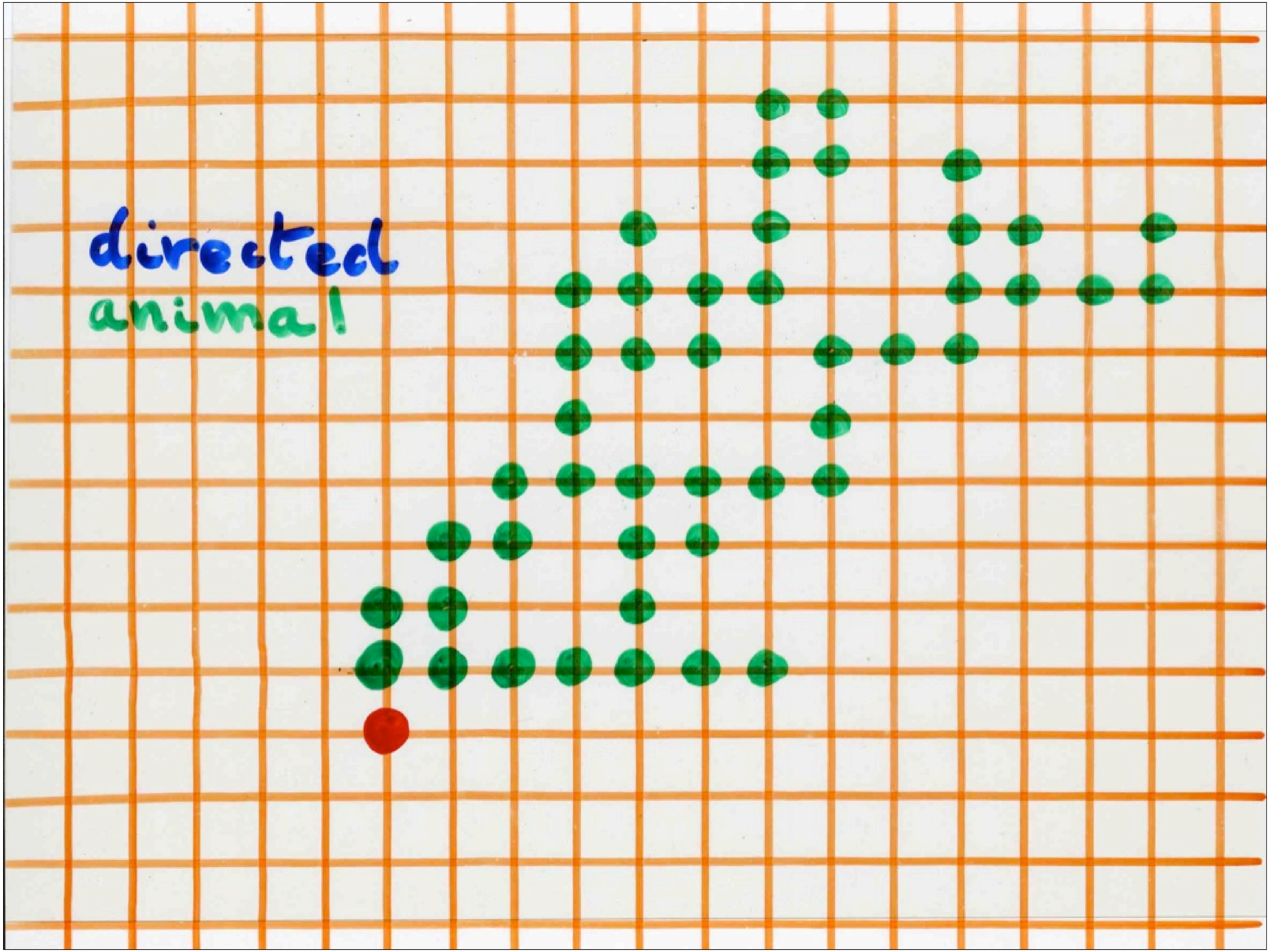
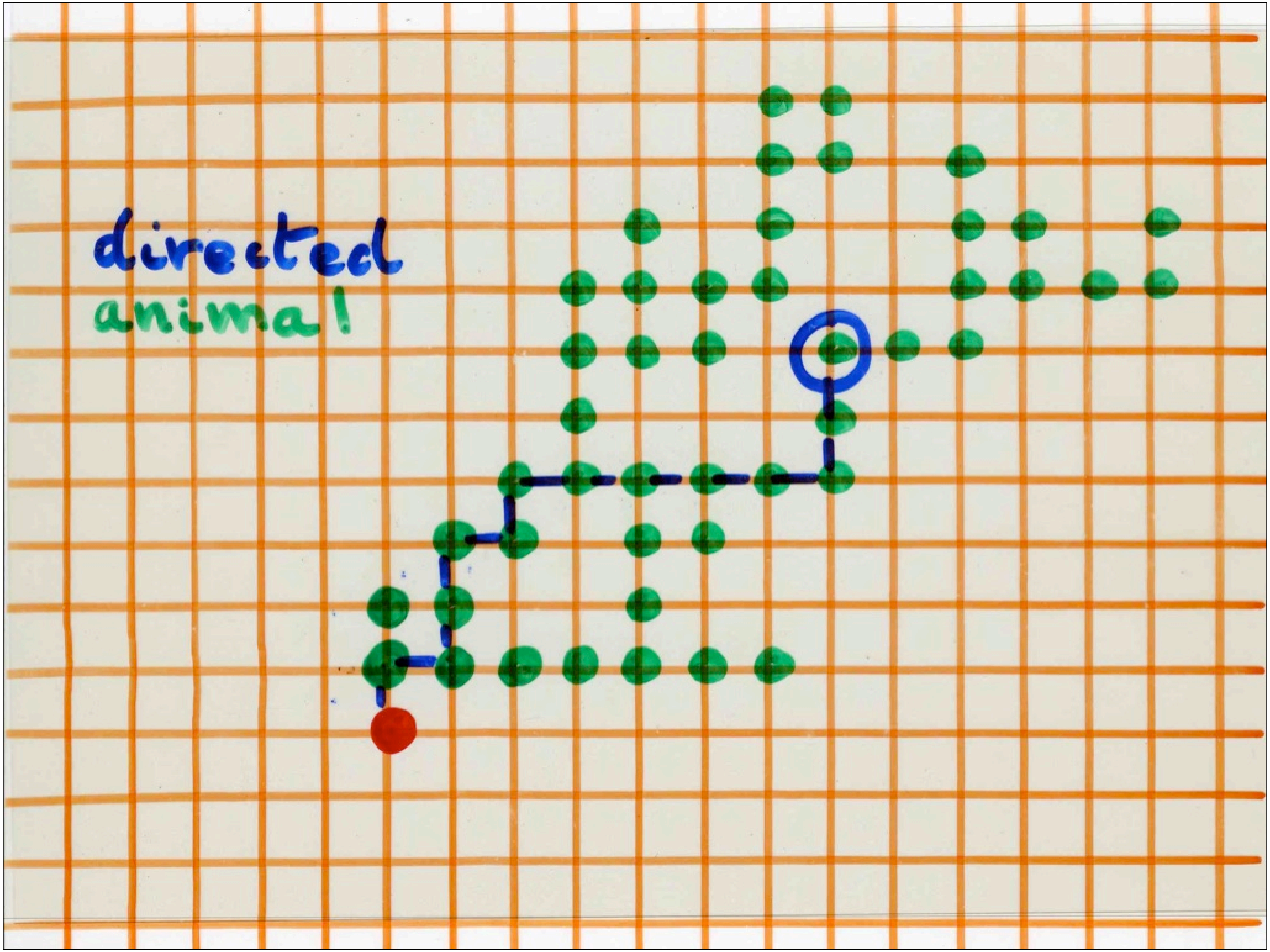
Multidirected animals, heaps of dimers and Lorentzian triangulations

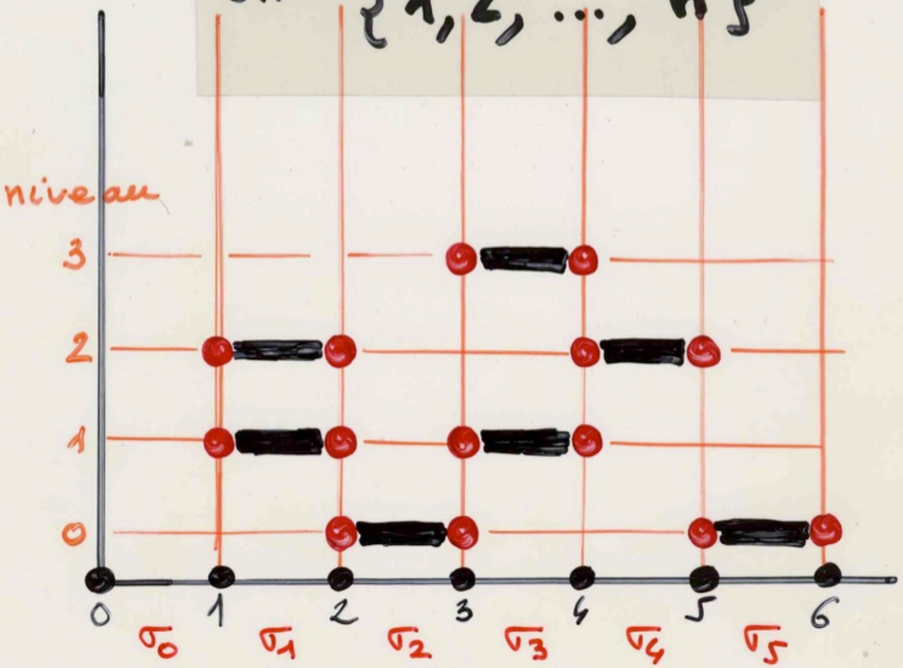
xavier viennot
LaBRI, CNRS, Bordeaux

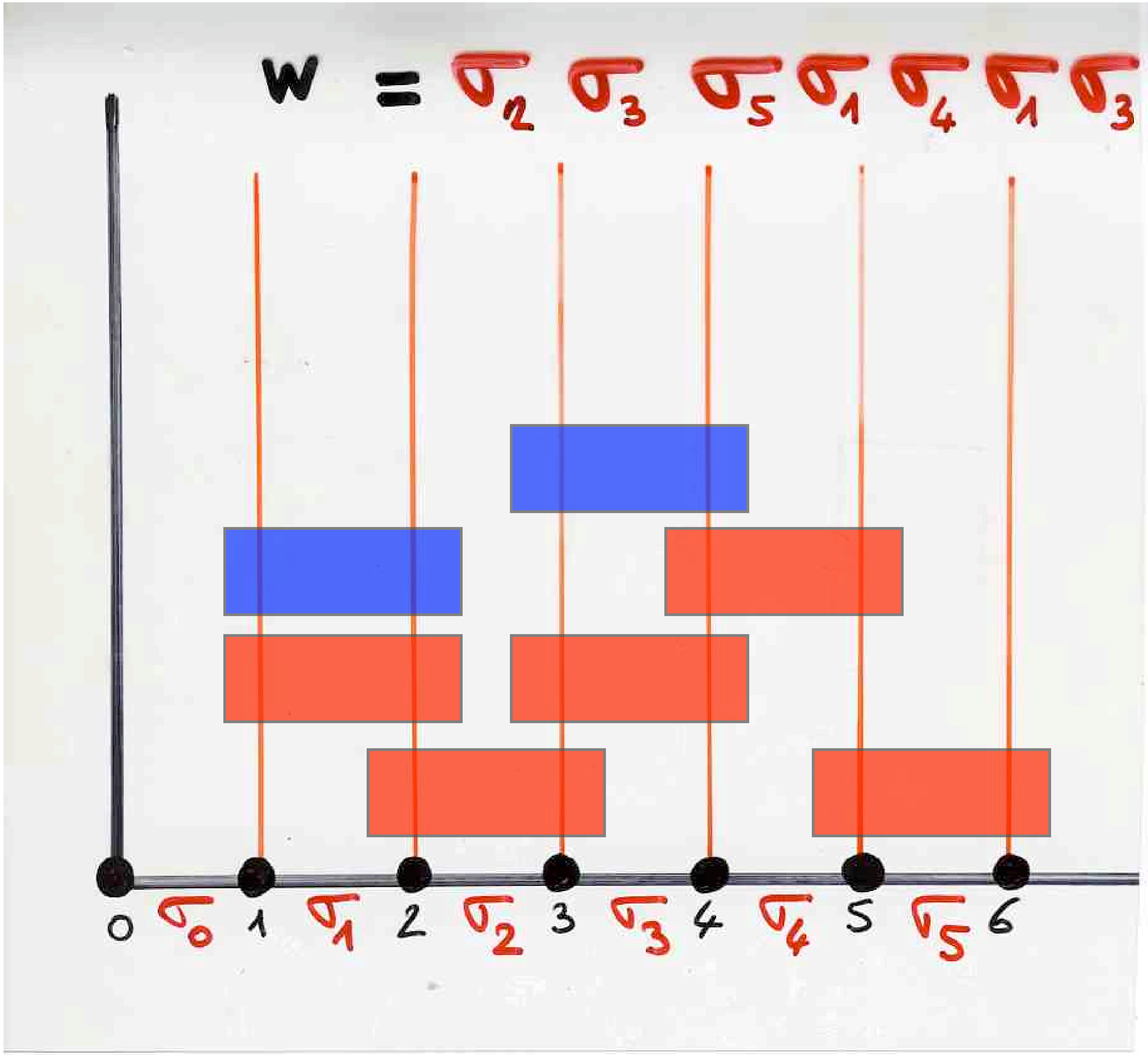
dedicated to
Tony Guttmann
for his 60th birthday


directed
animal

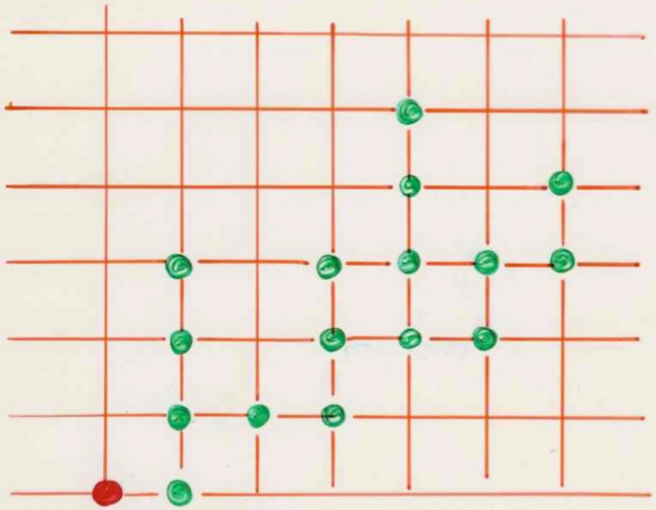

directed
animal

heap of dimers on $\{1, 2, \ldots, n\}$

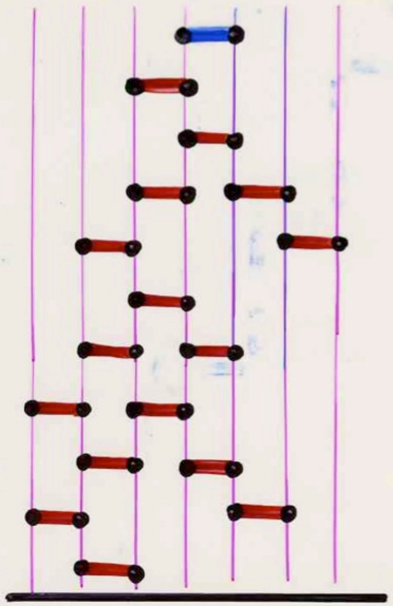




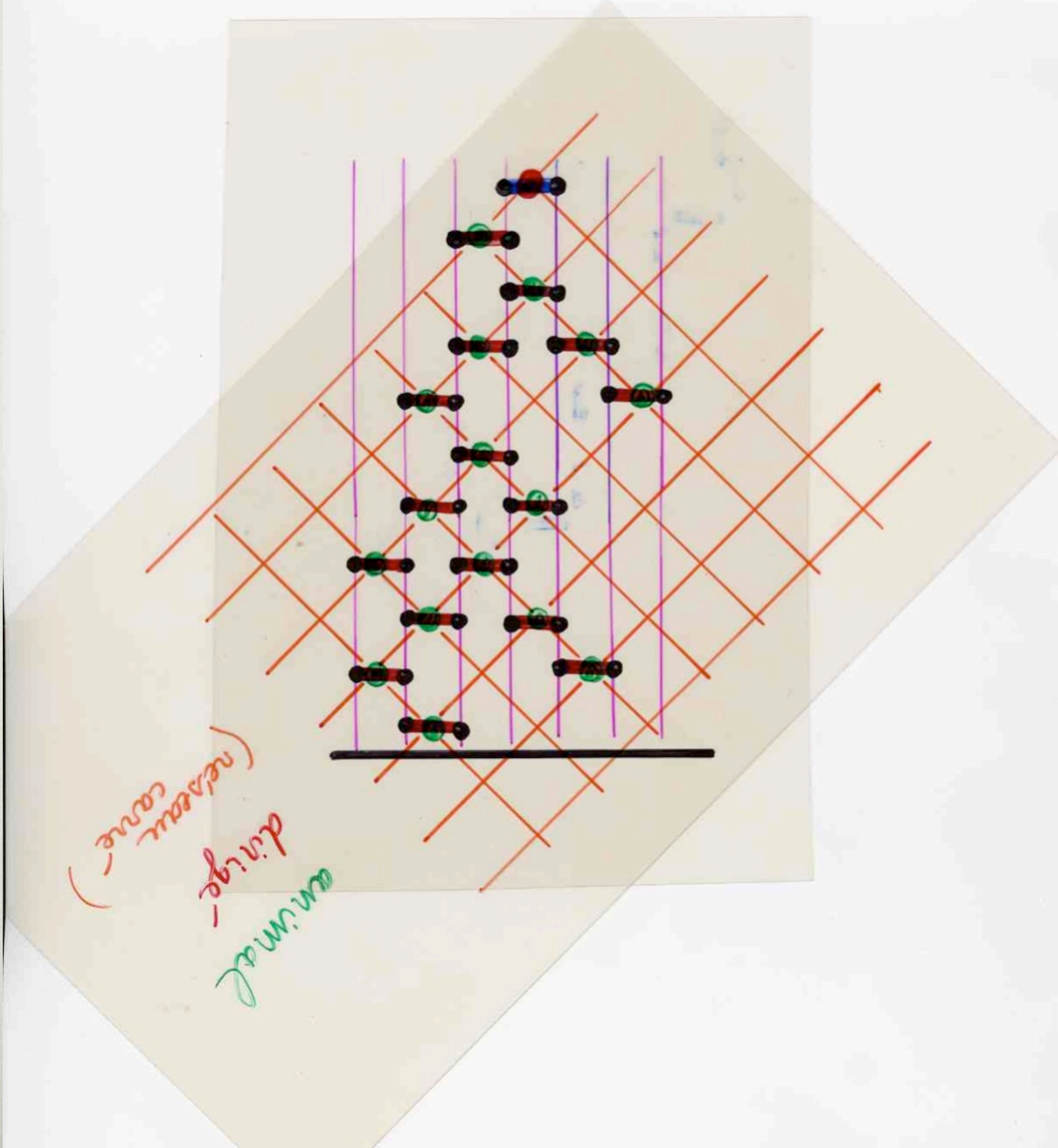
w = σ2 σ3 σ5 σ1 σ4 σ1 σ3
0 σ0 1 σ1 2 σ2 3 σ3 4 σ4 5 σ5 6

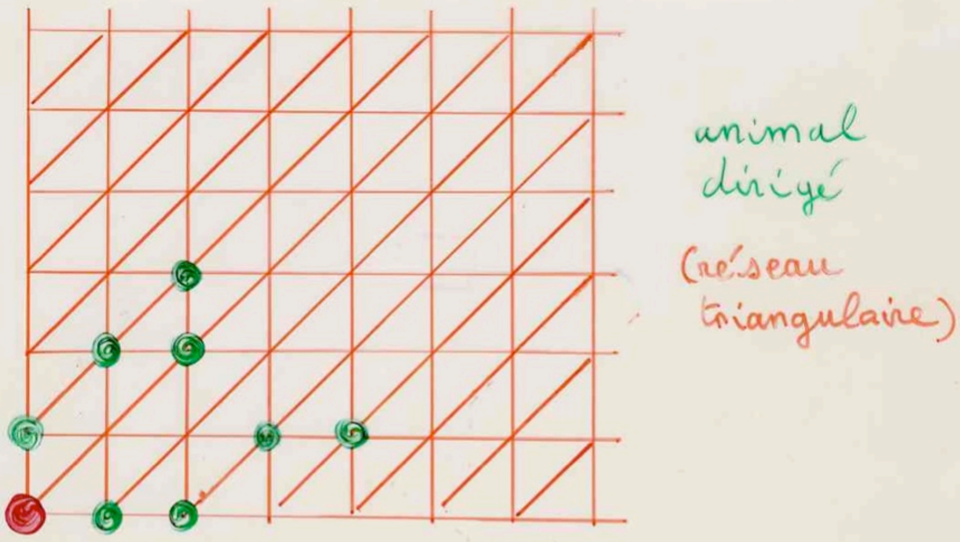


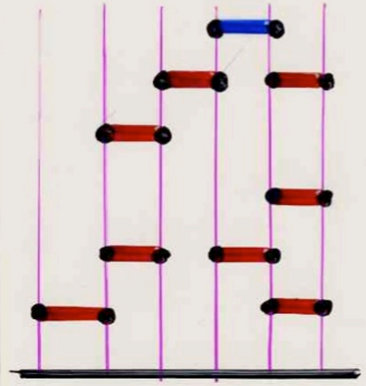
animal
dirigé
(réseau carré)

