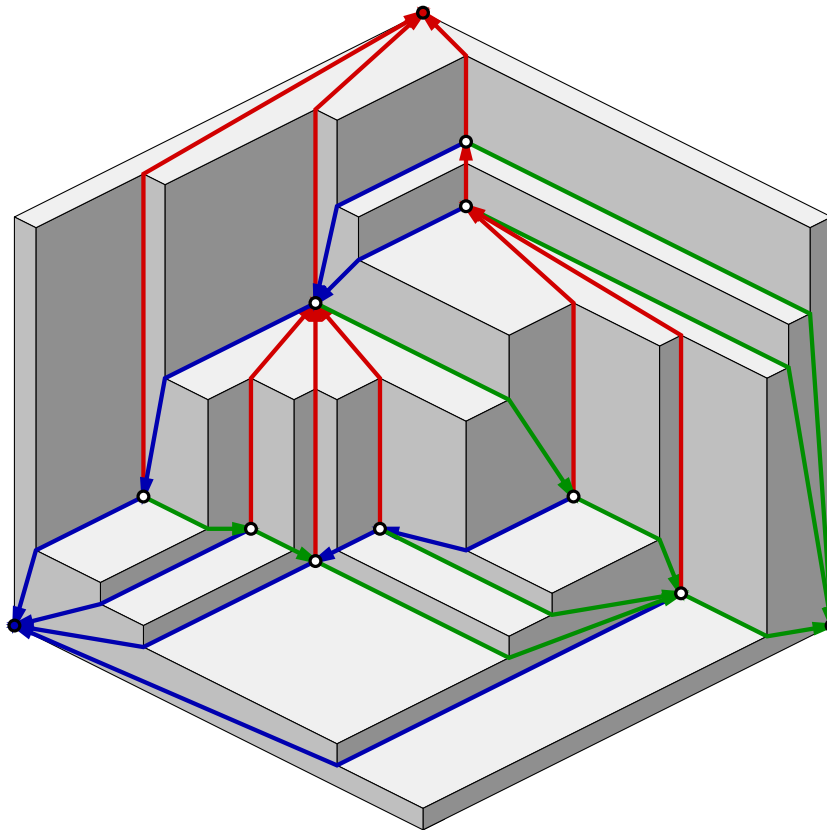
# Using 3 Coordinates

Take the count of faces in the three regions as a set  $P$  of points in  $\mathbb{R}^3$ . Let  $P$  generate an ideal  $I_P^{\leq}$  the boundary  $\partial I_P^{\leq}$  is the orthogonal surface generated by  $P$ .



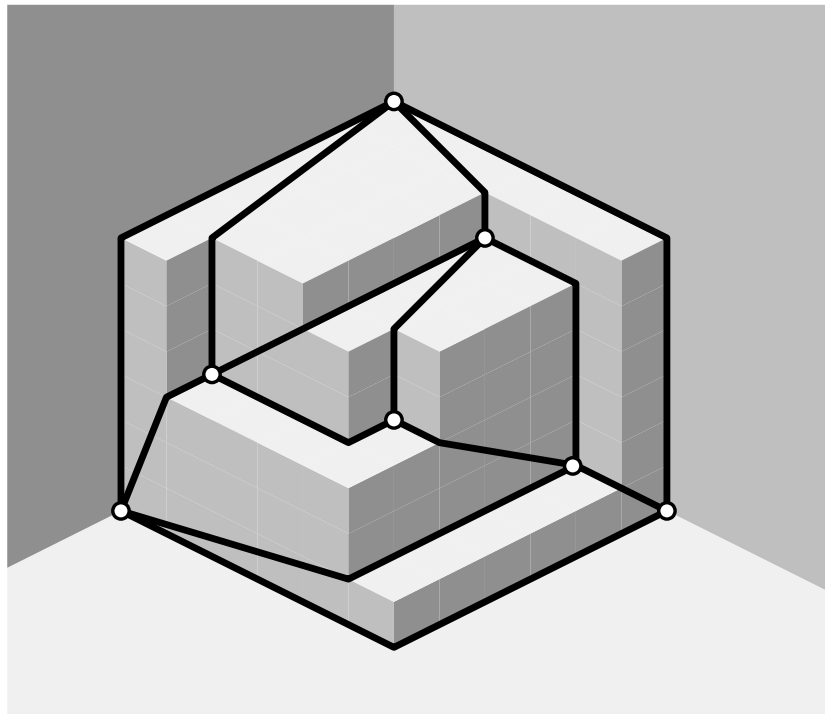
# Schnyder Woods on Orthogonal Surfaces

The Schnyder wood used for the count of faces is nicely represented on the orthogonal surface  $\partial I_P^{\leq}$ .



# From Orthogonal Surfaces to Graphs

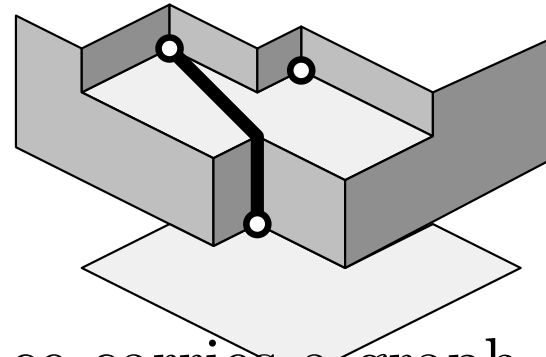
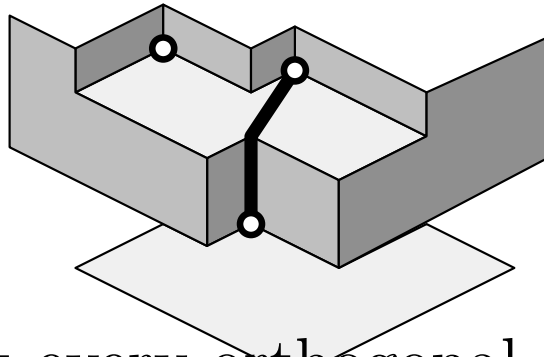
The graph of an orthogonal surface need not be a triangulation:



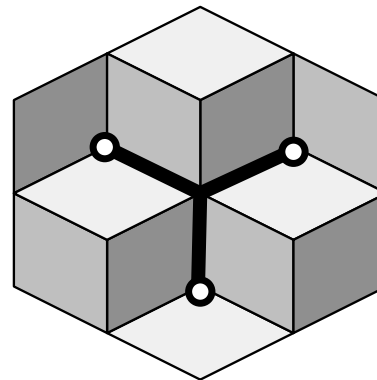
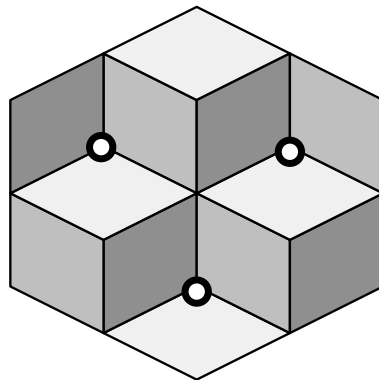


