

Orthogonal Surfaces

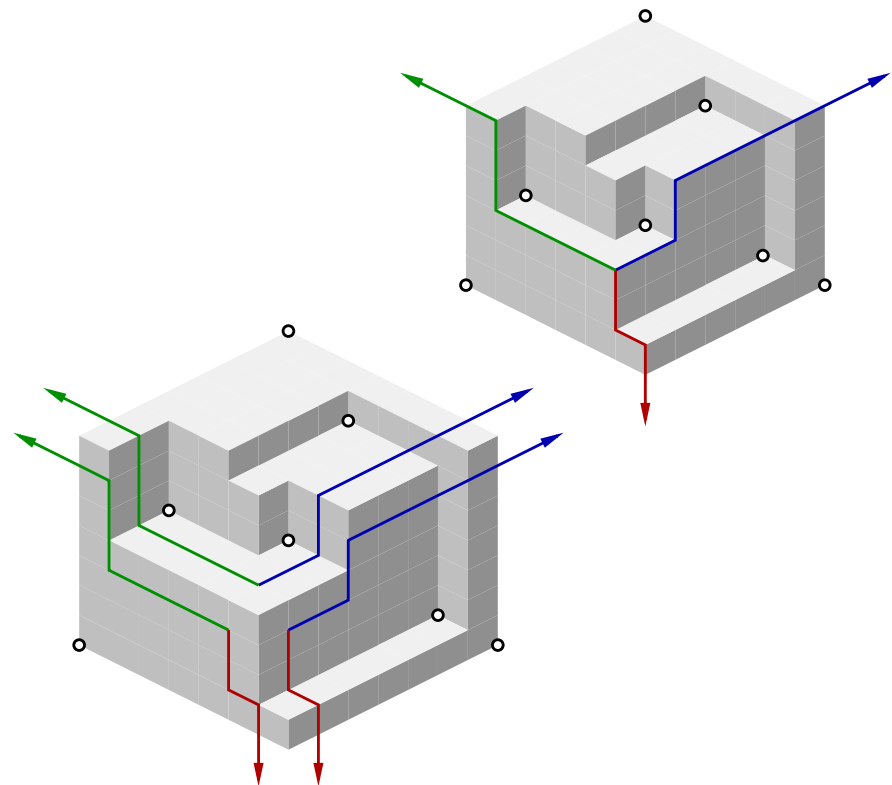
Flips and

Lattices

Petit École de Combinatoire
LaBRI, Bordeaux
March 19, 2004

Stefan Felsner

Technische Universität Berlin
felsner@math.tu-berlin.de



Content

Plane Triangulations

- (1) Schnyder Structures
- (2) Applications: Dimension, Drawing and Surfaces

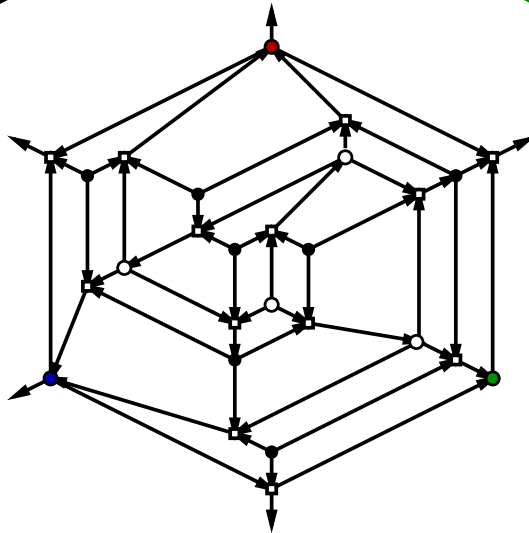
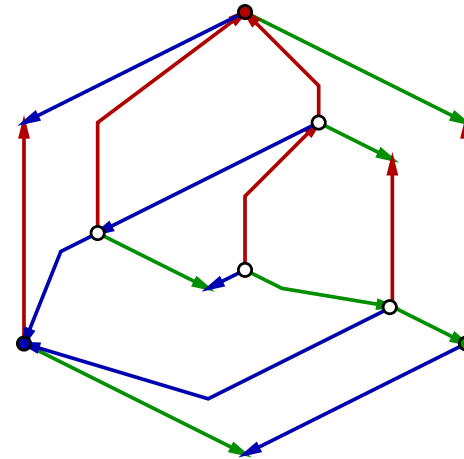
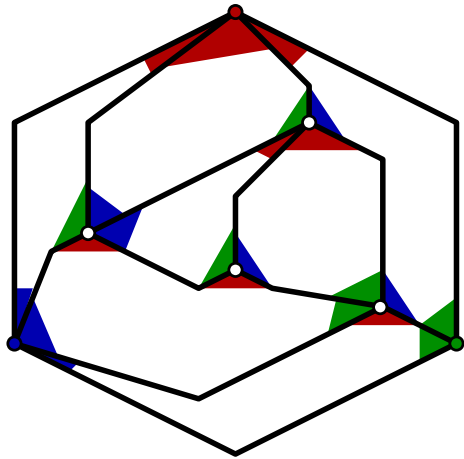
3-Connected Plane Graphs

- (3) Schnyder Structures Revisited
- (4) Applications and more on Orthogonal Surfaces
- (5) Flips and Lattices

Today's talk covers (4) and (5).