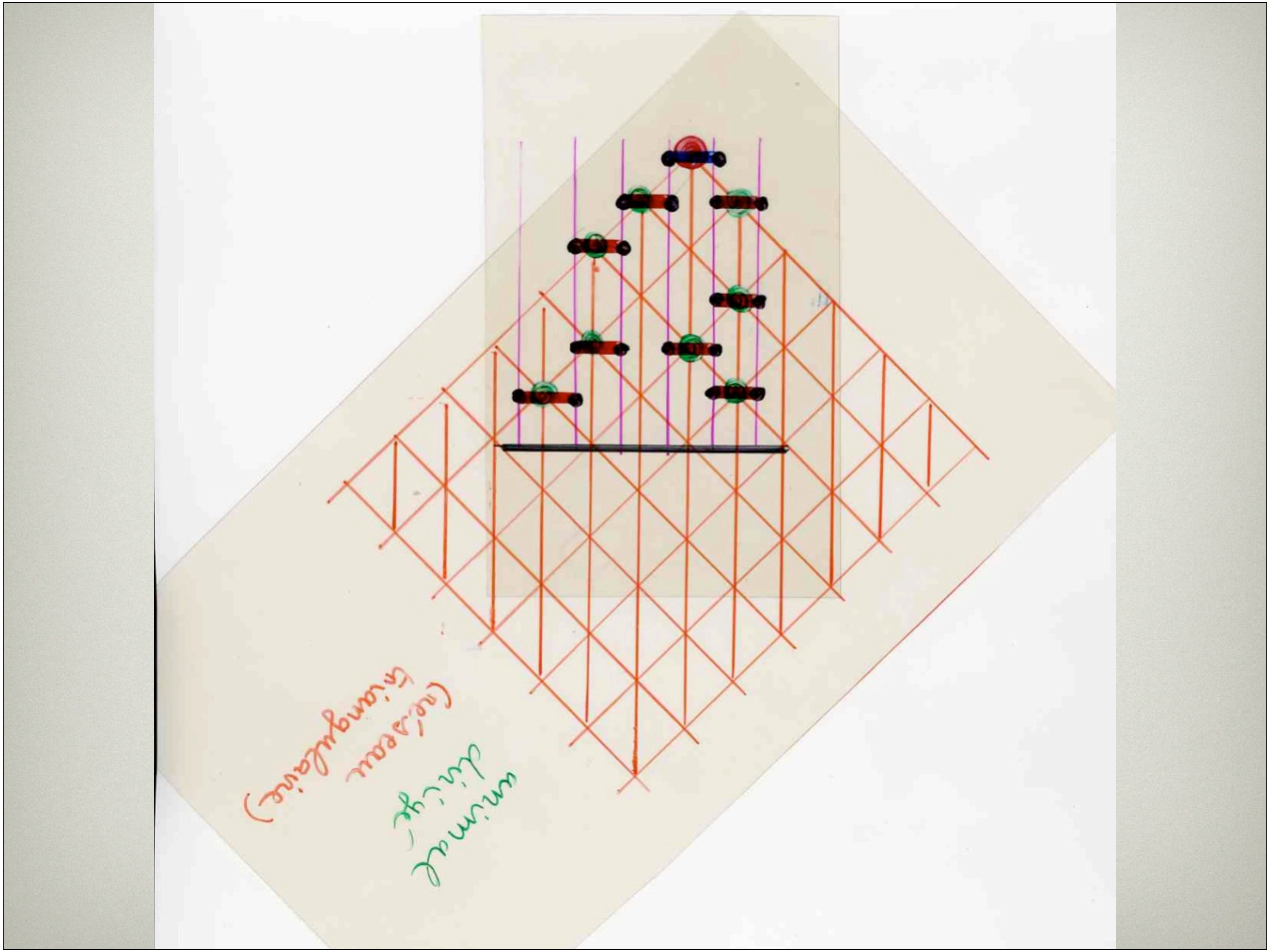


strict
heap


(réseau carré)
dirigé
animal

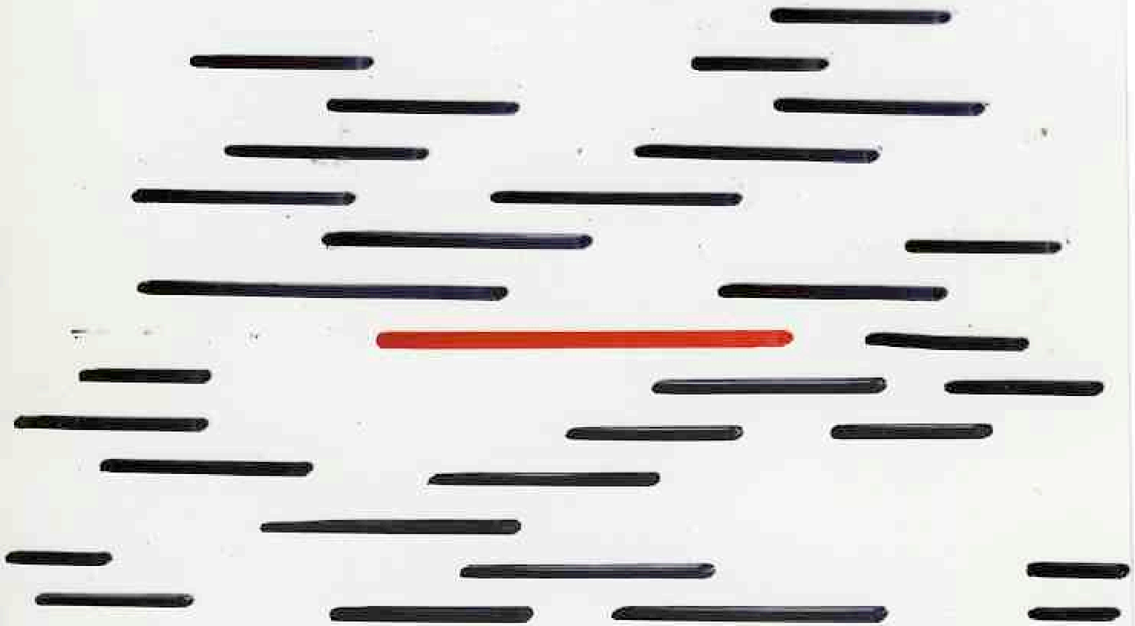

animal
dirigé
(réseau
triangulaire)




minimal dirigé
(réseau triangulaire)

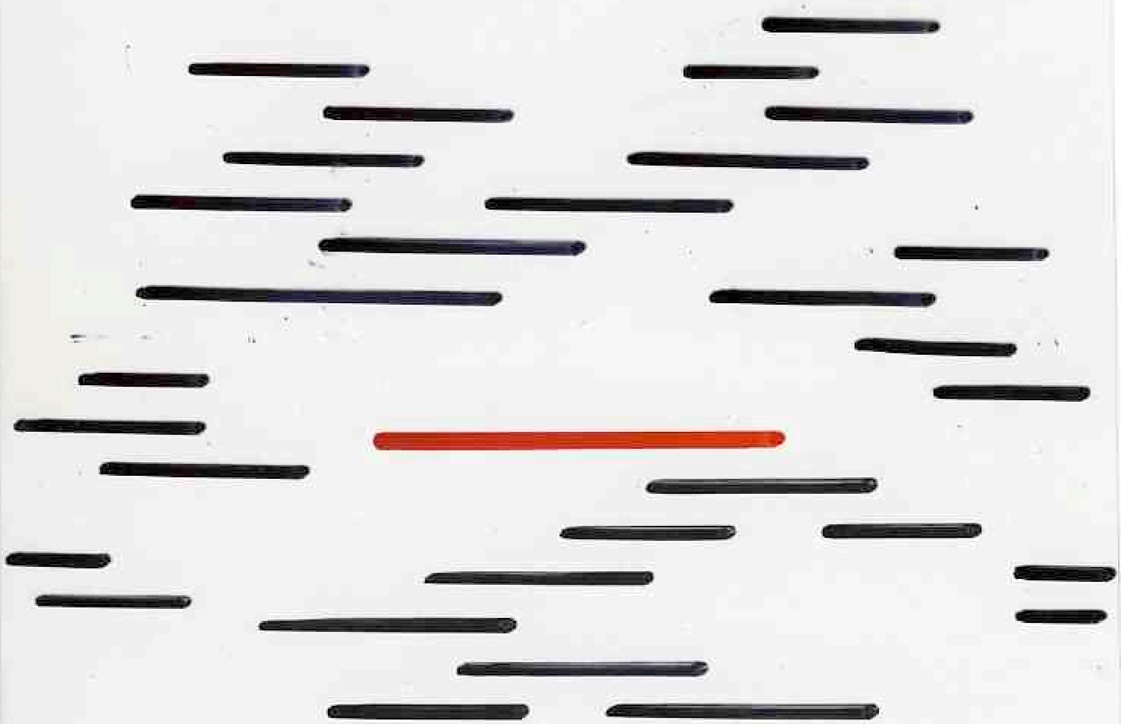
Opérateur

"Poussez" ....



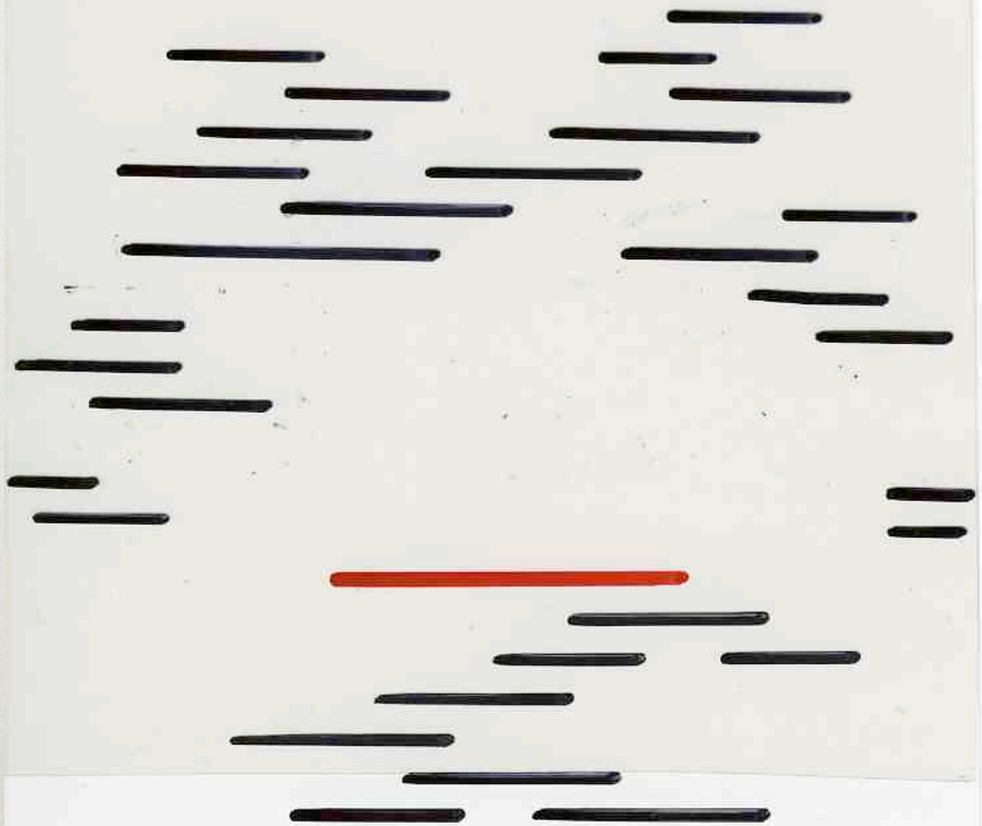
Opérateur

"Poussez" ...



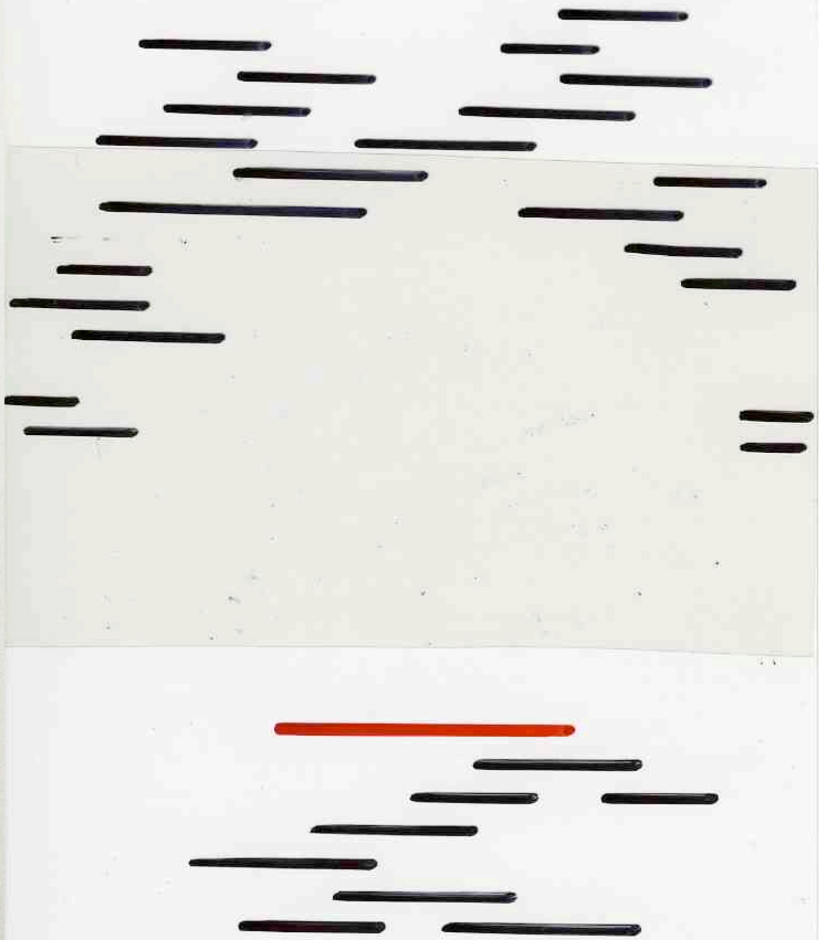
Opérateur

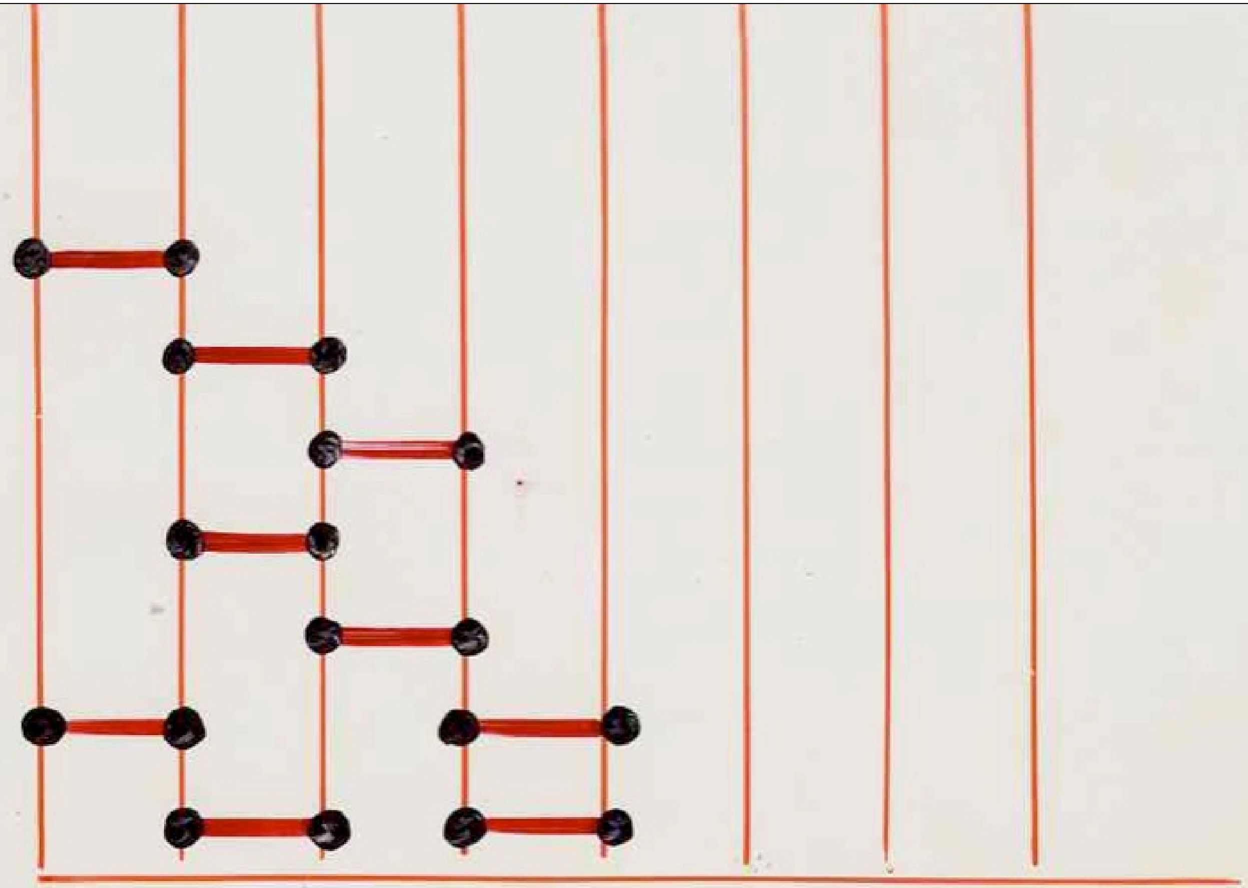
"Poussez" ..



Opérateur

"Poussez" ...





