Kepler towers
Catalan numbers
and
Strahler distribution

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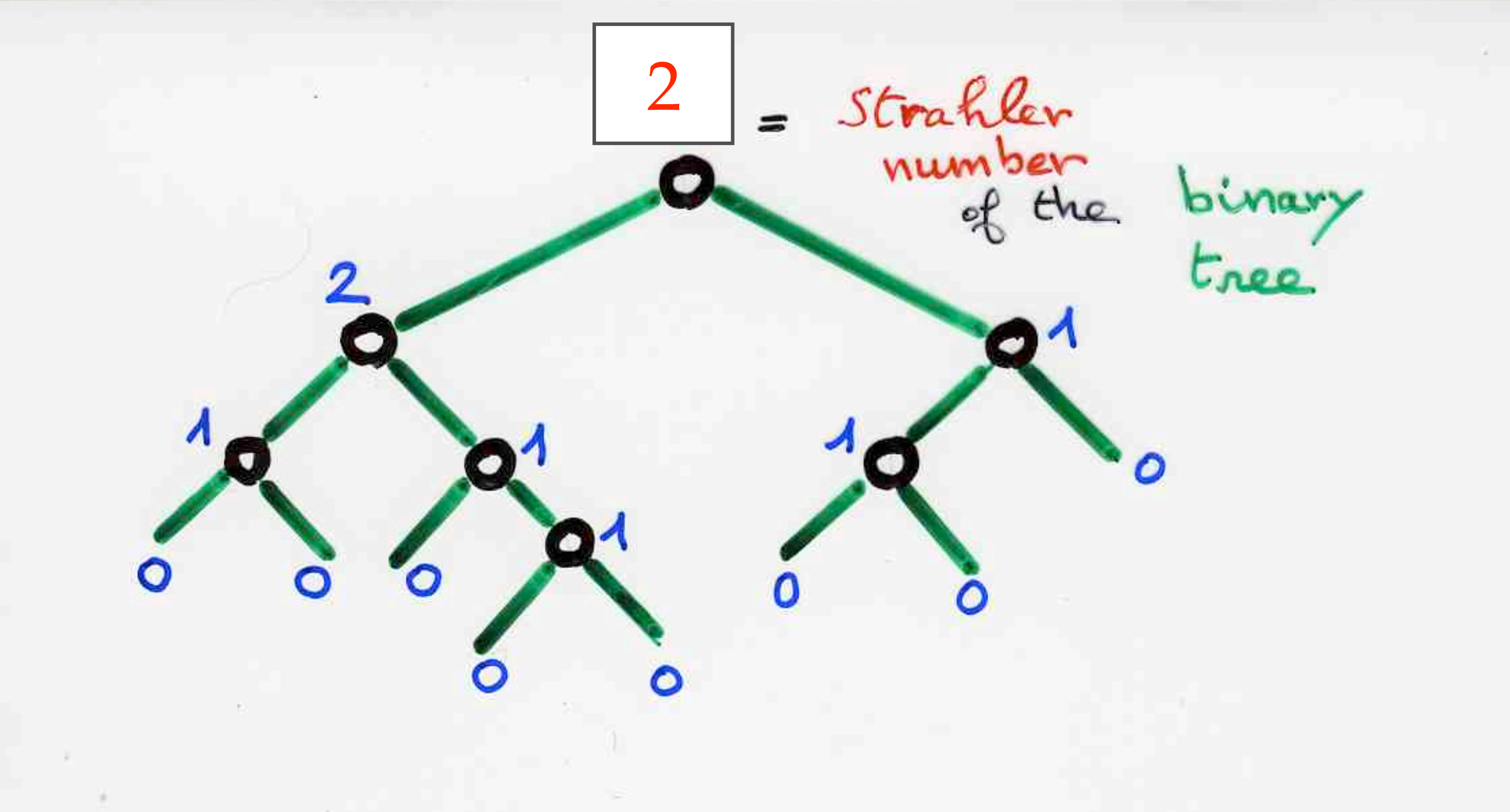
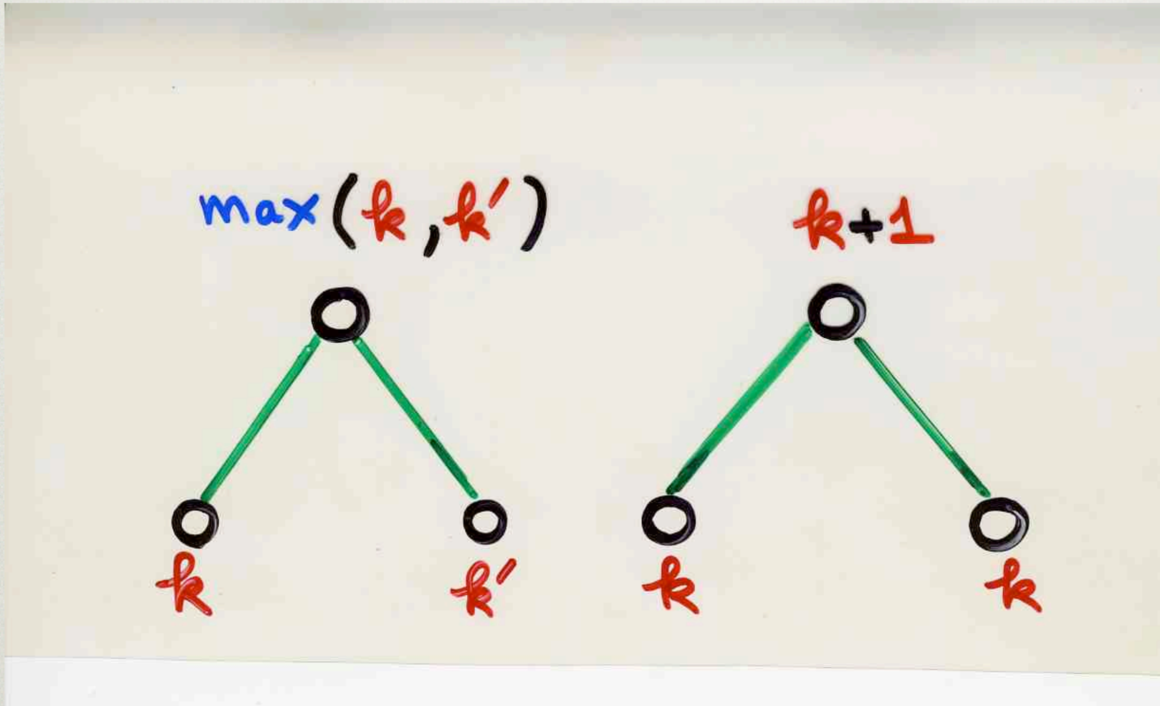
dedicated to my friend
Adriano Garsia
for his 75^{th} birthday

1. Introduction
Strahler



2
= Strahler number of the binary tree
2
1
1
1
1
1
0
0
0
0
0
0
0
0


max(k,k′)
k+1
k
k′
k
k

Horton (1945)

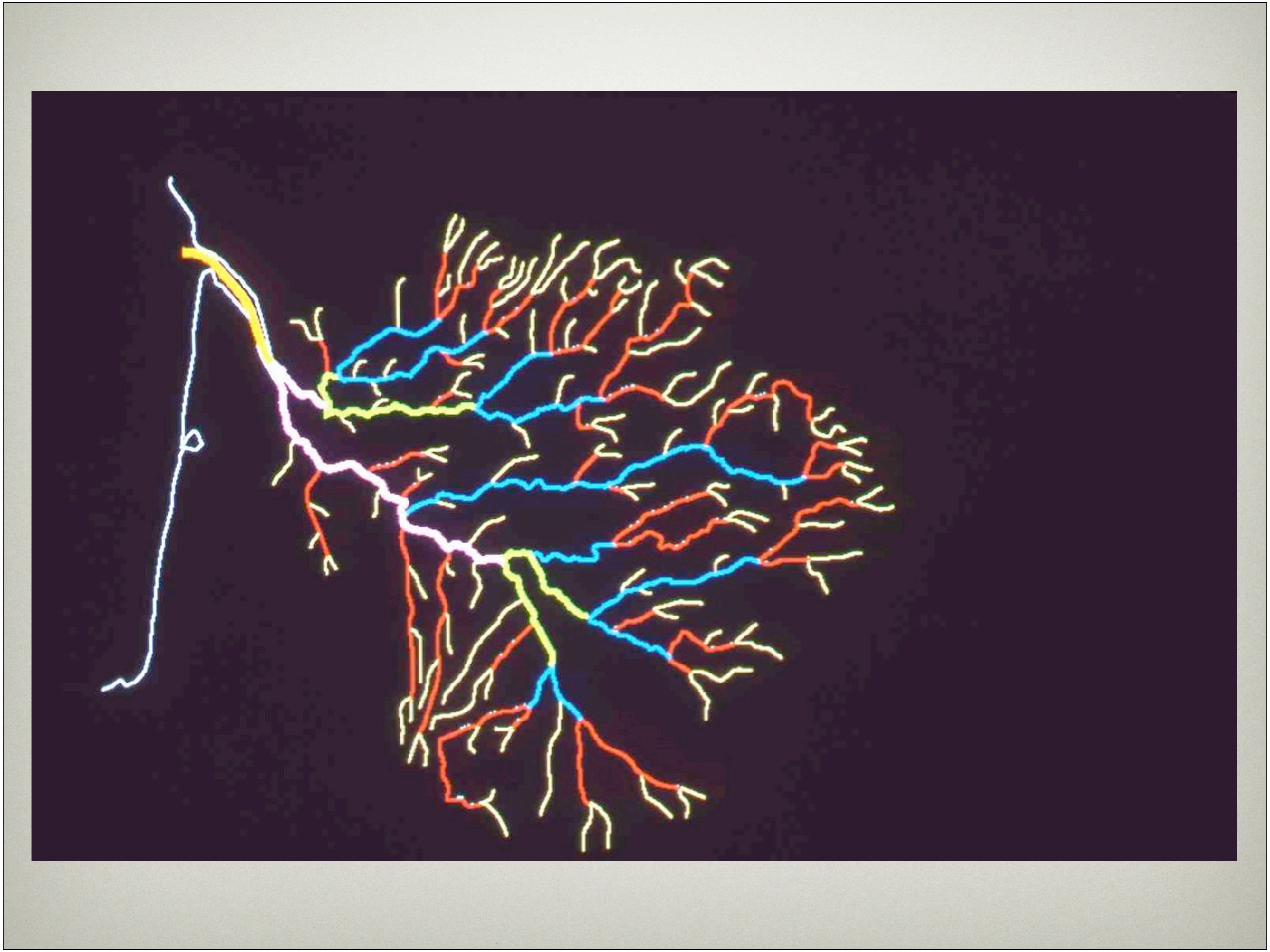
Strahler (1952)

Hydrogeology

morphology of rivers

of network

Order of a river

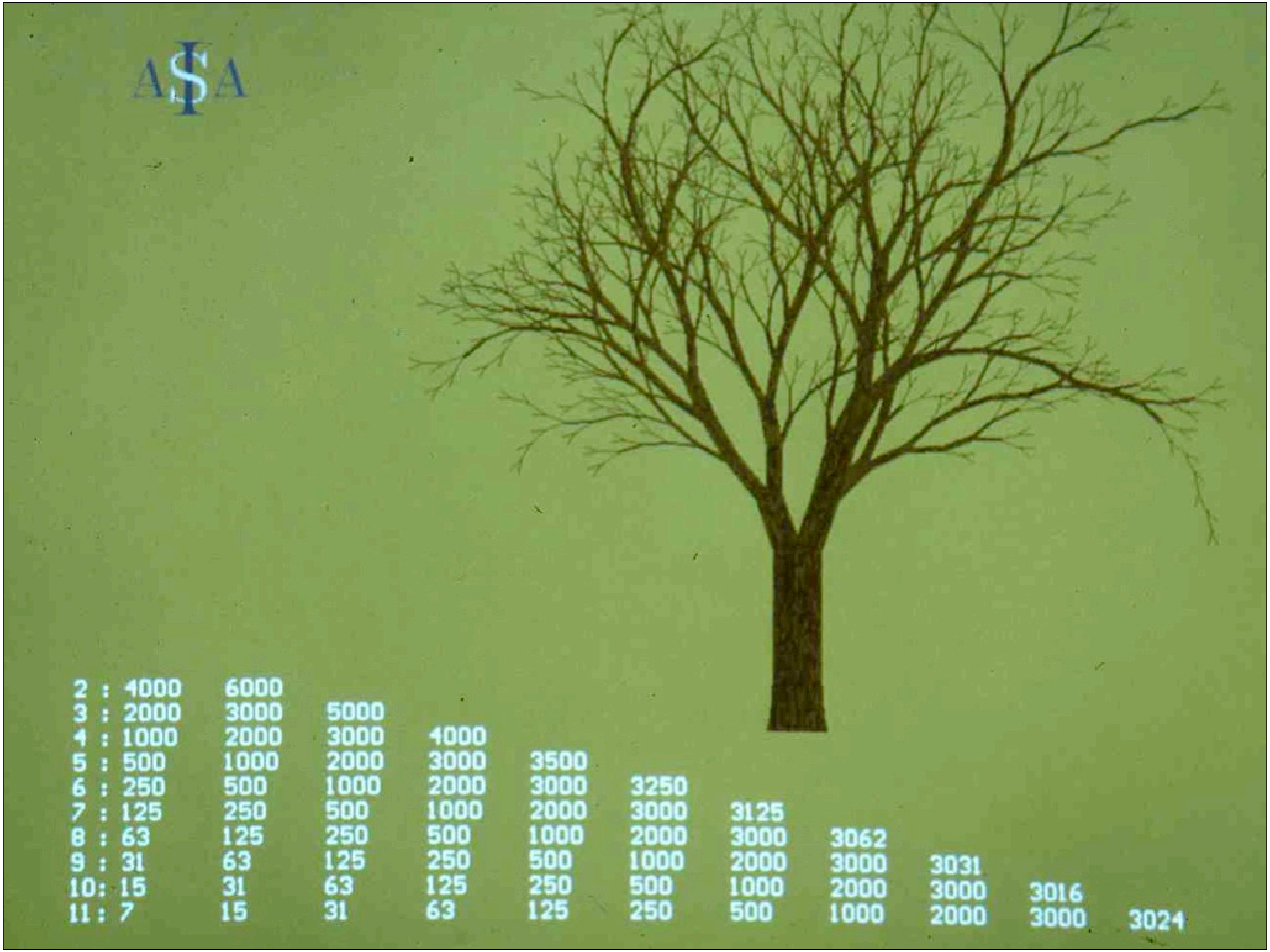


Synthetic images of

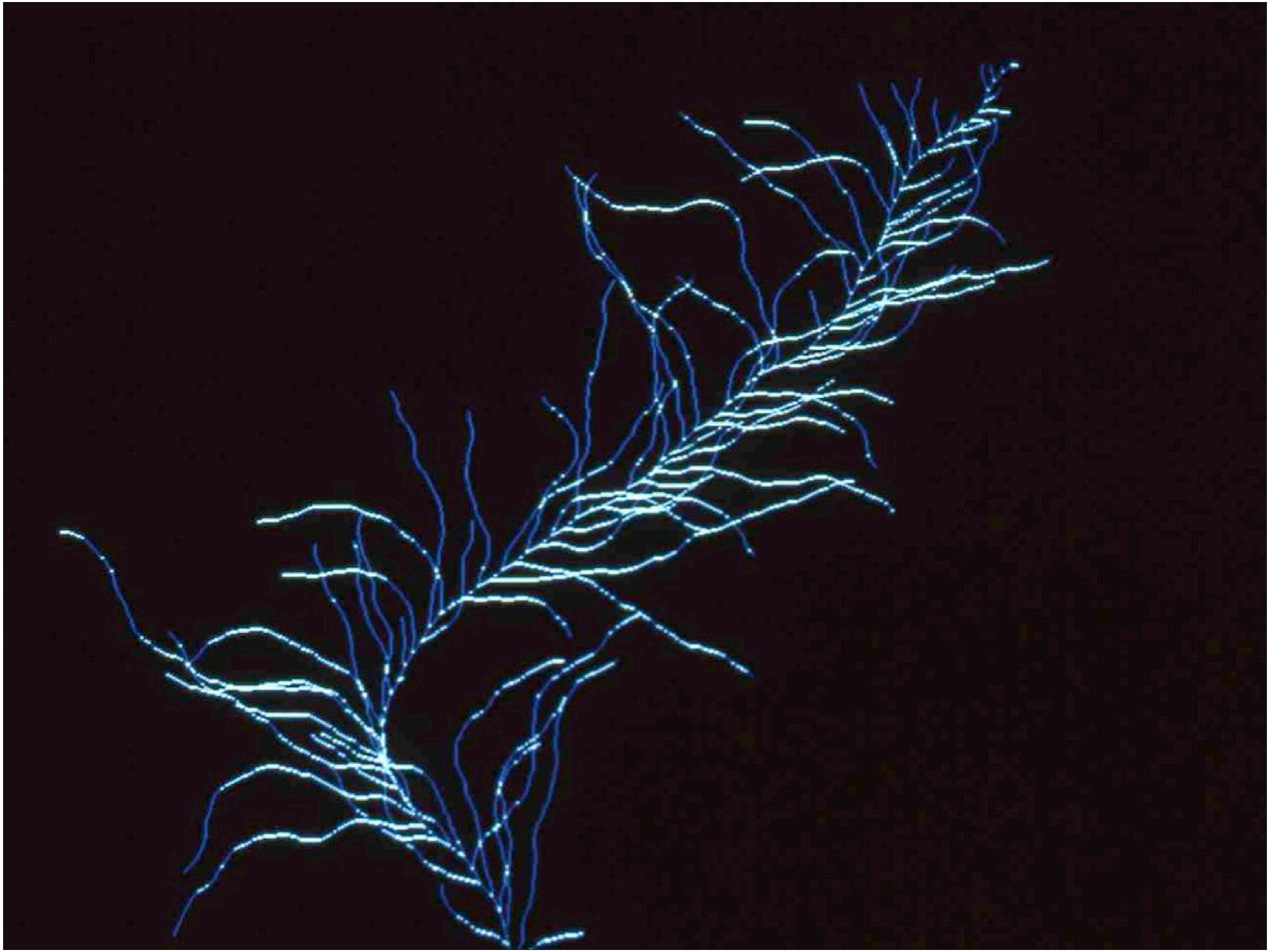
trees, leaves, landscapes ...