Schnyder Structures

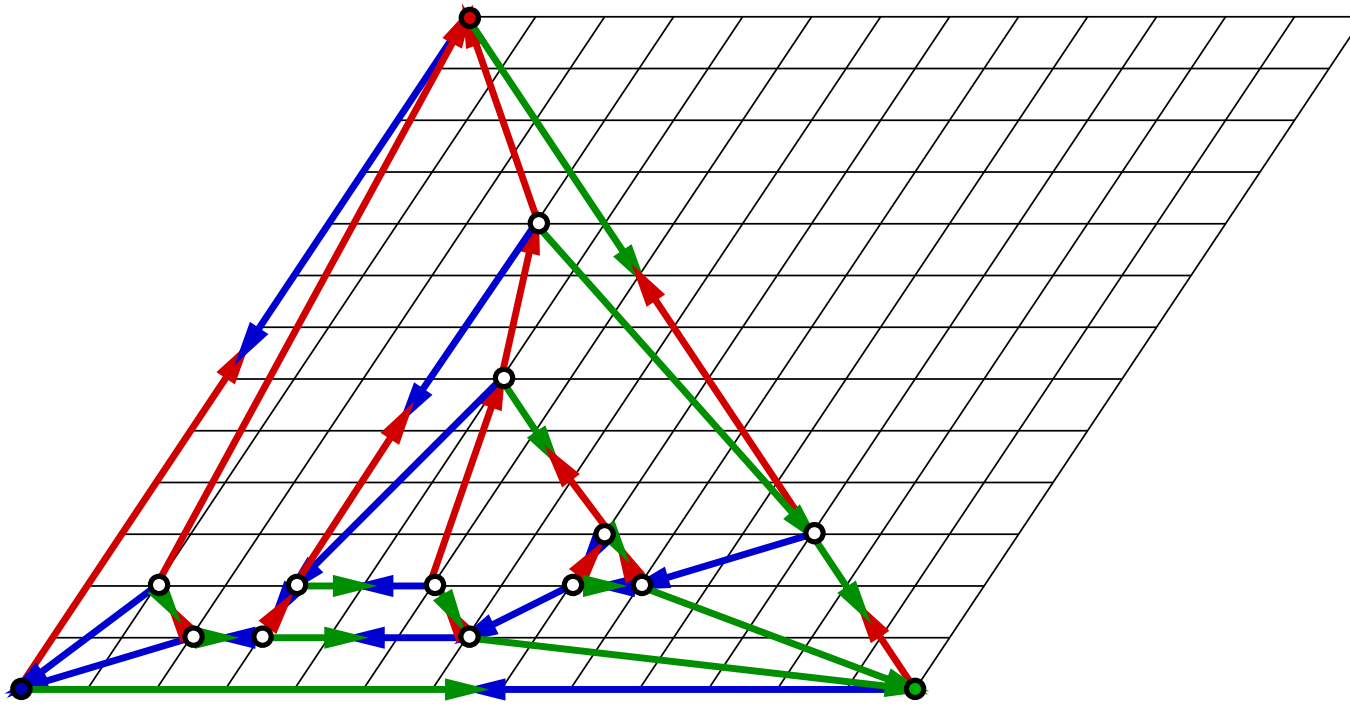
- Angle labelings.
- Woods.
- 3-Orientations.



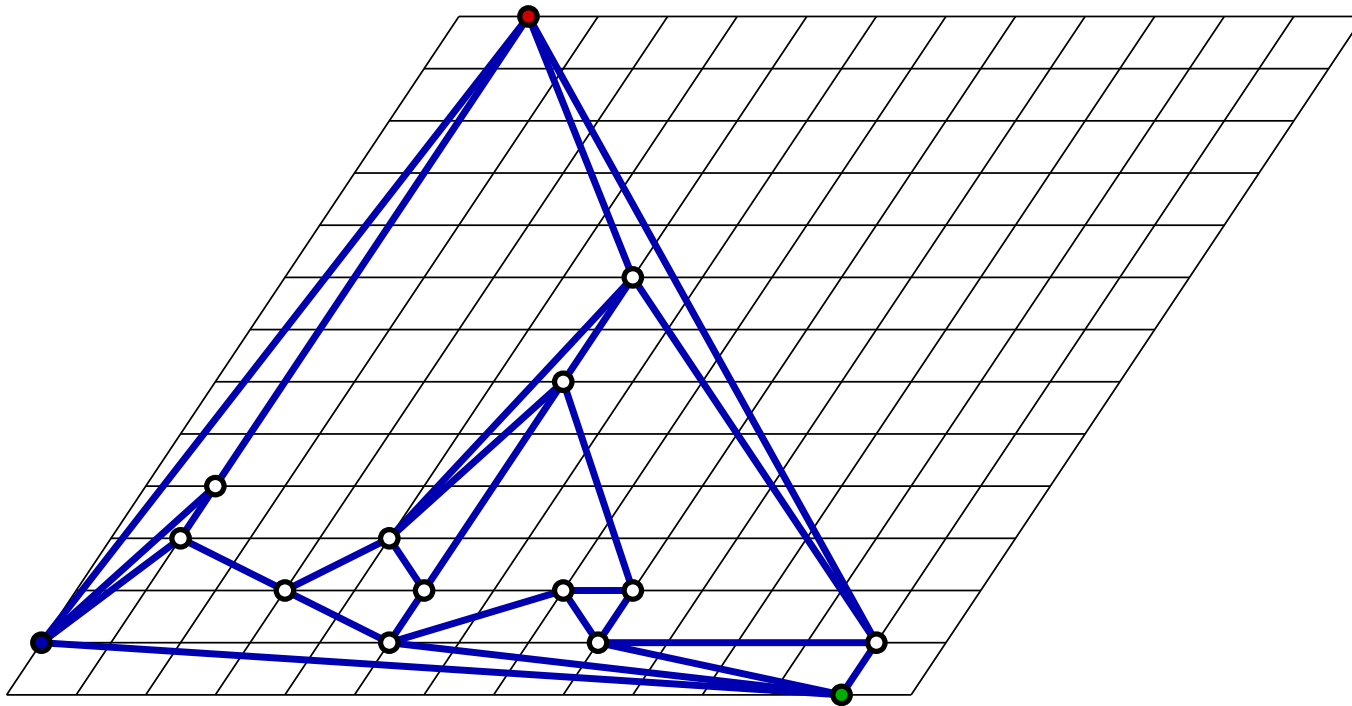
Drawings by Counting Faces

Counting faces in the three regions yields:

Theorem. 3-connected planar graphs admit convex drawings on the $(f - 1) \times (f - 1)$ grid.



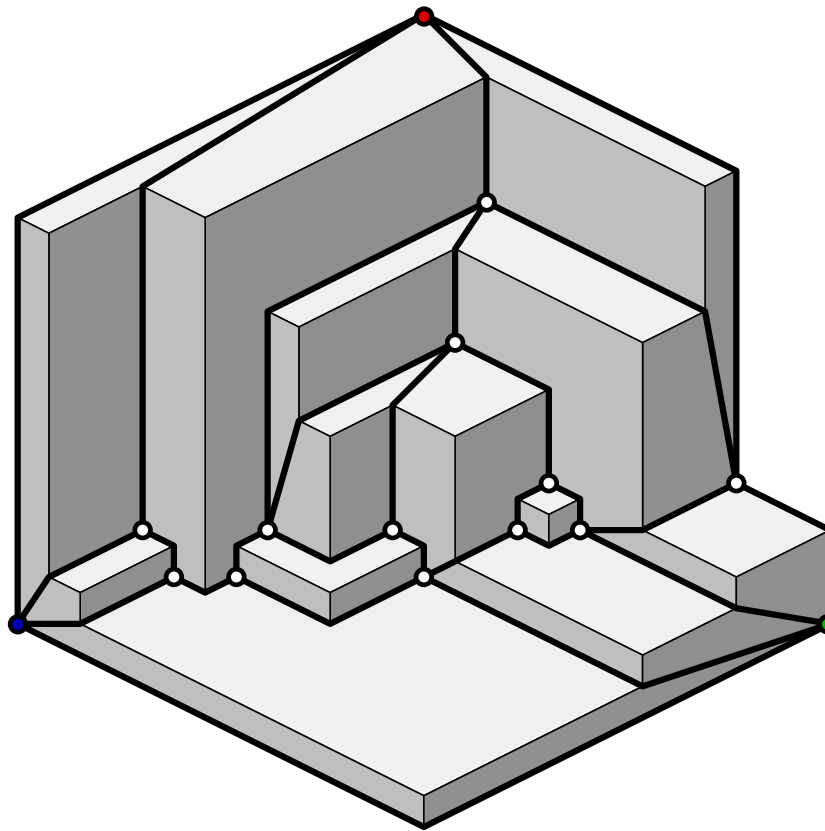
Drawings by Counting Vertices



This is not the right way for producing convex drawings for 3-connected planar graphs.