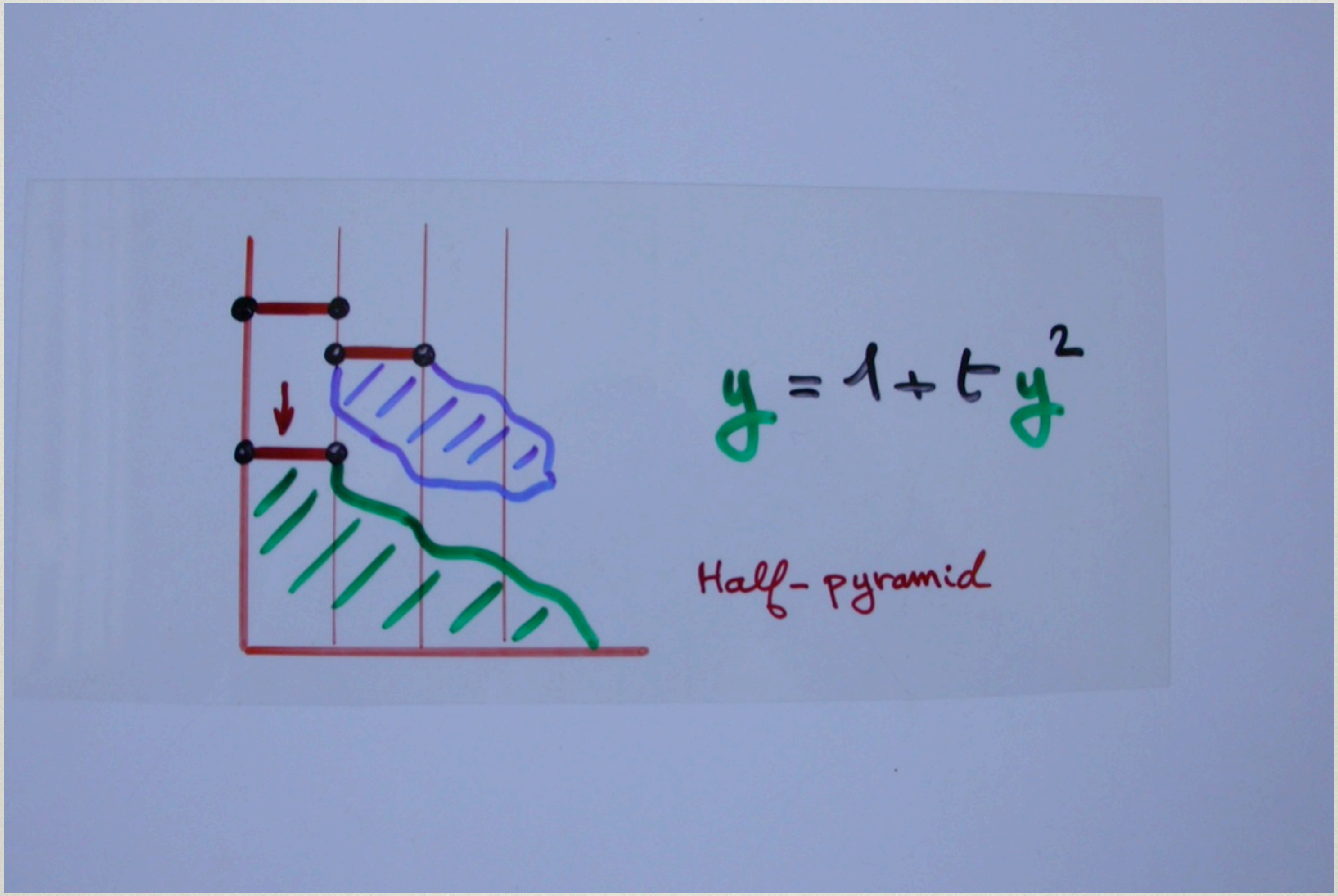
$$Q(t) = \frac{1-2t-\sqrt{1-4t}}{2t}$$

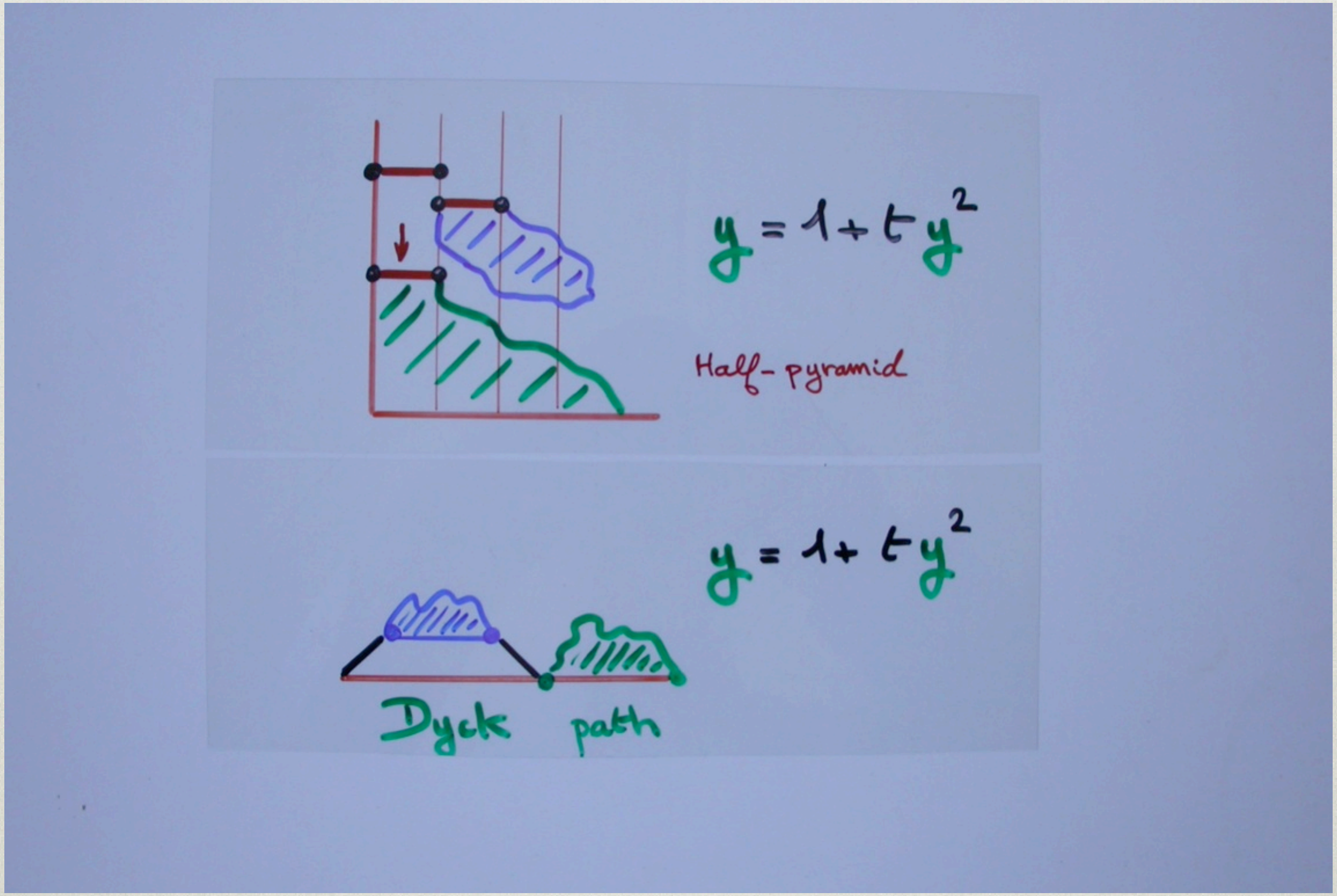
generating function for

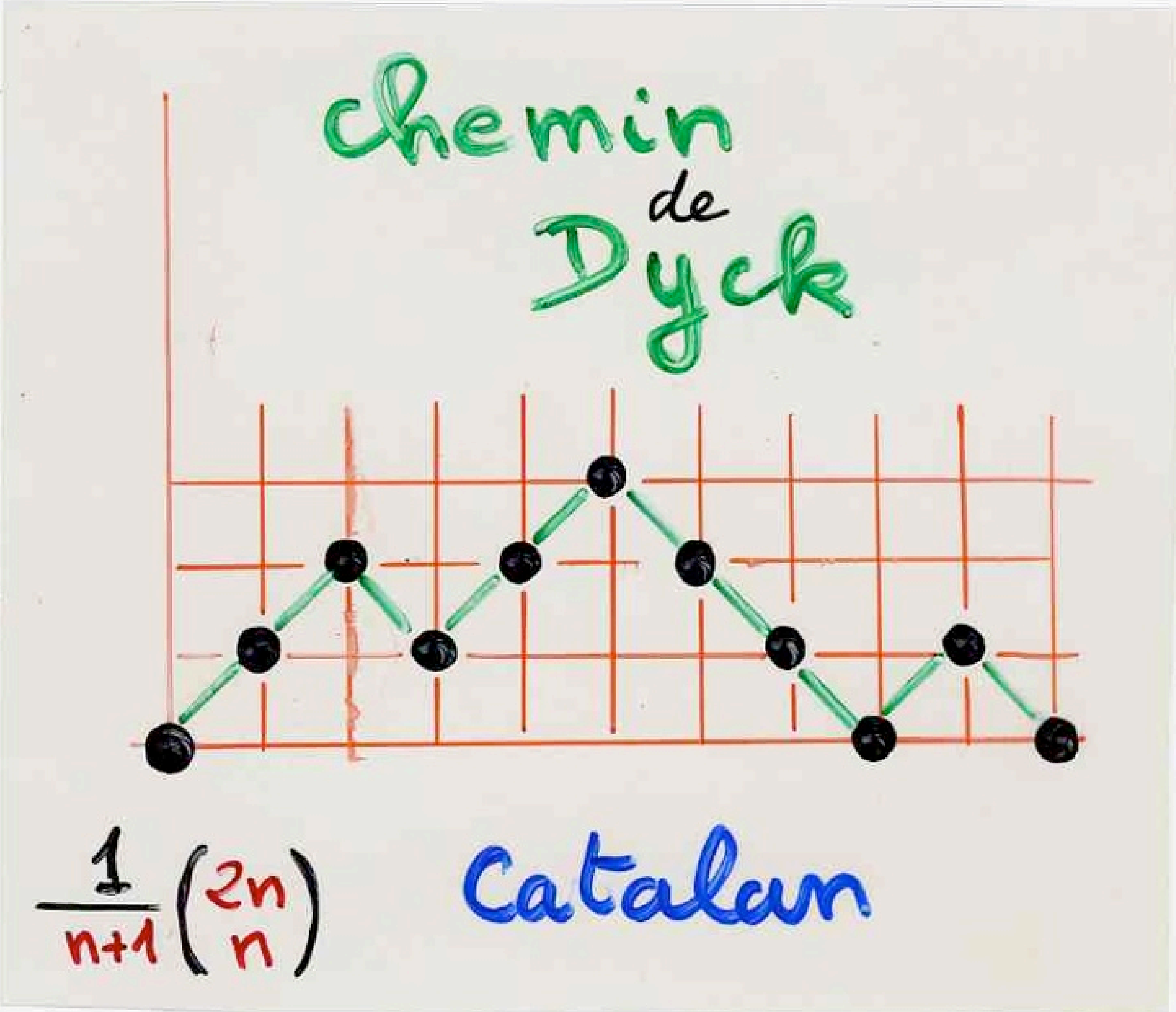
half-pyramid $(\neq \emptyset)$

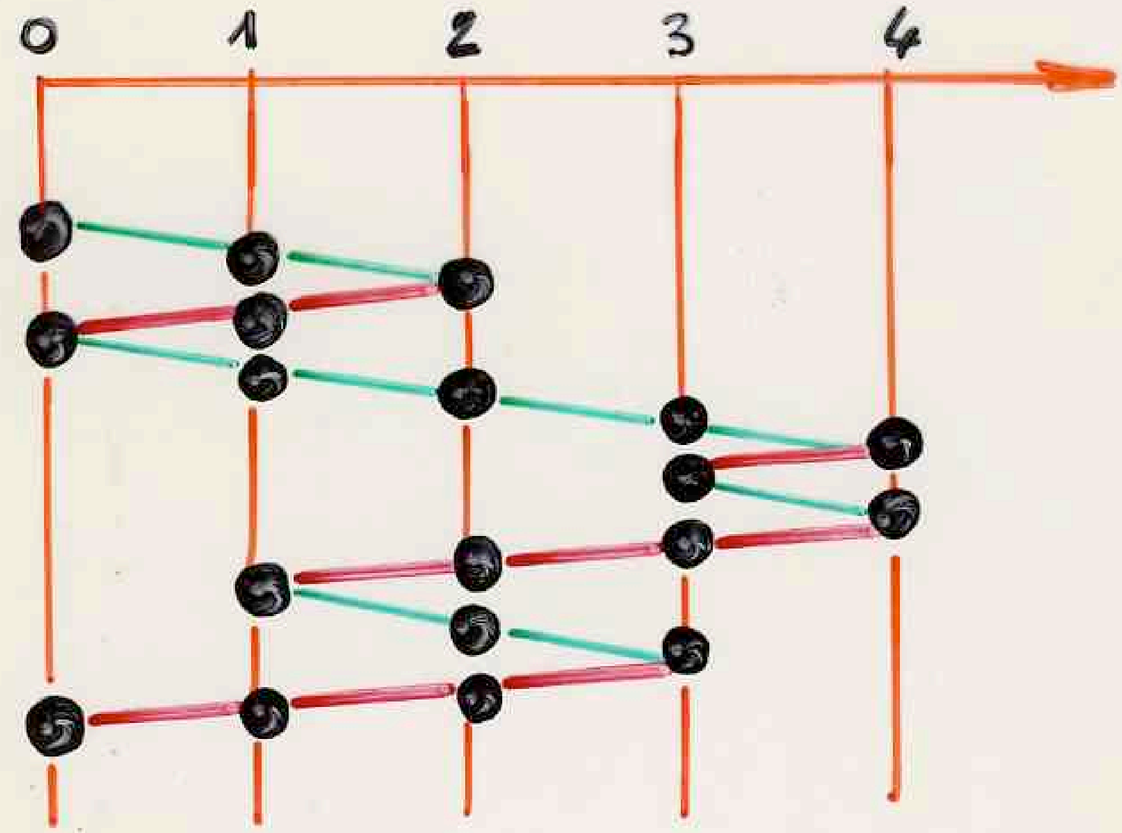
$$= \sum_{n \geqslant 1} C_n t^n$$

Catalan

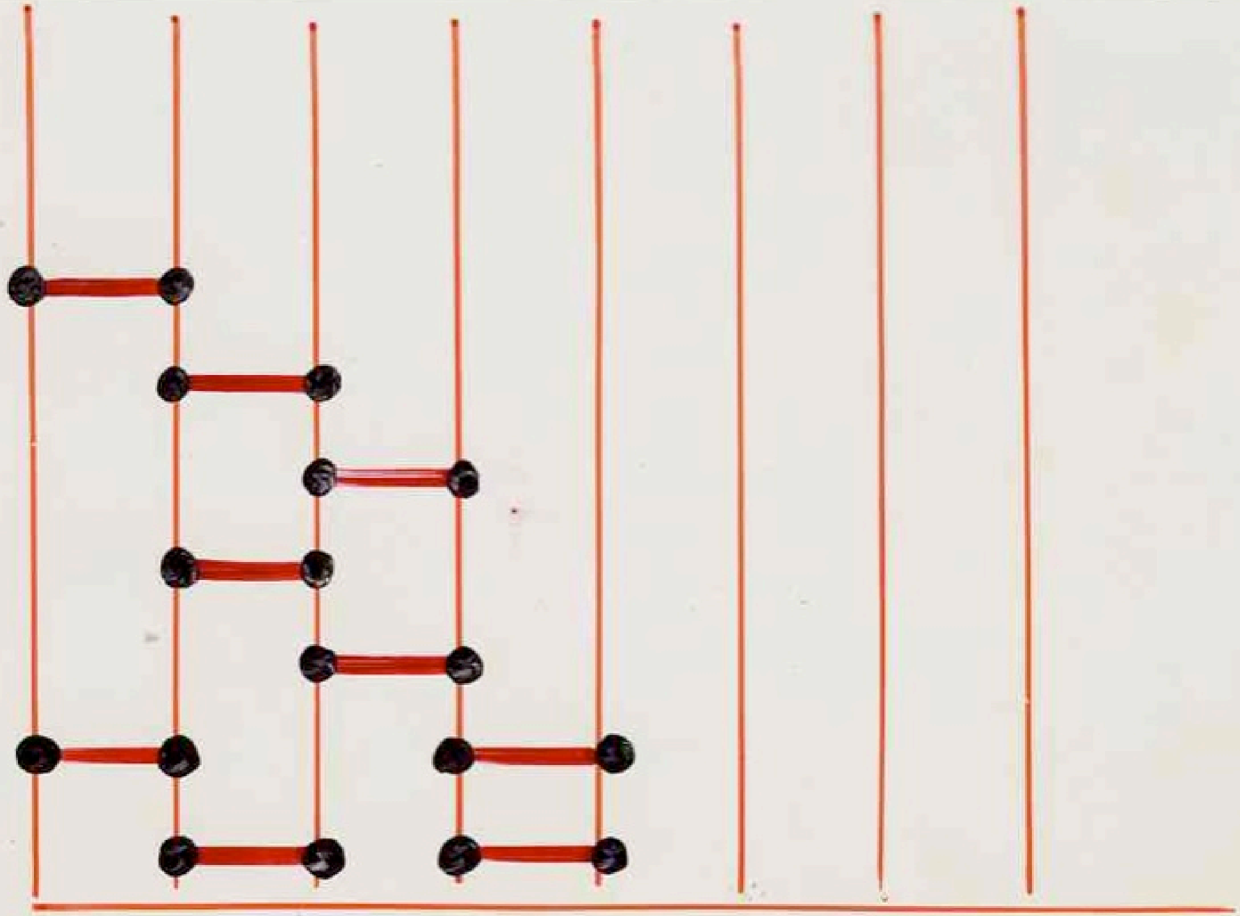

$y = 1 + t y^2$
Half-pyramid

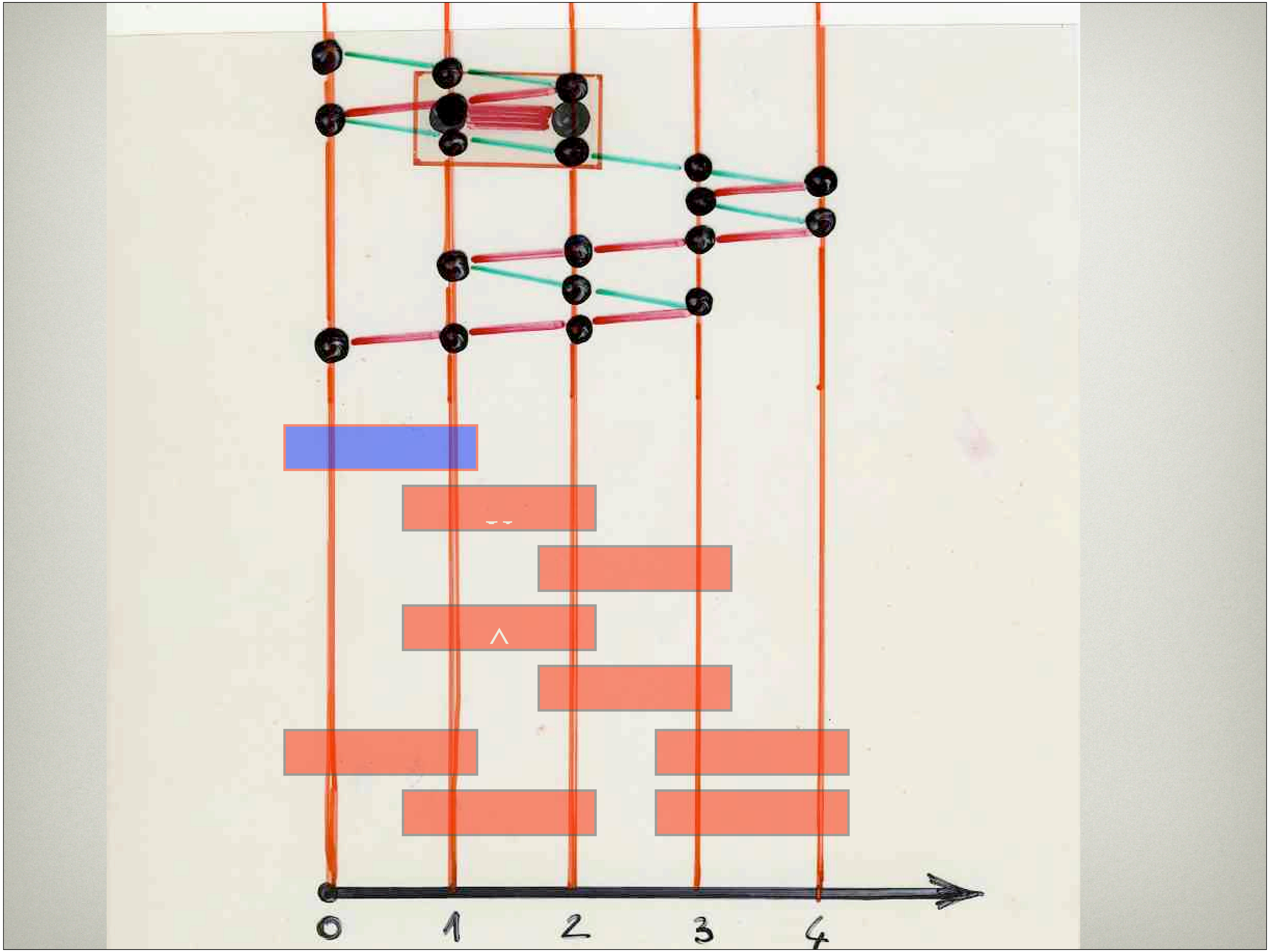

$y = 1 + t y^2$
Half-pyramid
$y = 1 + t y^2$
Dyck path

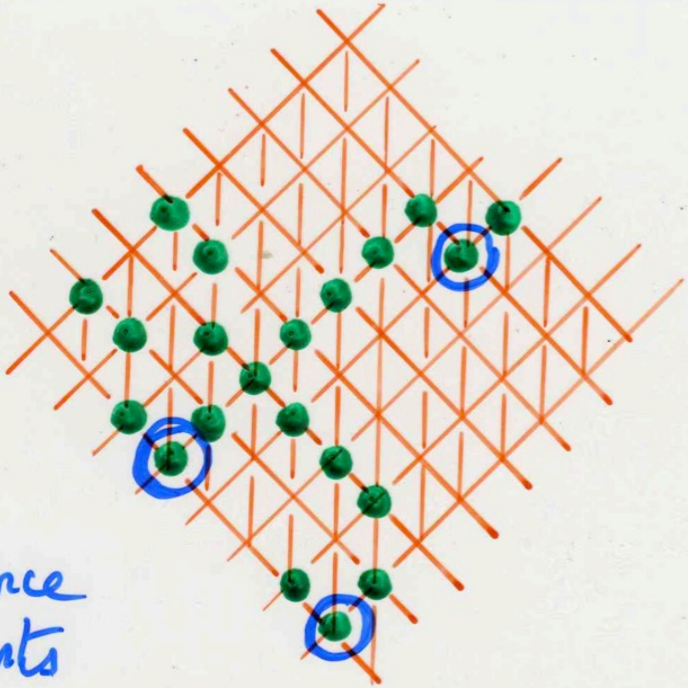

Chemin
de
Dyck
$\frac{1}{n+1}\binom{2n}{n}$
Catalan



0
1
2
3
4




0
1
2
3
4

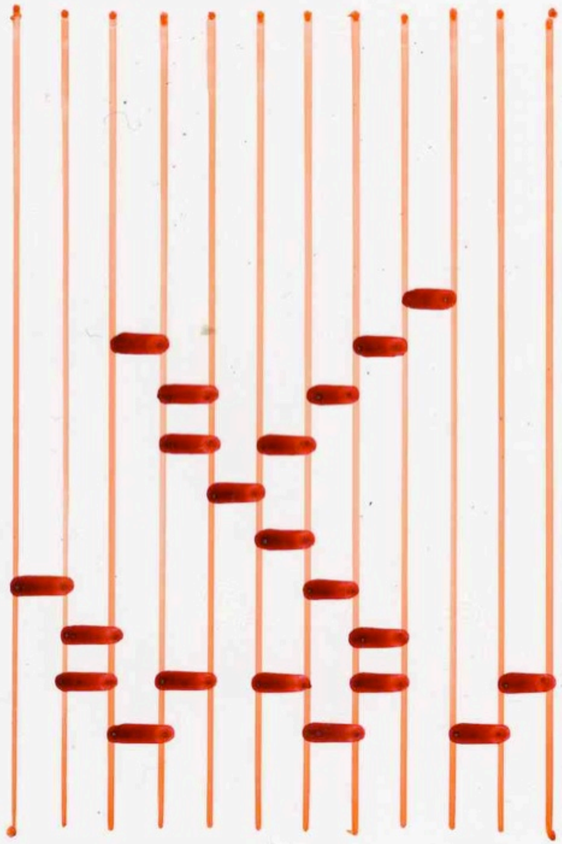


source points

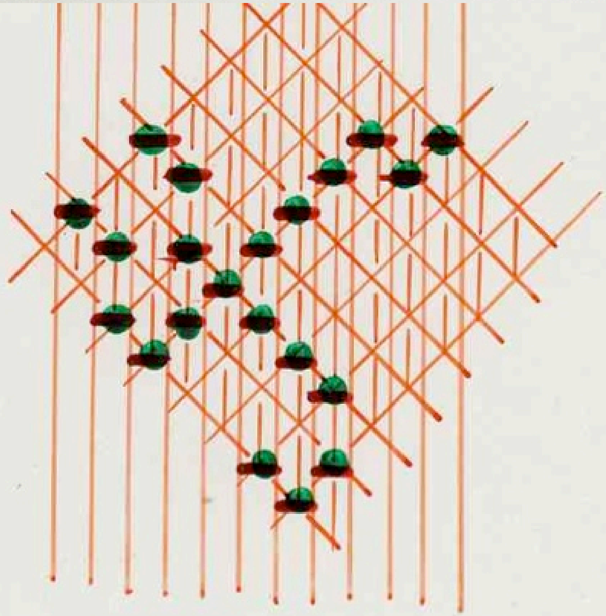
multidirected animal

(triangular lattice)

(Bousquet-Mélou, Rechnitzer, 2002)



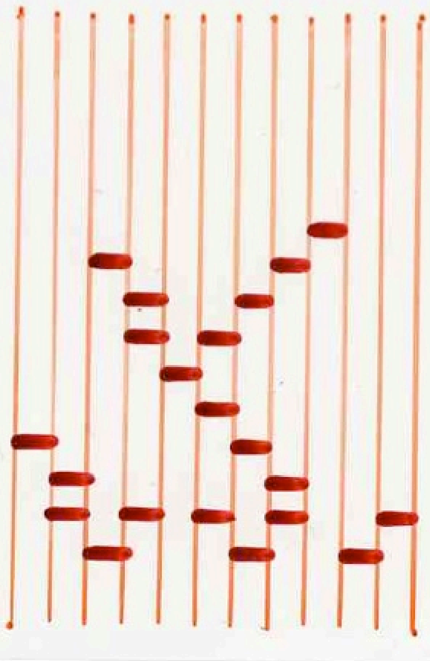
connected
heap
of
dimers



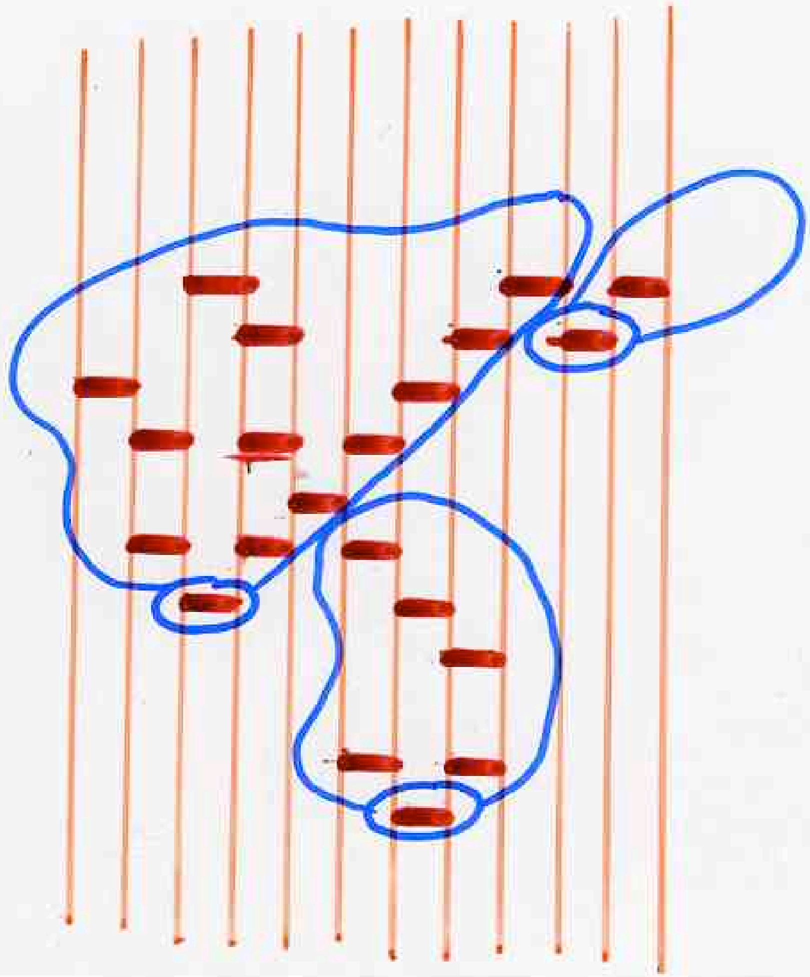
multidirected animal

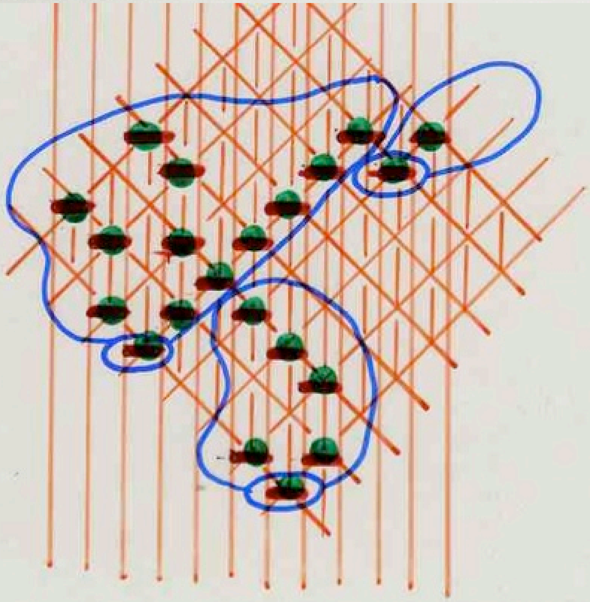
(triangular lattice)

(Bousquet-Mélou, Rechnitzer, 2002)



connected heap of dimers

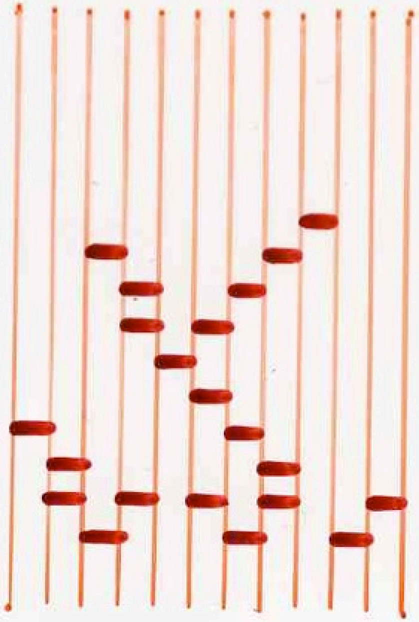




multidirected animal

(triangular lattice)