Arquès, Eyrolles, Janey, X.V.

A$\mathfrak{I}$A


2 : 4000 6000
3 : 2000 3000 5000
4 : 1000 2000 3000 4000
5 : 500 1000 2000 3000 3500
6 : 250 500 1000 2000 3000 3250
7 : 125 250 500 1000 2000 3000 3125
8 : 63 125 250 500 1000 2000 3000 3062
9 : 31 63 125 250 500 1000 2000 3000 3031
10: 15 31 63 125 250 500 1000 2000 3000 3016
11: 7 15 31 63 125 250 500 1000 2000 3000 3024

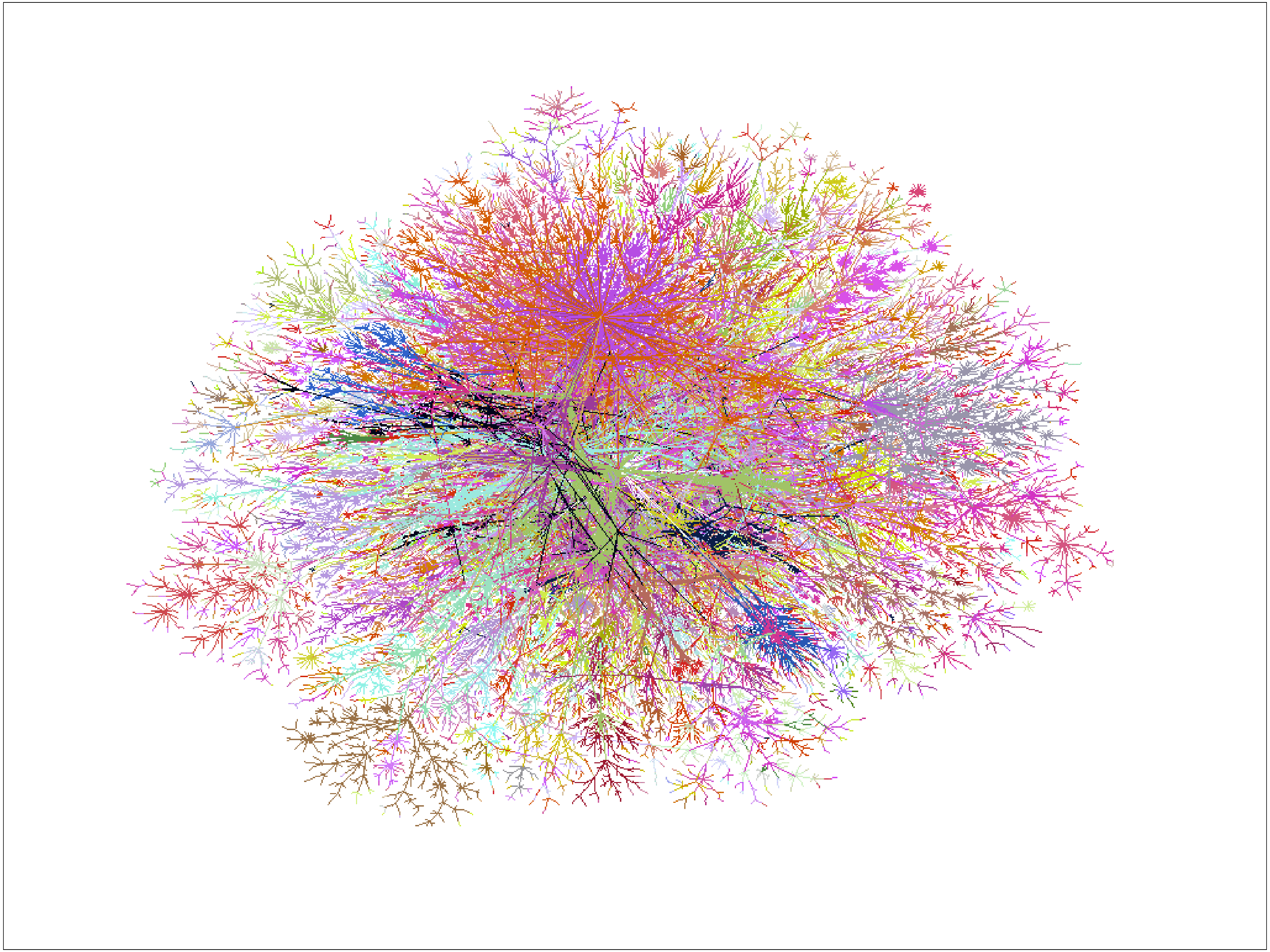

Arbre binaire aléatoire
en 3D



- Visualisation de l'information très grands graphes

D. Auber, M. Delest
Y. Chiricota, G. Mélançon, J.M. Fedou

analyse de Horton-Strahler



What is doing Kepler
in this Catalan - Strahler story?

2. Un peu d'histoire

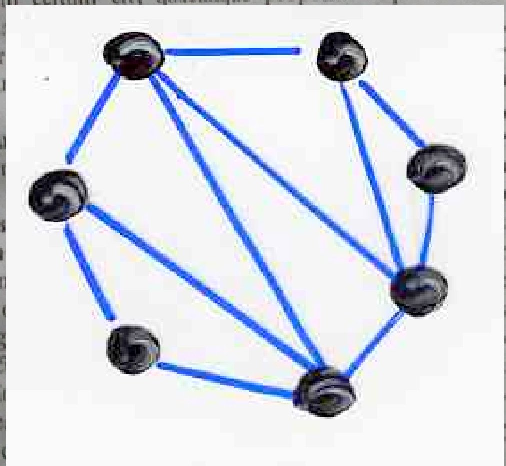
II	III	IIII	V	VI	VII	VIII	VIIII	X	XI	XII
0	1	2	3	4	5	6	7	8	9	10
1	1	2	4	10	28	84	264	858	2860	9724
.	.	.	1	4	10	28	84	264	858	2860
.	.	.	.	.	4	20	56	168	528	1716
.	.	.	.	.	.	. .	25	140	420	1320
.	.	.	.	.	. .	. .	. .	. .	196	1176
1	1	2	5	14	42	132	429	1430	4862	16796

XIII	XIIII	XV	XVI	XVII
11	12	13	14	15
33592	117572	416024	1503800	5384880
9724	33592	117572	416024	1503800
5720	19448	77184	235144	832048
4290	14300	48620	167960	587860
3696	12012	39040	136136	470288
1764	11088	36036	120120	408408
....	.....	17424	113256	377520
....	.....	.....	......	184041
58786	208012	751900	2692440	9748845

XVIII	XVIIII	XX
16	17	
19497690	69016140	252827580
5384880	19497690	69016140
3007600	10769760	38995380
2080120	7519000	26924400
1646008	5824336	21053200
1410864	4938024	17473008
1283568	4434144	15519504
197340	4171596	14410968
....	243100	13905320
34508070	126413790	469925500

METHO

diri nequeant, noua haec methodus, qua negotium per multiplicatores conficitur, non solum optimo cum successu adhibetur, sed etiam nullum est dubium, quin ea, si vberius excolatur, multo maiora commoda sit allatura. Pari autem quoque successu ad aequationes differentiales tertii et altiorum graduum extendi poterit, siquidem certum est, quacunque proposita aequatione dif-
ferenti
semper
tio mu
verum
et haru
gandi
totum
idoneus
quidem
erit fu
ipsum
inuestig
iste fac
ratio i
sontane
lenda



ENV-

ENVMERATIO

MODORVM QVIBVS FIGVRAE PLANAE RECTILINEAE PER DIAGONALES DIVIDVNTVR IN TRIANGVLA.

Auctore

IOH. ANDR. DE SEGNER.

Triangulum per diagonalem in alia solui non posse, vtpote quod ex se ipso vno tantum modo componitur, notum est: quadrilaterum autem ita diuidi duplicem in modum posse, mox apparet. Neque difficulter perspicitur, modos, quibus quinque laterum figura per diagonales in triangula soluitur, quinque esse, quorum quilibet discrepat ab altero. *Sex* autem, vel plurium laterum figurae, quot modis ita soluantur enumerare difficilius est; eoque difficilius, quo plura, sunt figurae latera. Soluitur autem hexagonum in triangula 14 modis diuersis, heptagonum 42, ogdogonum 132, enneagonum 429; quos numeros mecum beneuolus communicauit summus *Eulerus*; modo, quo eos reperit, atque progressionis ordine, celatis. Vtrumque perspiciendi cupido, post tentamina quaedam inania, eos numeros eliciendi methodus occurrit adeo simplex, vt in ea acquiescendum mihi putauerim, quam hic proponam.

Triangulum prima ordine est figurarum planarum rectilinearum, quadrilaterum secunda, et ita deinceps,

sic

NOVI COMMENTARII ACADEMIAE SCIENTIARVM IMPERIALIS PETROPOLITANAE

TOM. VII.

pro Annis MDCCLVIII. et MDCCLIX.

PETROPOLI
TYPIS ACADEMIAE SCIENTIARVM
MDCCLXI.

VI.

Enumeratio modorum, quibus figurae planae rectilineae per diagonales diuiduntur in triangula.

Auct. I. A. de Segner pag. 203.

Quando in Geometria area figurarum pluribus lateribus incluſarum definiri debet, eae per diagonales in triangula reſolui ſolent, quia tum cuiusque trianguli areae ex cognitis lateribus facile determinantur. Quo pluribus autem lateribus figura eſt praedita, eo pluribus modis eam hoc modo in triangula reſolui poſſe, vel leuiter attendenti ſtatim eſt manifeſtum. Ita cum in quadrilaterum duas diagonales ducere liceat, quadrilaterum duplici modo in bina triangula diuiditur. Pentagonum autem quintuplici modo, diagonalibus ducendis, in triangula reſolui poſſe reperitur, hexagonum vero 14 modis, et heptagonum 42 modis, octogonum 132 modis, enneagonum 429 modis etc. quae omnium modorum poſſibilium enumeratio, quo magis cum laterum numero eorum multitudo creſcit, eo fit difficilior et taedioſior. Quare quaeſtio omnino curioſa, et Geometrarum attentione digna videtur, qua lege iſti reſolutionum numeri pro laterum multitudine progrediantur, vt inde pro quouis polygono reſolutionum numerus rite definiri queat? Hinc Ill. Auctor modo prorſus ſingulari et ingenioſo legem progreſſionis horum numerorum exponit, ac rigoroſe demonſtrat, dum docet, quomodo pro quouis polygono reſolutionum numerus, ex cognita reſolutione polygonorum ſimpliciorum, quae paucioribus conſtant lateribus, colligi debeat. Hac ratione, ſi a ſimpliciſſimis incipiamus, hanc inueſtigationem continuo ad polygona plurium laterum extendere licet, ſicque Ill. Auctor ſub finem tabellam adiecit, in qua iſtiusmodi reſolutiones ad polygona 20 laterum vsque exhibet. Liceat autem nobis, a ſummo quodam Geometra, qui eandem hanc tabulam calculo ſubiecit, admonitis, obſeruare, hanc tabulam, ob quendam calculi errorem, tantum vsque ad polygona 15 laterum eſſe iuſtam, quippe pro hoc polygono reſolutionum numerus non eſt 751900, vt tabella habet, ſed 742900, ſequentes quoque numeri, dum forte nouus error irrepſit, primo ad 17 vsque latera nimis ſunt magni, deinde vero nimis parui, dum pro 20 lateribus reſolutionum numerus eſt 477638700. Facilius haec apparent, ſi ex lege primum obſeruata, qua quilibet numerus ex omnibus praecedentibus colligitur, alia ad computum facilior eliciatur, cuius ope quilibet numerus ex ſolo praecedente definiatur. Ita ſi pro polygono n laterum numerus reſolutionum ſit P, pro polygono ſequente $n+1$ laterum numerus reſolutionum erit $\frac{4n-6}{n}$ P. Quin etiam hinc, ſine conſideratione praecedentium, ſtatim indefinite pro polygono n laterum numerus reſolutionum ita per factores exprimitur, vt ſit:

$$\frac{2}{2}\cdot\frac{6}{3}\cdot\frac{10}{4}\cdot\frac{14}{5}\cdot\frac{18}{6}\cdot\frac{22}{7}\cdot\ \ldots\ldots\ \frac{4n-10}{n-1}$$

vbi numeratores quaternario, denominatores vero vnitate creſcunt. Hinc ſequentem nouam tabulam, beneuole

nobiscum communicatam, adiungere e re viſum eſt, quod Ill. Auctori huius ſchediasmatis non diſpliciturum eſſe ſperamus.

Num. laterum	numerus reſolutionum.	num. laterum	numerus reſolutionum.
III	1	XV	742900
IV	2	XVI	2674440
V	5	XVII	9694845
VI	14	XVIII	35357670
VII	42	XIX	129644790
VIII	132	XX	477638700
IX	429	XXI	1767263190
X	1430	XXII	6564120420
XI	4862	XXIII	24466267020
XII	16796	XXIV	91482563640
XIII	58786	XXV	343059613650.
XIV	208012		

VII.