# Orthogonal Surfaces and Graphs II

- The graph of an orthogonal surface is not necessarily unique



- Not every orthogonal surface carries a graph



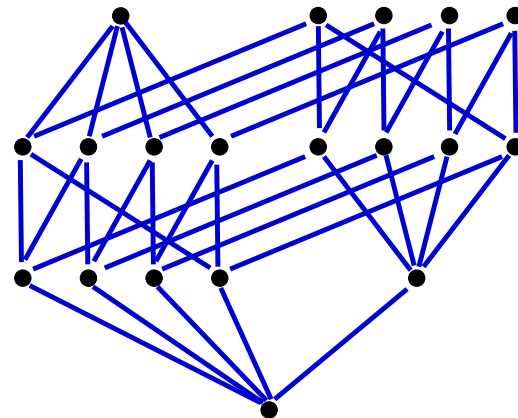
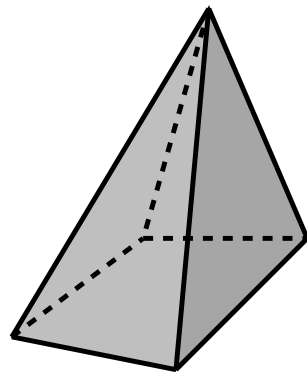
# Digression: More on Dimension

Let  $\mathcal{F}_P$  be the face lattice of a polytope and let  $\dim$  denote order dimension

**Theorem** [Schnyder].

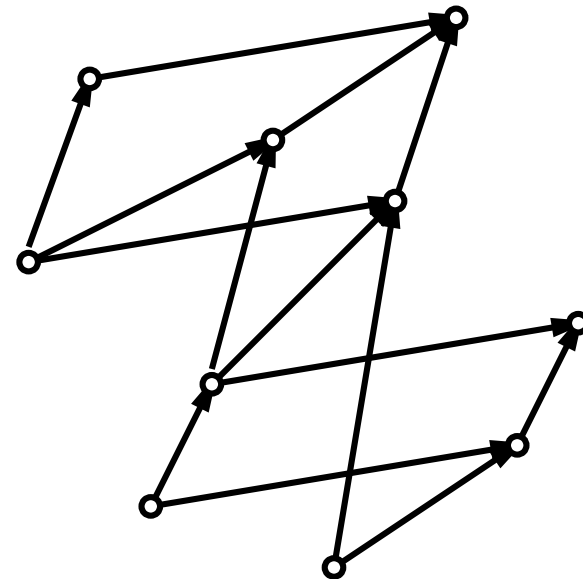
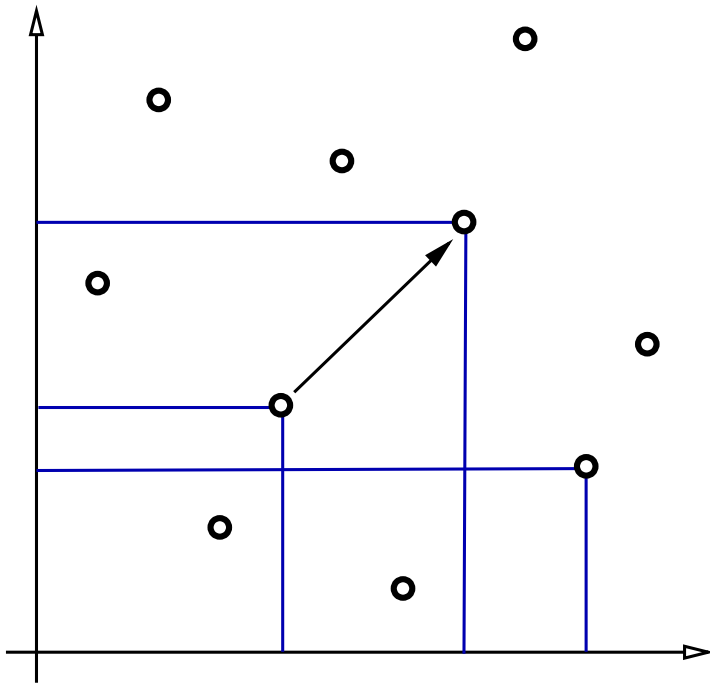
If  $P$  is a simplicial 3-polytope and  $F$  is any face of  $P$ , then

- $\dim(\mathcal{F}_P \setminus F) = 3$
- $\dim(\mathcal{F}_P) = 4$

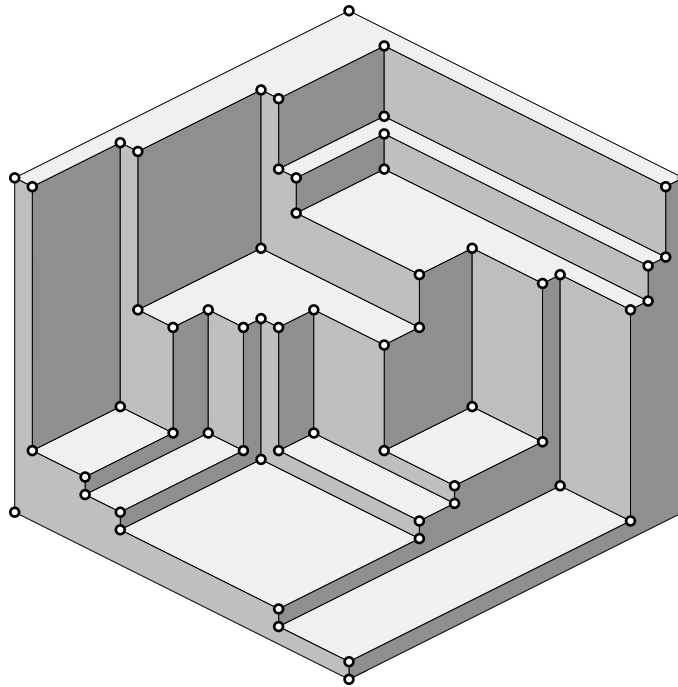


# Dimension of Orders

The *dimension* of an order  $\mathbf{P} = (\mathbf{X}, <)$  is the least  $\mathbf{t}$ , such that  $\mathbf{P}$  is isomorphic to a suborder of  $\mathbf{IR}^{\mathbf{t}}$  with the product ordering.