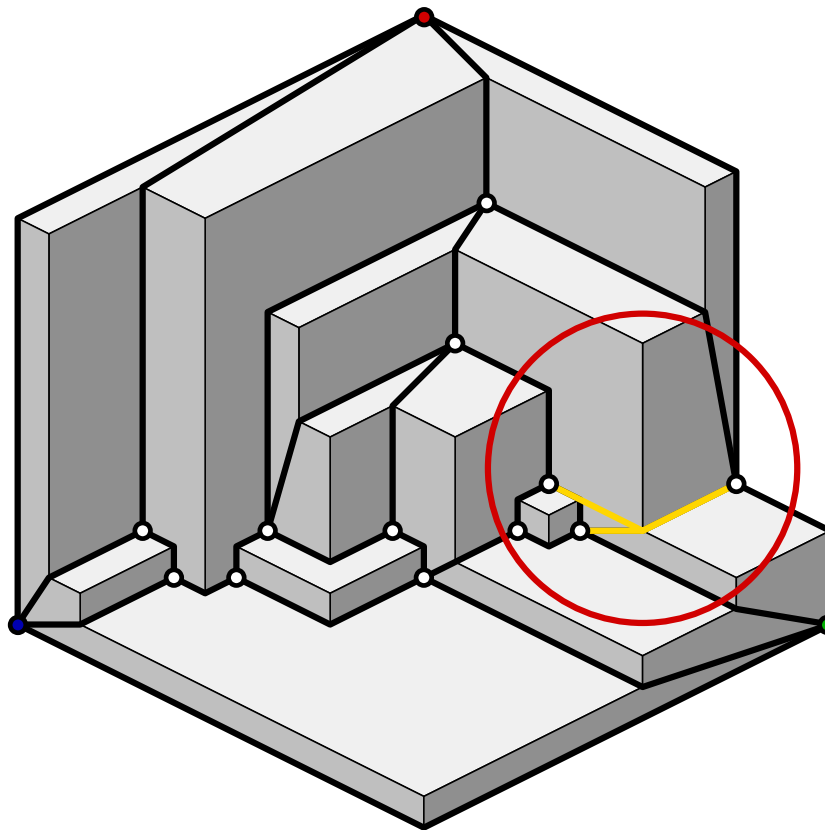
Graphs and Orthogonal Surfaces

Every 3-connected planar graph is generated by some orthogonal surface.



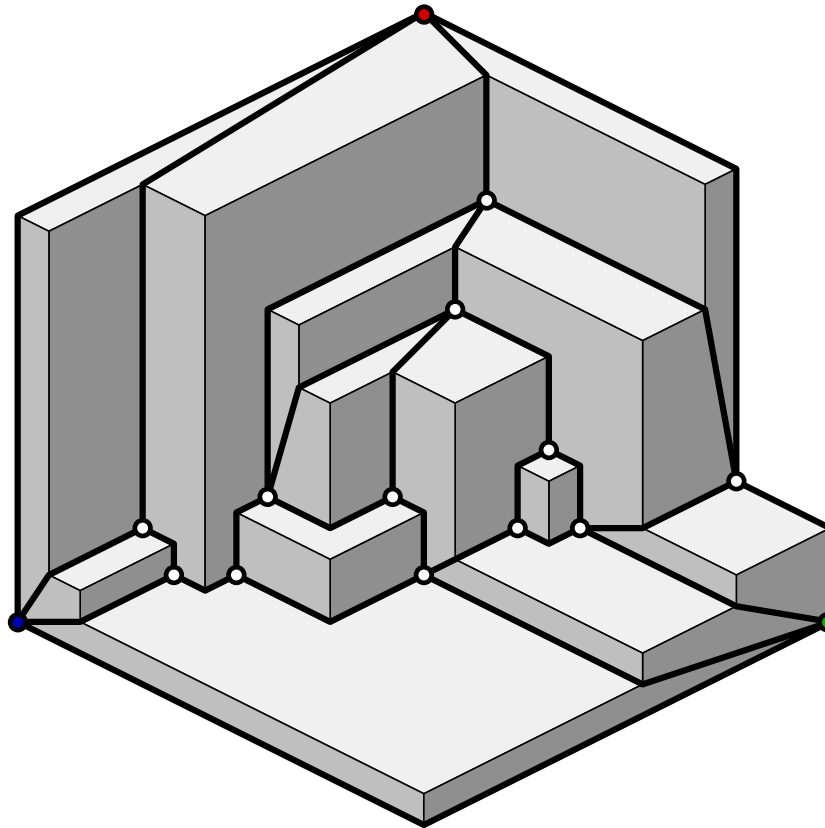
Graphs and Orthogonal Surfaces

Every 3-connected planar graph (more precisely every Schnyder wood) is generated by some orthogonal surface.



Rigid Orthogonal Surfaces

A modified definition of regions \Rightarrow
every 3-connected planar graph is generated by
a *rigid* orthogonal surface.



Rigid Orthogonal Surfaces

Theorem [Felsner'01].

G 3-connected and planar.

There is a bijection between

Schnyder woods of G

and

rigid orthogonal surfaces generating G .

Dimension of Polytopes

A rigid orthogonal surface generating G induces three permutations (projections) of vertices, edges and faces.

Theorem [Brightwell+Trotter'92,Felsner 01].

The hypergraph of vertices, edges and bounded faces of a 3-connected planar graph G is of dimension 3.

Equivalently

Theorem.