Methodus ſimplex et vniuerſalis omnes omnium aequationum radices determinandi.

$$2(2n+1)\,C_n = (n+2)\,C_{n+1}$$

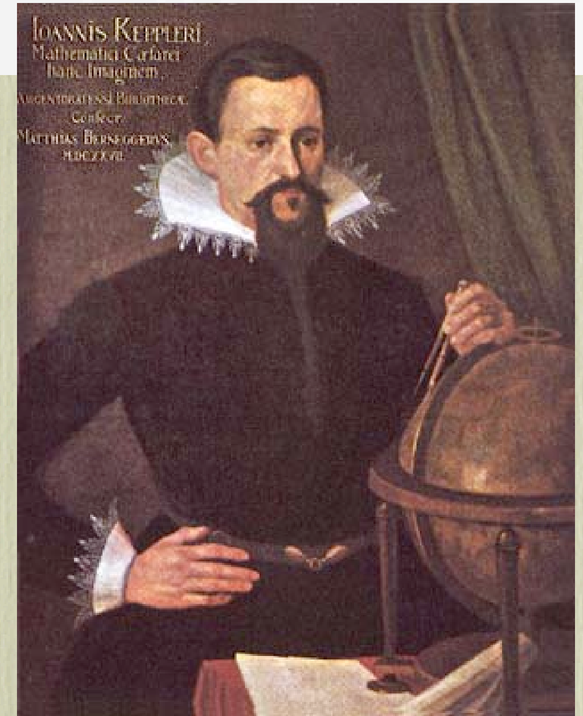
Complures iam a Geometris excogitatae ſunt methodi aequationum algebraicarum radices, vel accurate, vel proxime ſaltem, determinandi: omnes autem fere poſtulant, vt valores radicum, quae quaeruntur,

$$\frac{1}{n+1}\binom{2n}{n}$$

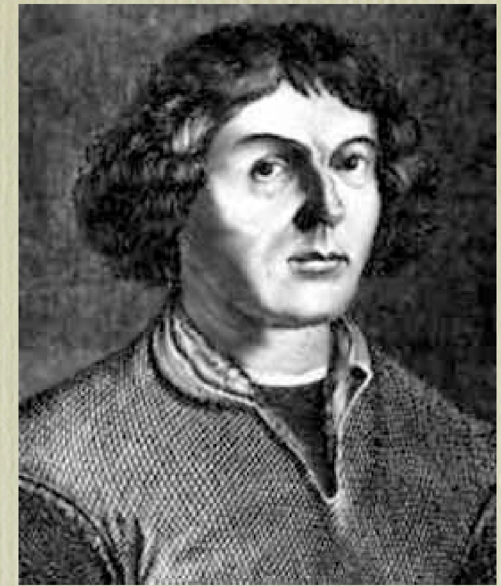
Kepler

3 laws

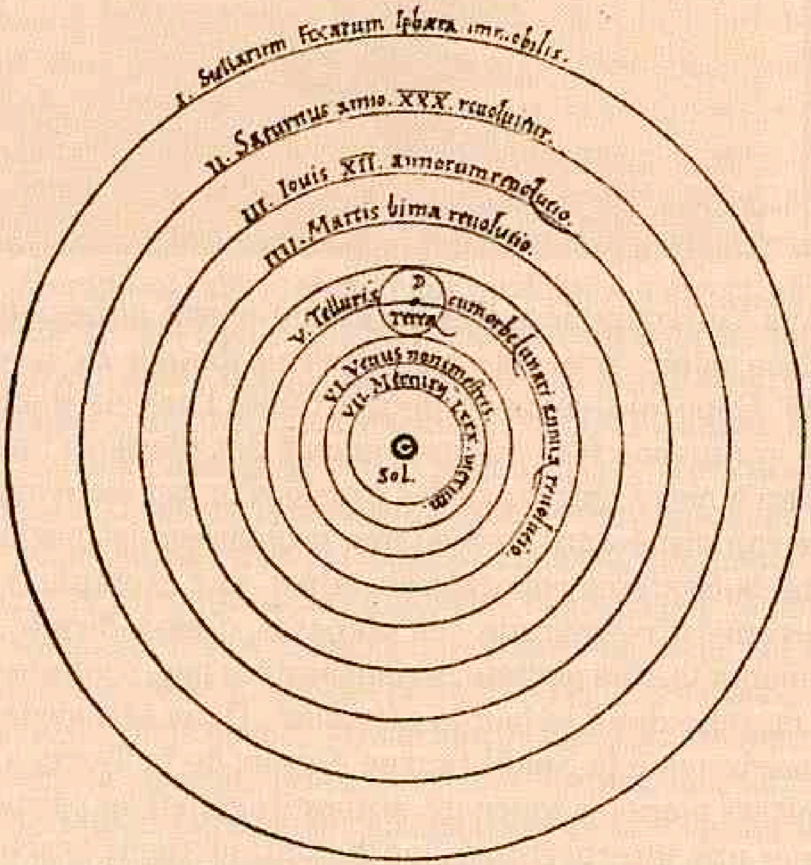
- Astronomia nova (1609)
- Harmonice mundi (1619)

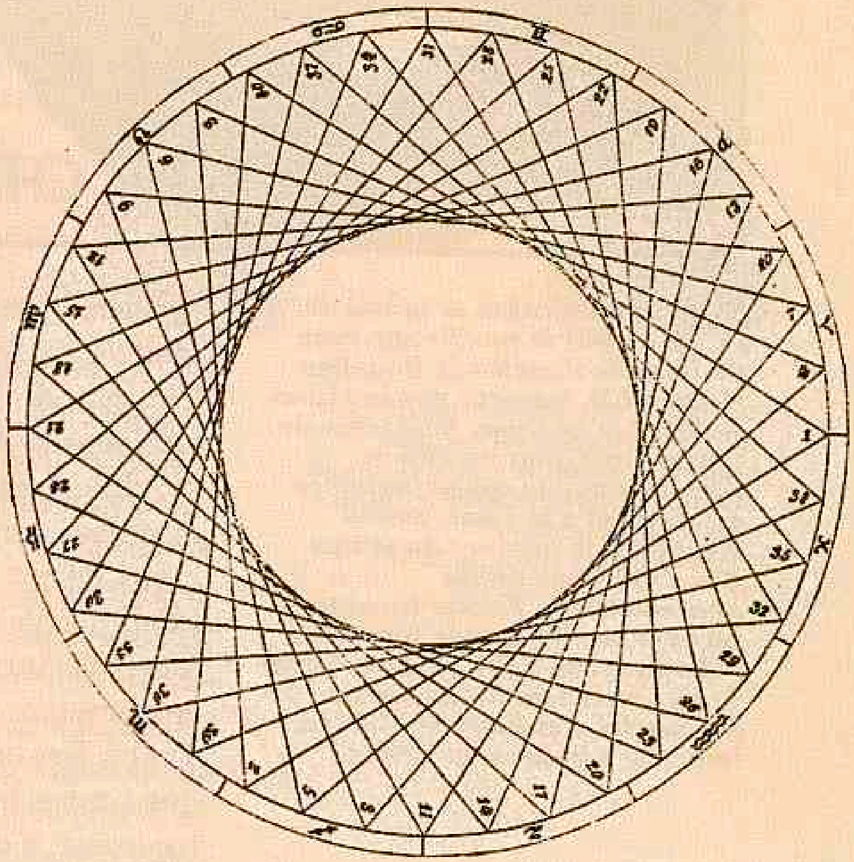


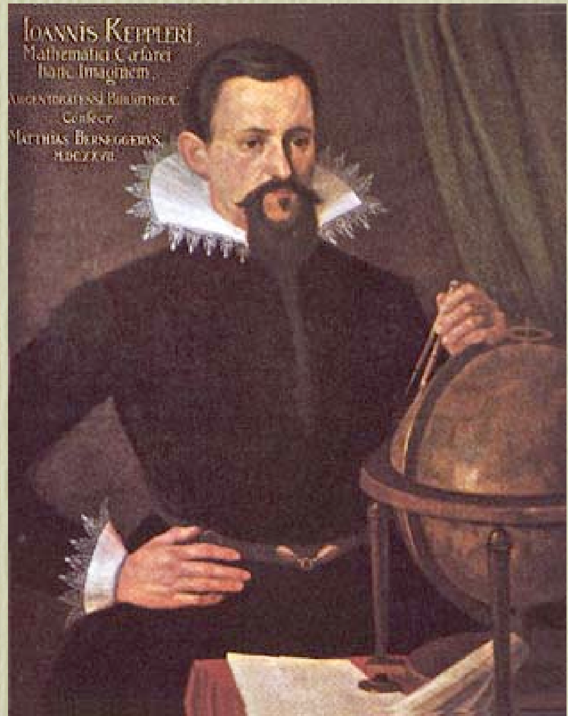
Mysterium cosmographicum (1596)

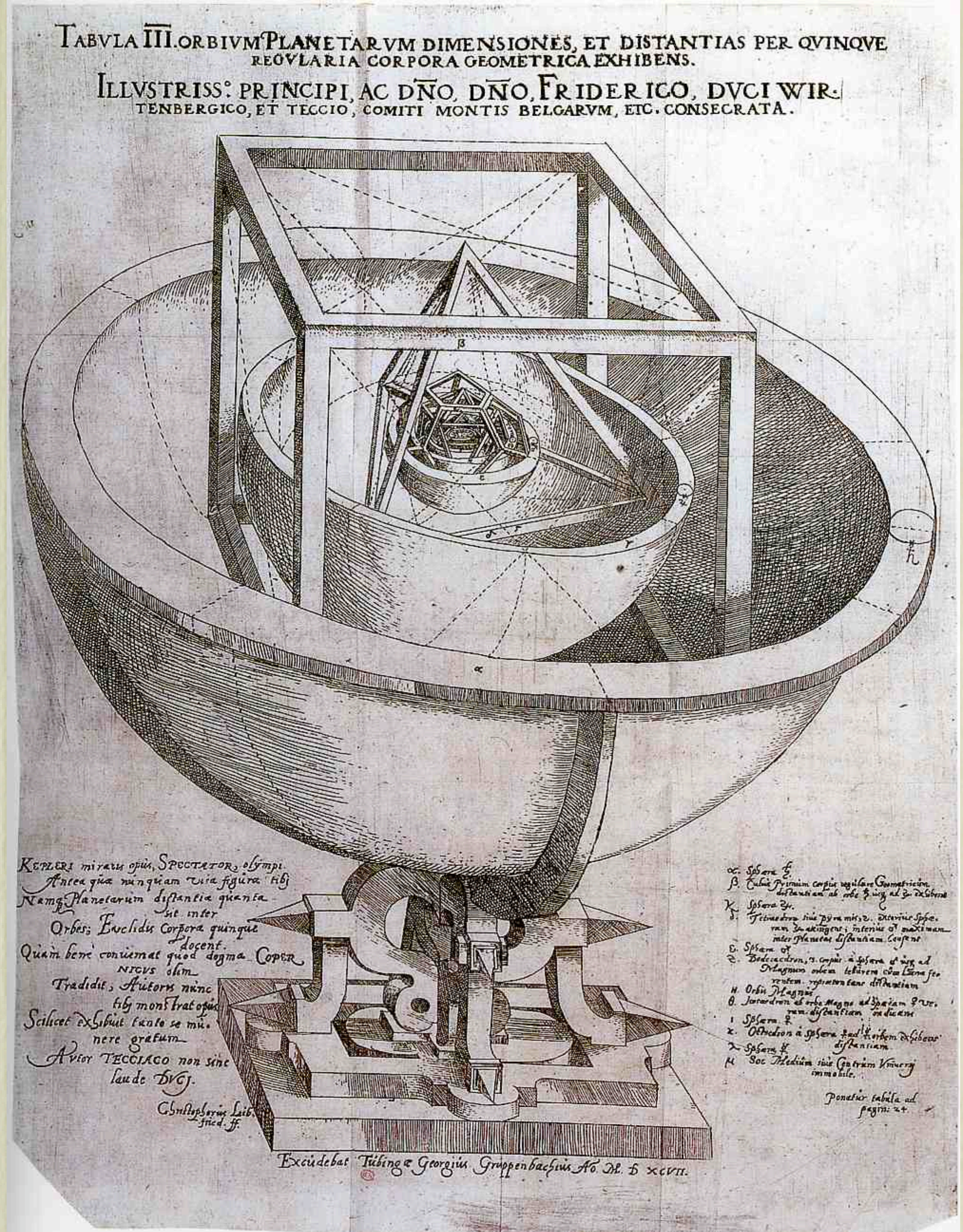


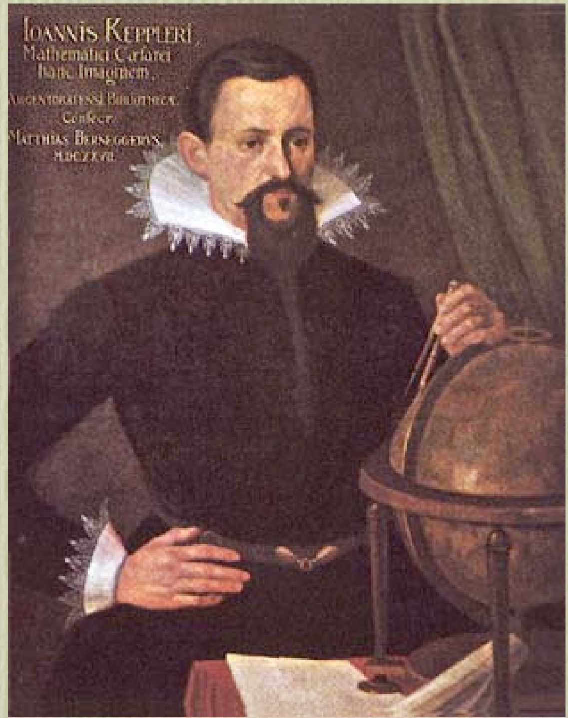
Copernic De Revolutionibus (1543)

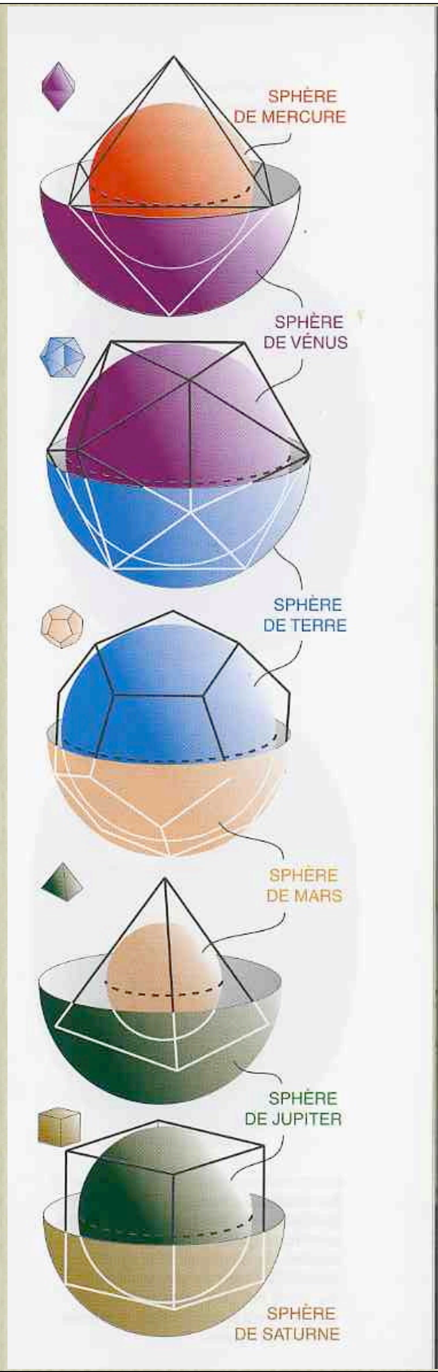

I. Stellarum fixarum sphaera immobilis.
II. Saturnus anno. XXX. revolvitur.
III. Iovis XII. annorum revolutio.
IIII. Martis bima revolutio.
V. Telluris cum orbe lunari annua revolutio
Terra
VI. Venus nono mense.
VII. Mercurius. LXXX. dierum.
Sol.

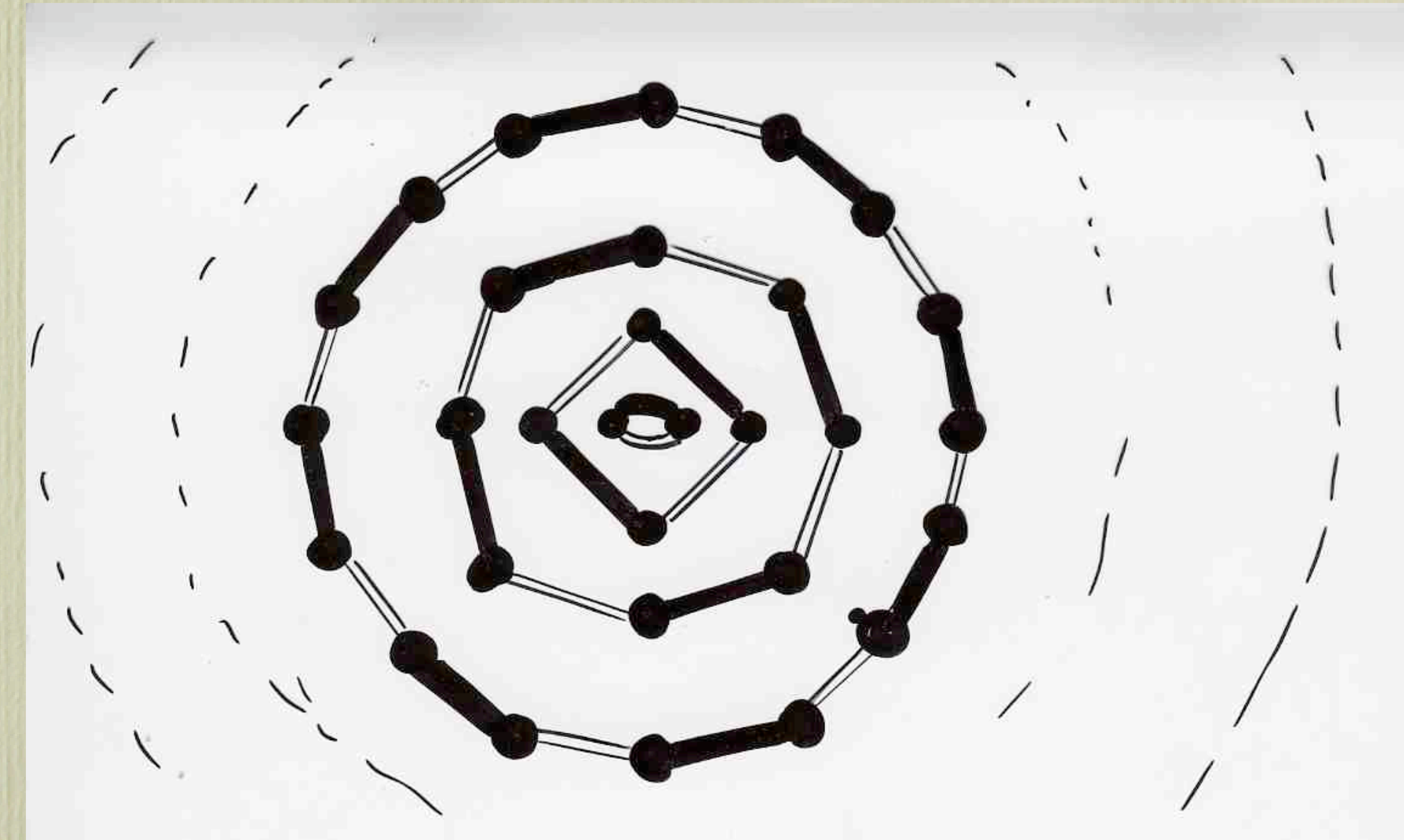



IOANNIS KEPPLERI,
Mathematici Caesarei
hanc Imaginem,
Consecr.