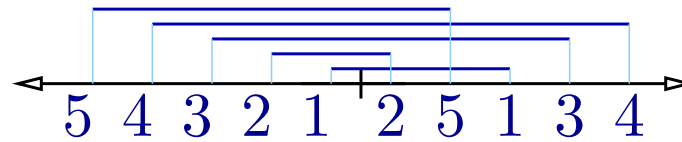
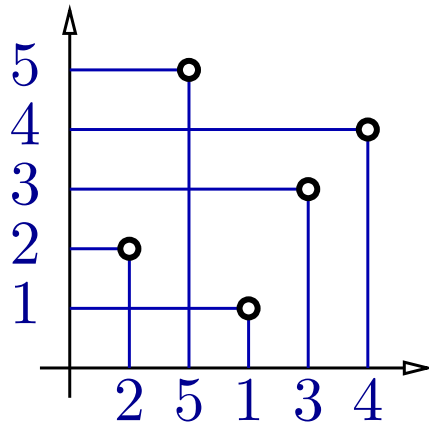


# An Example in Dimension 3

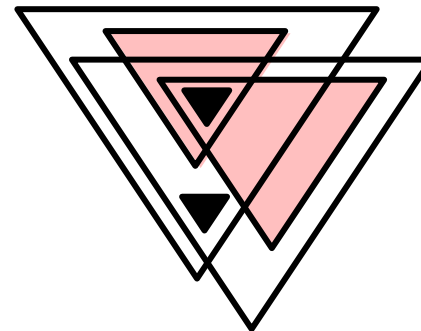
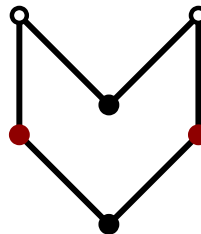
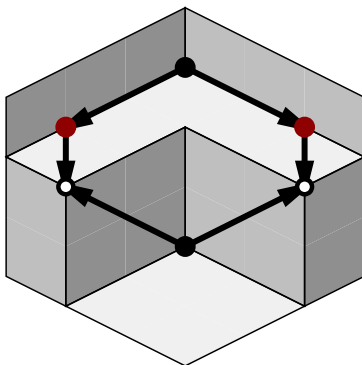


# Characterizations

- Dimension 2: Containment orders of intervals.

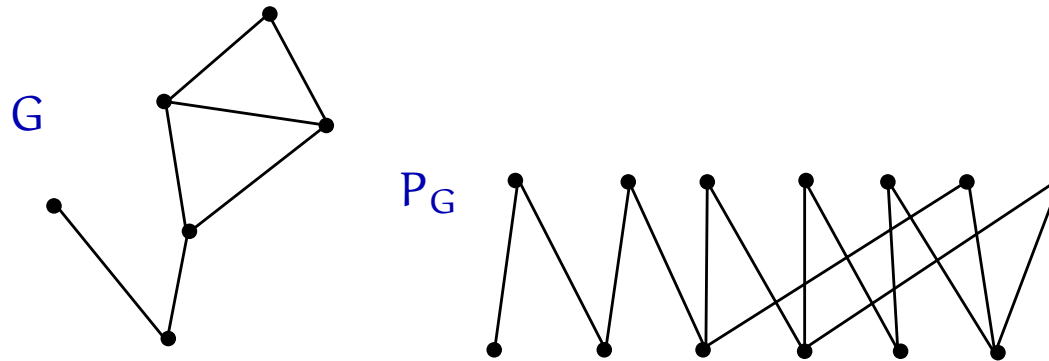


- Dimension 3: Containment orders of triangles.



# Incidence Orders and Dimension

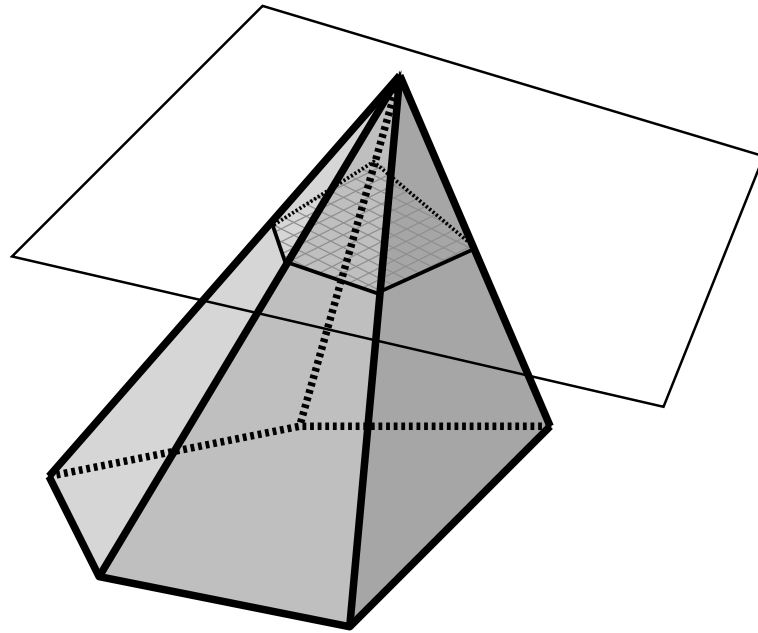
The *incidence order*  $P_G$  of  $G$



1.  $\dim(G) \leq \dim(P_G) \leq \dim(G) + 1$ .
2. If  $\delta(G) \geq 2$ , then  $\dim(G) = \dim(P_G)$ .

# Dimension of Polytopes: Lower Bound

**Theorem.** If  $P$  is a  $d$ -polytope, then  $\dim(P) \geq d + 1$ .



# More Examples

- Boolean lattice:  $\dim(\mathbf{B}_n) = n$
- Standard example of an  $n$  dimensional order:  
Atoms and coatoms of a Boolean lattice:

$$\dim(\mathbf{B}_n[1, n-1]) = n$$

- First two levels of a Boolean lattice:

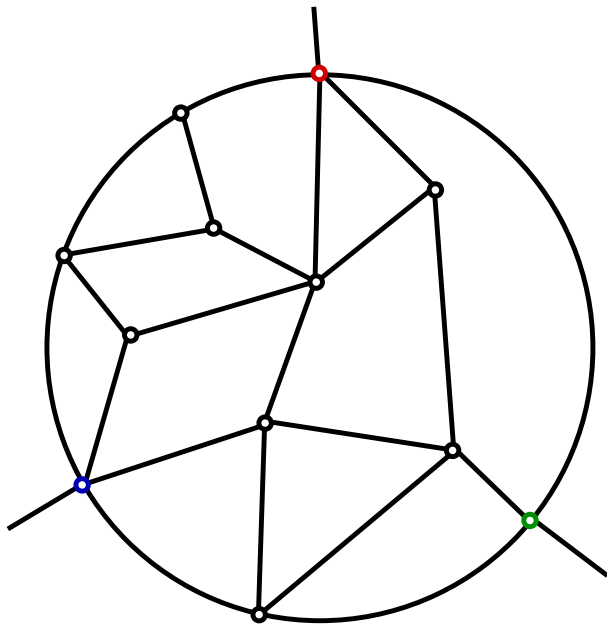
$$\dim(\mathbf{B}_n[1, 2]) \sim \log \log n$$

$\dim(K_n) \leq$	2	3	4	5	6	7	8
$n \leq$	2	4	12	81	2646	1422564	229809982112



# Chapter 3: Schnyder Structures for 3-Connected Plane Graphs

We adapt Schnyder's structures for 'suspended' 3-connected planar graphs.