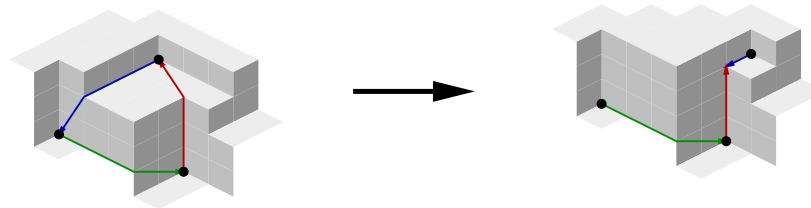
If P is a 3-polytope and F is any face of P , then

- $\dim(\mathcal{F}_P \setminus F) = 3$
- $\dim(\mathcal{F}_P) = 4$

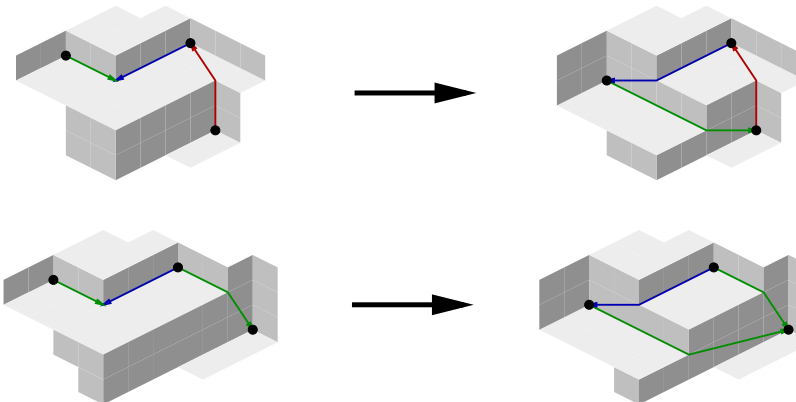
This Weeks Work

With Nicolas Bonichon we have been discussing **orthogonal surfaces** and **semi-flips**:

Merge:



Split:

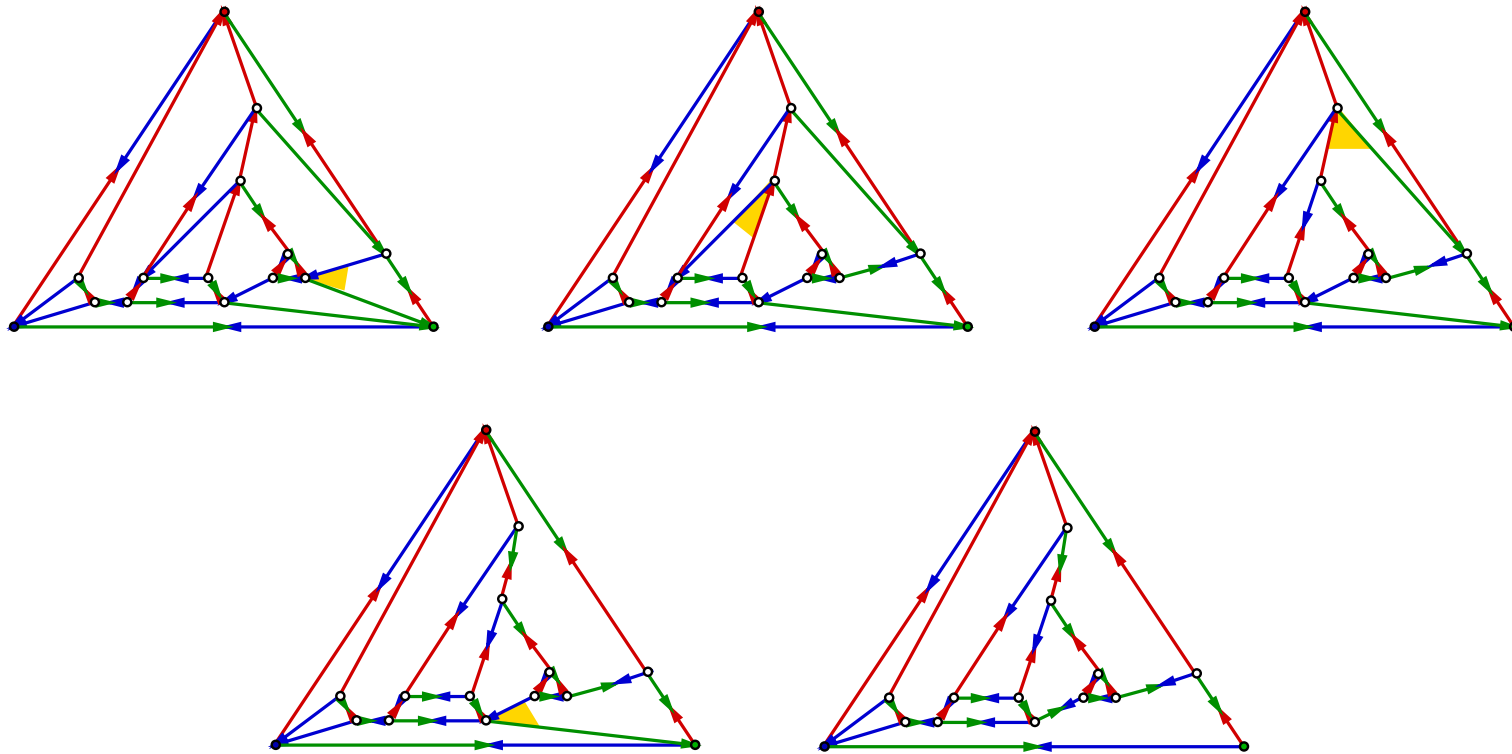


New Results

- Rigid embeddings [simplified proofs]
- Flat embeddings [good description]
- Drawings of 3-connected plane graphs [more compactly]