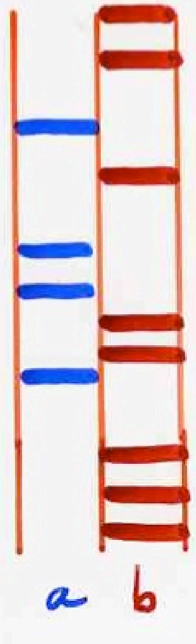
(Bousquet-Mélou, Rechnitzer, 2002)



connected heap of dimers

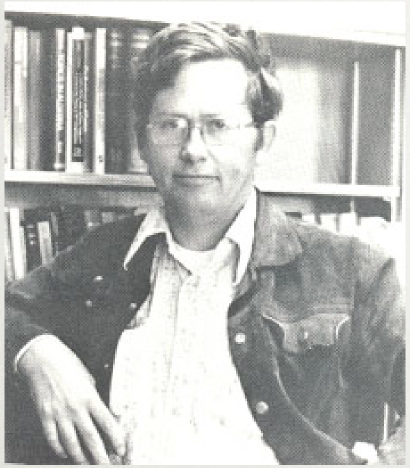
Klarner, 1965

$$\sum_{a_1+a_2+\cdots+a_k=n} f(a_1,a_2)\cdots f(a_{k-1},a_k)$$

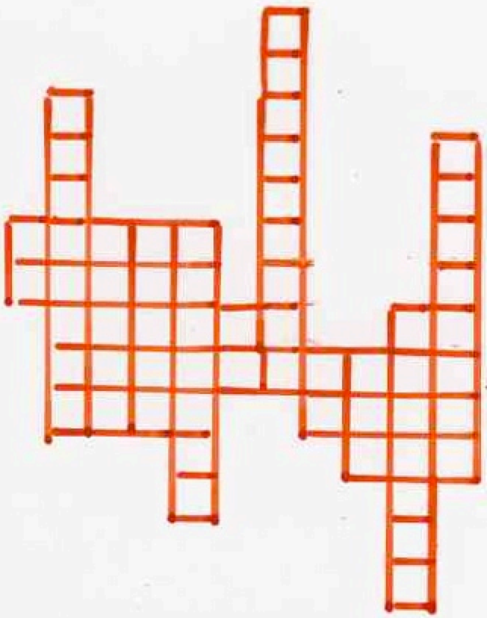


$$f(a,b)=\binom{a+b}{a}$$

animal
triangular lattice
polyomino
hexagonal cells

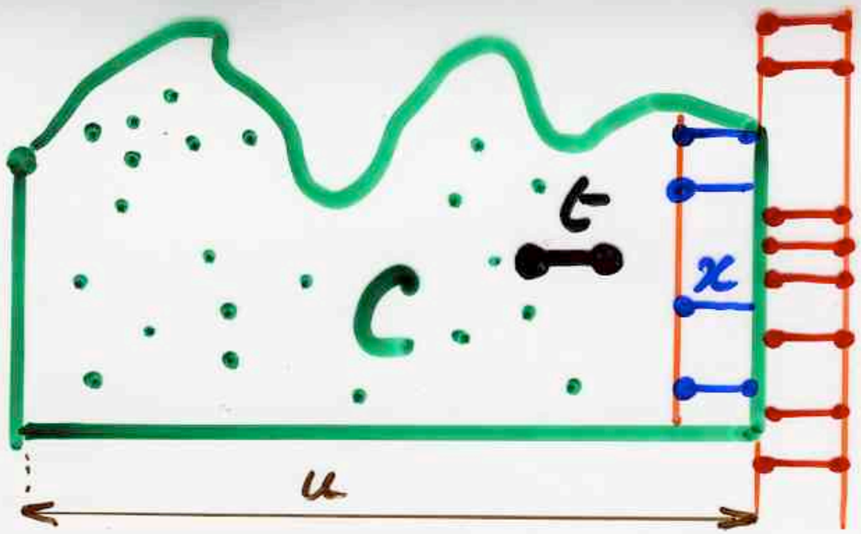


$$f(a,b) = a+b-1$$



vertically-
column- convex
polyomino

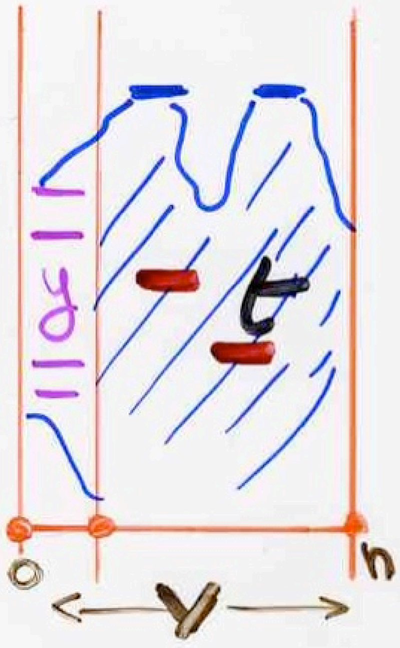
Fredholm integral equation



$$C(t,x,u) = \frac{utx}{1-tx} + \frac{u}{1-tx}\, C\!\left(t, \frac{t}{1-tx}, u\right) - u\, C(t,1,u)$$

$C(t, y, v)$ connected heaps

$$= \frac{\displaystyle\sum_{n\geq 0} \frac{v^n}{\tilde{F}_{n+1}}}{\displaystyle\sum_{n\geq 0} \frac{v^n}{\tilde{F}_n}} - 1$$



(BM., R.)

$$\sin(n+1)\theta = (\sin\theta)\, U_n(\cos\theta)$$

$U_n(t/2)$ reciprocal

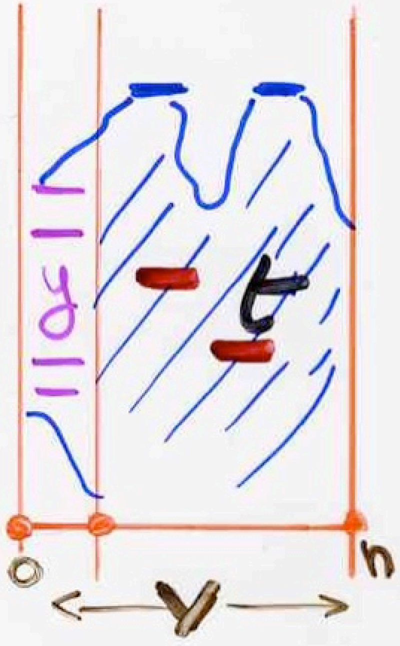
$F_n(t^2)$

Fibonacci polynomial

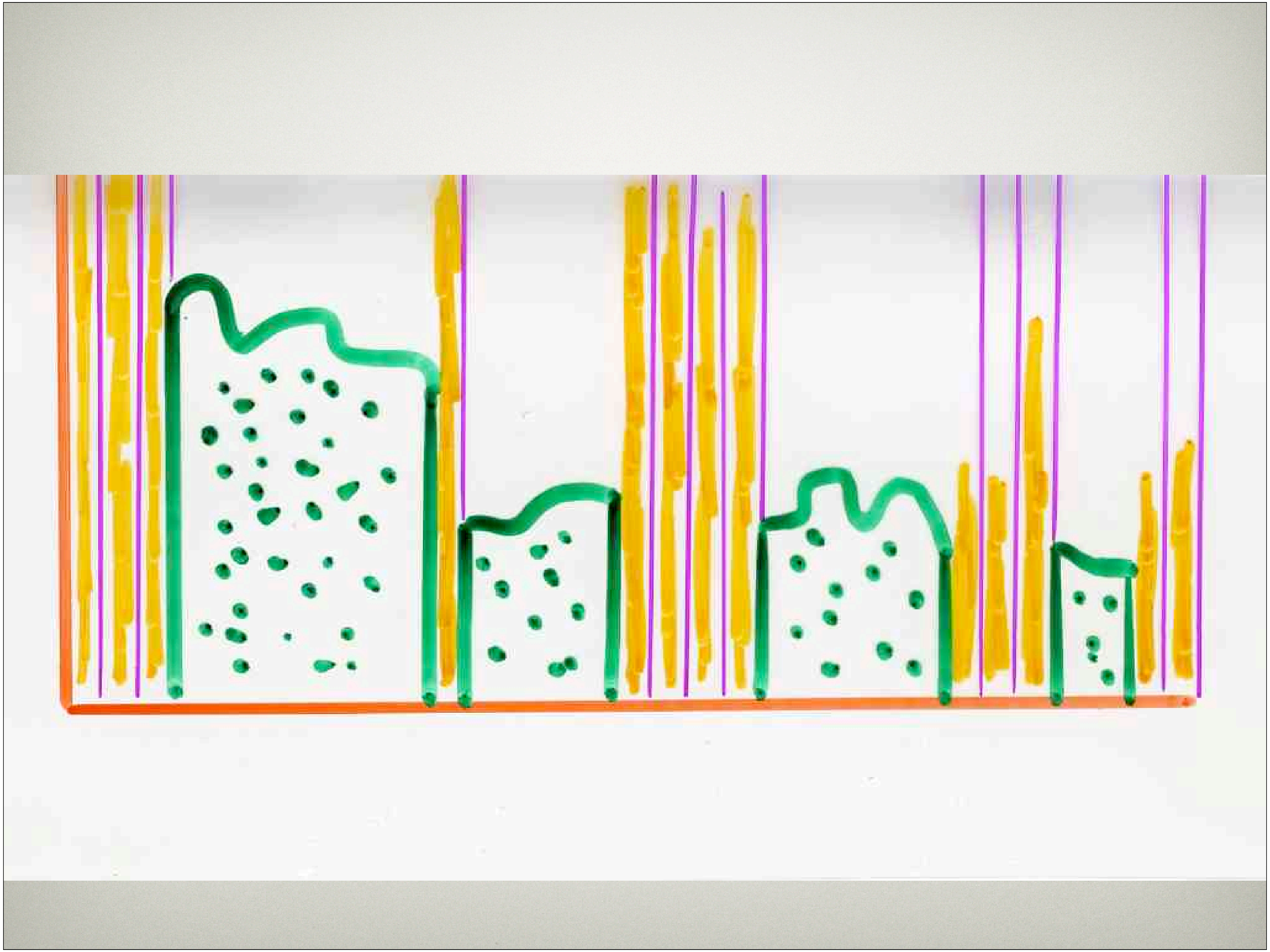
$$F_{n+1}(t) = F_n(t) - t\, F_{n-1}(t) \qquad F_0 = F_1 = 1$$

$C(t, y, v)$ connected heaps

$$= \frac{\displaystyle\sum_{n\geq 0} \frac{v^n}{\tilde{F}_{n+1}}}{\displaystyle\sum_{n\geq 0} \frac{v^n}{\tilde{F}_n}} - 1$$



(BM., R.)



$$F_n = \frac{(1-Q^{n+1})}{(1-Q)(1+Q)^n}$$

$$Q(t) = \frac{1-2t-\sqrt{1-4t}}{2t}$$

generating function for

half-pyramid $(\neq \emptyset)$

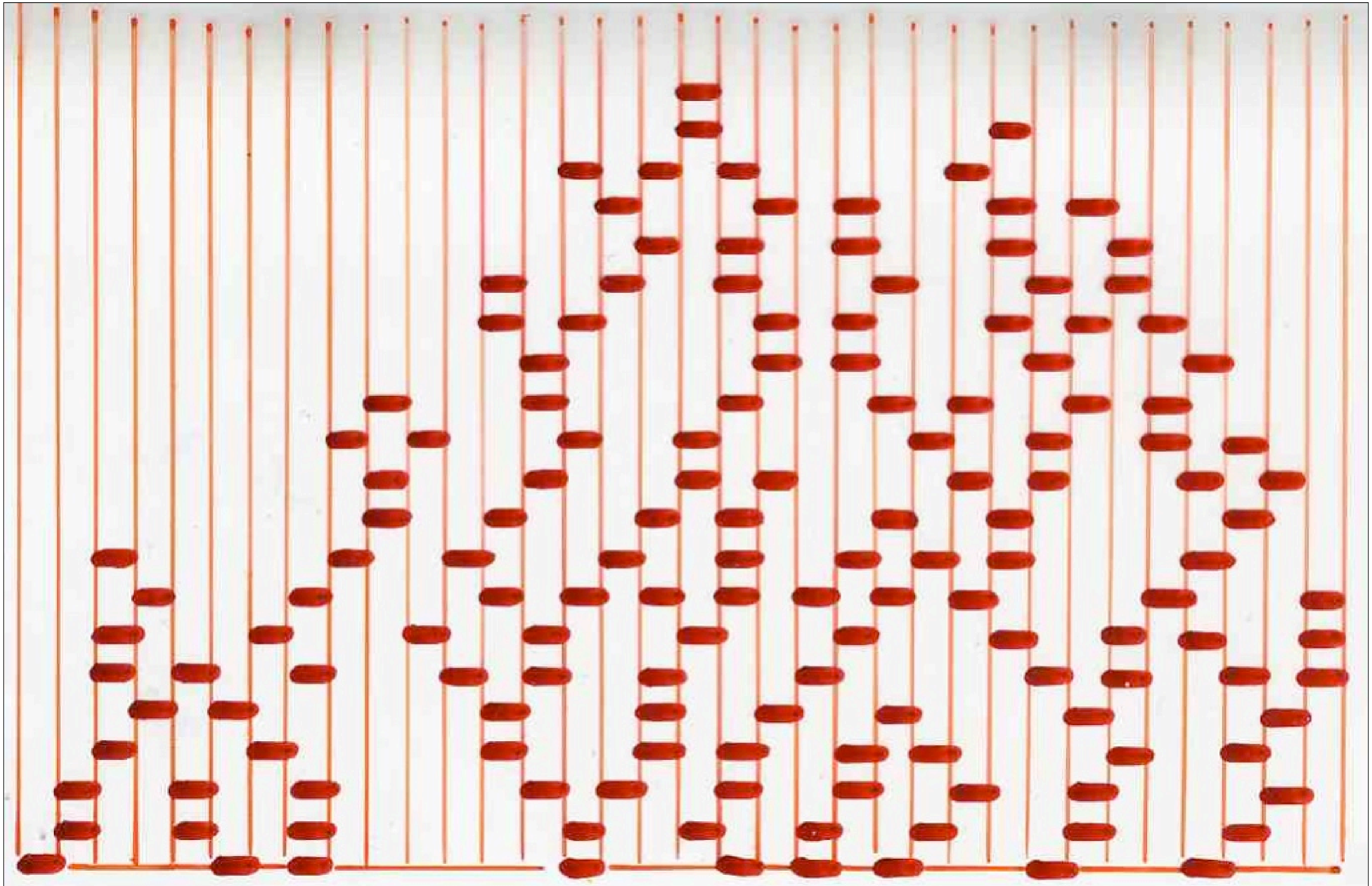
$$= \sum_{n \geqslant 1} C_n \, t^n$$

Catalan

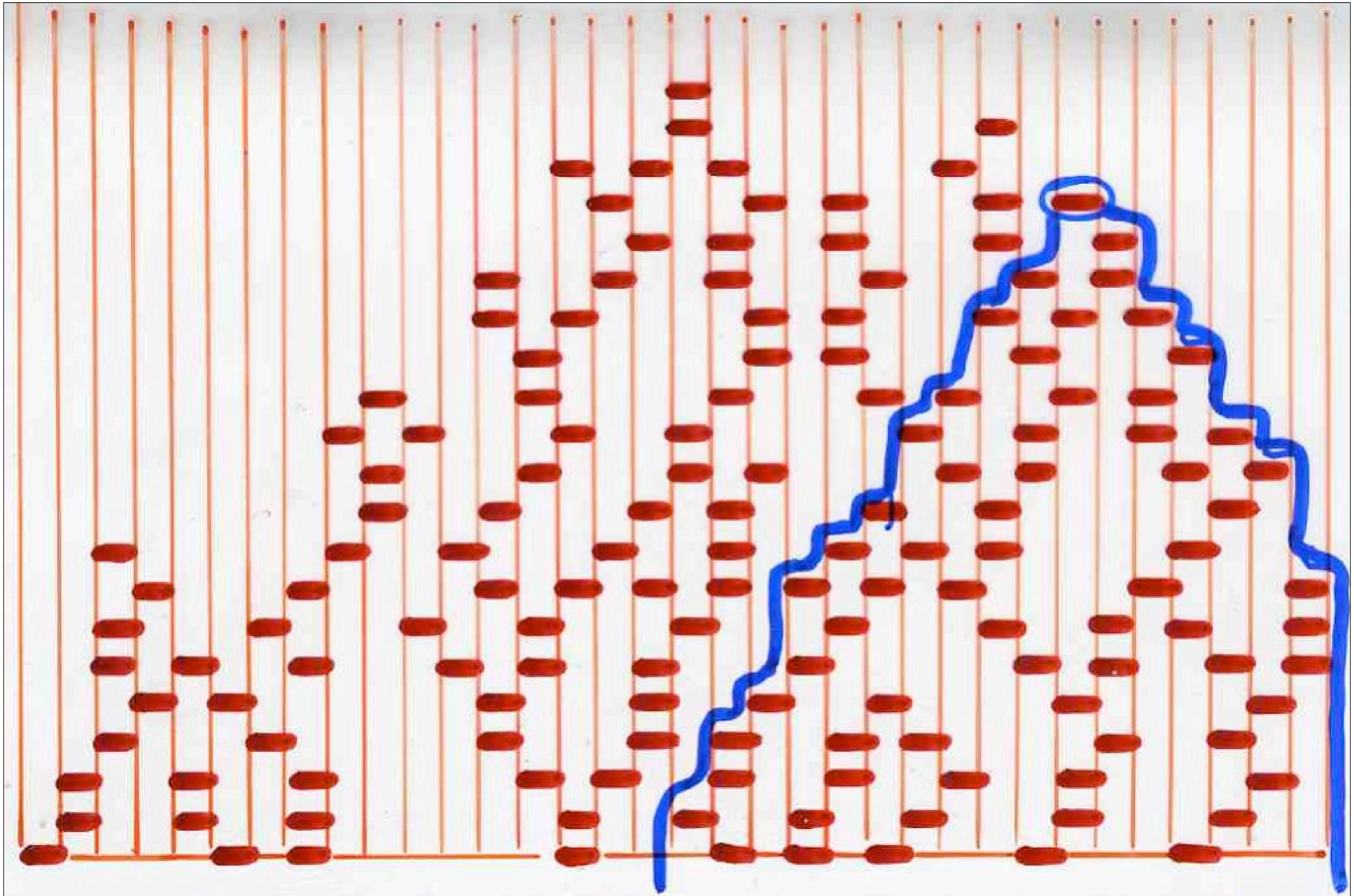
$C(t)$ g.f. connected heap

(Bousquet-Mélou, Rechnitzer) 2002

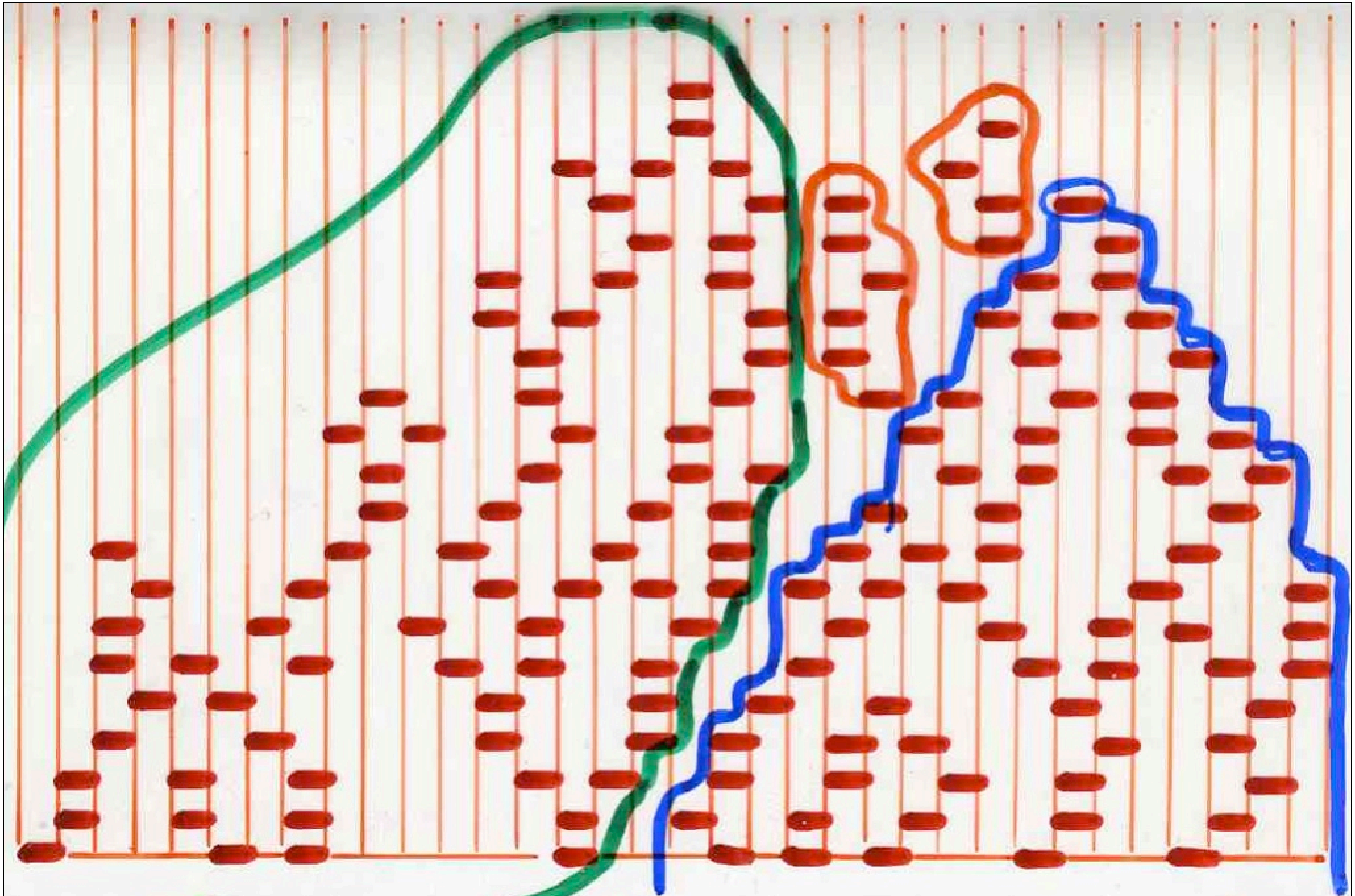
$$C(t) = \frac{Q}{(1-Q)\left[1 - \sum_{k\geq 1} \frac{Q^{k+1}}{1-Q^k(1+Q)}\right]}$$