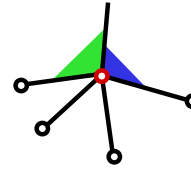


- Schnyder angle labelings,
- Schnyder woods,
- 3-orientations.

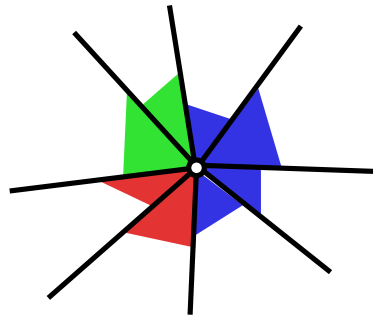
# Schnyder Angle Labelings

Axioms for the 3-coloring of angles:

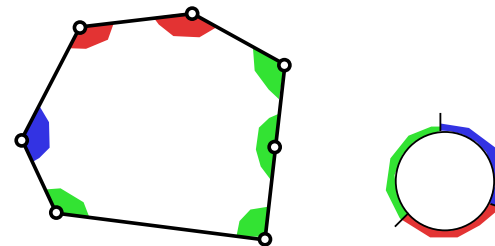
(A1) Angles at the half-edges:



(A2) Rule of vertices:



(A3) Rule of faces:



# Angles at an Edge

**Lemma.** Let  $G$  be a plane graph with a Schnyder angle labeling, then the four angles of each edge contain all three colors.

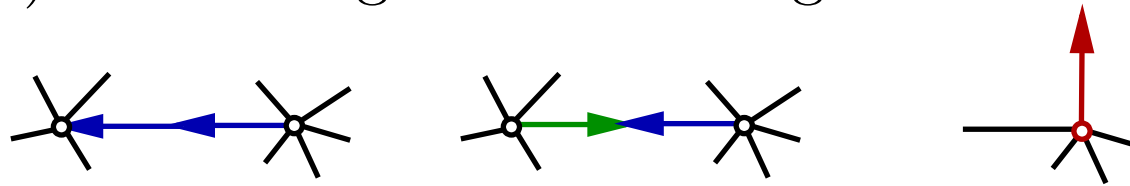
Consequently, there are only two types of edges:



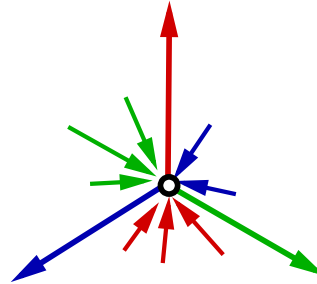
# Schnyder Woods

Axioms for 3-coloring and orientation of bi-edges:

(W1 - W2) Rule of edges and half-edges:

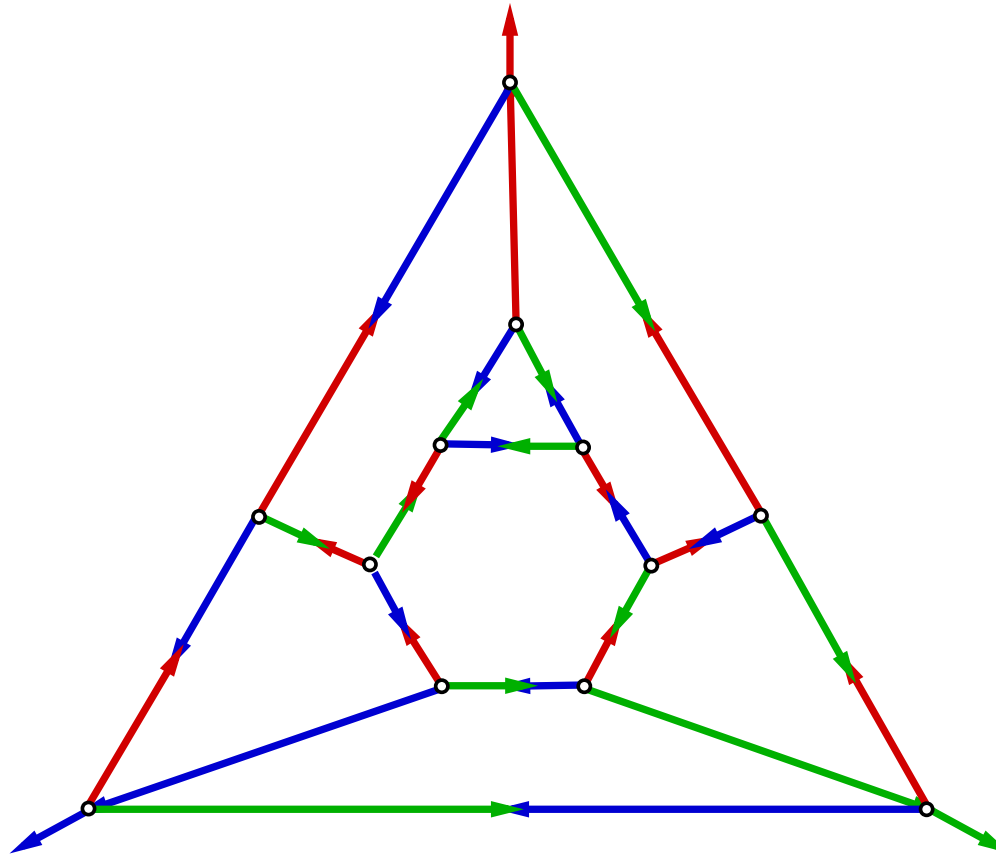


(W3) Rule of vertices:

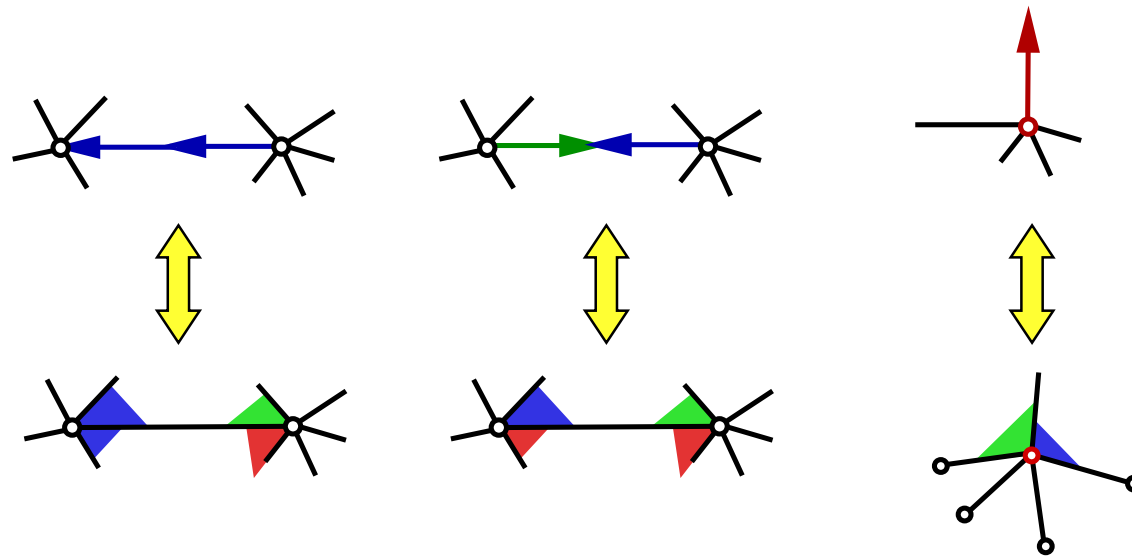


(W4) There is no interior face whose boundary is a directed cycle in one color.

We need W4!



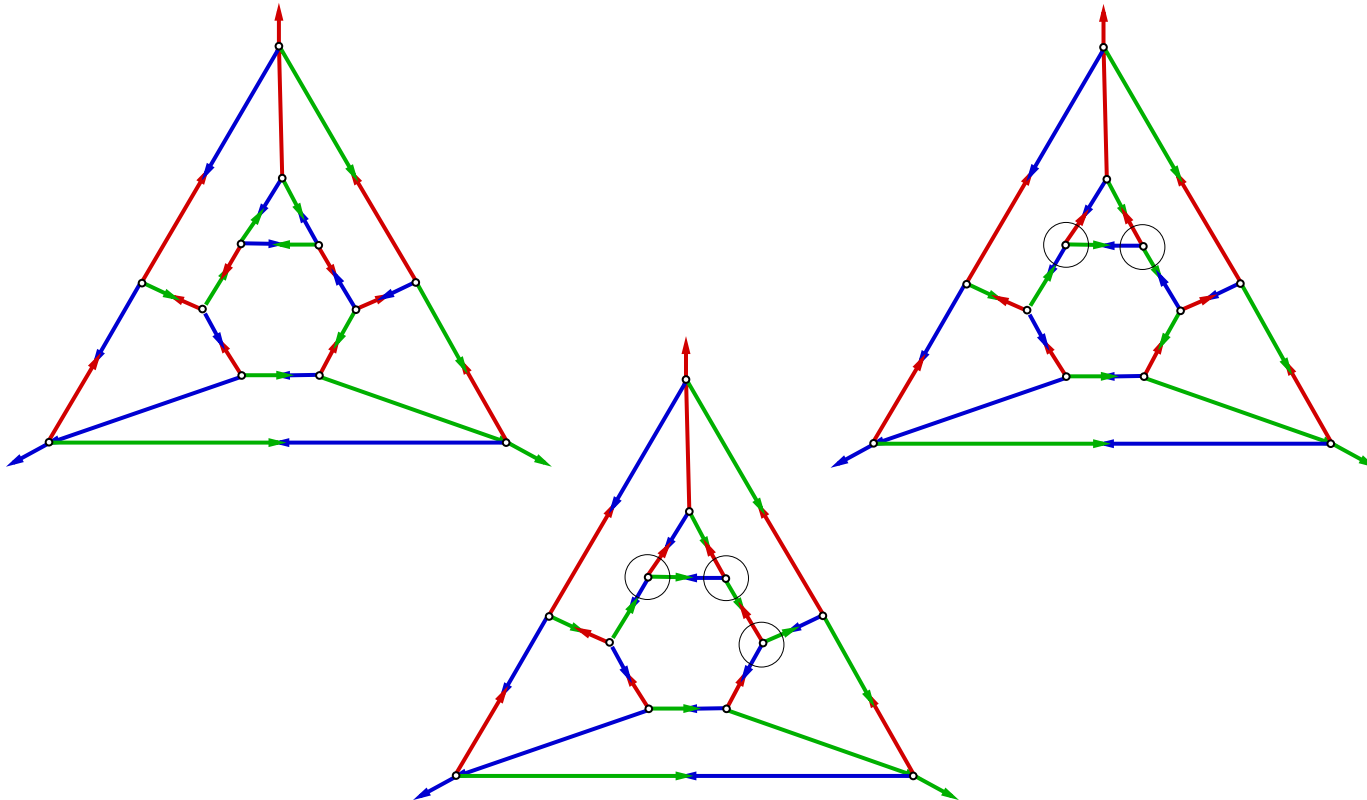
# Equivalence of Schnyder Woods and Angle Labelings