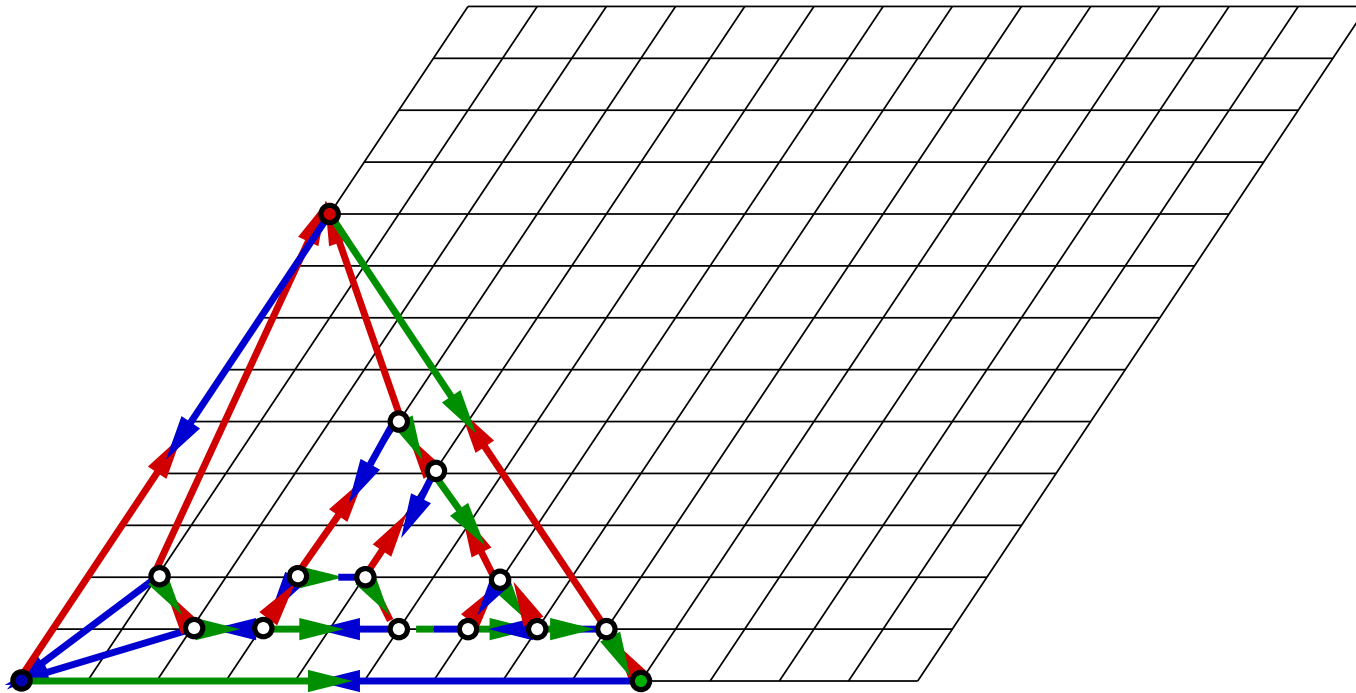
Compact Drawings I

Reduce the graph with semi-merges.



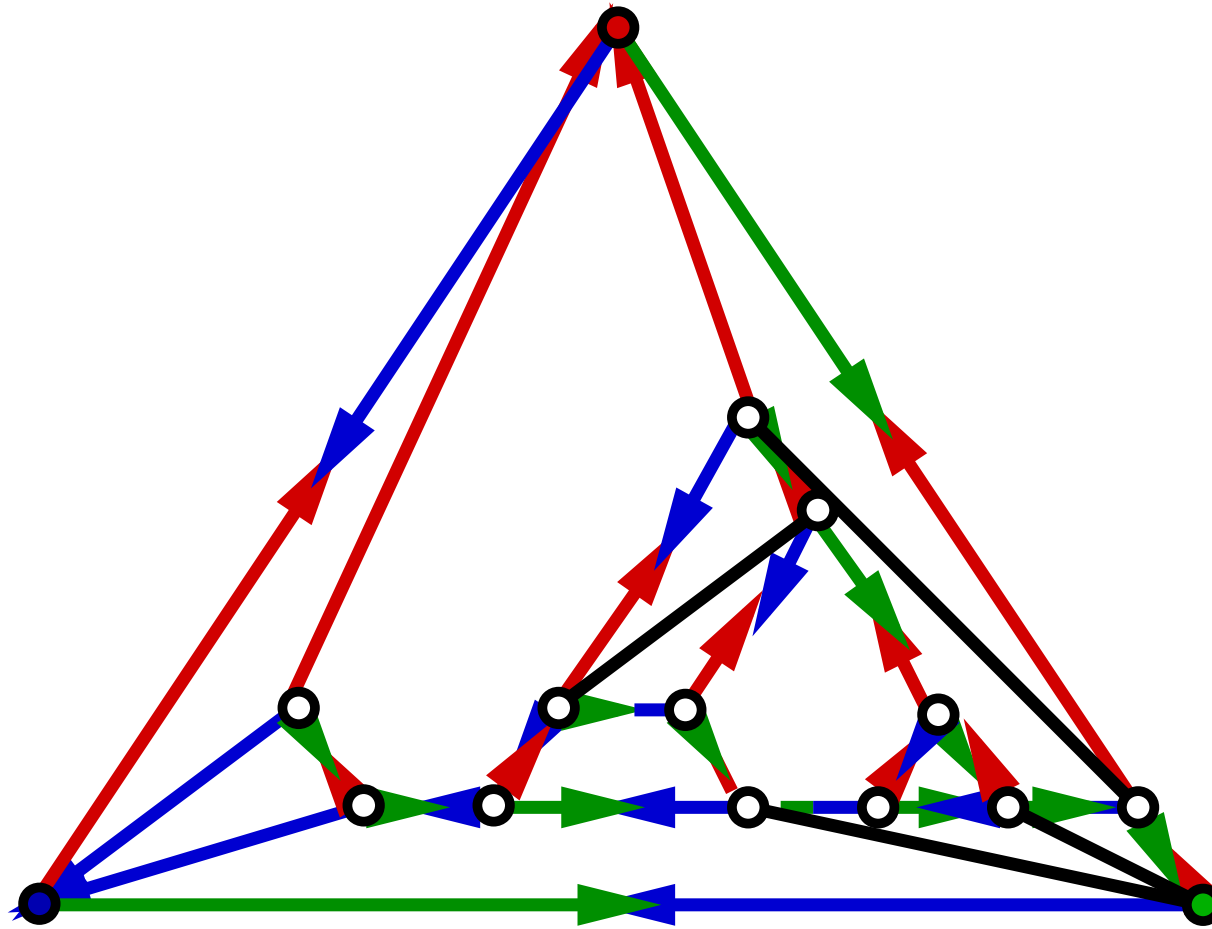
Compact Drawings II

Draw the graph on the $(f-1) \times (f-1)$ grid.