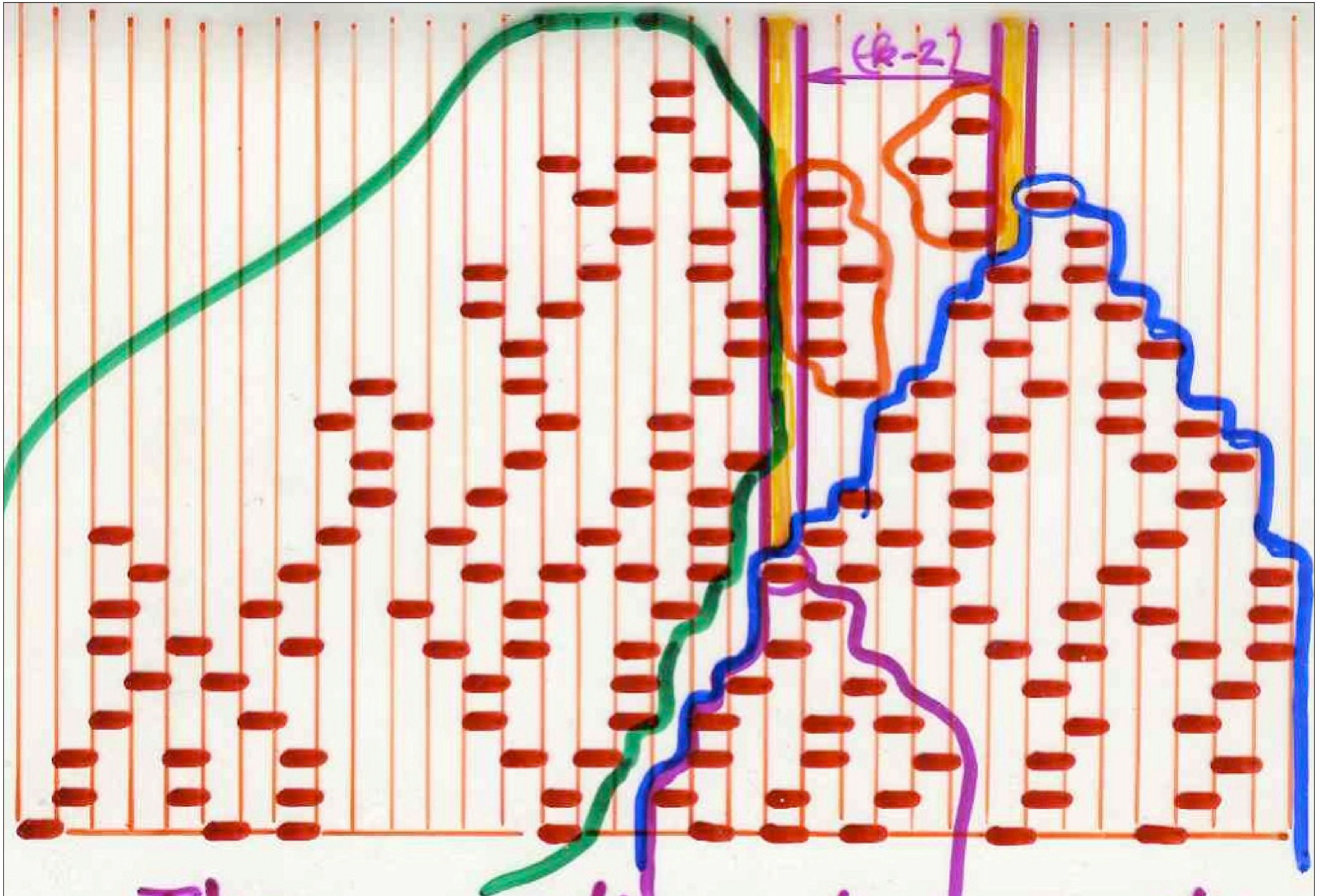
The nordic decomposition



The nordic decomposition

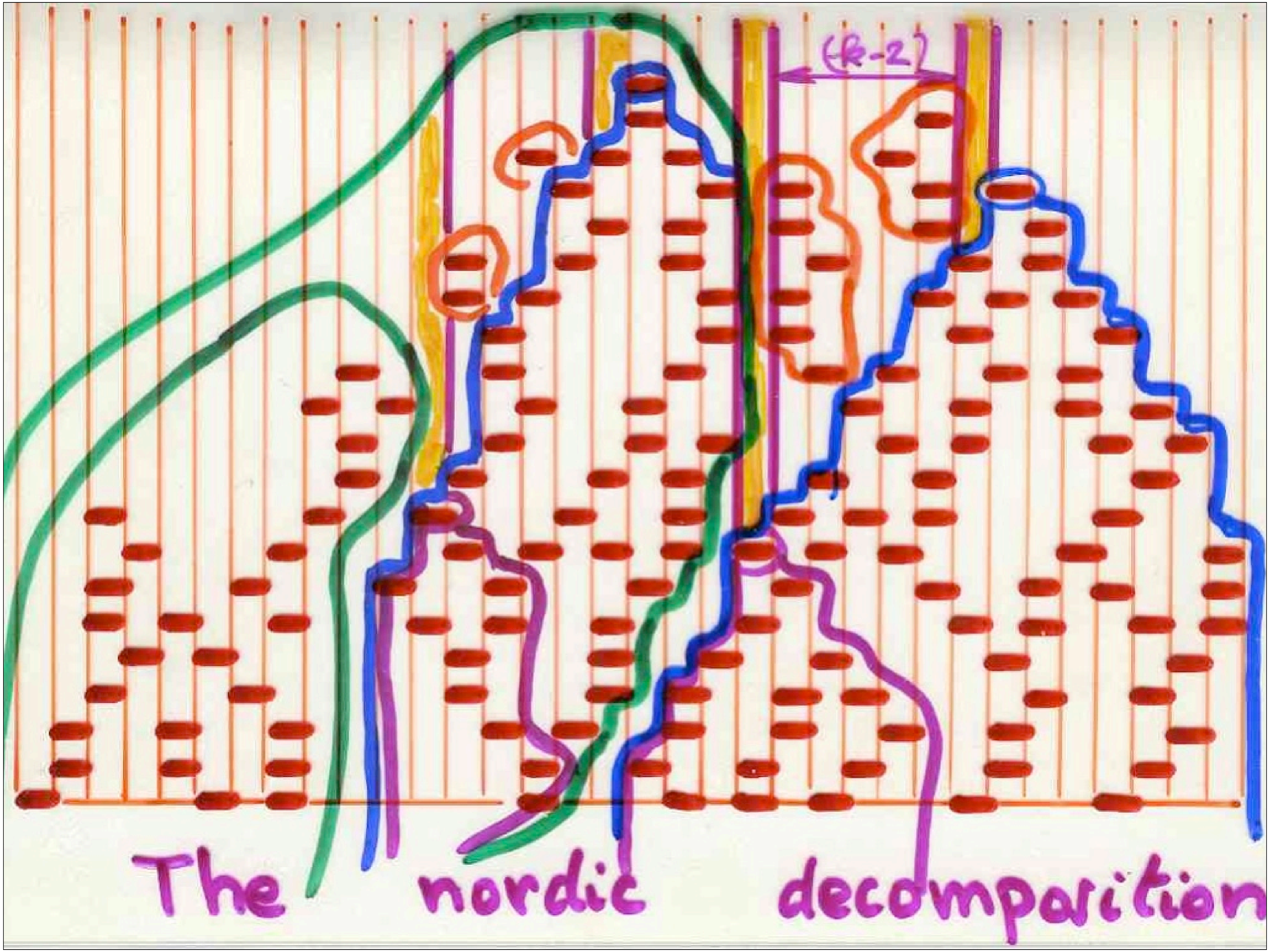


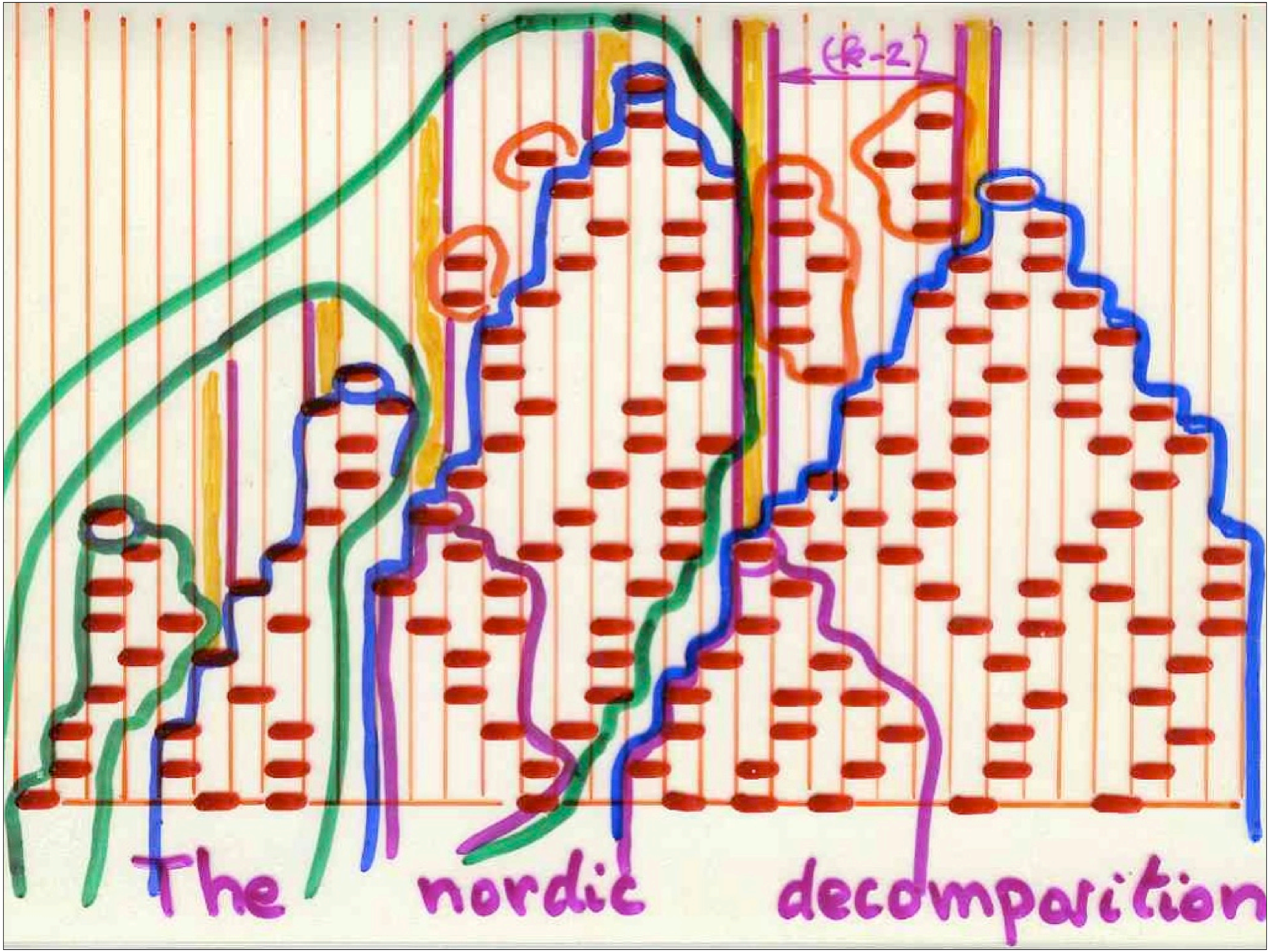
The nordic decomposition



The nordic decomposition

$$C = \frac{Q}{1-Q} + C \sum_{k \geq 1} \frac{Q}{1-Q} \times Q^{k} \times \frac{1}{F_{k-1}}$$


(k-2)
The nordic decomposition


(k-2)
The nordic decomposition

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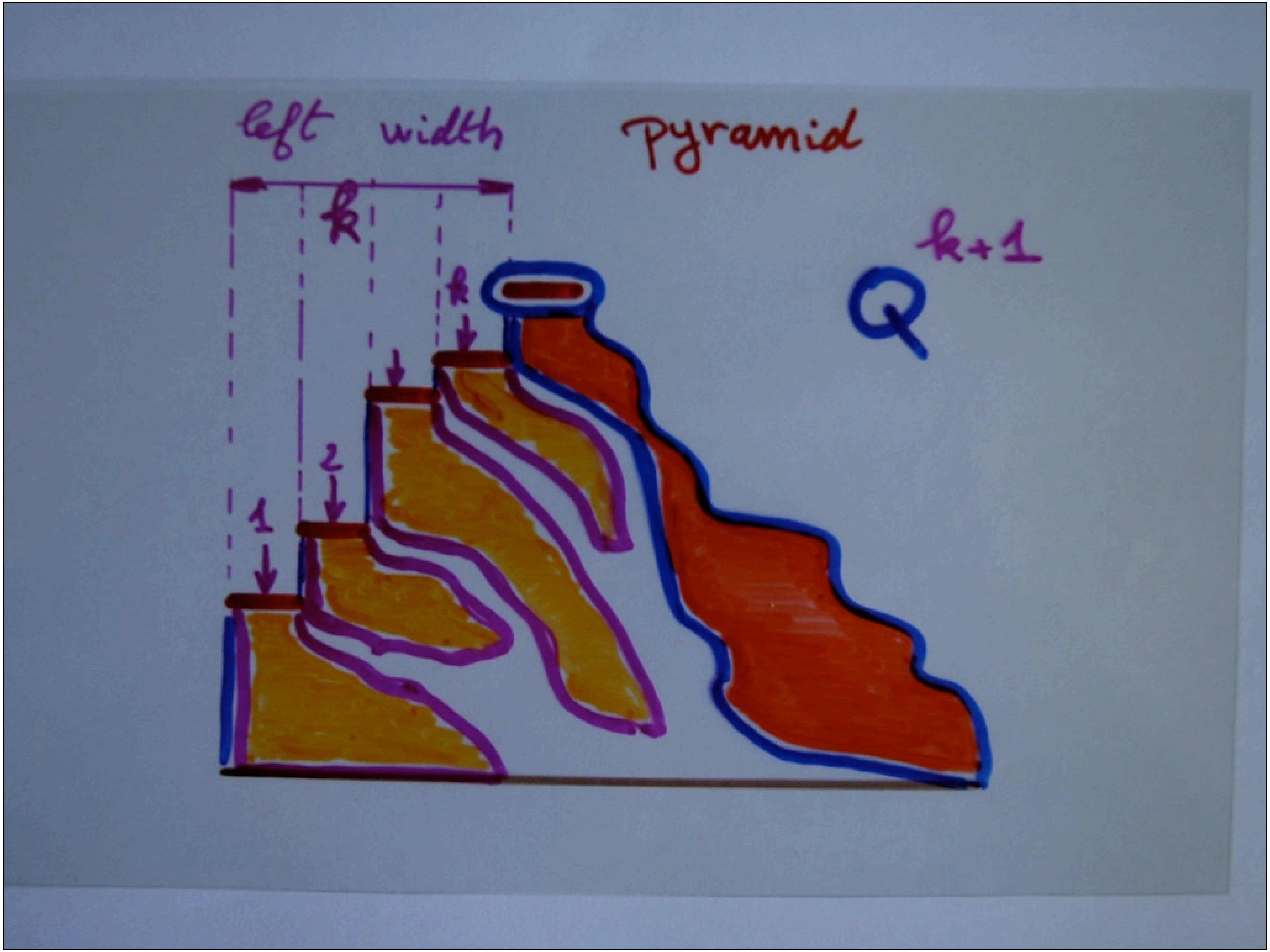
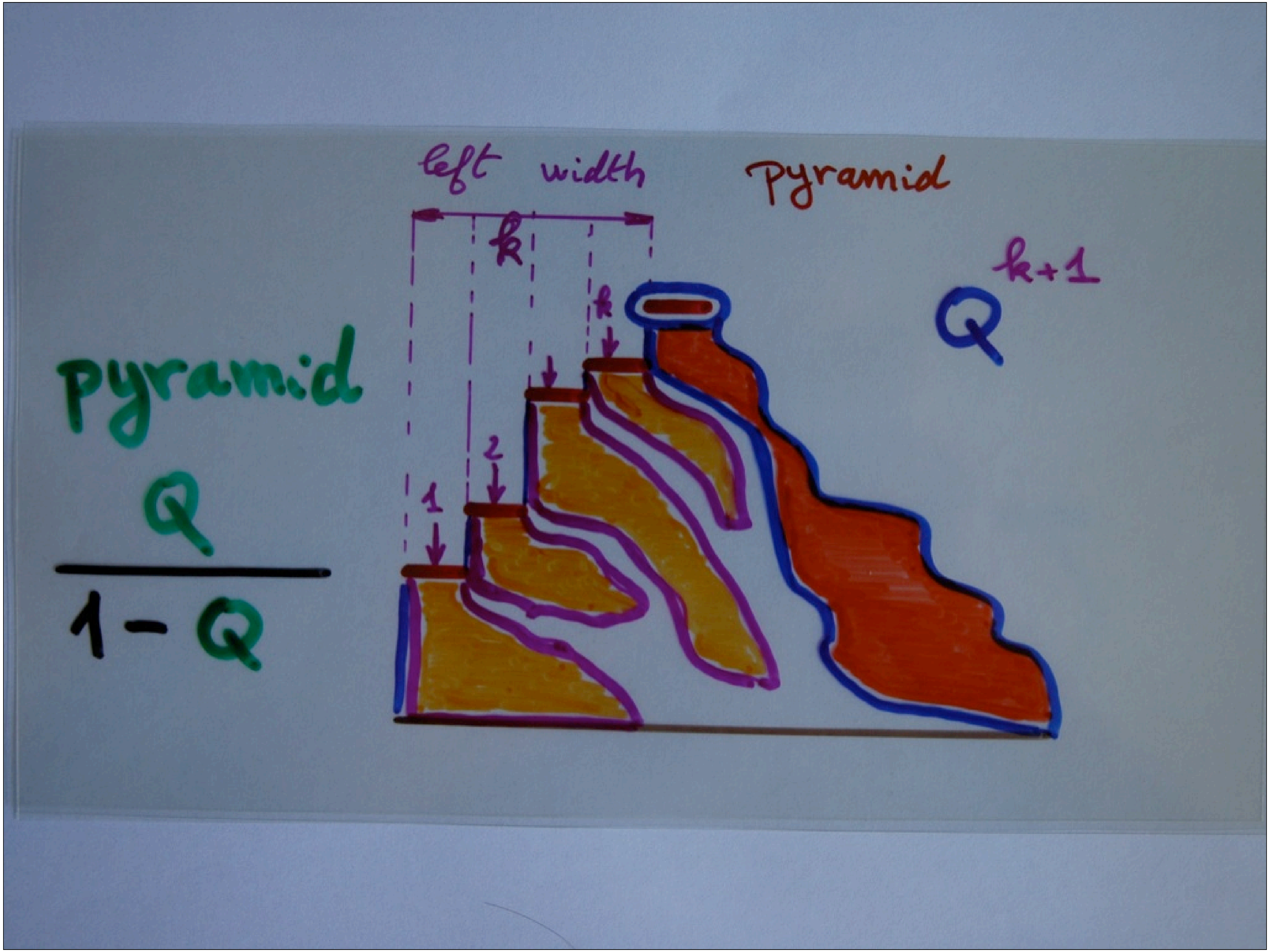
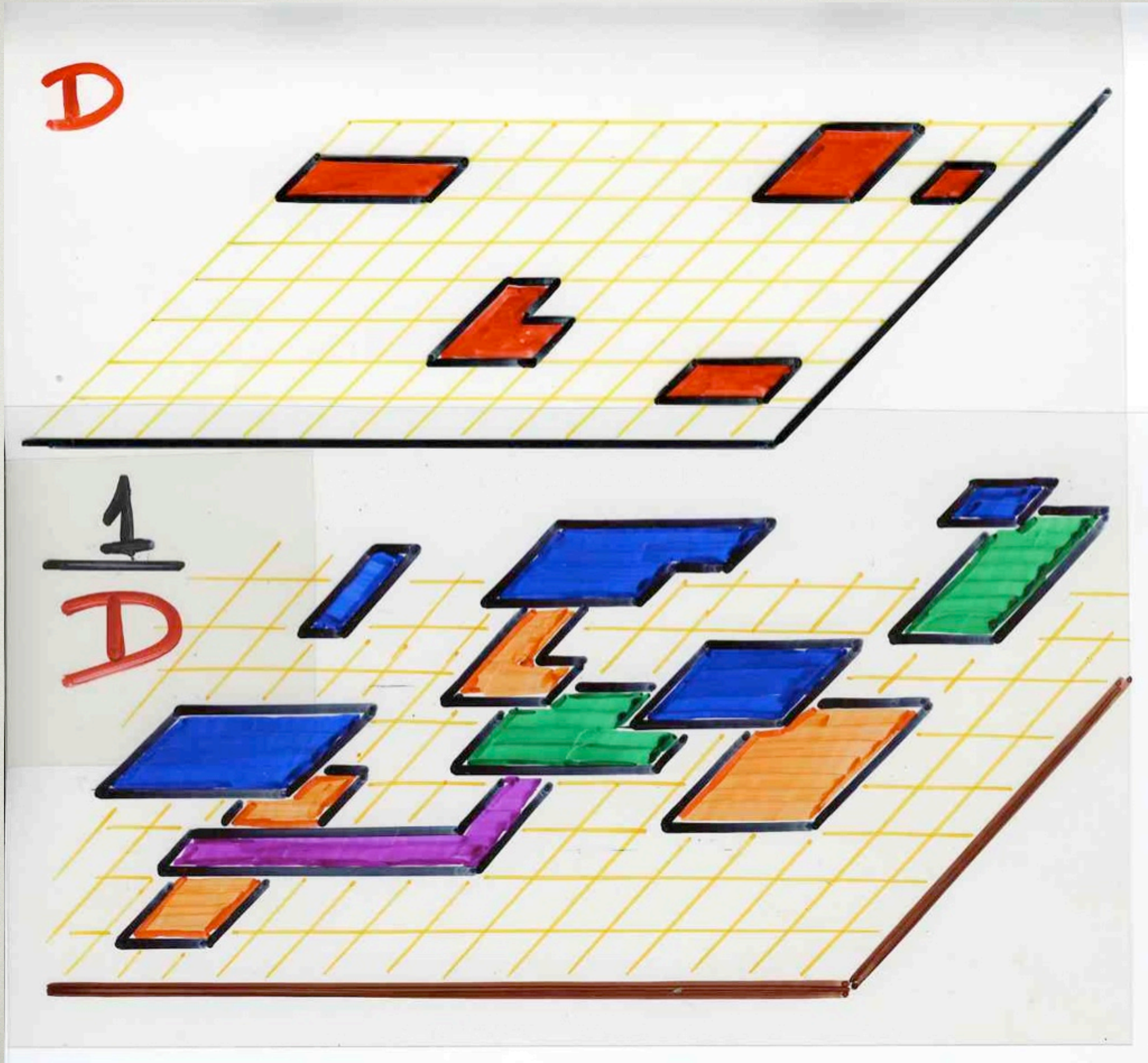
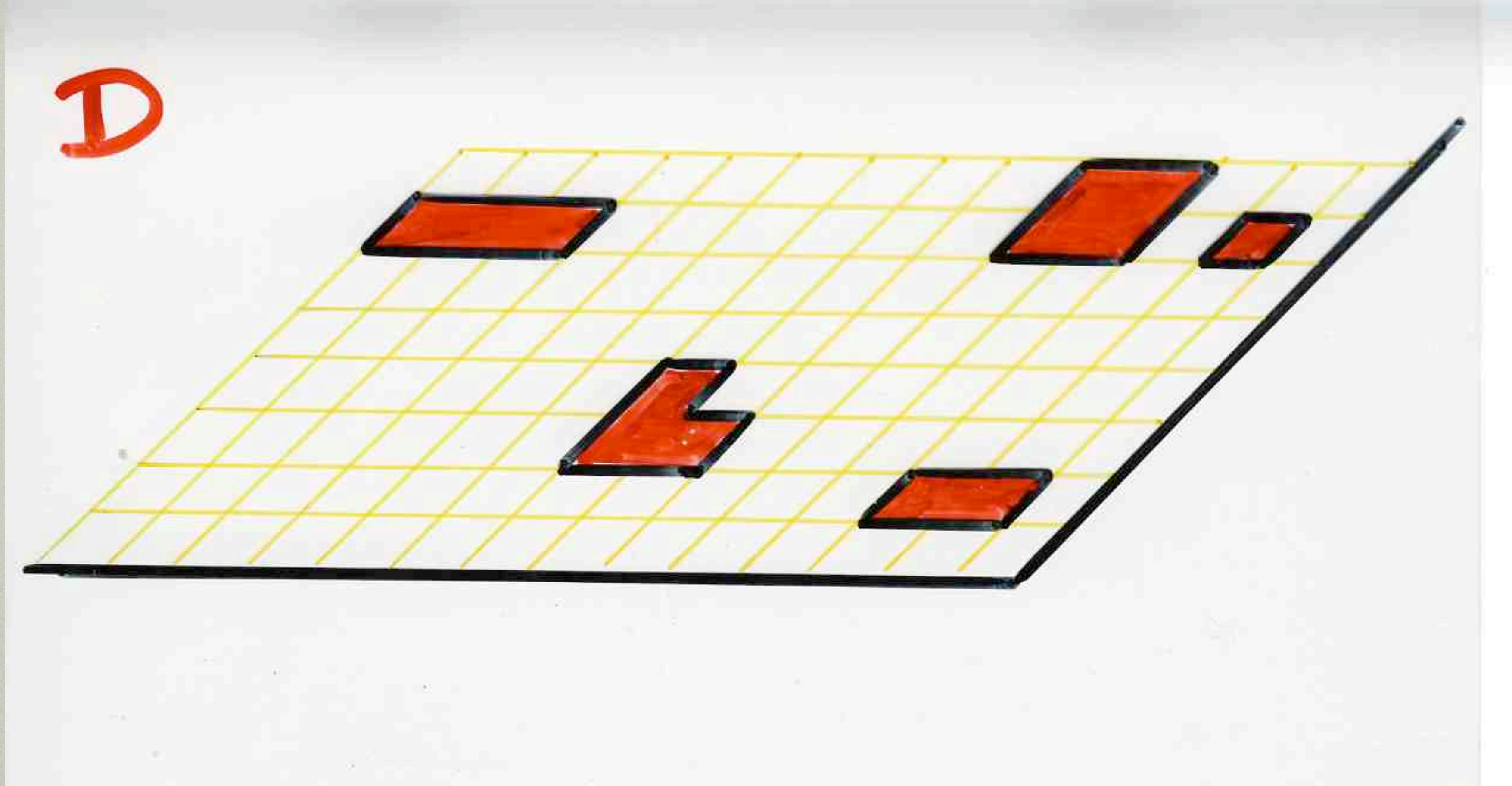

left width
k
Pyramid
Q
k+1
1
2
k



left width
Pyramid
k
1
2
k
Q^{k+1}
pyramid
$\frac{Q}{1-Q}$


D
1
D


D

$$\sin(n+1)\theta = (\sin\theta)\, U_n(\cos\theta)$$

$U_n(t/2)$ reciprocal

$F_n(t^2)$

Fibonacci polynomial

$$F_{n+1}(t) = F_n(t) - t\, F_{n-1}(t) \qquad F_0 = F_1 = 1$$

alternating generating polynomial for trivial heaps of dimers on the graph segment ~~length~~ n points