This mapping preserves the axioms,

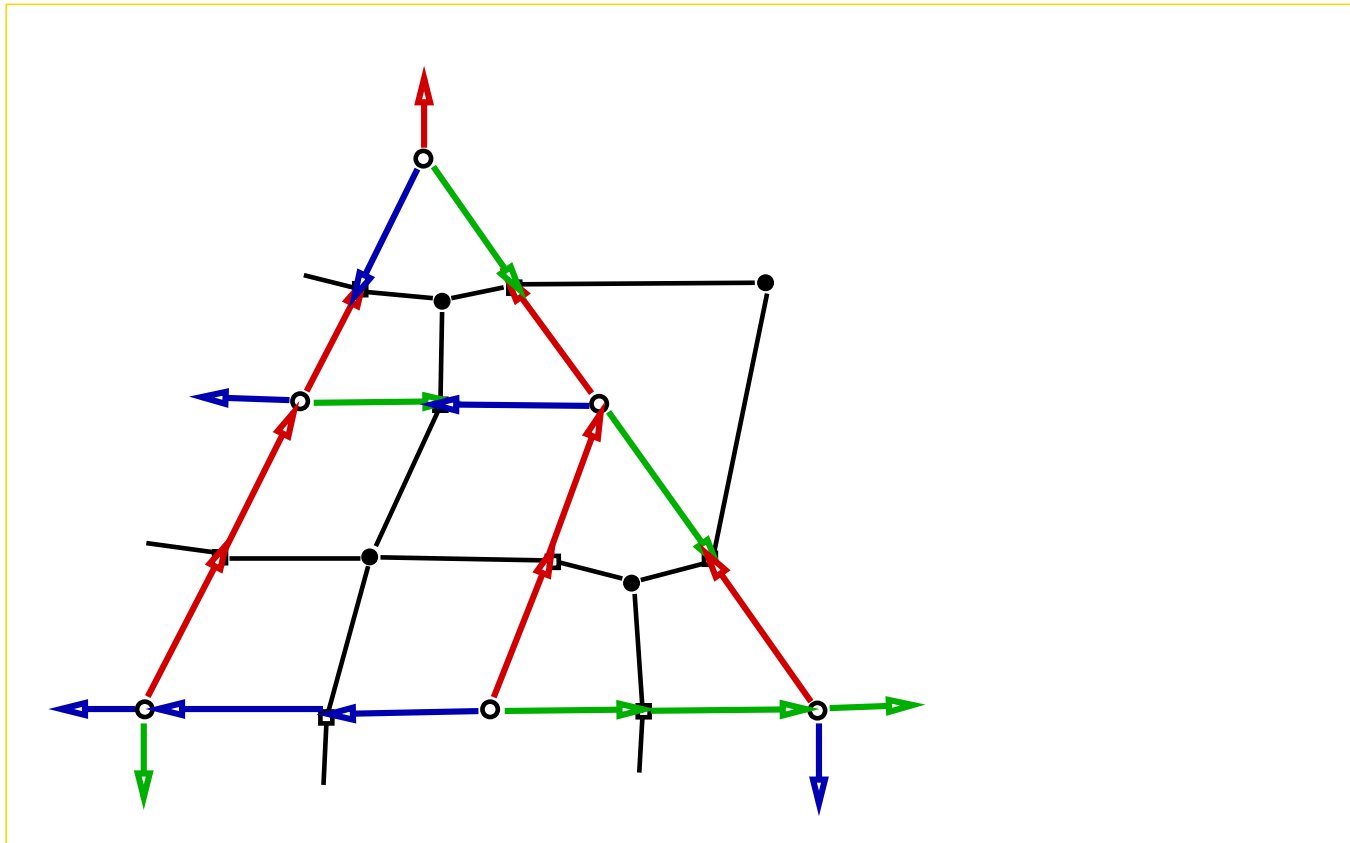
$$(A1), (A2), (A3) \iff (W1), (W2), (W3), (W4)$$

# 3-Orientations?



# Primal and Dual

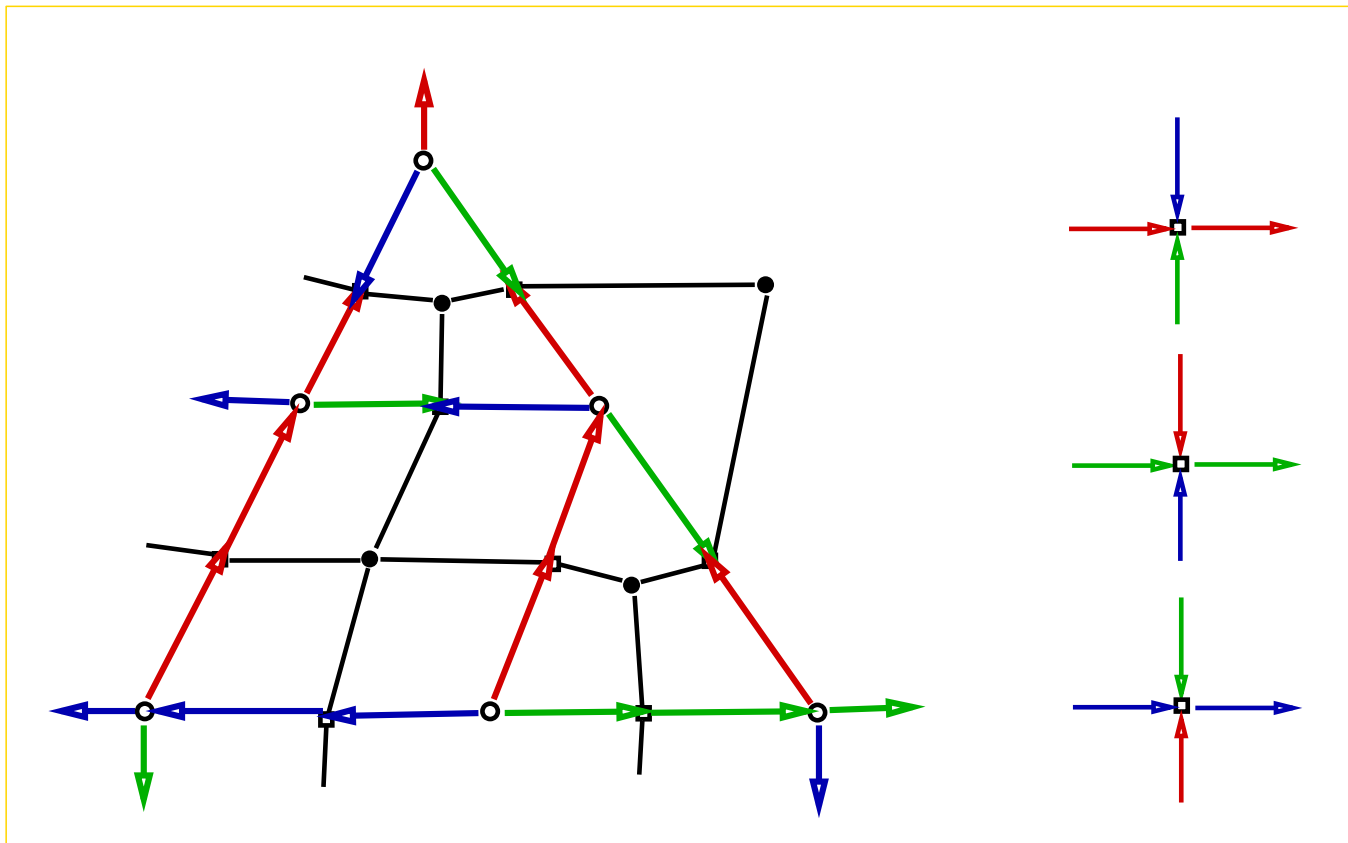
A Schnyder coloring of  $G$  induces a Schnyder coloring of the dual of  $G$ .