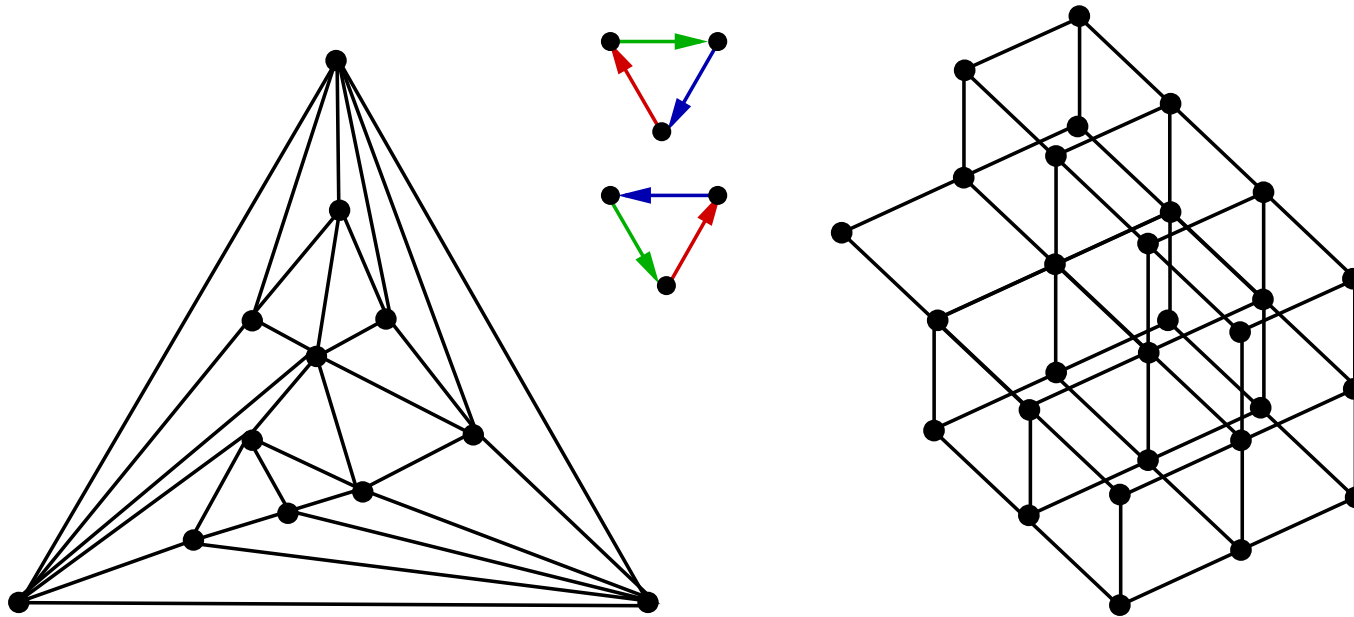
Compact Drawings III

Reinsert the merge-edges.



The Lattice of Schnyder Woods

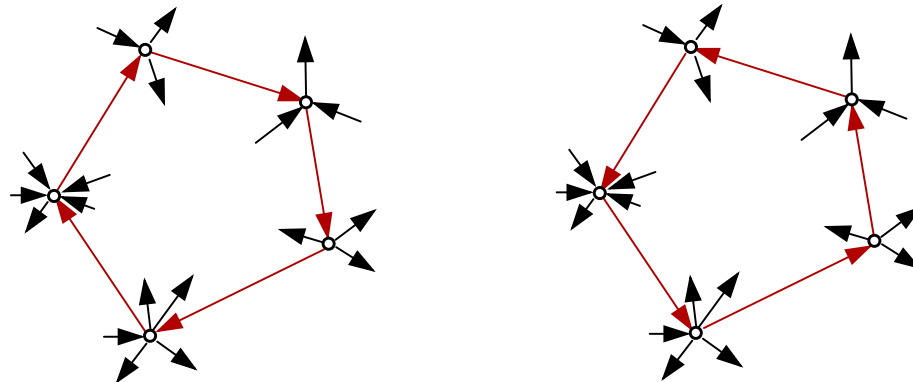
Theorem [Mendez & Brehm]. The set of Schnyder woods of a plane triangulation G has the structure of a distributive lattice.



A General Theorem

Definition. Given $G = (V, E)$ and $\alpha : V \rightarrow \mathbb{N}$.
An α -orientation of G is an orientation with $\text{outdeg}(v) = \alpha(v)$ for all v .

- Reverting directed cycles preserves α -orientations.



Theorem. The set of α -orientations of a planar graph G has the structure of a distributive lattice.

Proof I: Essential Cycles

Definition.

A cycle C of G is an *essential cycle* if

- C is chord-free and simple,
- the interior cut of C is *rigid*,
- there exists an α -orientation X such that C is a directed cycle in X .

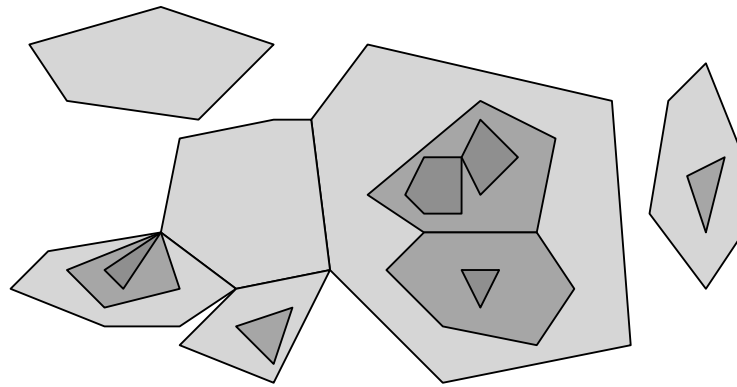
Lemma.

If e is oppositely directed in two α -orientations then there is a simple cycle C with $e \in C$ which is oppositely directed in both.

Proof II

Lemma.

Essential cycles are interiorly disjoint or contained.



Lemma.

If (C_1, \dots, C_k) is a flip sequence (ccw \rightarrow cw) on X then for every edge e the essential cycles $C^{l(e)}$ and $C^{r(e)}$ alternate in the sequence.

Proof III: Flip Sequences

Lemma.

The length of any flip sequence ($\text{ccw} \rightarrow \text{cw}$) is bounded and there is a unique α -orientation X_{\min} with the property that all cycles in X_{\min} are cw-cycles.

- $Y \prec X$ if a flip sequence $X \rightarrow Y$ exists.

Lemma.

Let $Y \prec X$ and C be an essential cycle. Every sequence $S = (C_1, \dots, C_k)$ of flips that transforms X into Y contains the same number of flips at C .

Proof IV: Potentials

Definition. An α -potential for G is a mapping $\wp : \text{Ess}_\alpha \rightarrow \mathbb{N}$ such that

- $|\wp(C) - \wp(C')| \leq 1$, if C and C' share an edge e .
- $\wp(C^{l(e)}) \leq \wp(C^{r(e)})$ for all e (orientation from X_{\min})