(matching polynomial)

$F_3(t) = 1 - 2t$

Fibonacci polynomials

$$F_0 = F_1 = 1$$

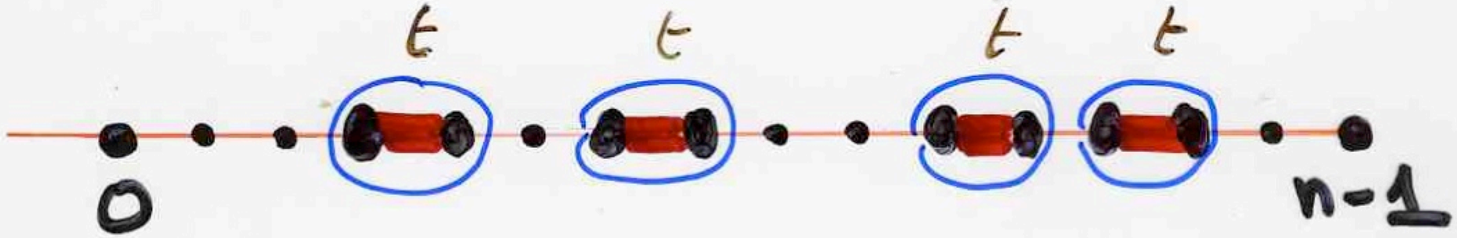
$$F_n = F_{n-1} - t F_{n-2}$$

1

$1 - t$

$1 - 2t$

$1 - 3t + t^2$



trivial heap of dimers

g.f heaps of dimers on $[0, n]$

$$= \frac{1}{F_{n+1}(t)}$$

Inversion lemma

Cartier, Foata 1968

X.V. 1985

$$C = \frac{Q}{1-Q} + C \sum_{k \geq 1} \frac{Q}{1-Q} \times Q^{k} \times \frac{1}{F_{k-1}}$$

$$C = \frac{Q}{1-Q} \times \frac{1}{\left[1-\left(\sum_{k\geqslant 1} \frac{Q}{1-Q} \times Q^{k} \times \frac{1}{F_{k-1}}\right)\right]}$$

connected heap

$$F_n = \frac{(1 - Q^{n+1})}{(1-Q)(1+Q)^n}$$

$$\underbrace{(1+Q)^n}_{D^n} = \frac{1}{F_n} \times (1 + Q + \ldots + Q^n)$$

$$C = \frac{Q}{(1-Q)\left[1-\sum_{k\geq 1}\frac{Q^{k+1}(1+Q)^{-k-1}}{1-Q^{k}}\right]}$$

$$C = \frac{Q}{(1-Q)\left[1-\boxed{\sum_{k\geq 1}\frac{Q^{k+1}(1+Q)^{-k-1}}{1-Q^{k}}}\right]}$$

$$= \sum_{k\geq 1}\frac{Q^{k+1}}{1-Q^{k}(1+Q)}$$

$C(t)$ g.f. connected heap

(Bousquet-Mélou, Rechnitzer) 2002

$$C(t) = \frac{Q}{(1-Q)\left[1 - \sum_{k\geq 1} \frac{Q^{k+1}}{1-Q^{k}(1+Q)}\right]}$$

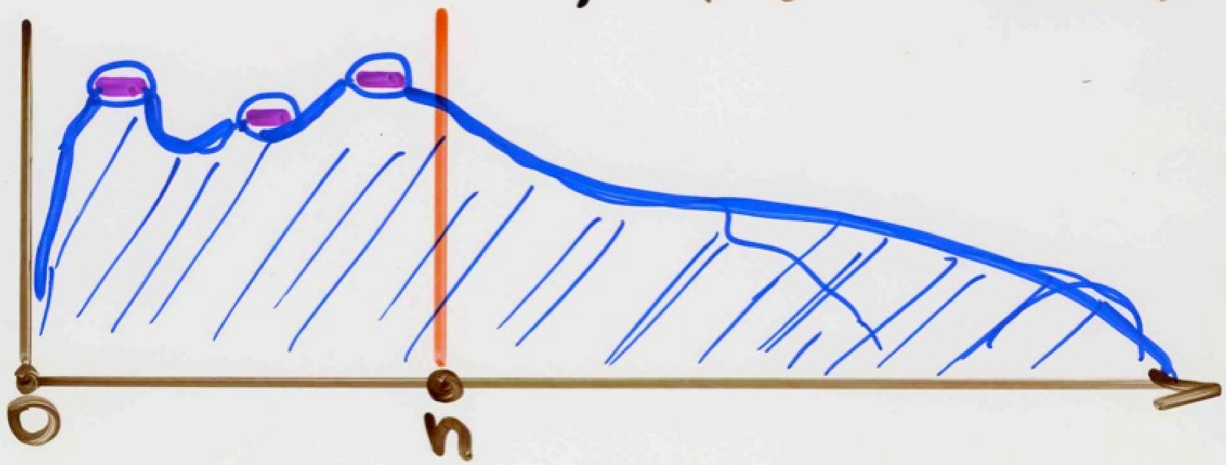
$$F_n = \frac{(1-Q^{n+1})}{(1-Q)(1+Q)^n}$$

$$\underbrace{(1+Q)^n}_{D^n} = \frac{1}{F_n} \times (1+Q+\ldots\ldots+Q^n)$$

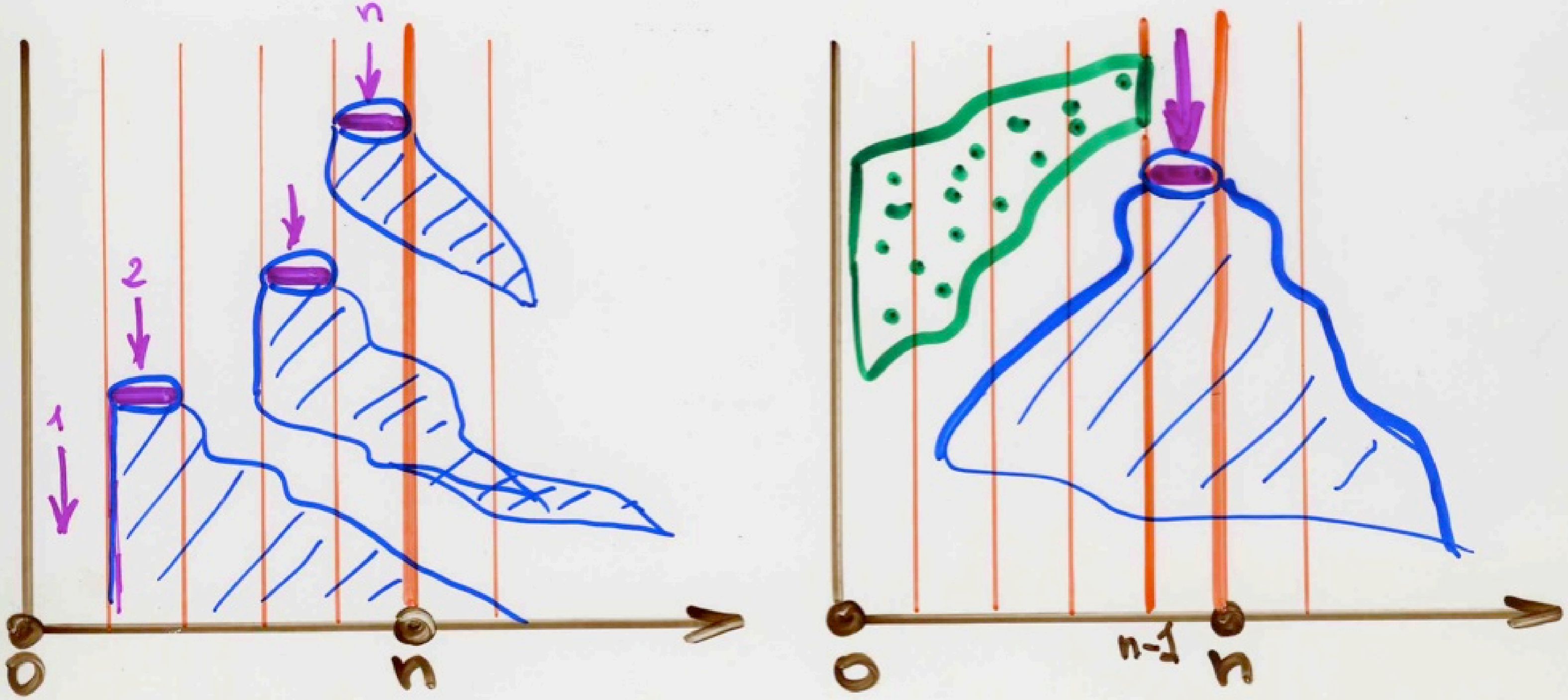
$$\underbrace{(1+Q)^n}_{D^n} = \frac{1}{F_n} \times (1+Q+\dots+Q^n)$$

heaps of dimers on $[0,\infty[$

maximal pieces, projection $\subseteq [0,n]$



$$\underbrace{(1+Q)^n}_{D^n} = \frac{1}{F_n} \times (1+Q+\dots+Q^n)$$



Lorentzian
quantum
gravity

$\underbrace{+++}_{\text{space}}\underbrace{-}_{\text{time}}$

2D

$\underset{\text{space}}{1} + \underset{\text{time}}{1}$

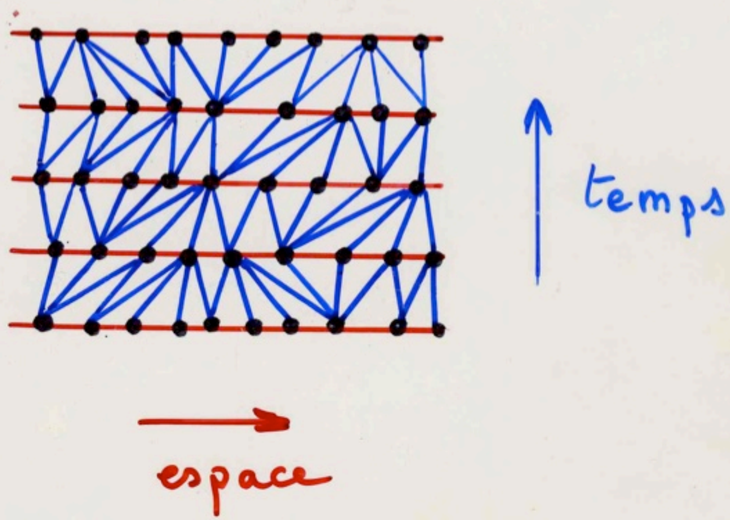
- J. Ambjørn, R. Loll, "Non-perturbative Lorentzian quantum gravity and topology change", Nucl. Phys. B536 (1998) 407-436
  arXiv: hep-th/9805108

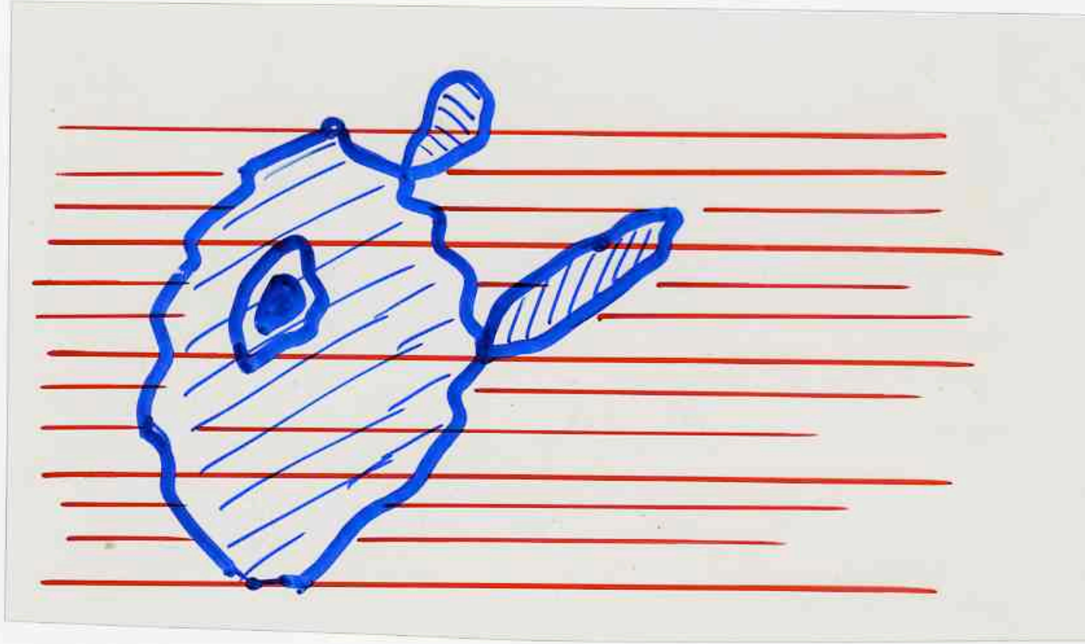
- P. Di Francesco, E. Guitter, C. Kristjansen, Integrable 2D Lorentzian gravity and random walks", Nucl. Phys. B 567 (2000) 515-553
  arXiv: hep-th/9907084