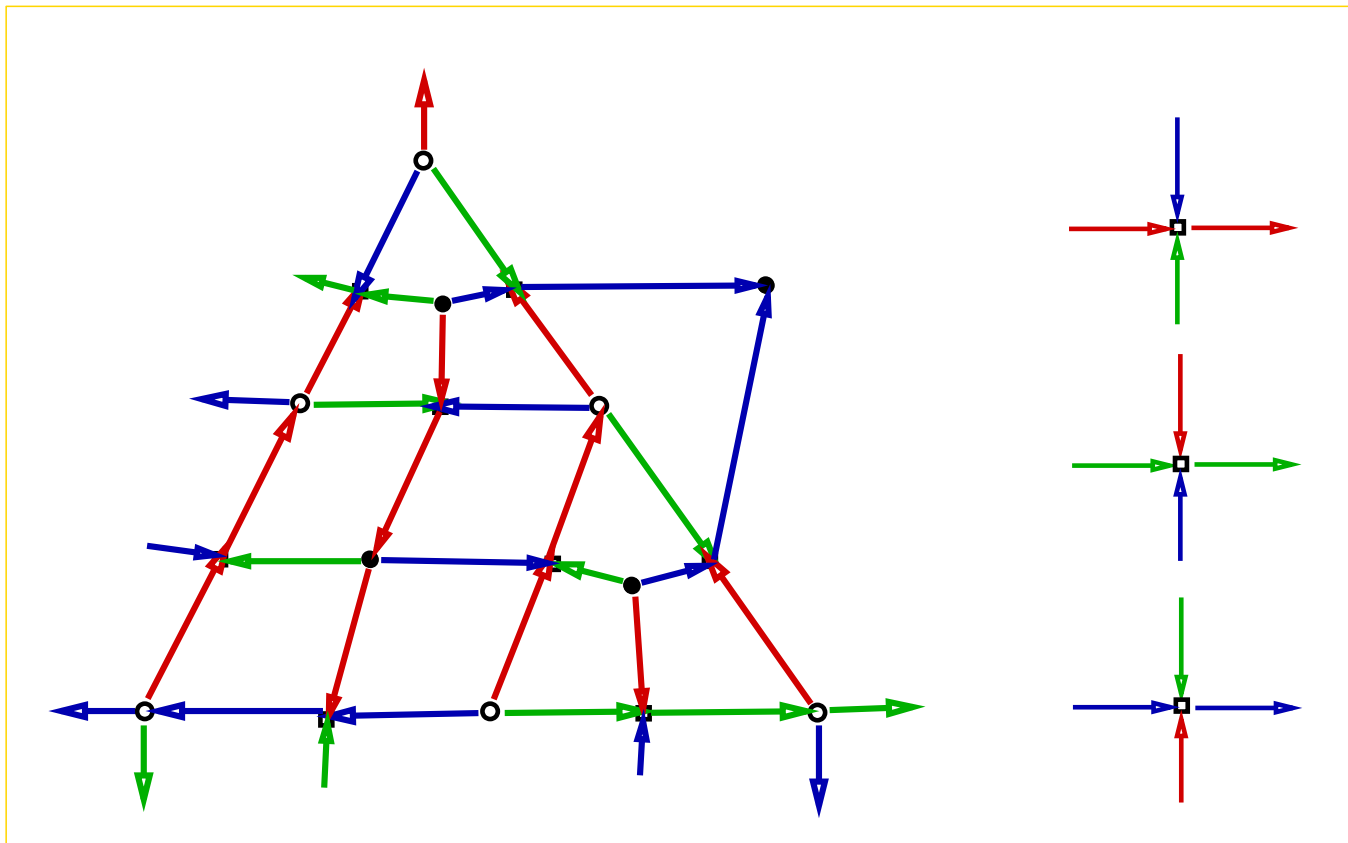
# Primal and Dual

A Schnyder coloring of  $G$  induces a Schnyder coloring of the dual of  $G$ .



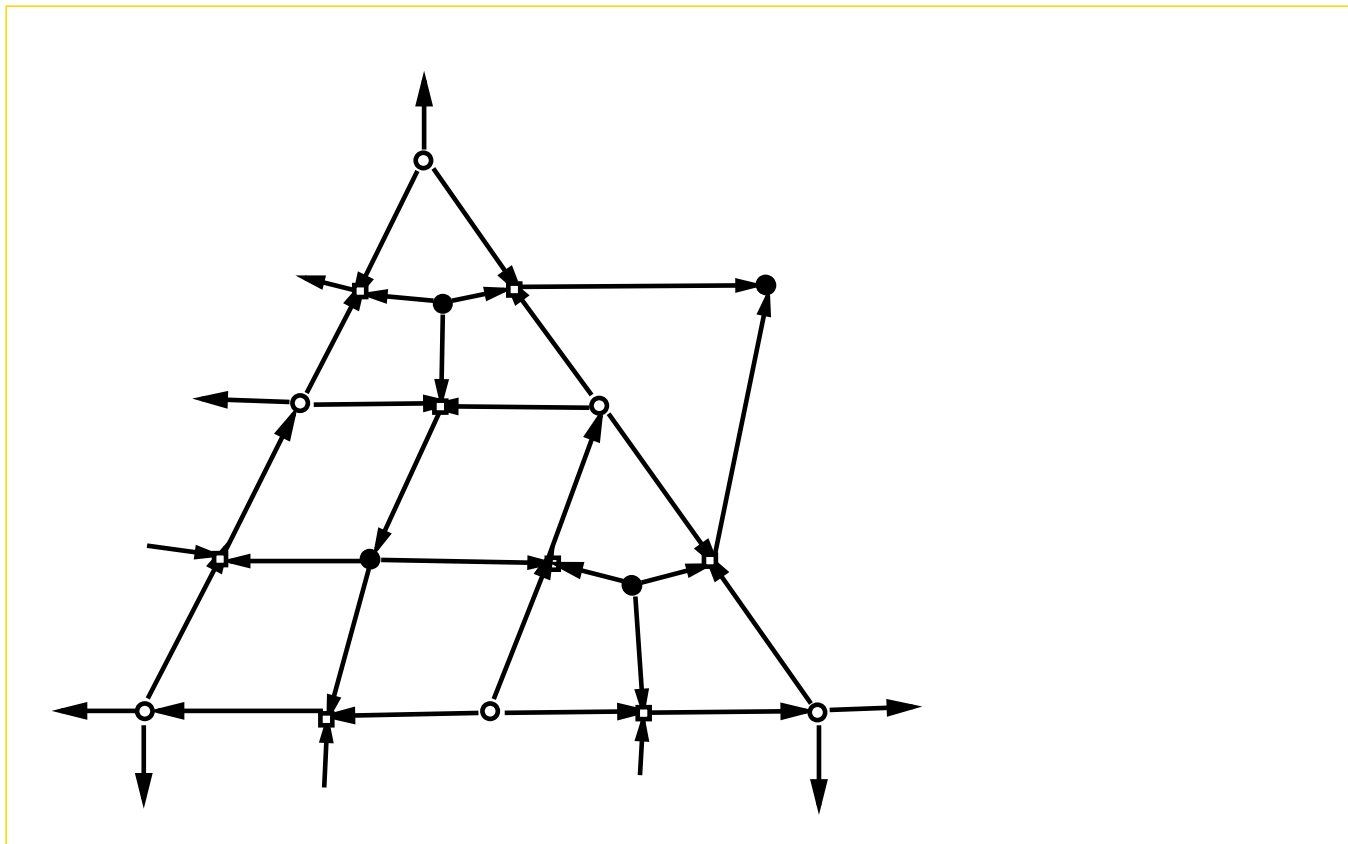
# Primal and Dual

A Schnyder coloring of  $G$  induces a Schnyder coloring of the dual of  $G$ .