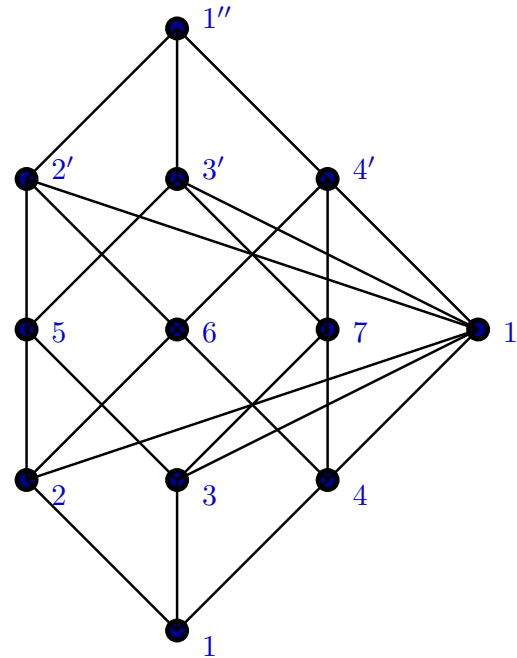
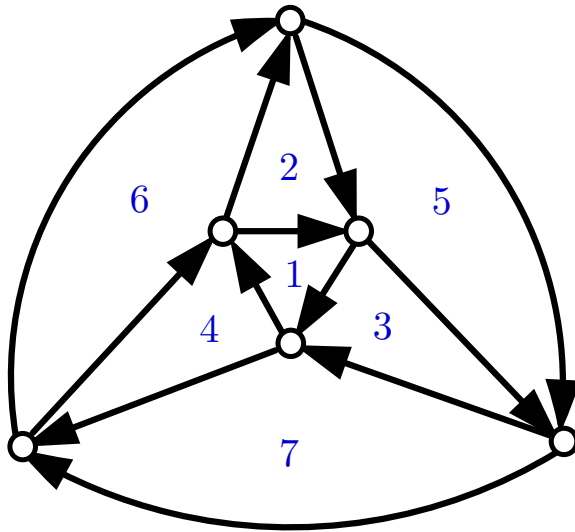
Lemma. There is a bijection between α -potentials and α -orientations.

Theorem. α -potentials are a distributive lattice with

- $(\wp_1 \vee \wp_2)(C) = \max\{\wp_1(C), \wp_2(C)\}$ and
- $(\wp_1 \wedge \wp_2)(C) = \min\{\wp_1(C), \wp_2(C)\}$ for all essential C .

Application I: Eulerian Orientations

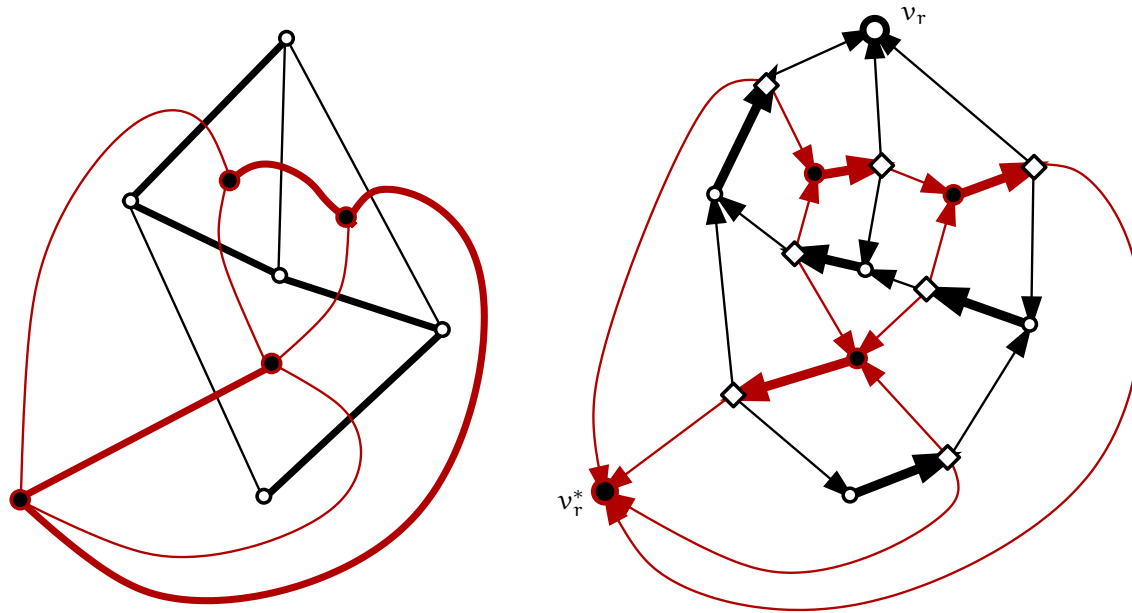
- Orientations with $\text{outdeg}(v) = \text{indeg}(v)$ for all v ,
i.e. $\alpha(v) = \frac{d(v)}{2}$



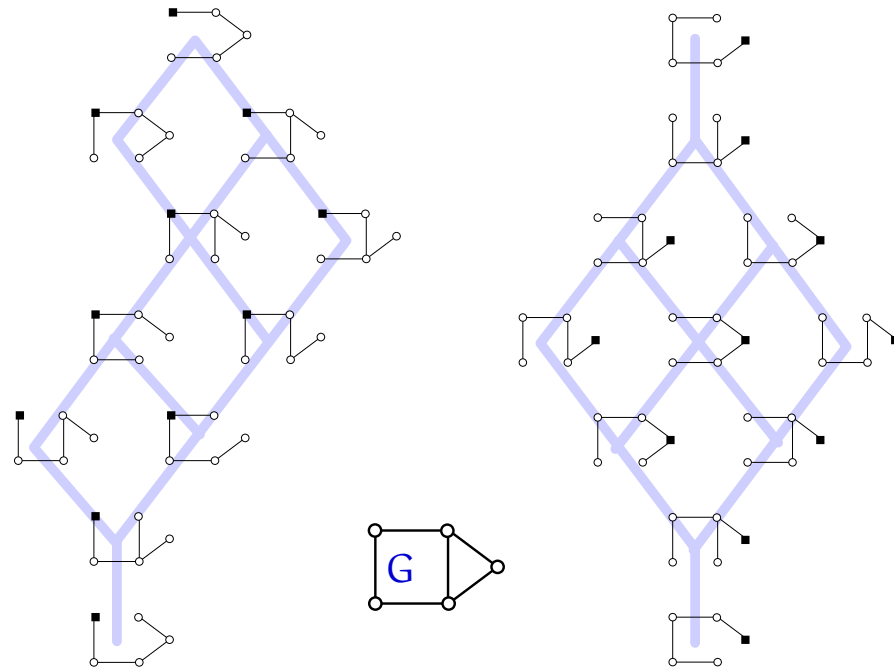
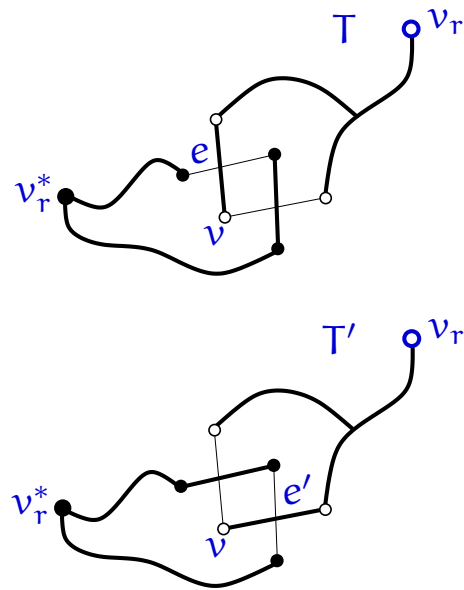
Application II: Spanning Trees

Spanning trees are in bijection with α_T orientations of a rooted primal-dual completion \tilde{G} of G

- $\alpha_T(v) = 1$ for a non-root vertex v and $\alpha_T(v_e) = 3$ for an edge-vertex v_e and $\alpha_T(v_r) = 0$ and $\alpha_T(v_r^*) = 0$.