TABVLA III. ORBIVM PLANETARVM DIMENSIONES, ET DISTANTIAS PER QVINQVE REGVLARIA CORPORA GEOMETRICA EXHIBENS.
ILLVSTRISS^o. PRINCIPI, AC DNO. DNO, FRIDERICO, DVCI WIRTENBERGICO, ET TECCIO, COMITI MONTIS BELGARVM, ETC. CONSECRATA.
KEPLERI miratus opus, SPECTATOR, olympi
Antea quæ nunquam visa figura tibi
Namq; Planetarum distantia quanta sit inter
Orbes; Euclidis Corpora quinque docent.
Quam bene conueniat quod dogma COPERNICUS olim
Tradidit, Autoris nunc tibi monstrat opus.
Scilicet exhibuit tanto se munere gratum
Autor TECCIACO non sine laude DVCI.
Ponatur tabula ad pagin. 24.
Excudebat Tubingæ Georgius Gruppenbachius Ao. M. D XCVII.


IOANNIS KEPPLERI,


SPHÈRE
DE MERCURE
SPHÈRE
DE VÉNUS
SPHÈRE
DE TERRE
SPHÈRE
DE MARS
SPHÈRE
DE JUPITER
SPHÈRE
DE SATURNE

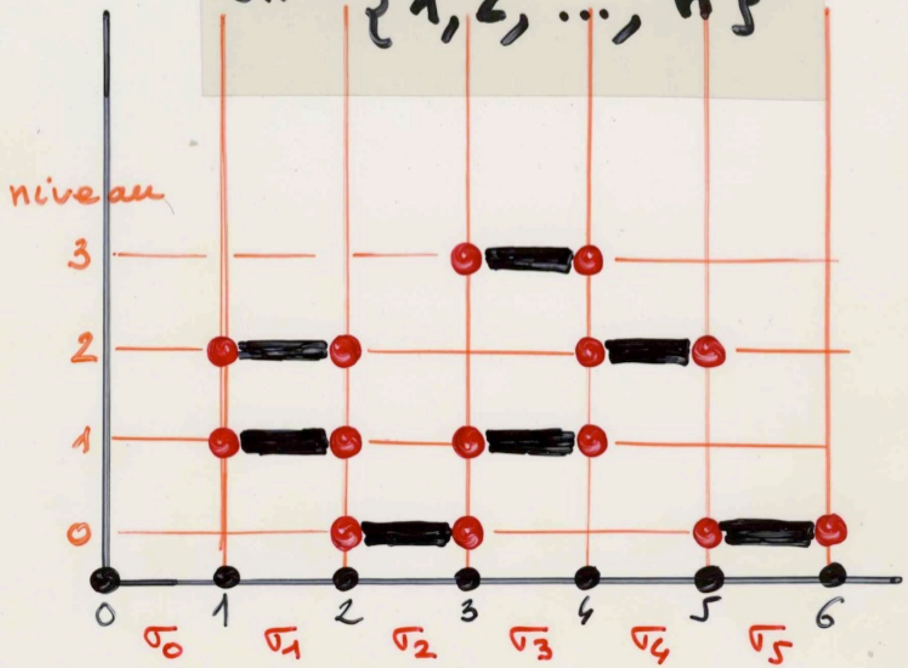


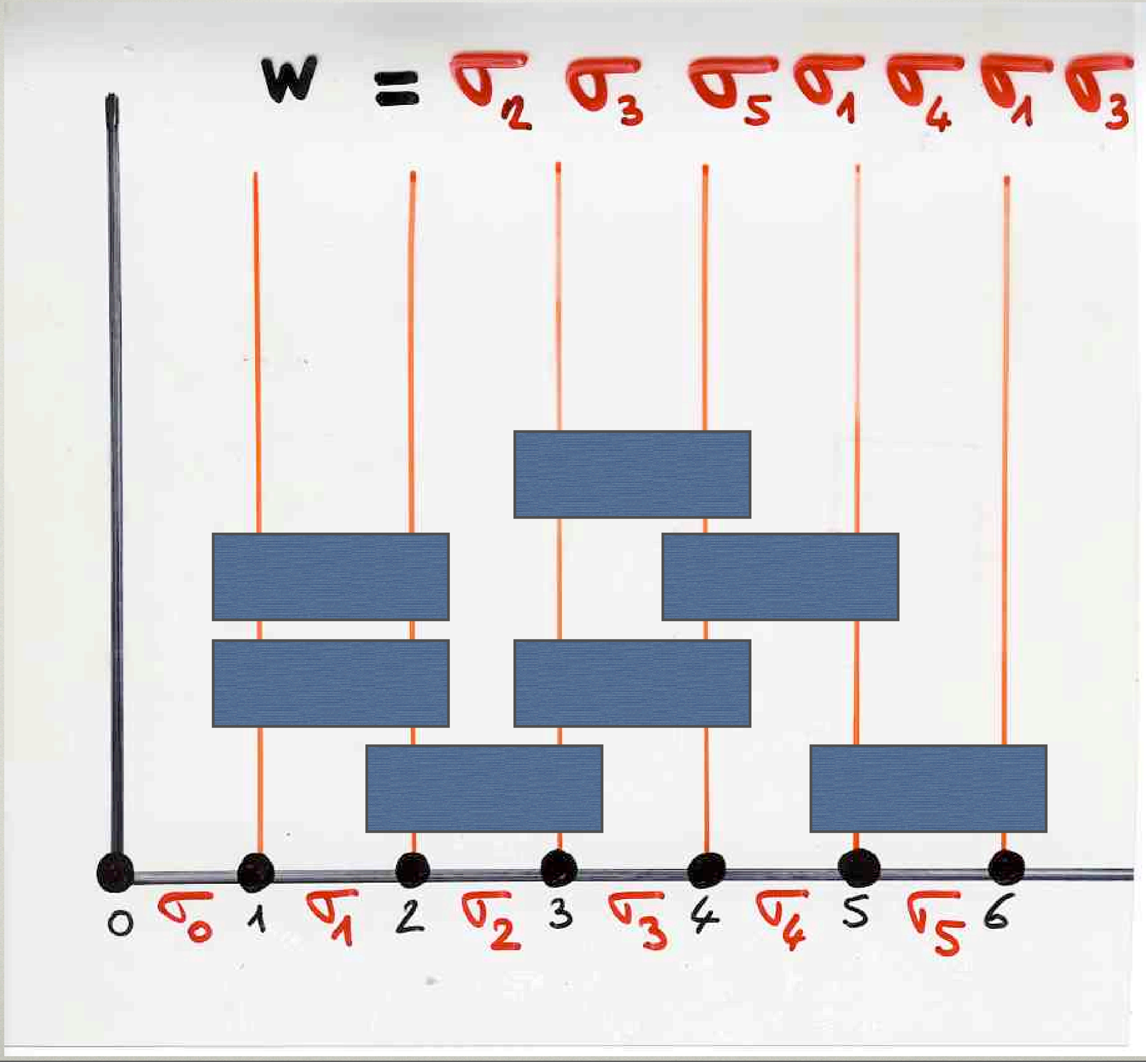
Galileo (1597)

"Figurarum regolorium"

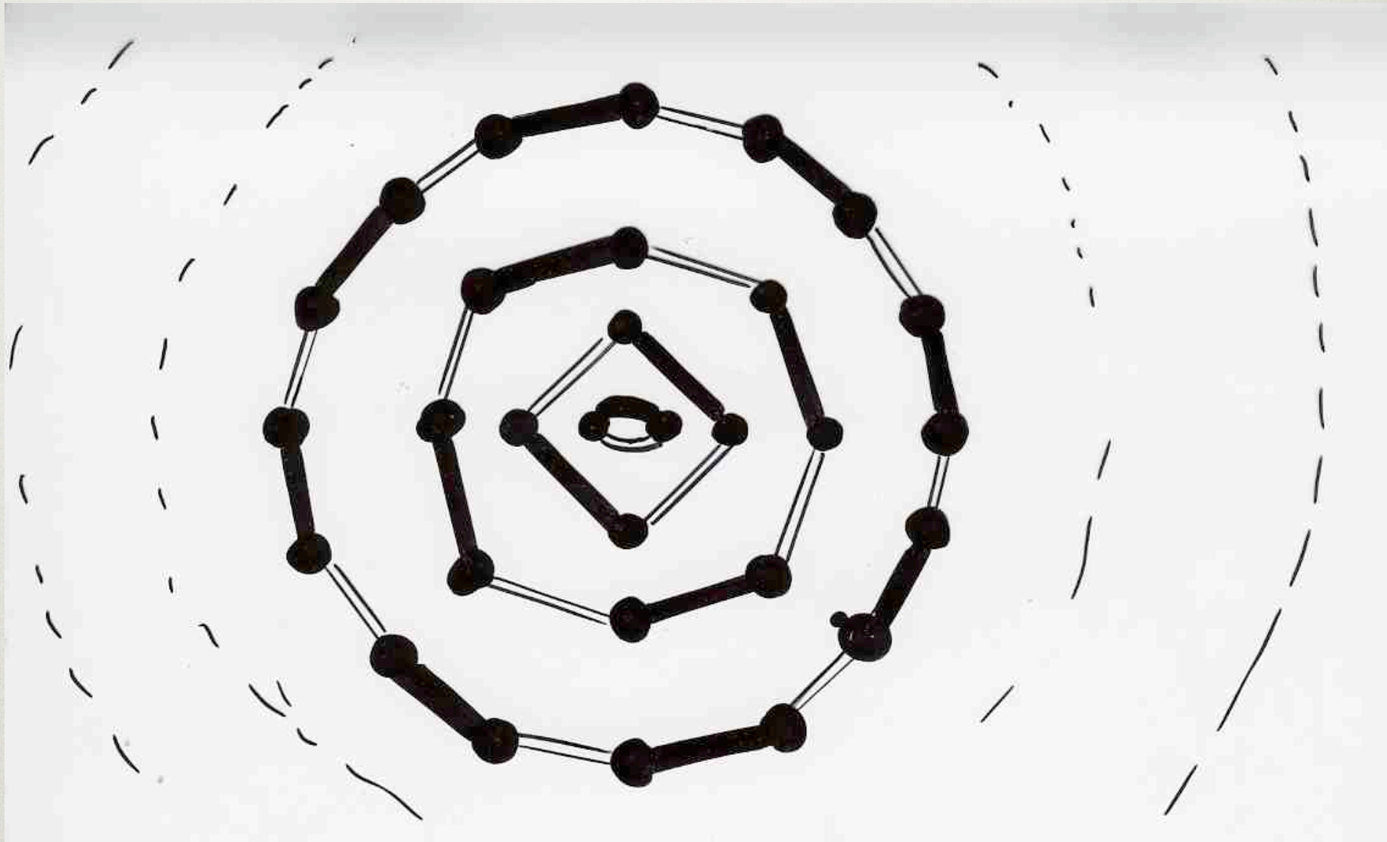
3. Les tours de Kepler

heap of dimers on $\{1, 2, \ldots, n\}$





$w = \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_4 \sigma_1 \sigma_3$
0 σ_0 1 σ_1 2 σ_2 3 σ_3 4 σ_4 5 σ_5 6



System of Kepler towers

- regular polygons $P_2, P_4, P_8, \ldots$

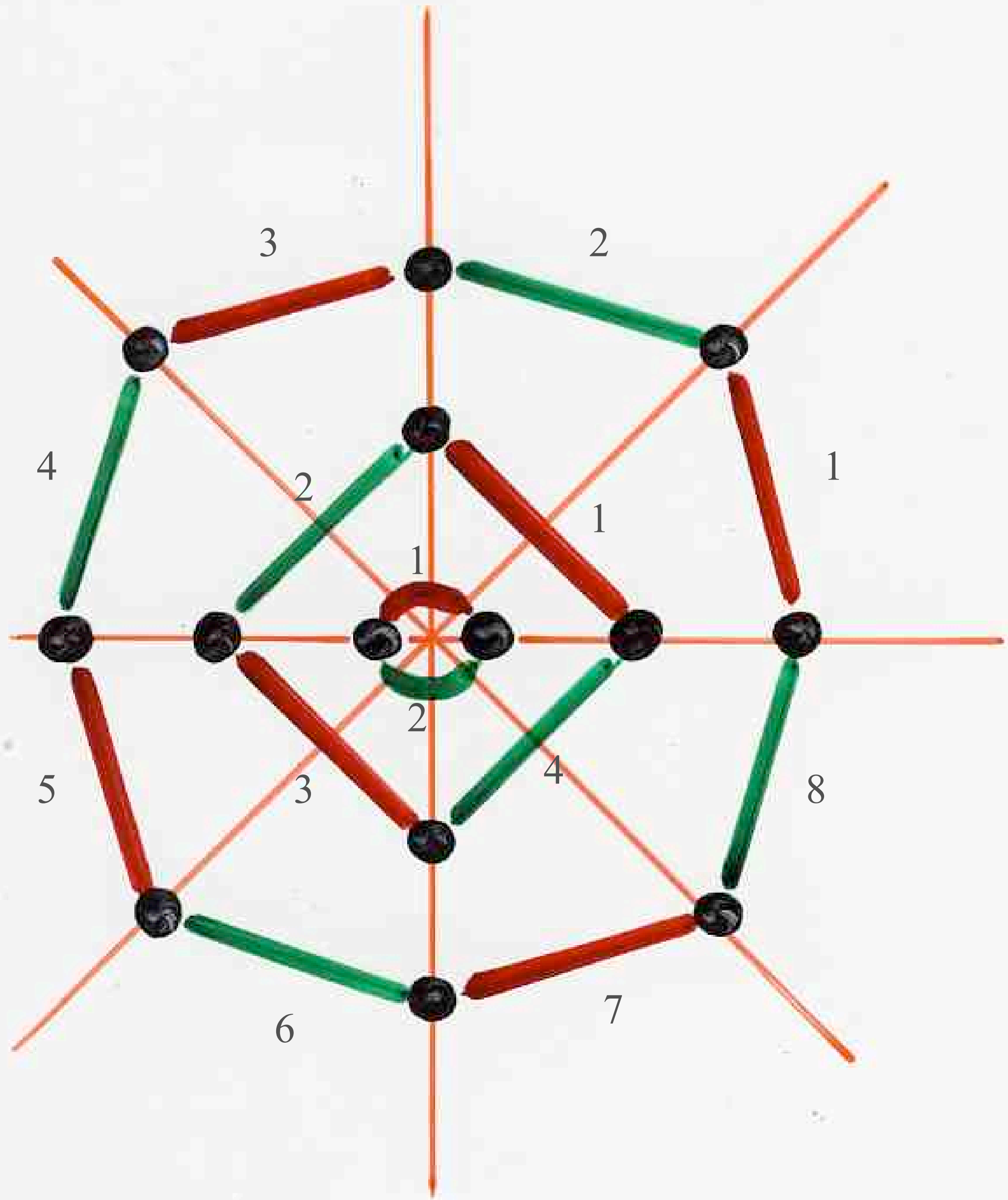
  P_i $\quad 2^i$ edges

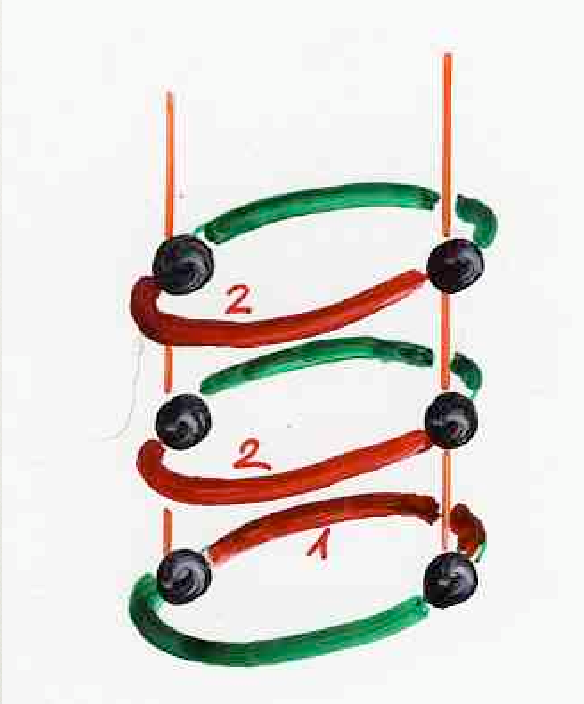
- heaps $H_1, \ldots, H_k$

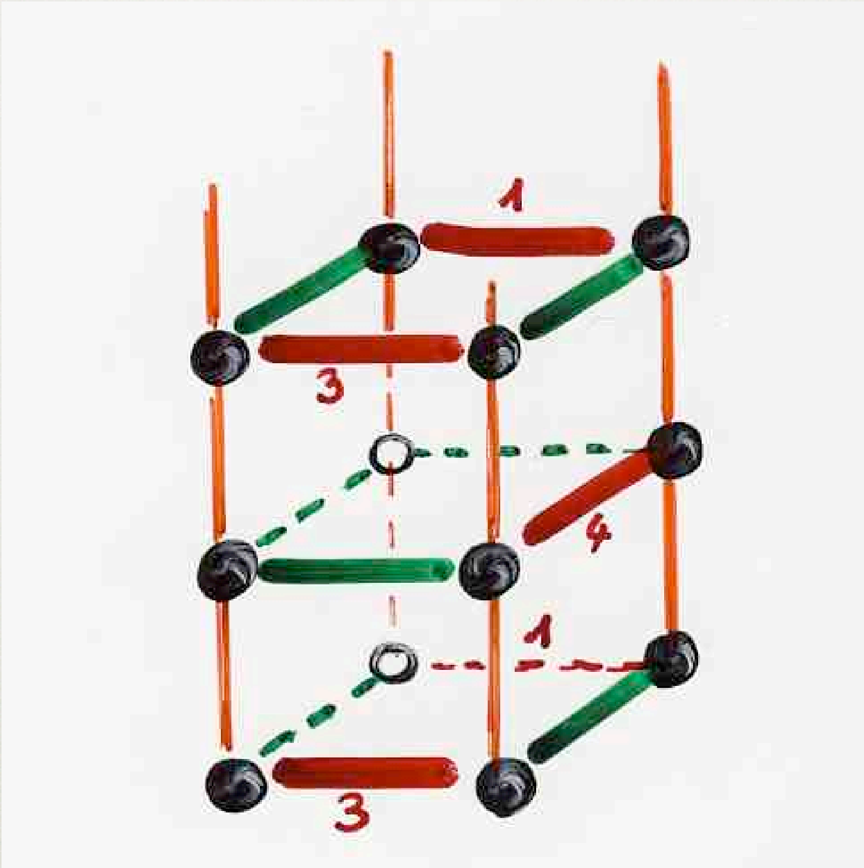
  H_i heap of dimers above P_i (= tower)