gravitation quantique

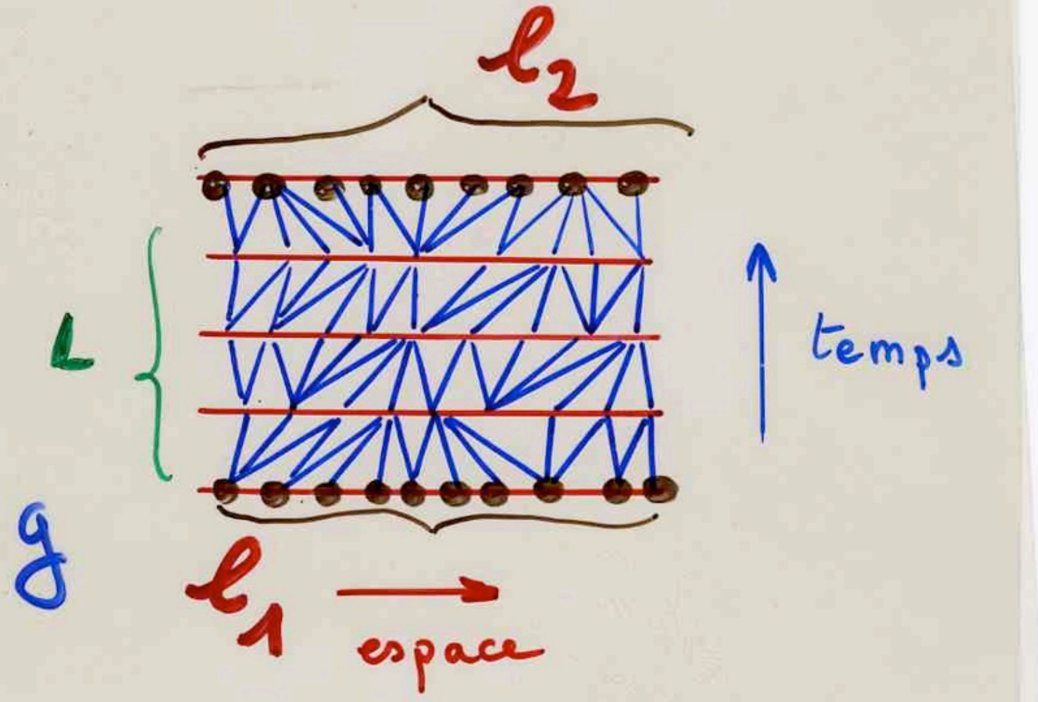
Lorentzian

triangulation

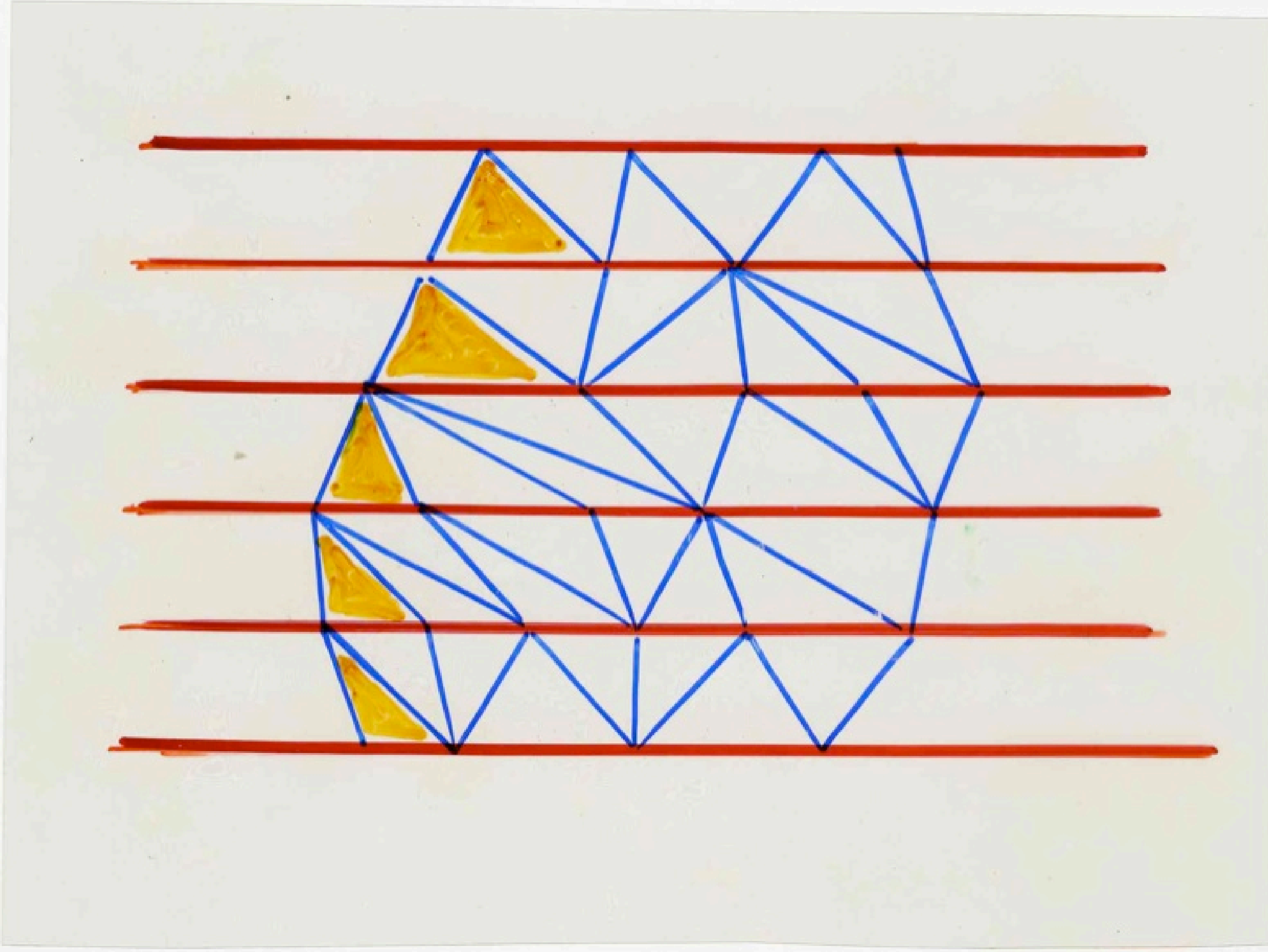


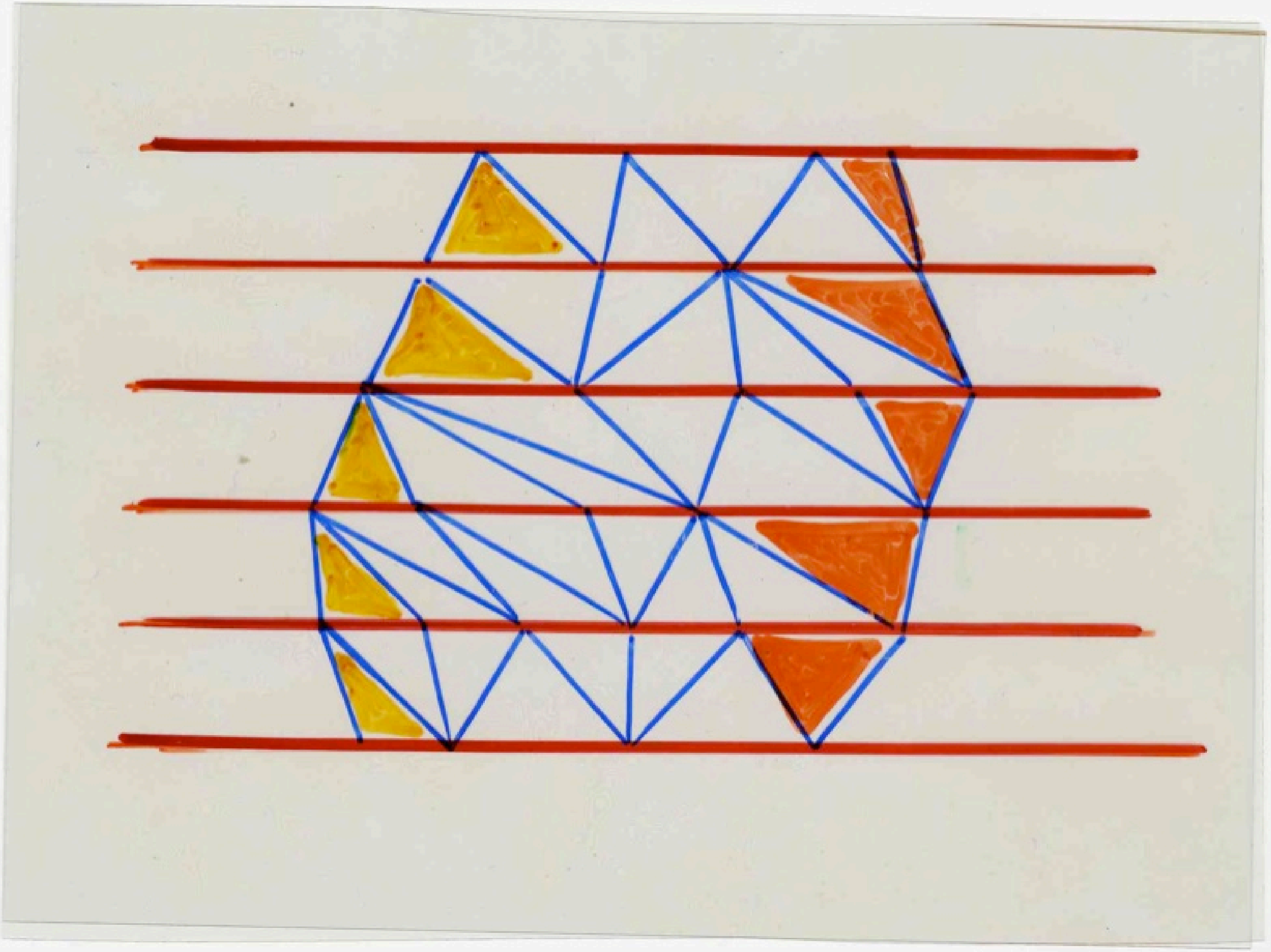


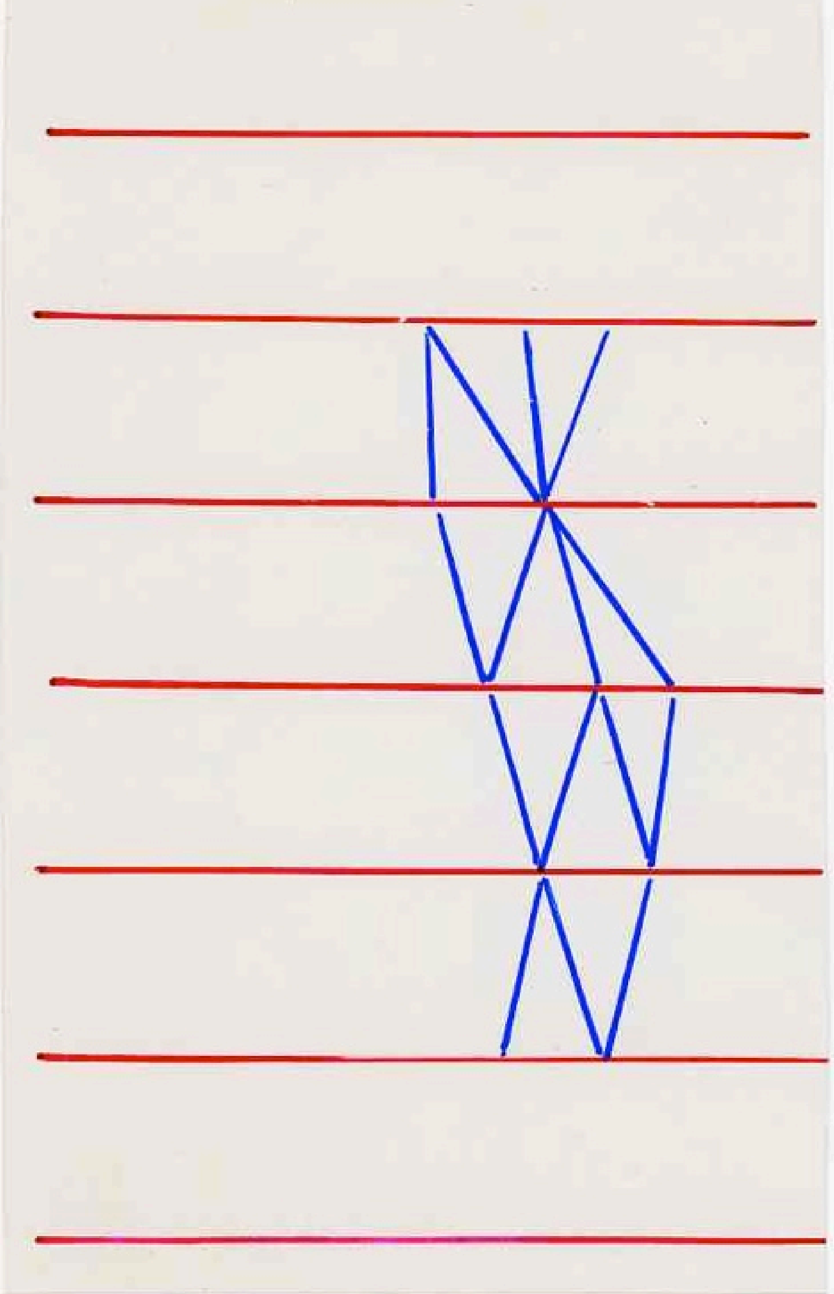
bébés
univers

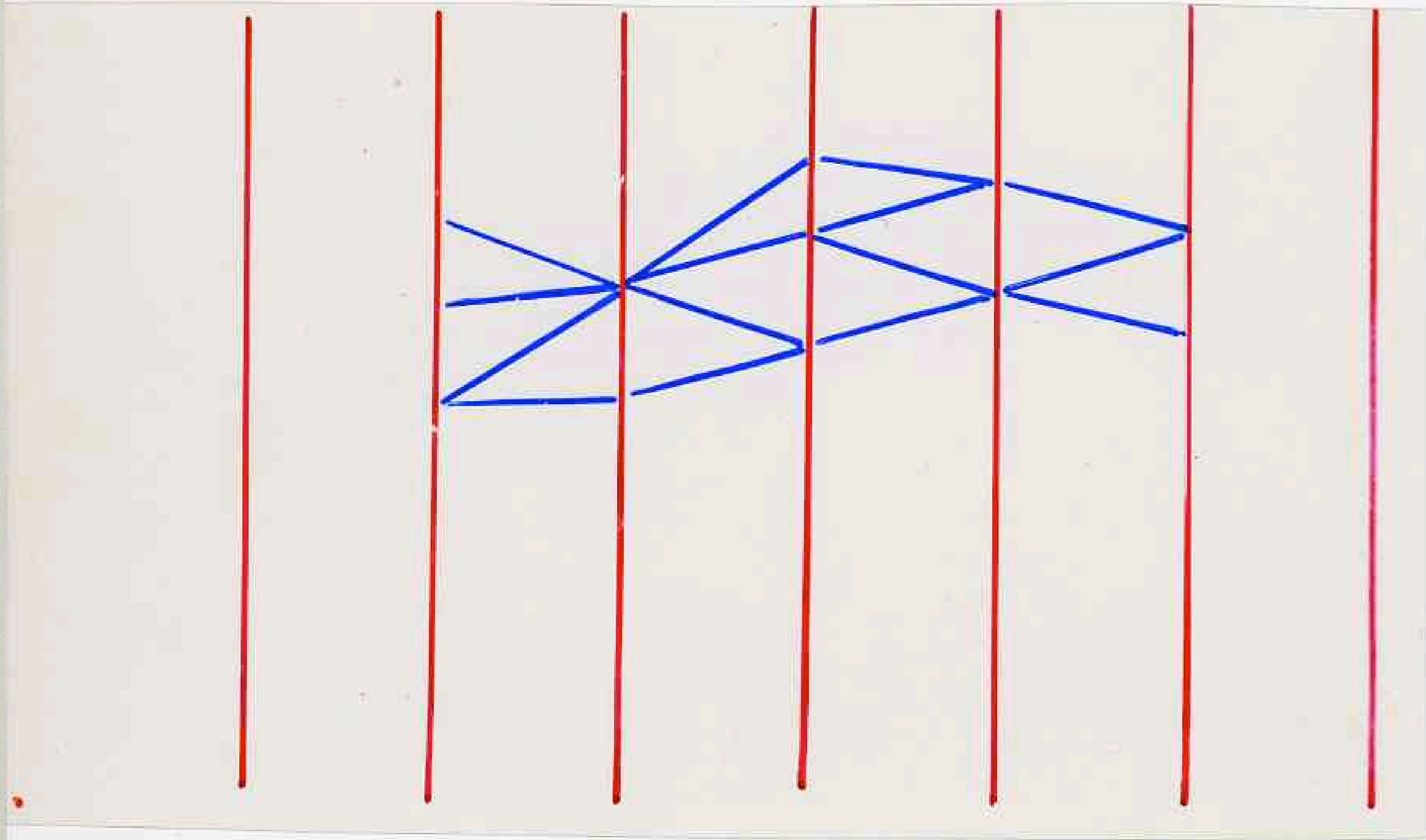


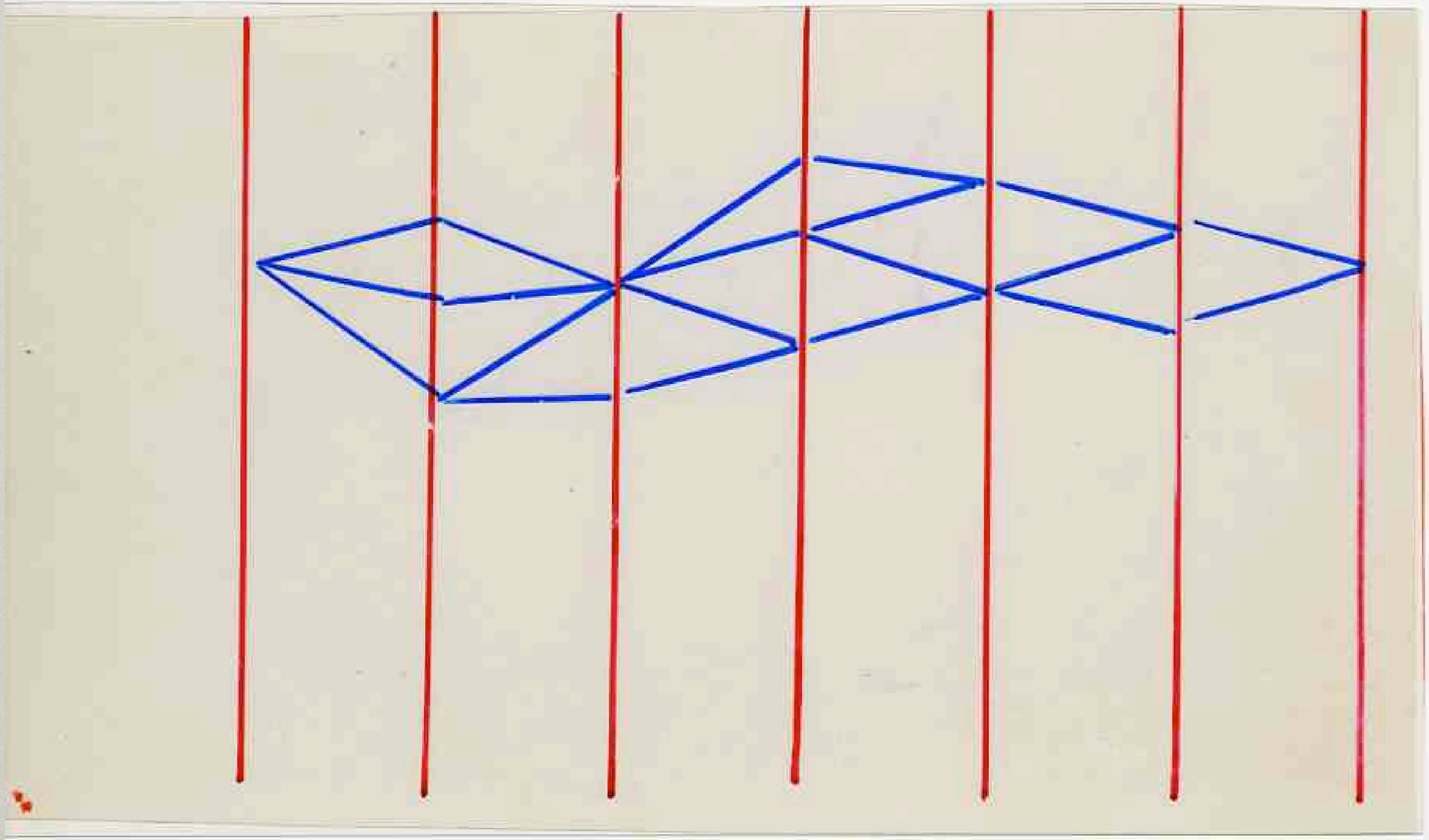
Path integral amplitude
for the propagation from
geometry ℓ_1 to ℓ_2