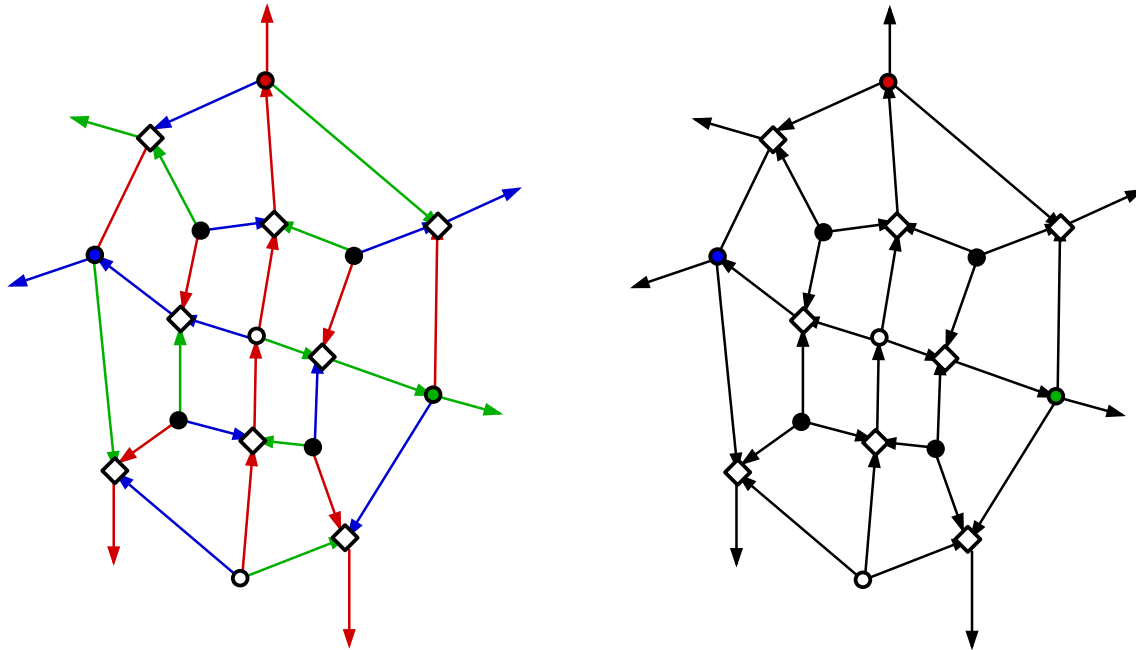
Lattice of Spanning Trees



Application III: Schnyder Woods

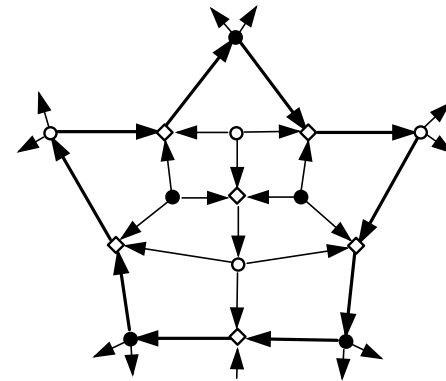
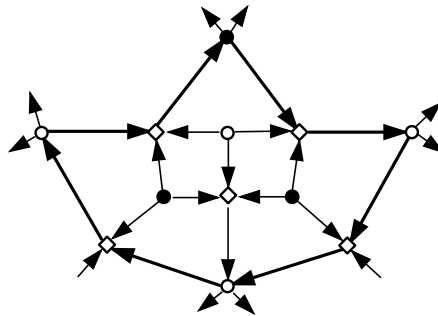
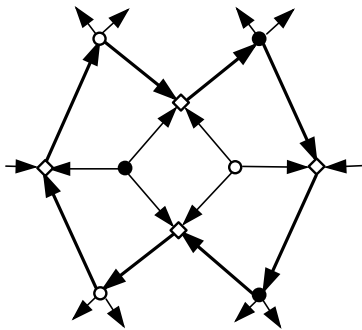
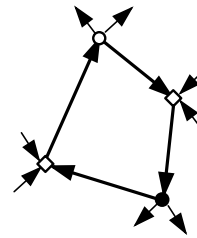
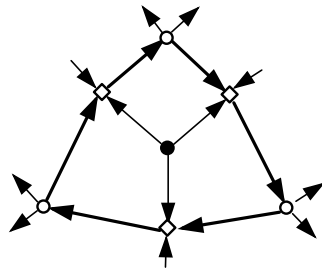
Schnyder woods are in bijection with α_S orientations of a primal-dual completion \tilde{G} of G

- $\alpha_S(v) = 3$ for a vertex v and $\alpha_S(v_e) = 1$ for an edge-vertex v_e and exterior rays.



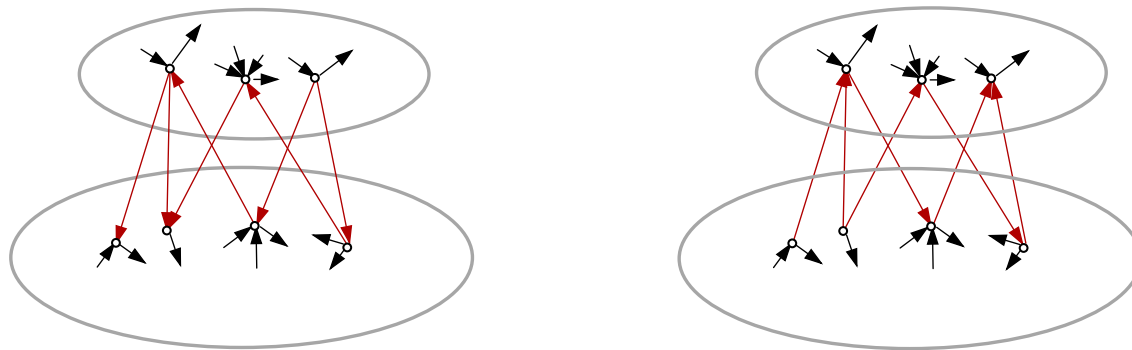
Essential Cycles

- Essential cycles for α_S .



A Dual Construction for Lattices

- Reorientations of cuts preserve flow-differences along cycles.



Theorem [Propp].

The set of all orientations of a graph G with prescribed flow-differences for all cycles has the structure of a distributive lattice.

In the future we will understand more.
For now this is

THE END

Thank you.