  $1 \le i \le k$

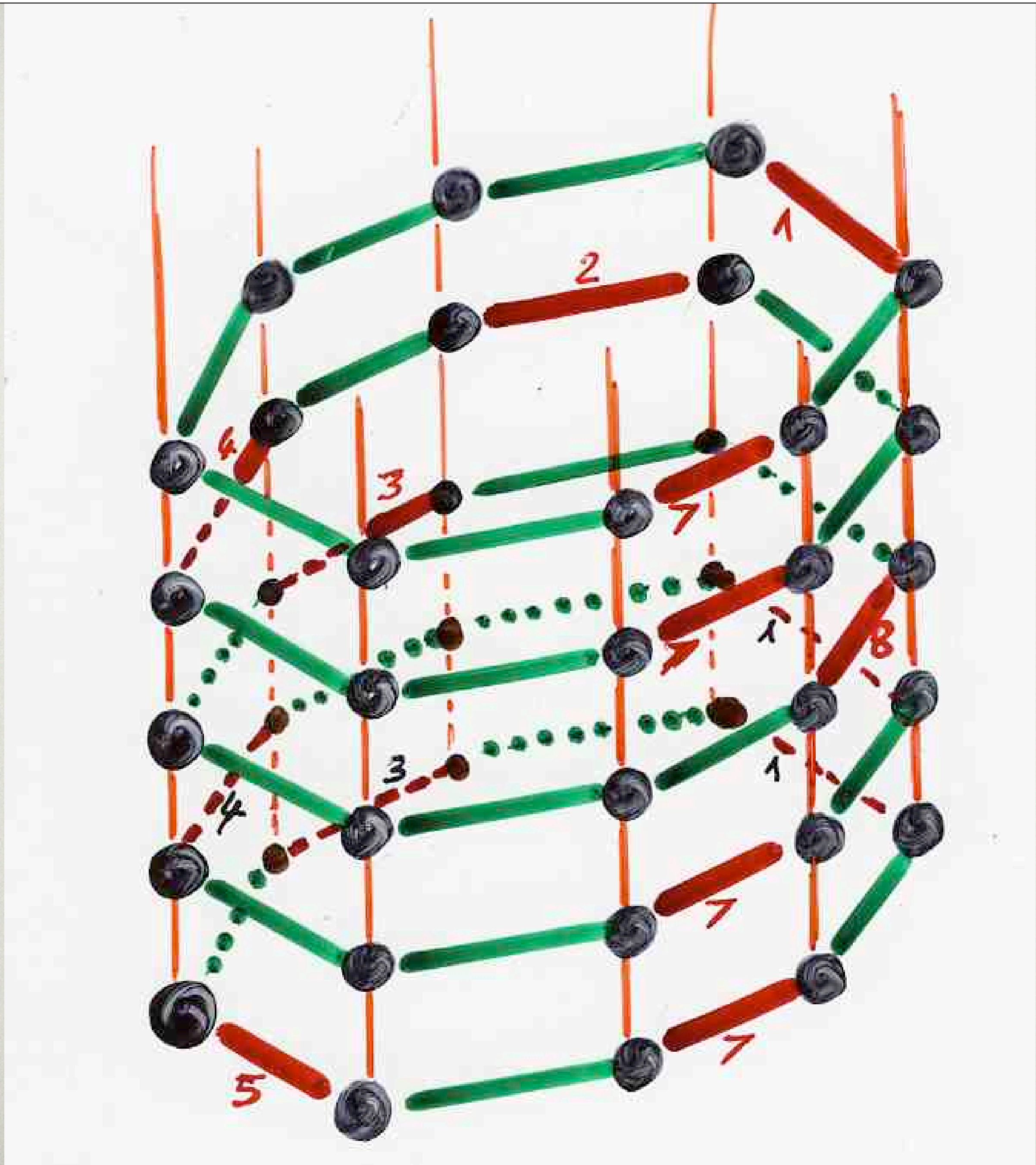
(*) at level 0, H_i contains all 2^{i-1} black edges of P_i