# 3-Orientations – Rescued

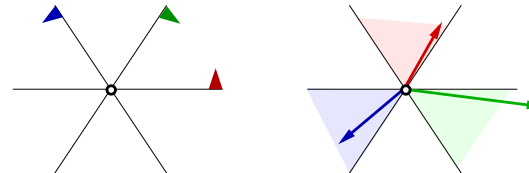
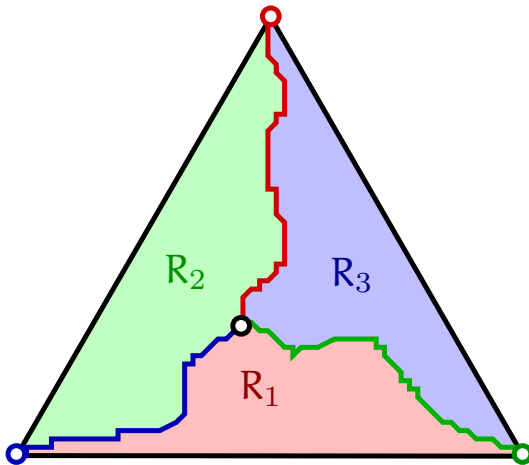
**Theorem.** Primal dual Schnyder woods are in bijection with primal dual 3-orientations.



# Regions and Applications

$\phi : \text{regions} \rightarrow \mathbb{R}^+$  such that

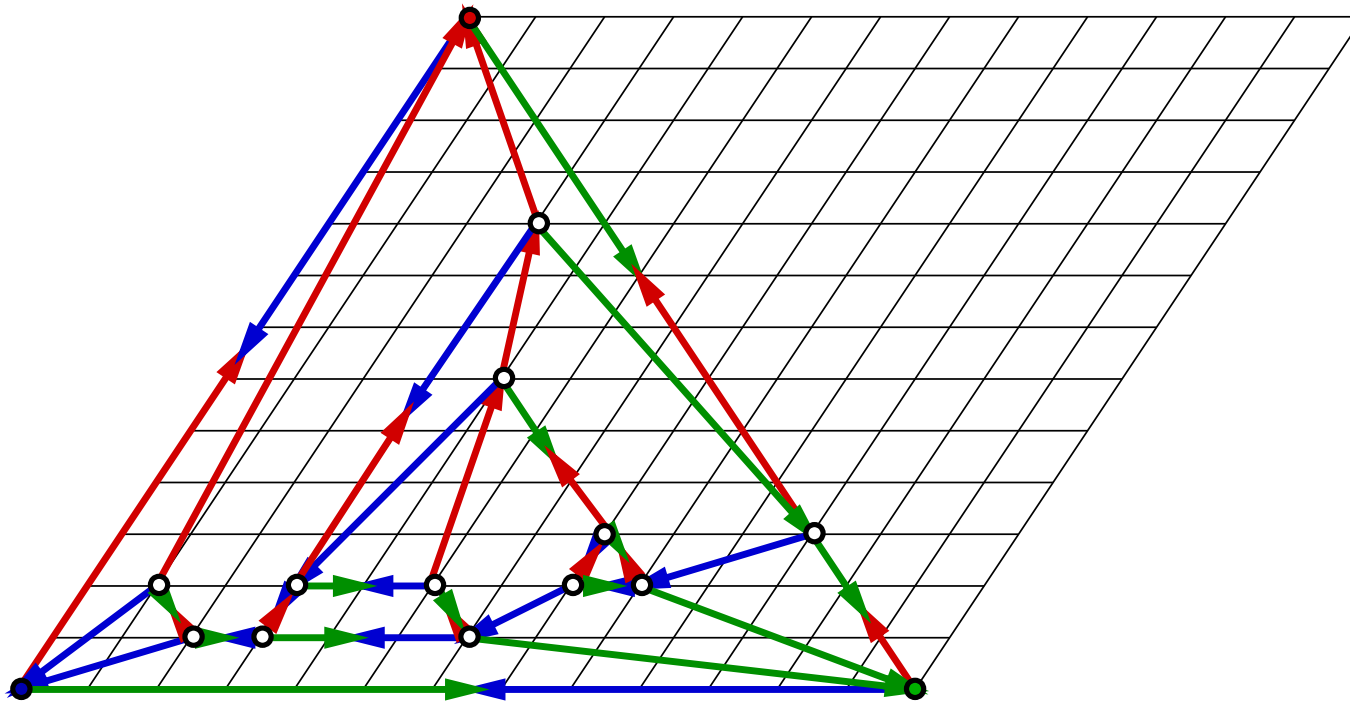
- $R_i(u) \subseteq R_i(v) \implies \phi(R_i(u)) \leq \phi(R_i(v))$
- $\phi(R_1(v)) + \phi(R_2(v)) + \phi(R_3(v)) = C$



# Drawings by Counting Faces

Counting faces in the three regions yields:

**Theorem.** 3-connected planar graphs admit convex drawings on the  $(f - 1) \times (f - 1)$  grid.



# Idea of the Proof

(1) No vertex on an edge.

(2) Edges are disjoint

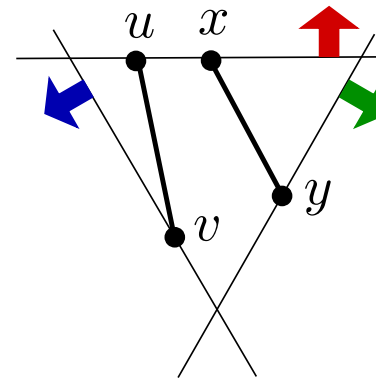
$\{u, v\}, \{x, y\} \in E$

$x, y \in R_i(u)$ ,  $x, y \in R_j(v)$ ,  $u, v \in R_k(x)$ ,  $u, v \in R_l(y)$

does **not** imply  $i = j$  or  $k = l$

The hard case:

$x, y \in R_1(u)$  and  $u, v \in R_1(x)$



(3) Convex: Inner vertices are never pointed.