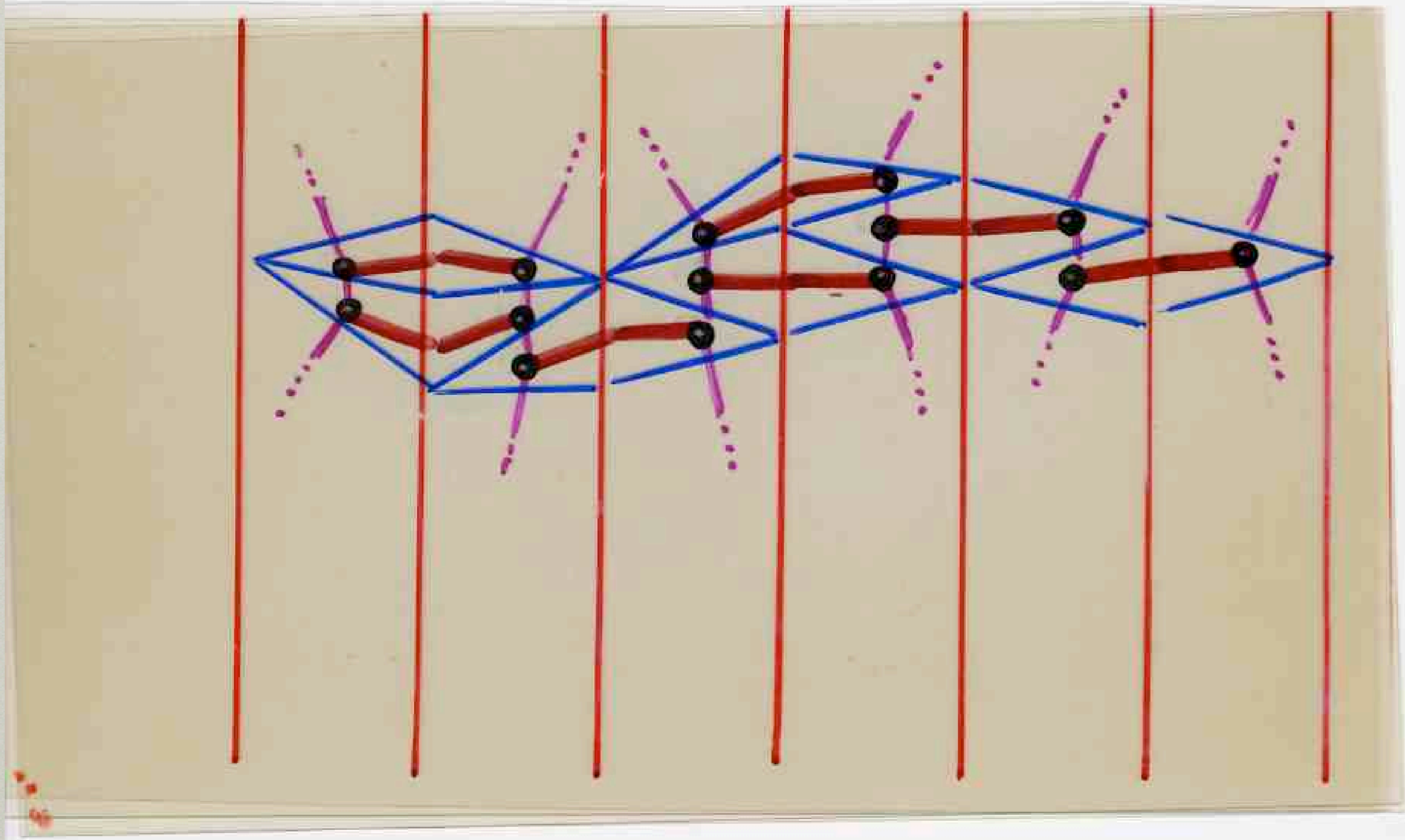


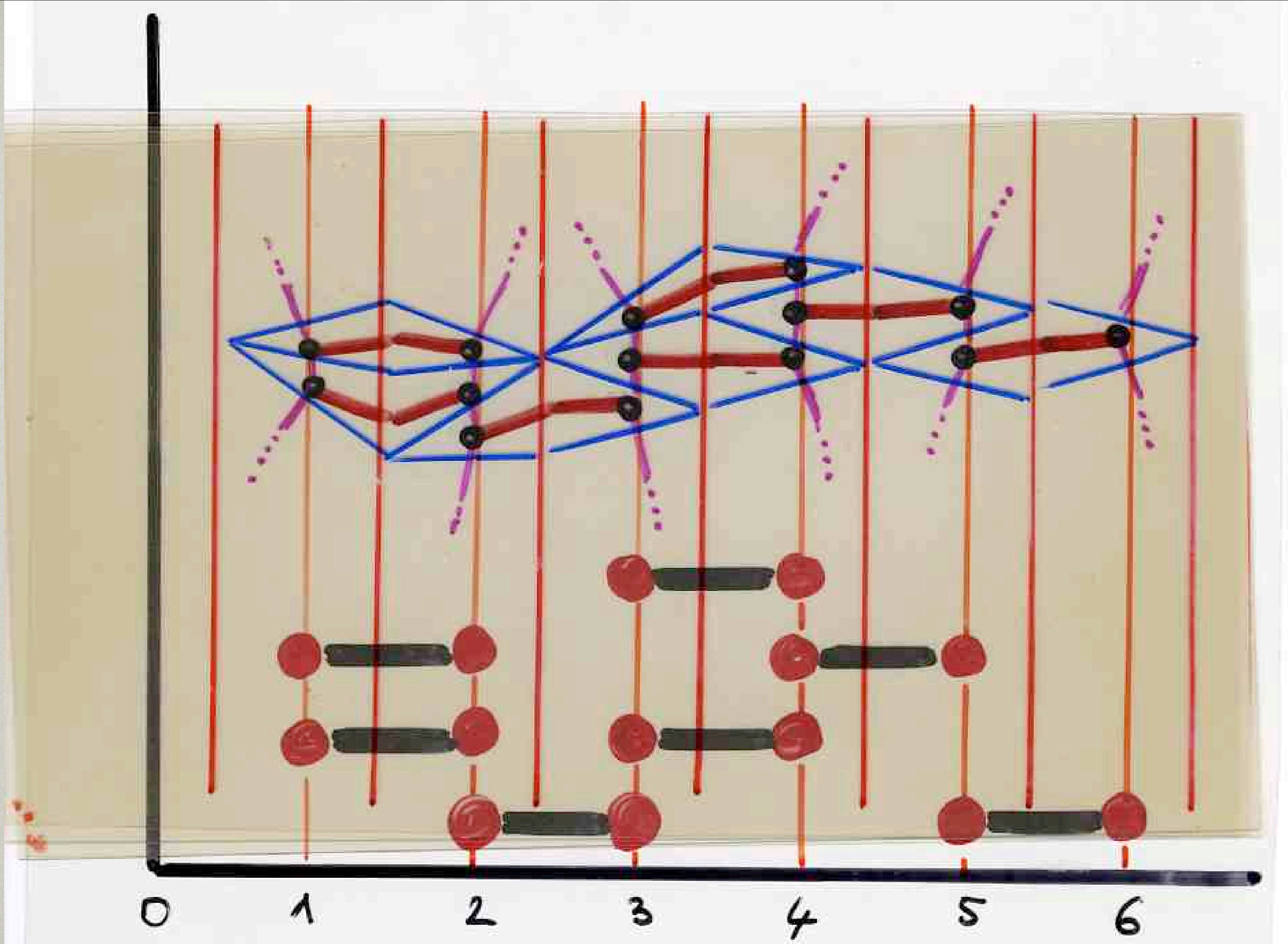








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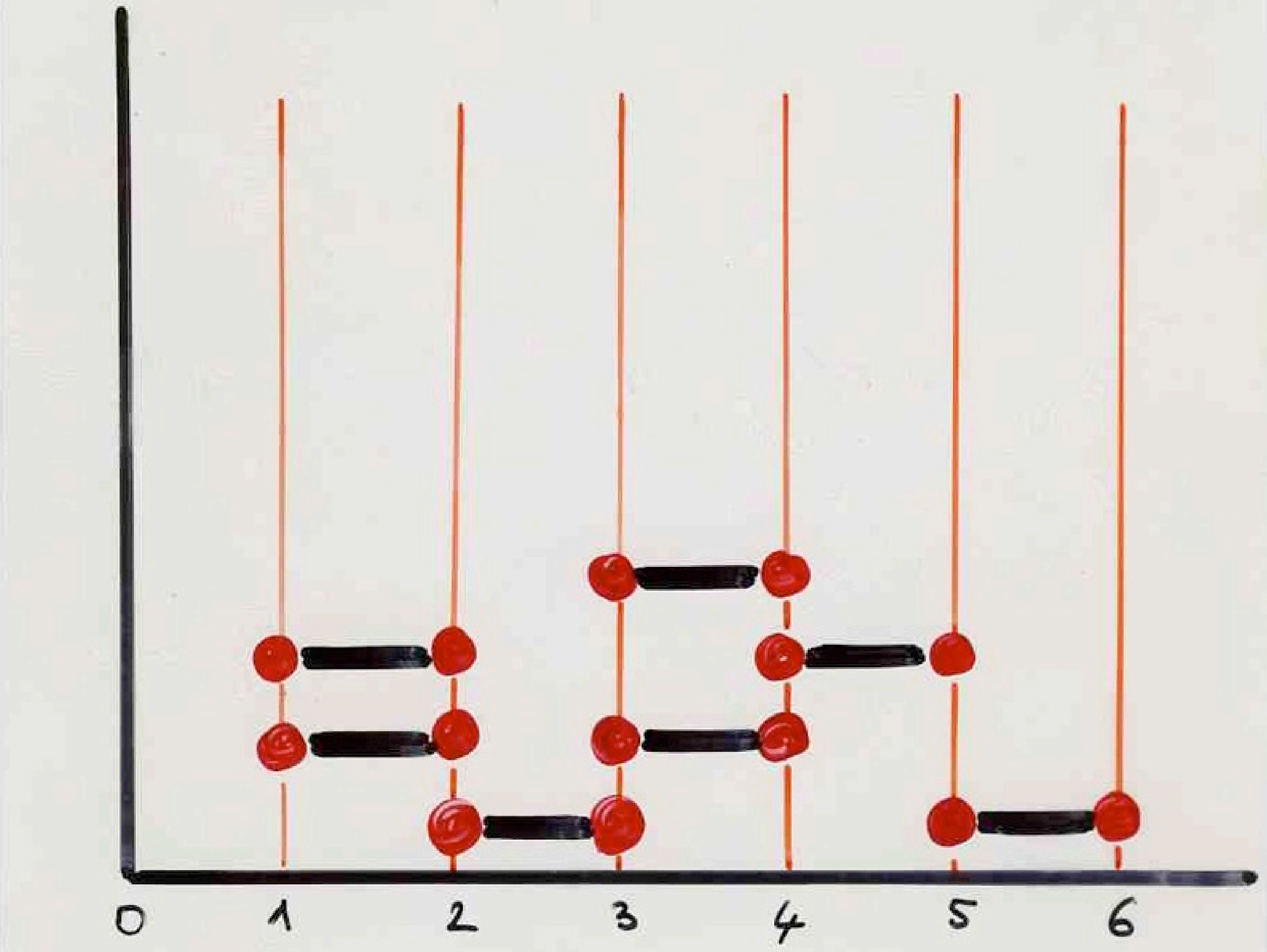
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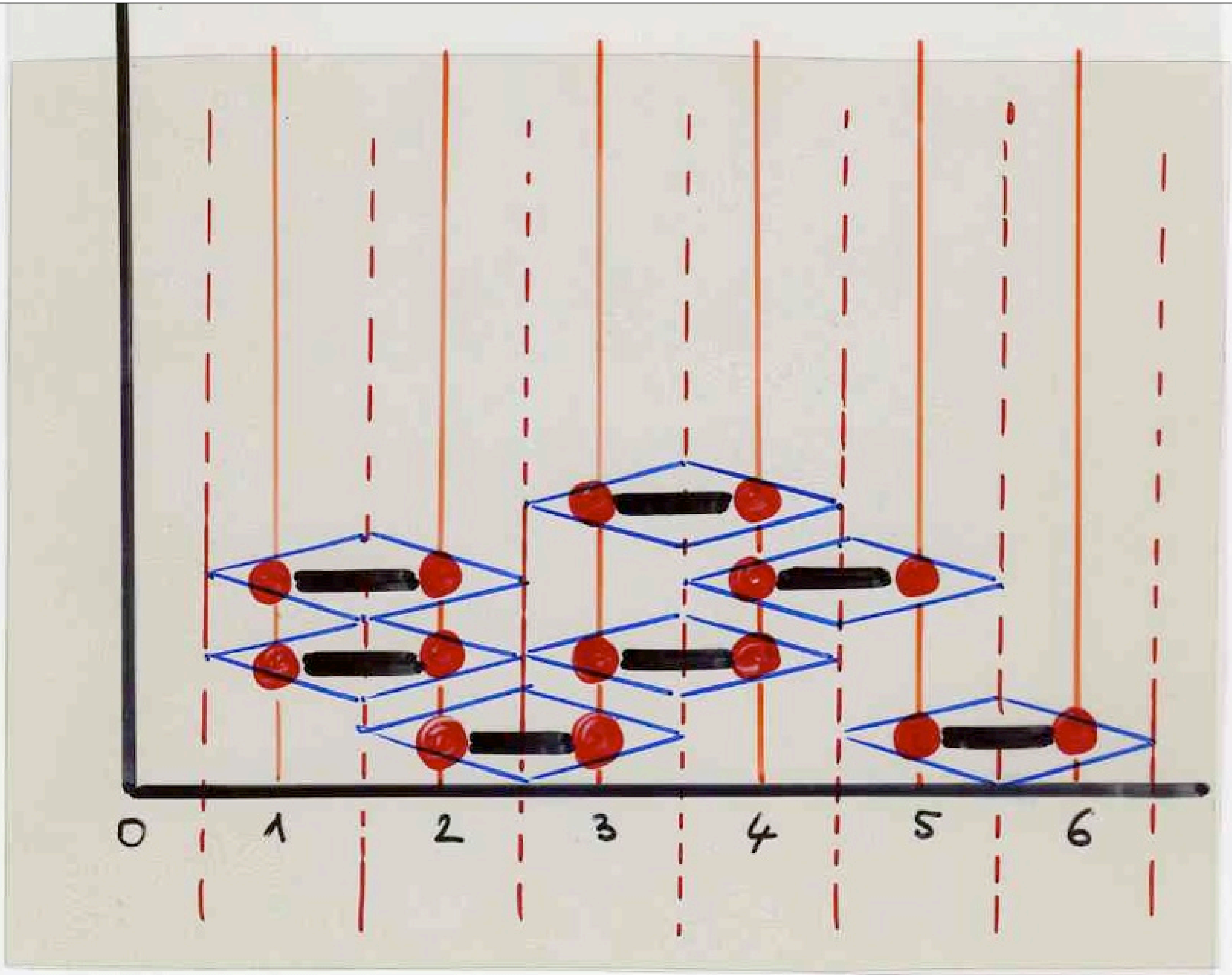
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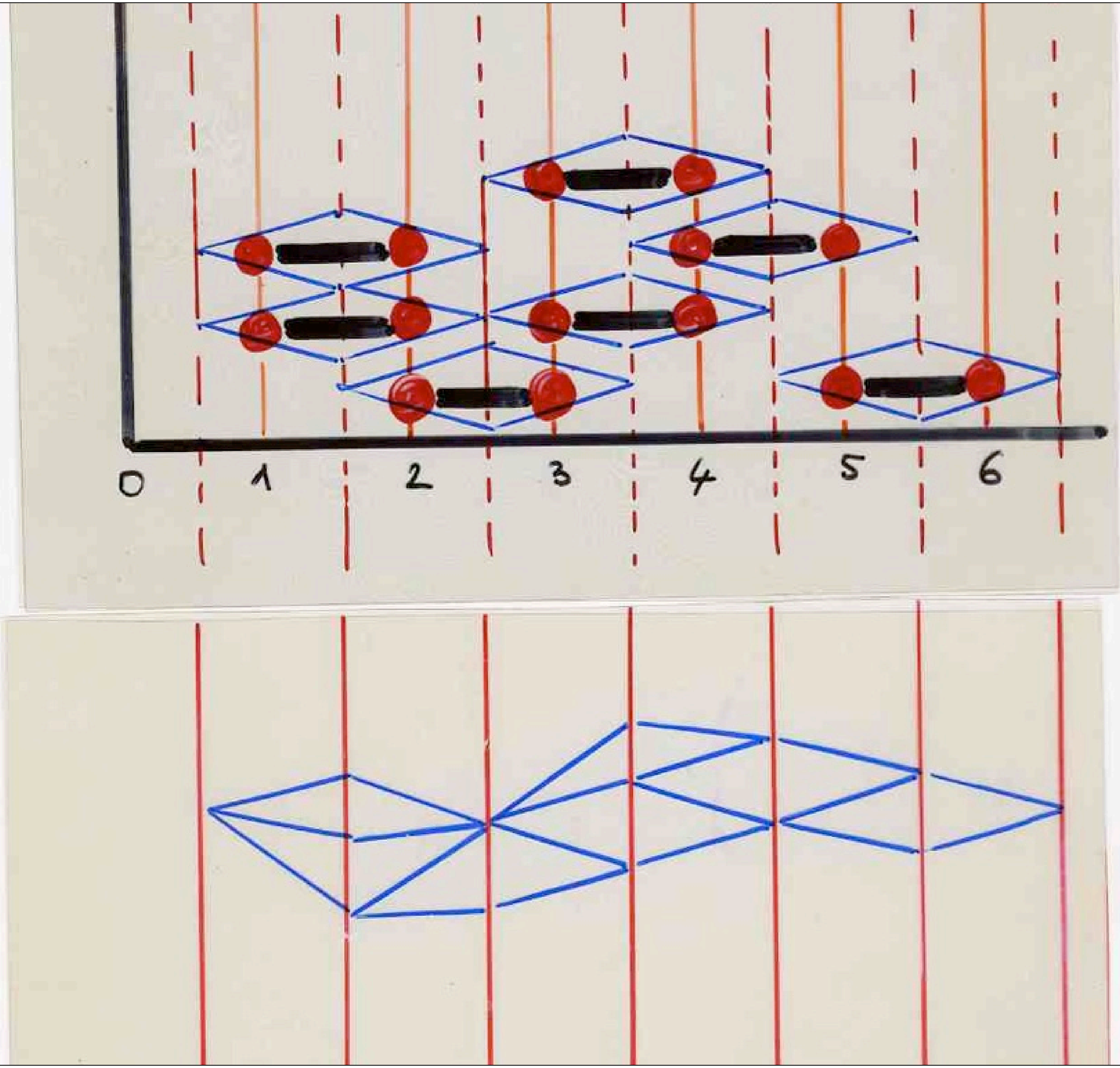
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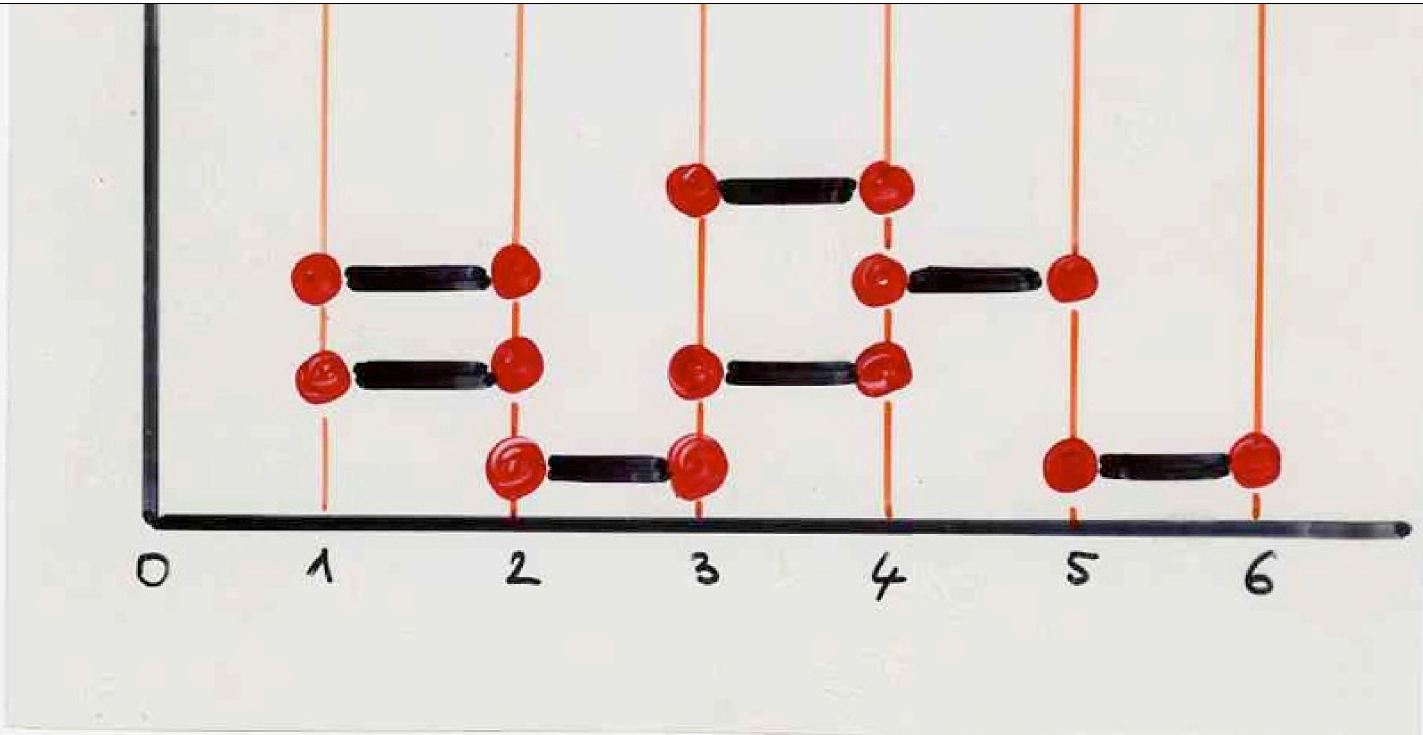
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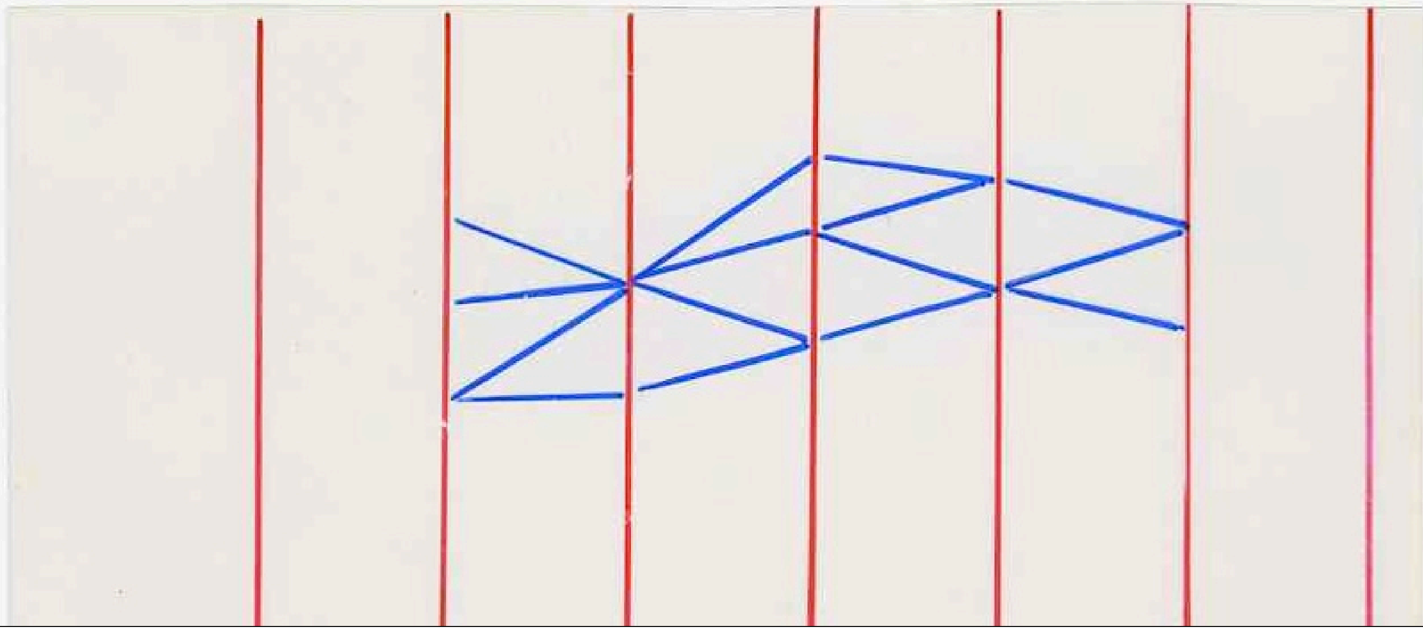

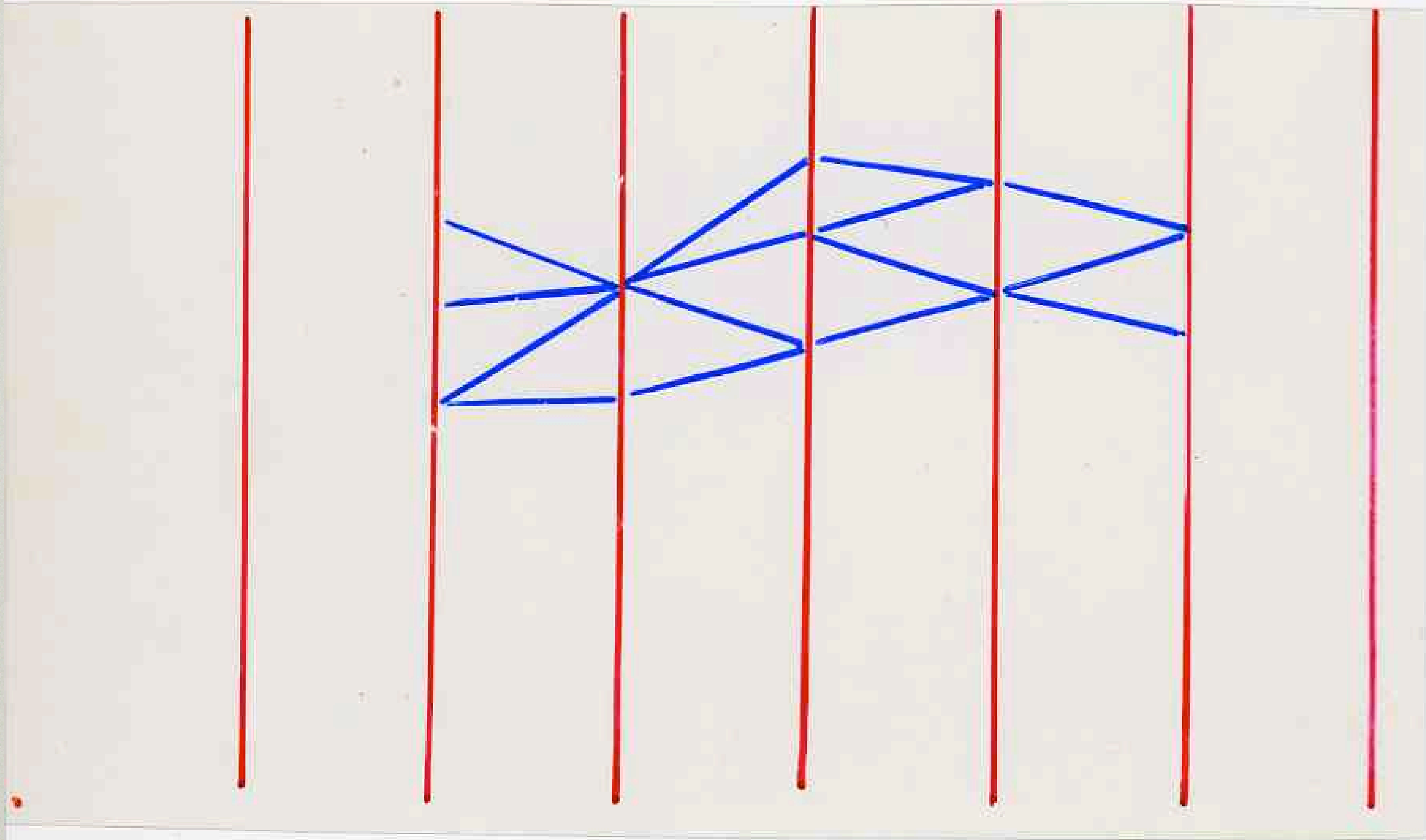

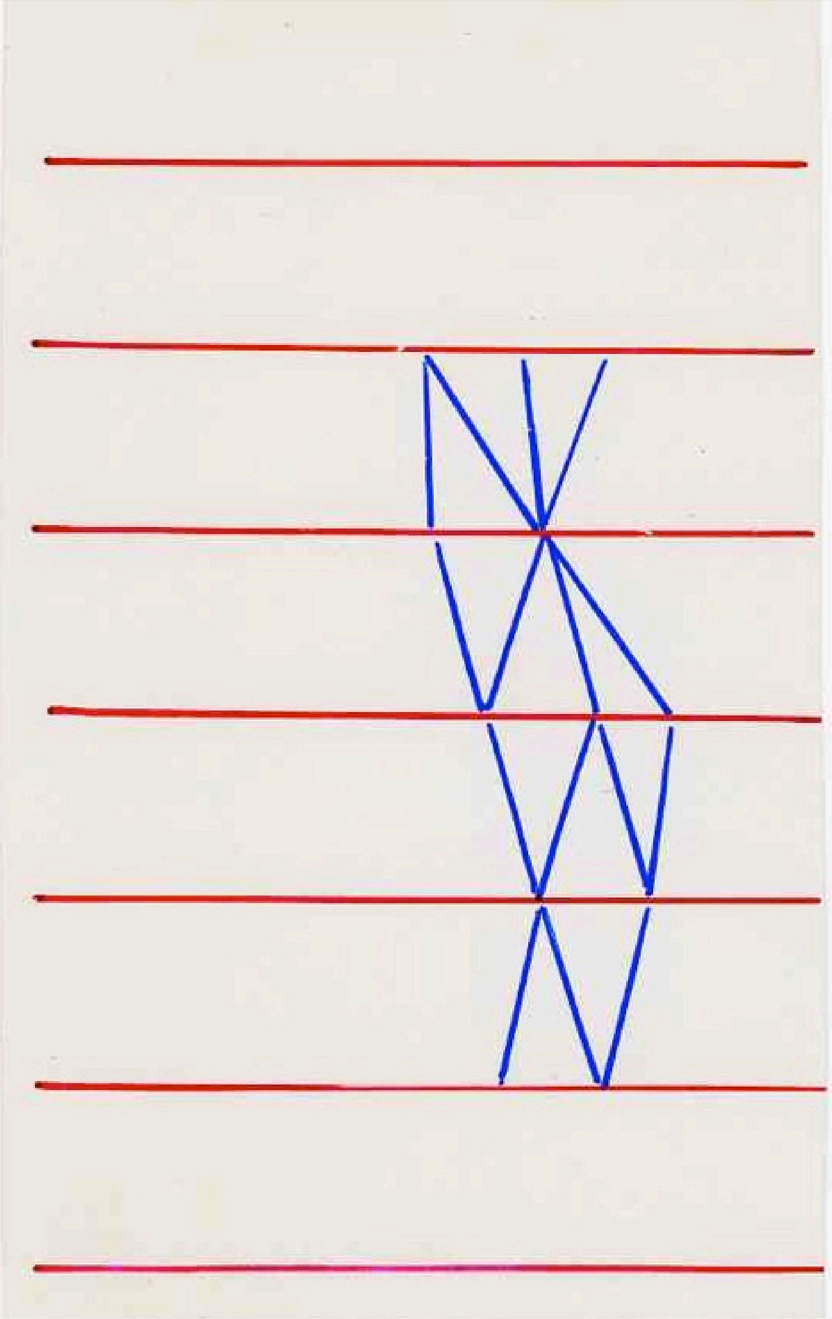

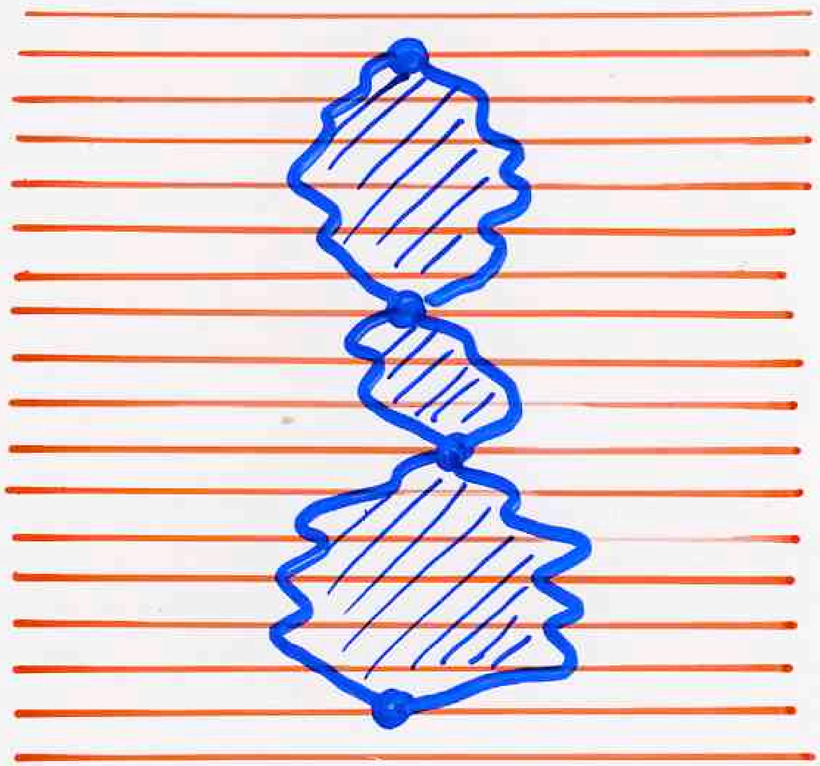

(general)
Lorentzian
triangulation

Lorentzian triangulation with no articulation points

$\Longleftrightarrow$

connected heap of dimers

merci beaucoup
à vous tous