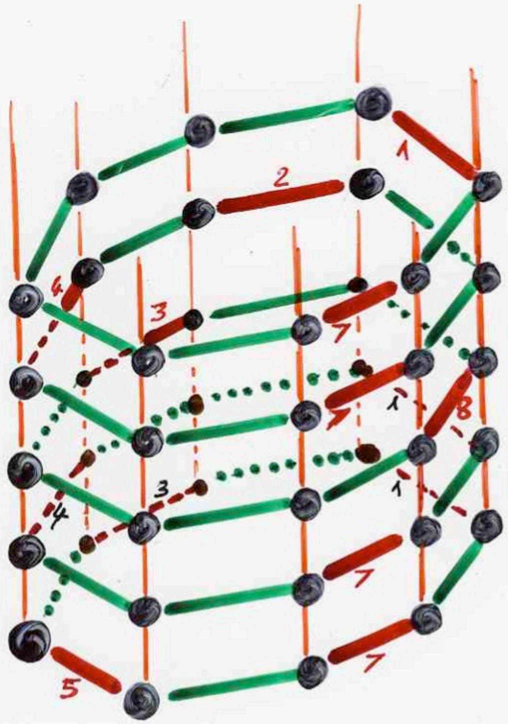

3
2
4
2
1
1
1
2
5
3
4
8
6
7

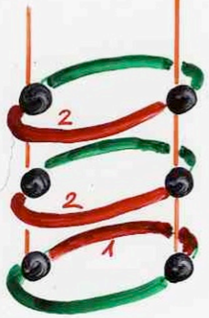

2
2
1

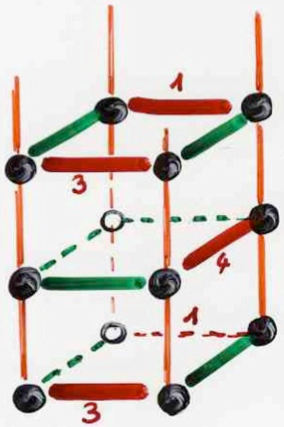


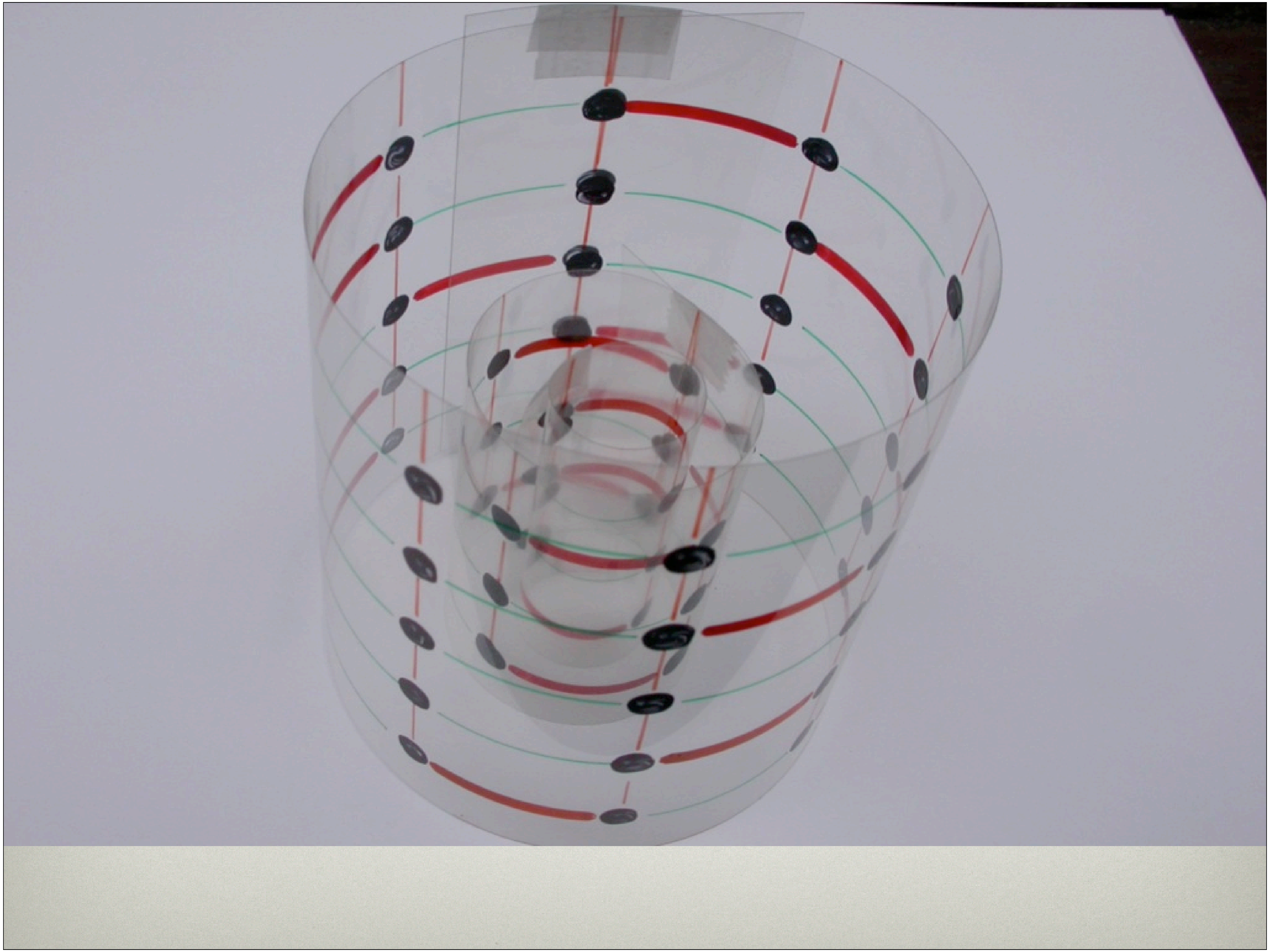
1
3
4
1
3

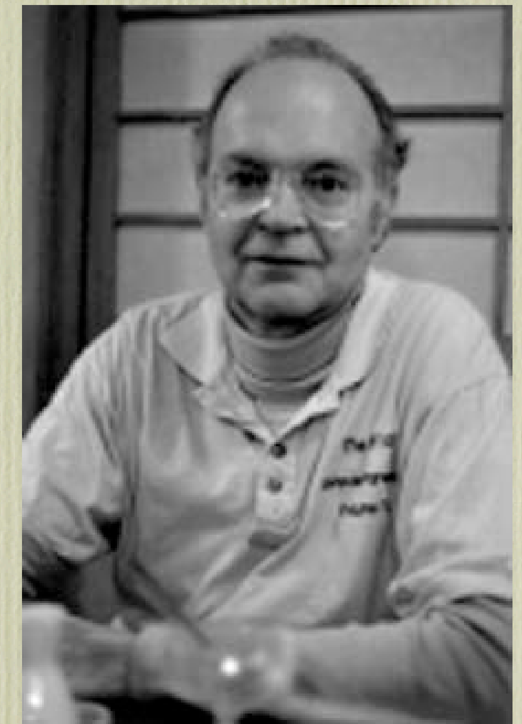

1
2
4
3
7
1
8
7
3
1
4
7
5
7

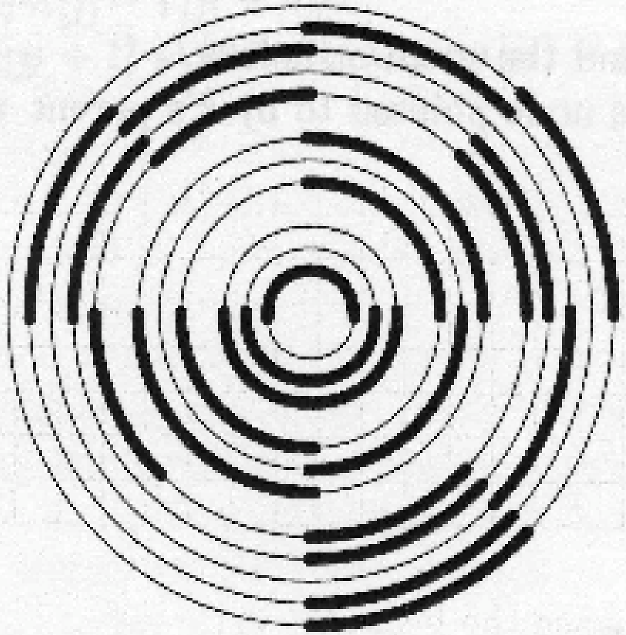

1
2
4
3
7
1
8
3
1
4
7
5
7


2
2
1


1
3
4
1
3





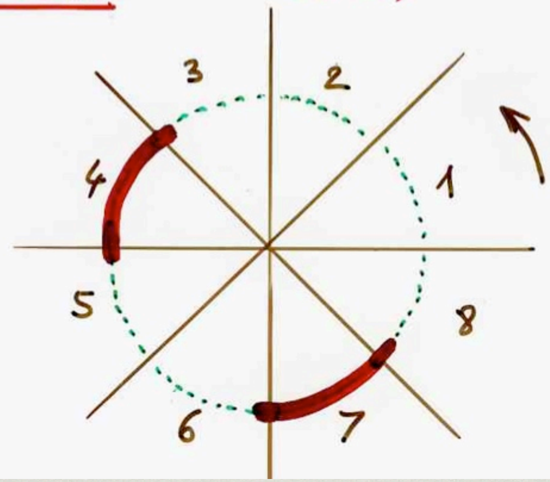


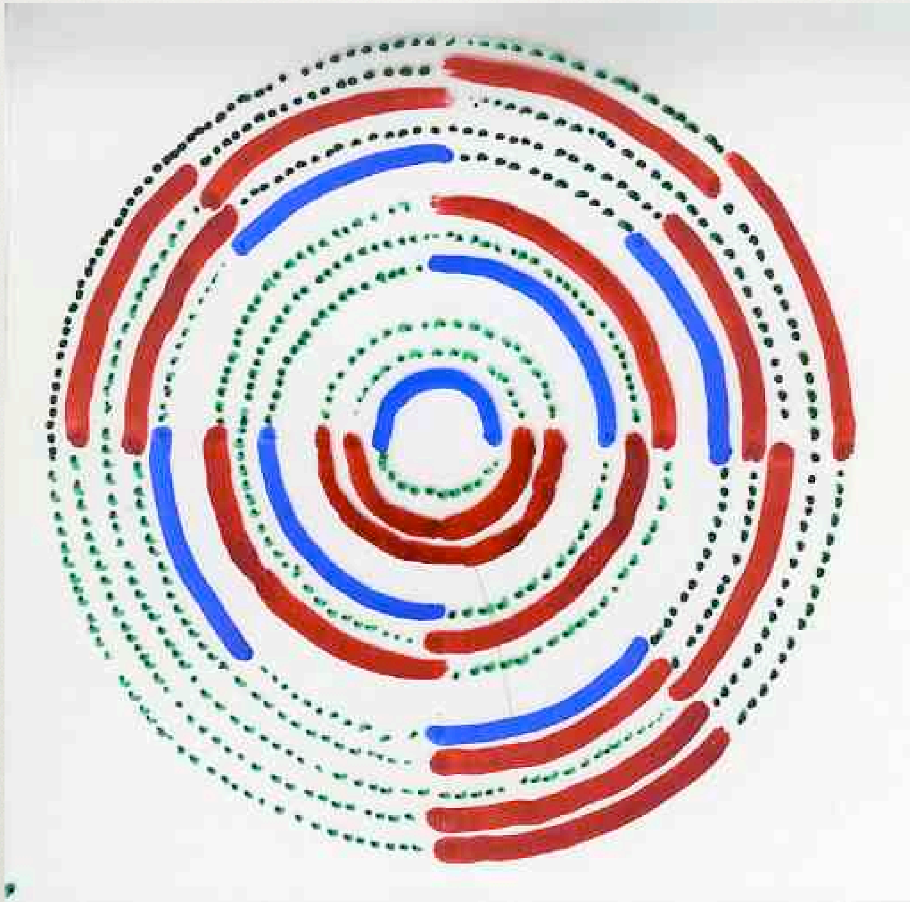
Kepler disk

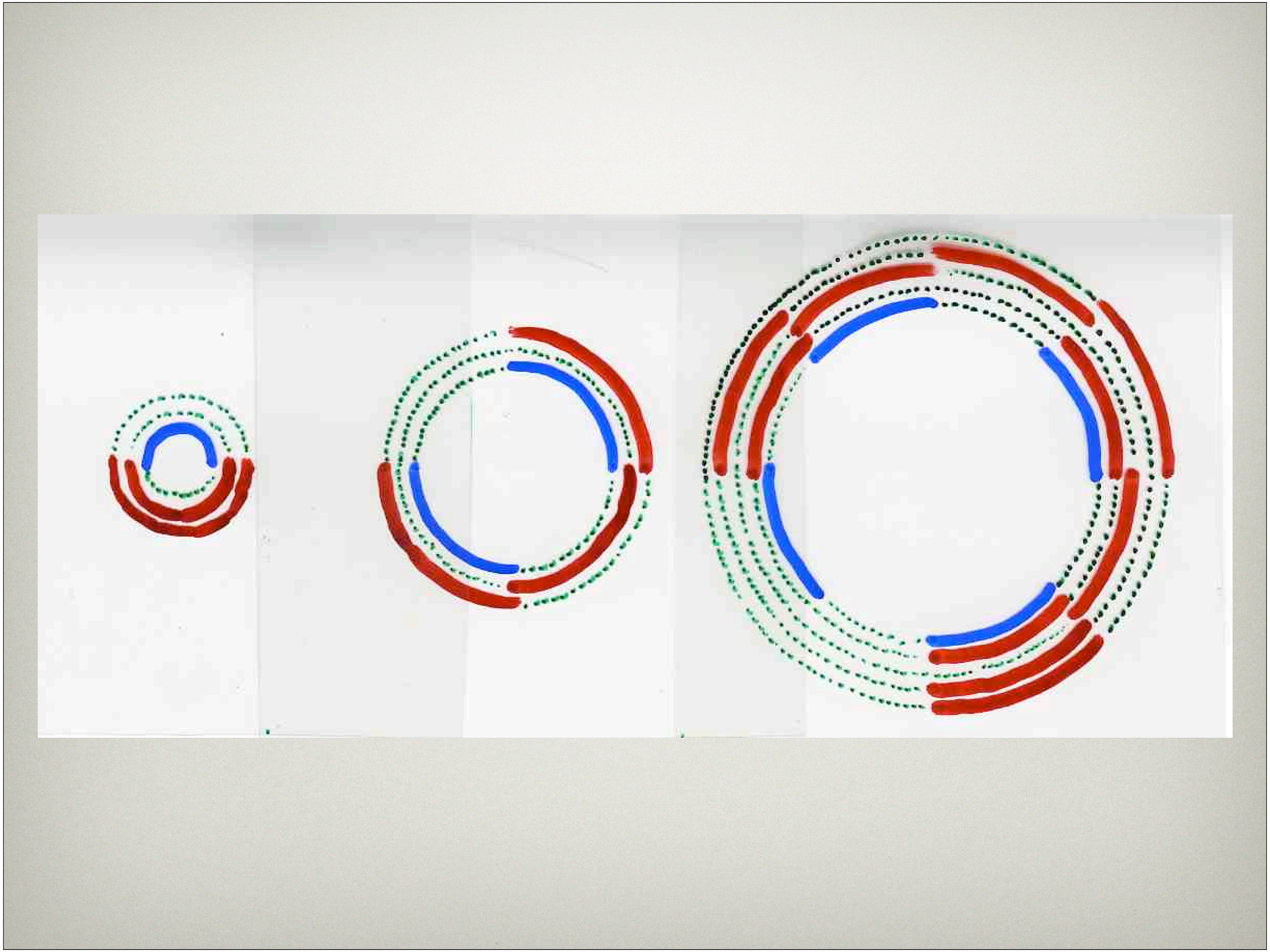
walls $W_1, \ldots, W_k$ (= tower)

W_i : one (or more) rings (circle) divided into 2^i segments (= polygon) equal length

bricks (= dimers)



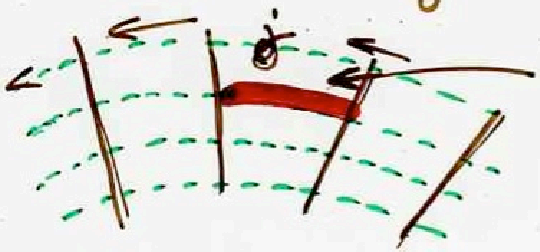




W_i wall

(i) first ring, positions of brick $1, 3, 5, \ldots, 2^i - 1$

(ii) bricks cannot occupy adjacent segments of a ring

(iii) 

brick

position j, $1 \leq j \leq 2^i$

(iv)

W_i wall

(i) first ring, positions of brick
1, 3, 5, . . . , $2^i - 1$

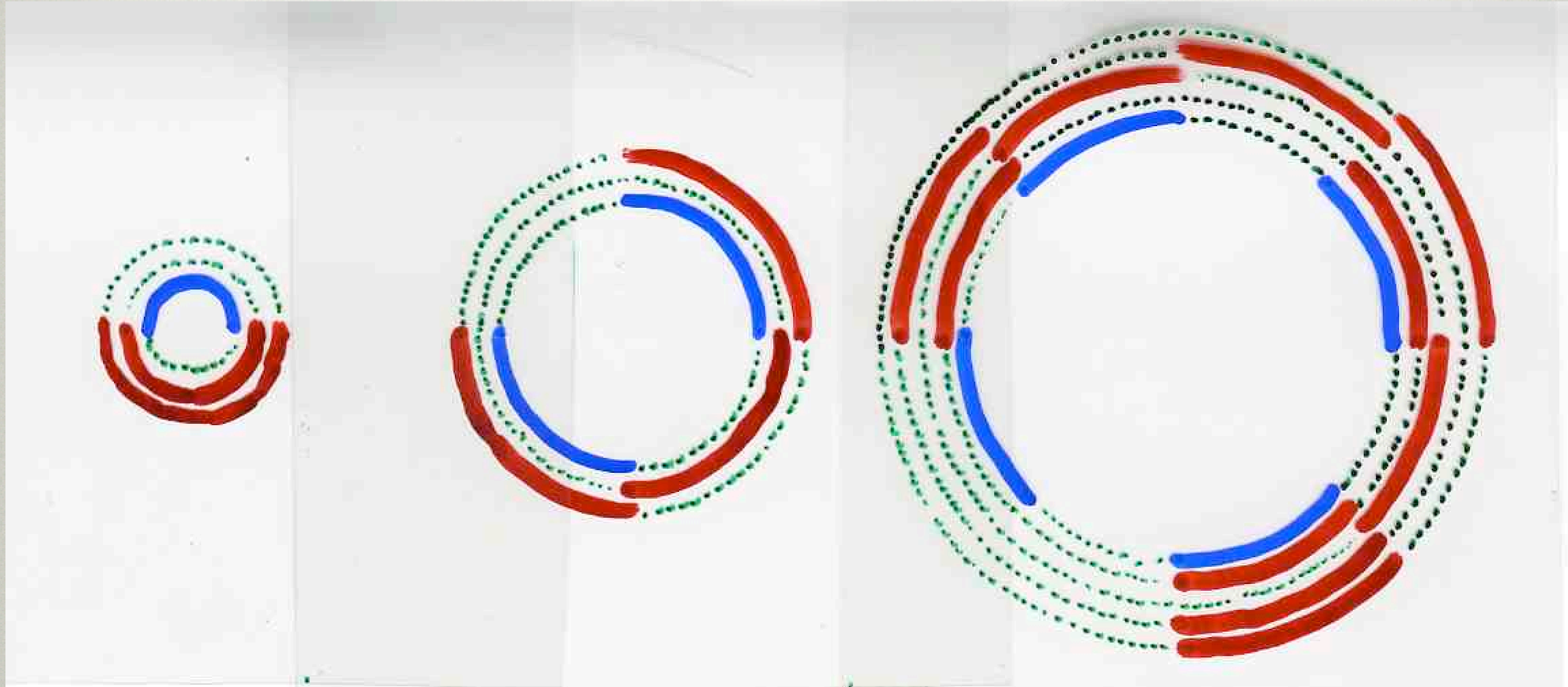
(ii) bricks cannot occupy adjacent segments of a ring

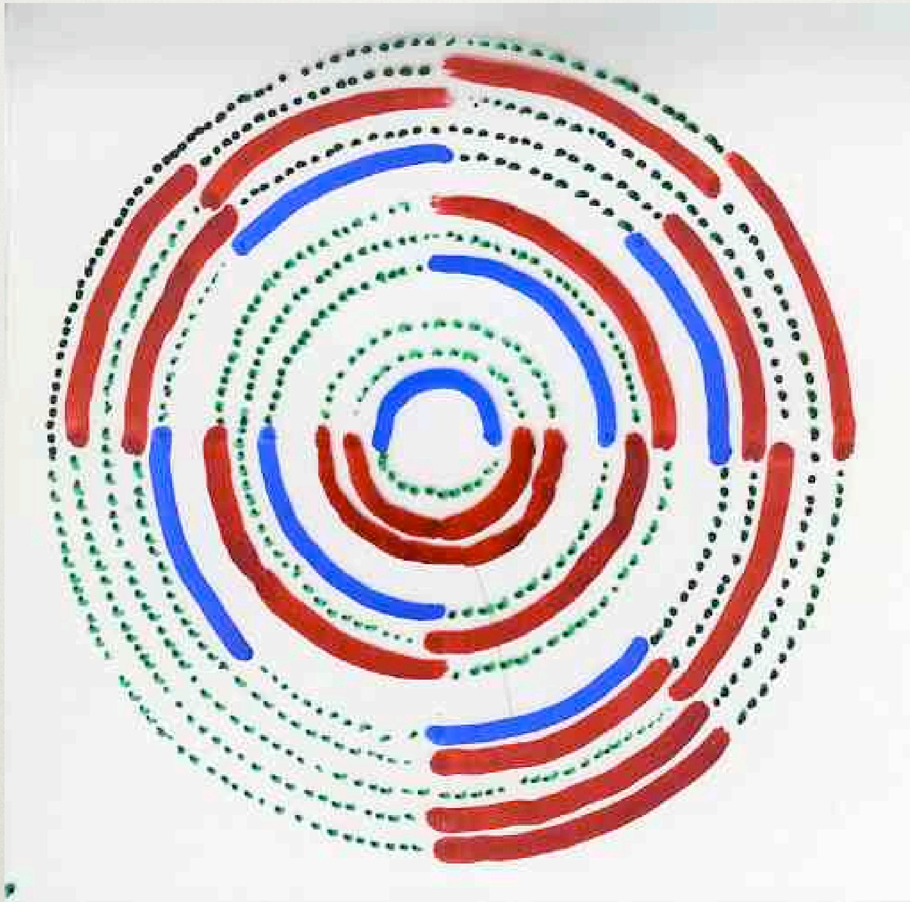
(iii) 

brick
position j, $1 \le j \le 2^i$

$\exists$ (at least) one brick in position $j-1$, j, or $j+1$ (modulo 2^i)

(iv)



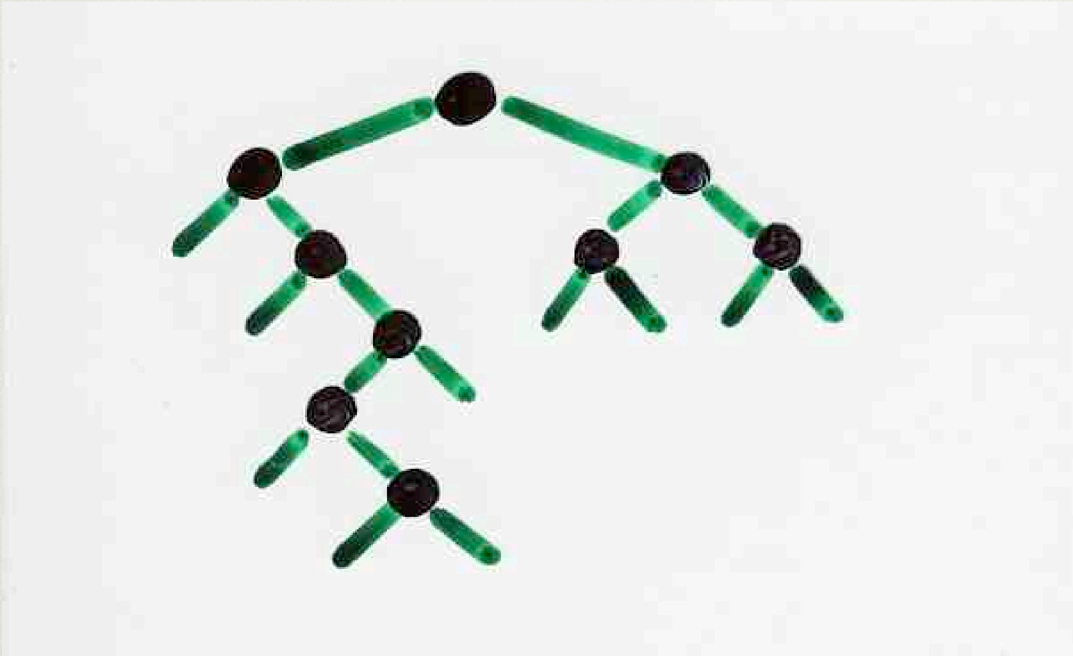


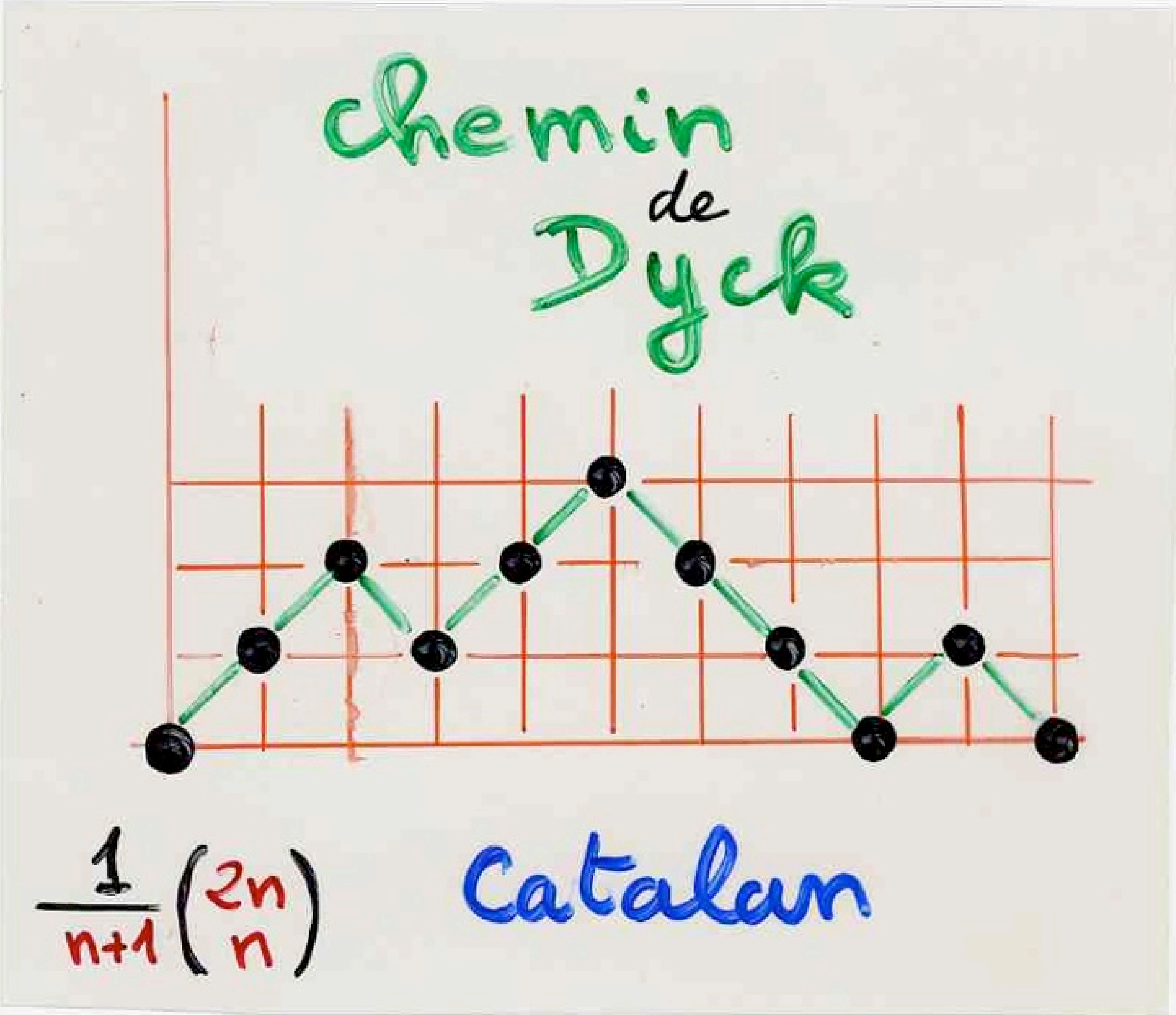
<u>Prop.</u> The number of system of Kepler towers having n dimers is

Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$

The distribution of system of Kepler towers according to the number of towers is the Strahler distribution

4. Hauteur logarithmique




Chemin
de
Dyck
$\frac{1}{n+1}\binom{2n}{n}$
Catalan

Dyck path w

Height $h(w)$

logarithmic height $lh(w)$

$$= \lfloor \log_2(1+h(w)) \rfloor$$

$$lh(w) = k$$

$$\Leftrightarrow \quad 2^k - 1 \leq h(w) < 2^{k+1} - 1$$

(complete)

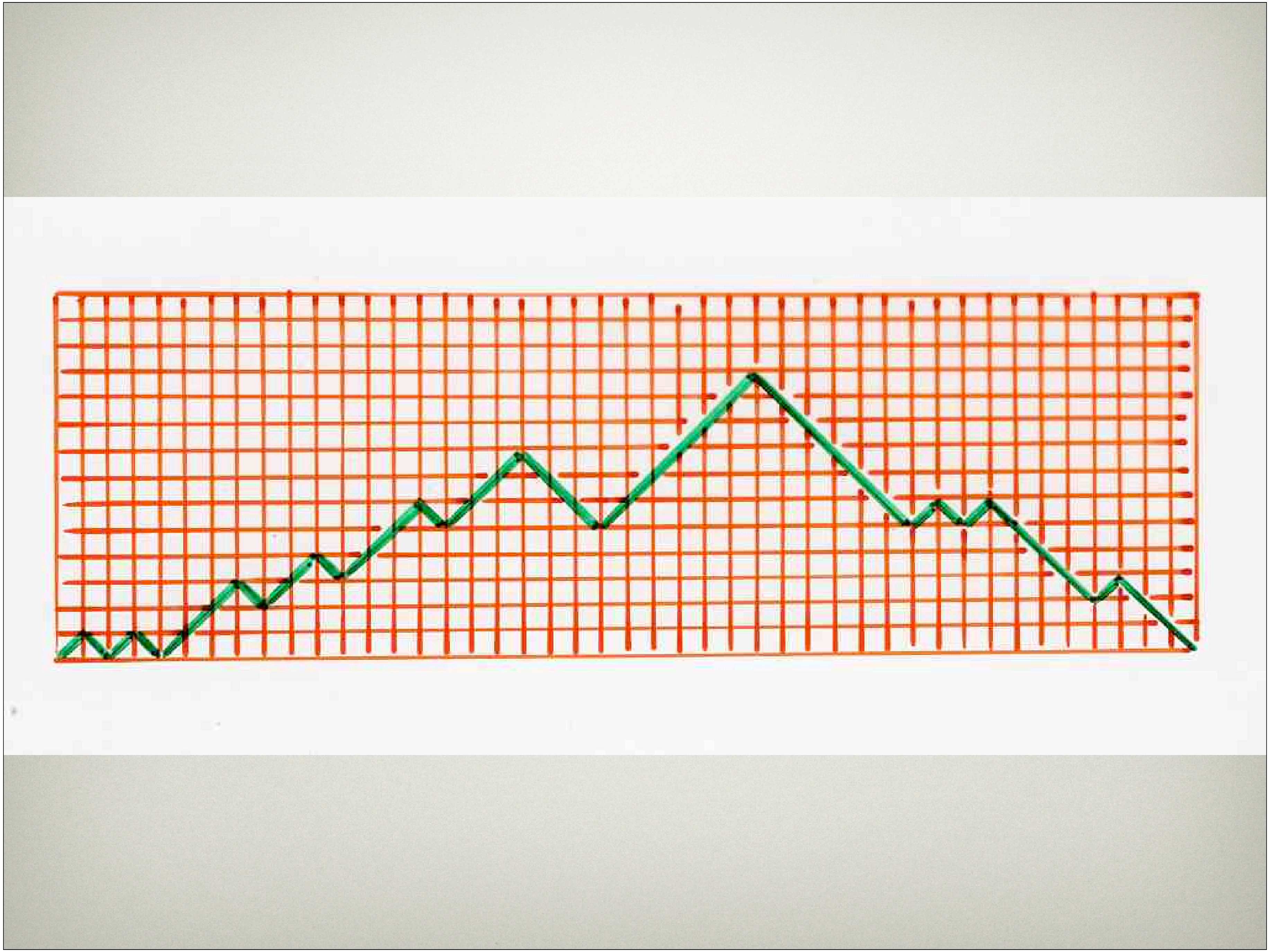
binary trees $\xleftrightarrow[\text{(1984)}]{\text{Françon}}$ Dyck paths

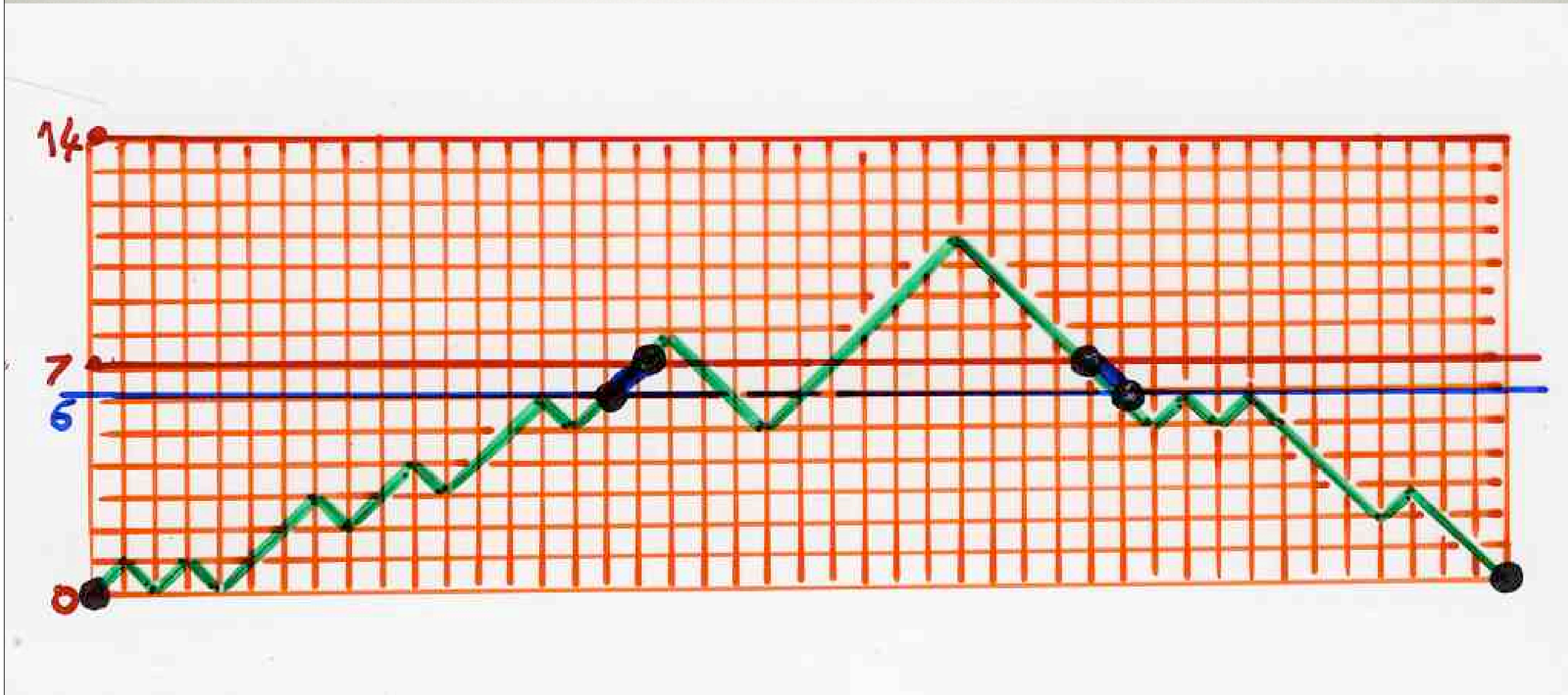
n (internal) vertices — length $2n$

Strahler nb $= k$ — log. height $lh(w) = k$

$$\xleftrightarrow[\text{(2005)}]{\text{Knuth}}$$

5. Bijection
chemin de Dyck
système de tours de Kepler

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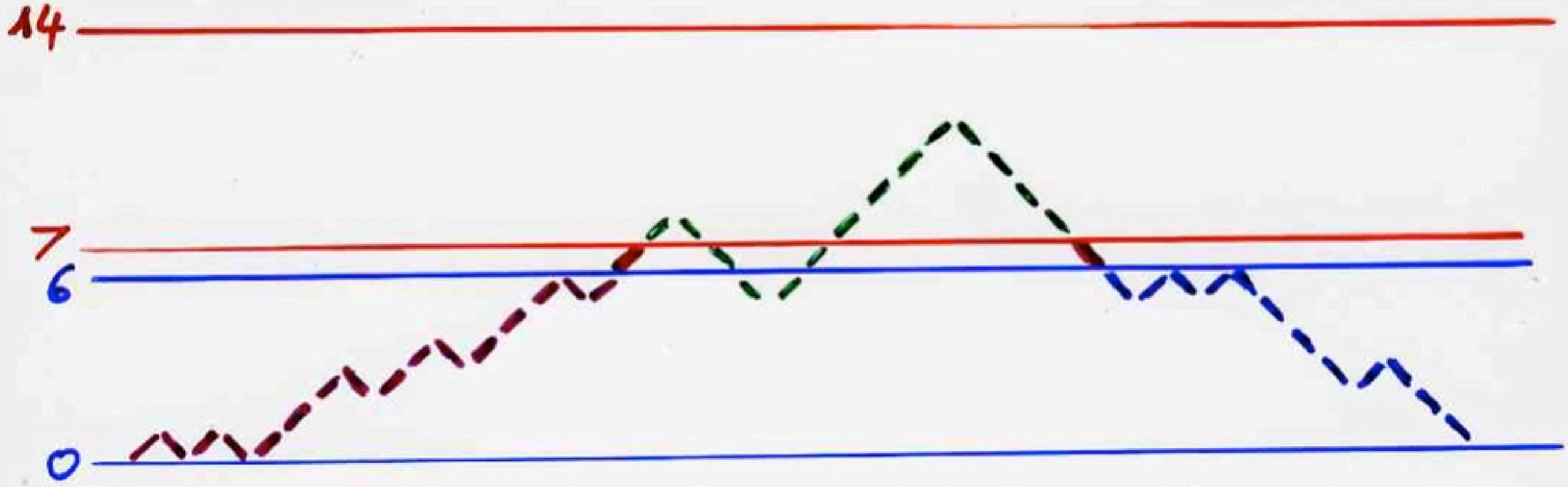

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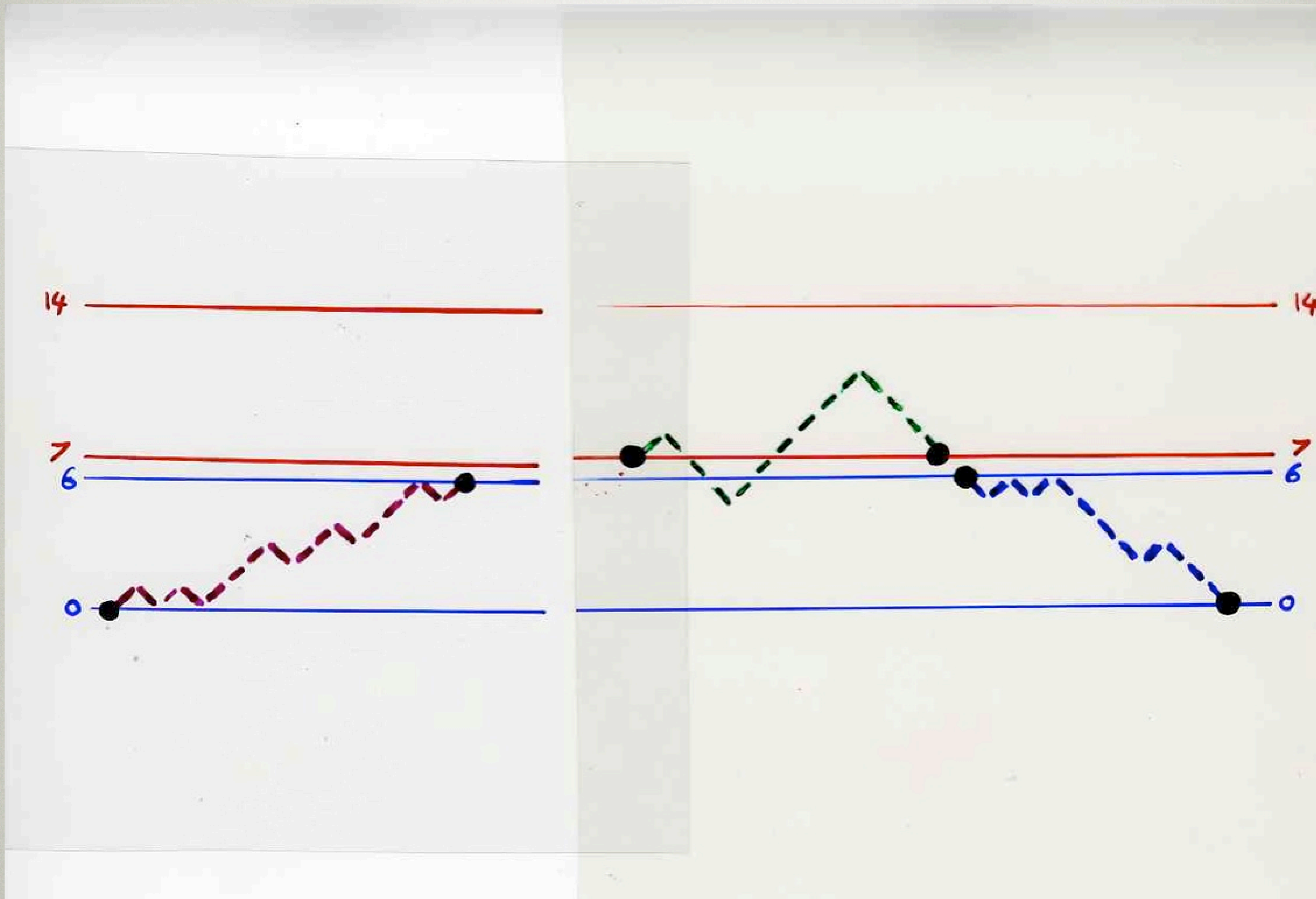
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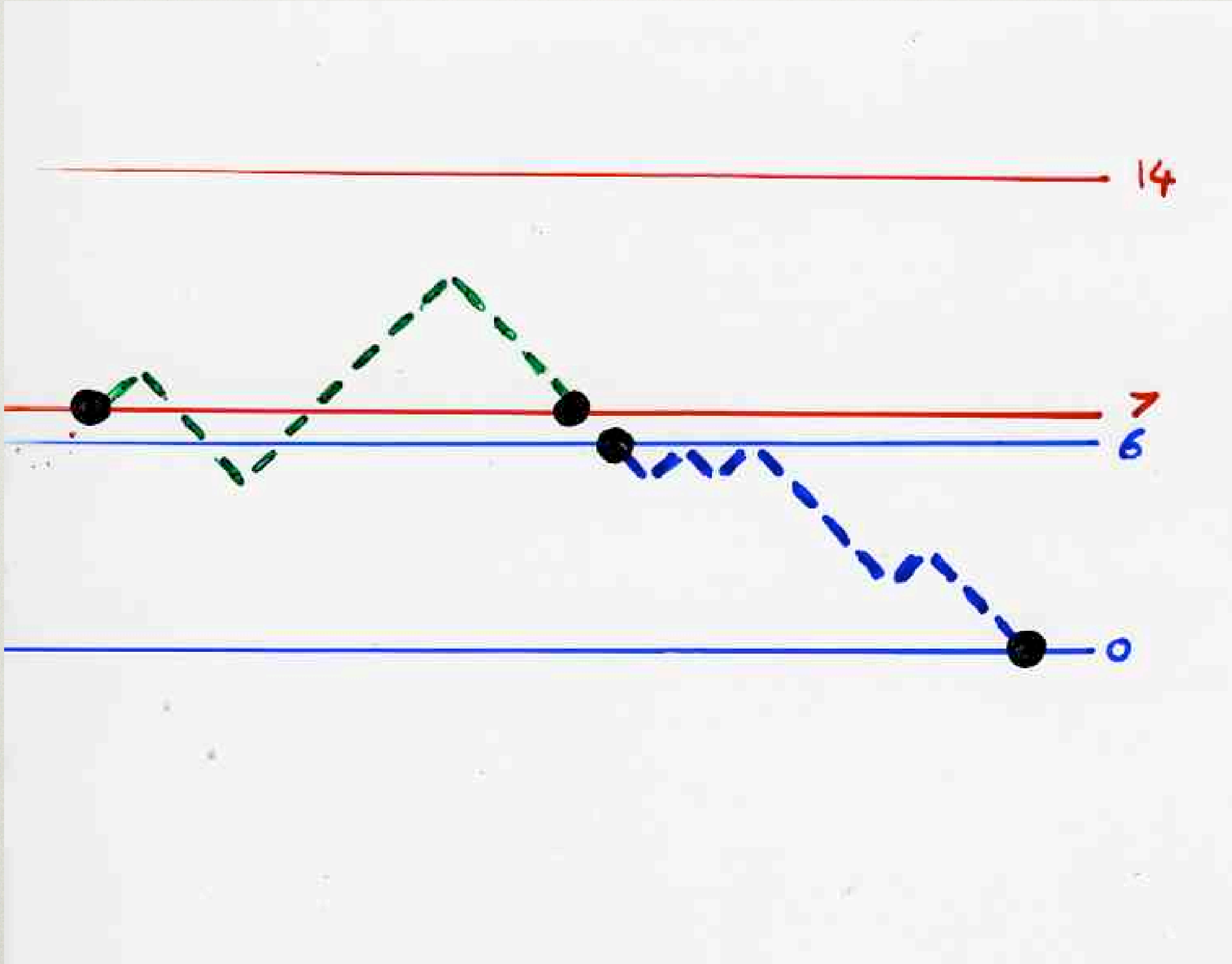
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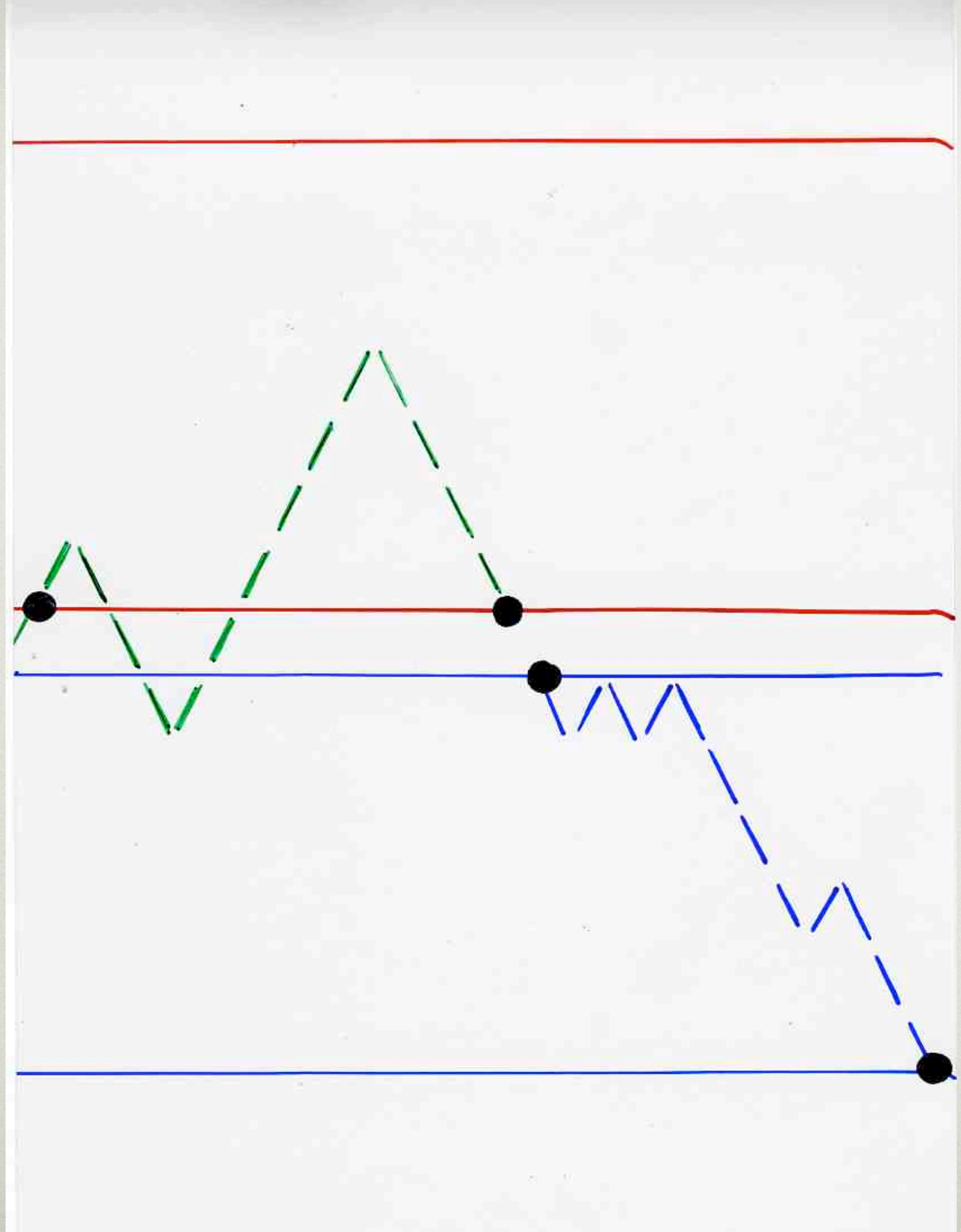
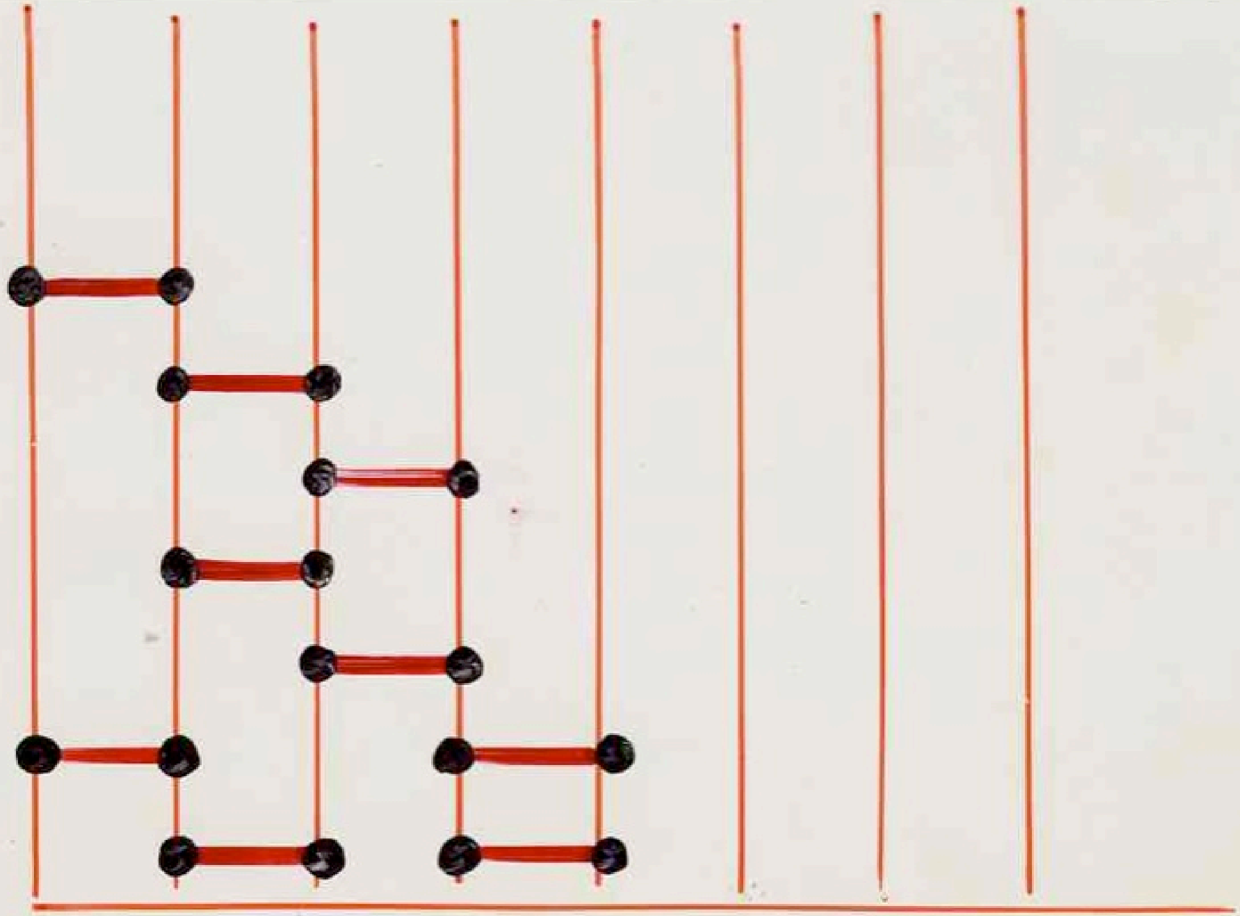
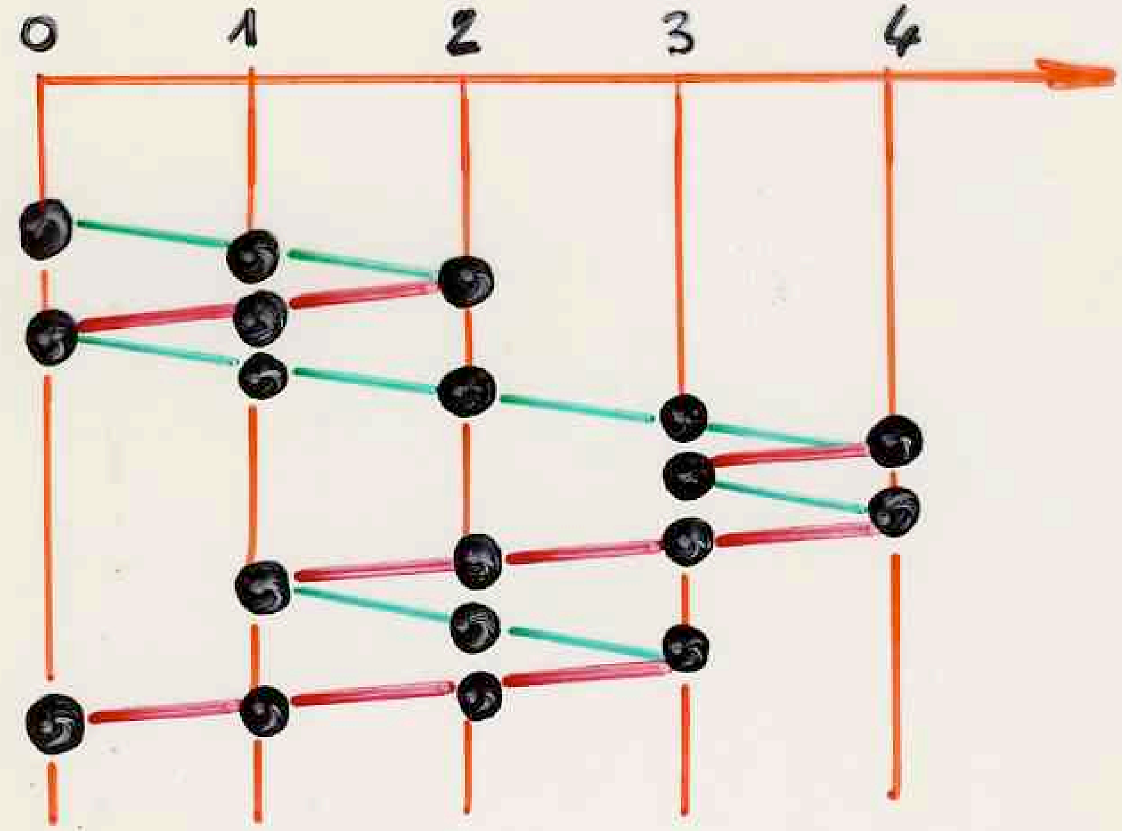


rappel: bijection
chemin de Dyck
demi-pyramide de dominos

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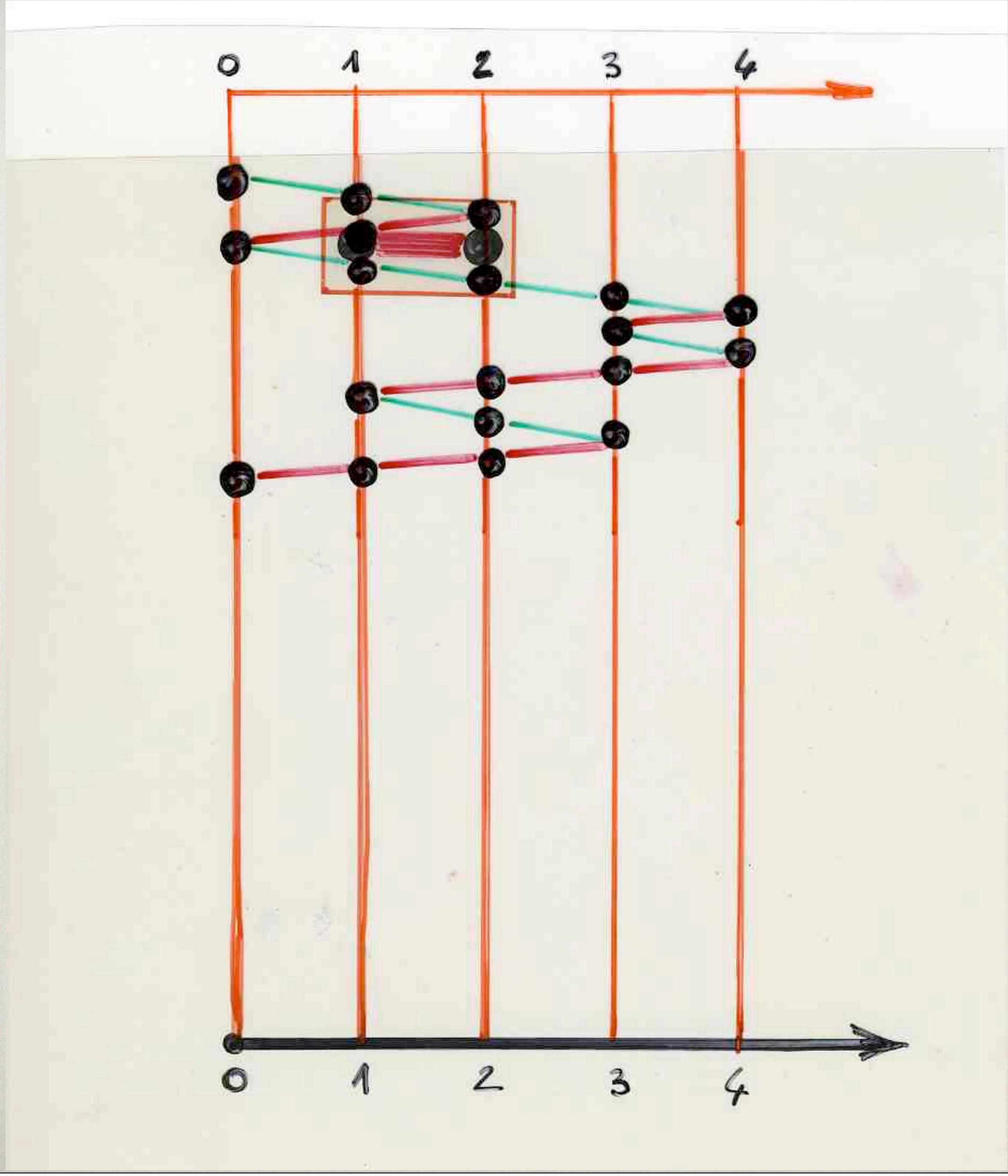

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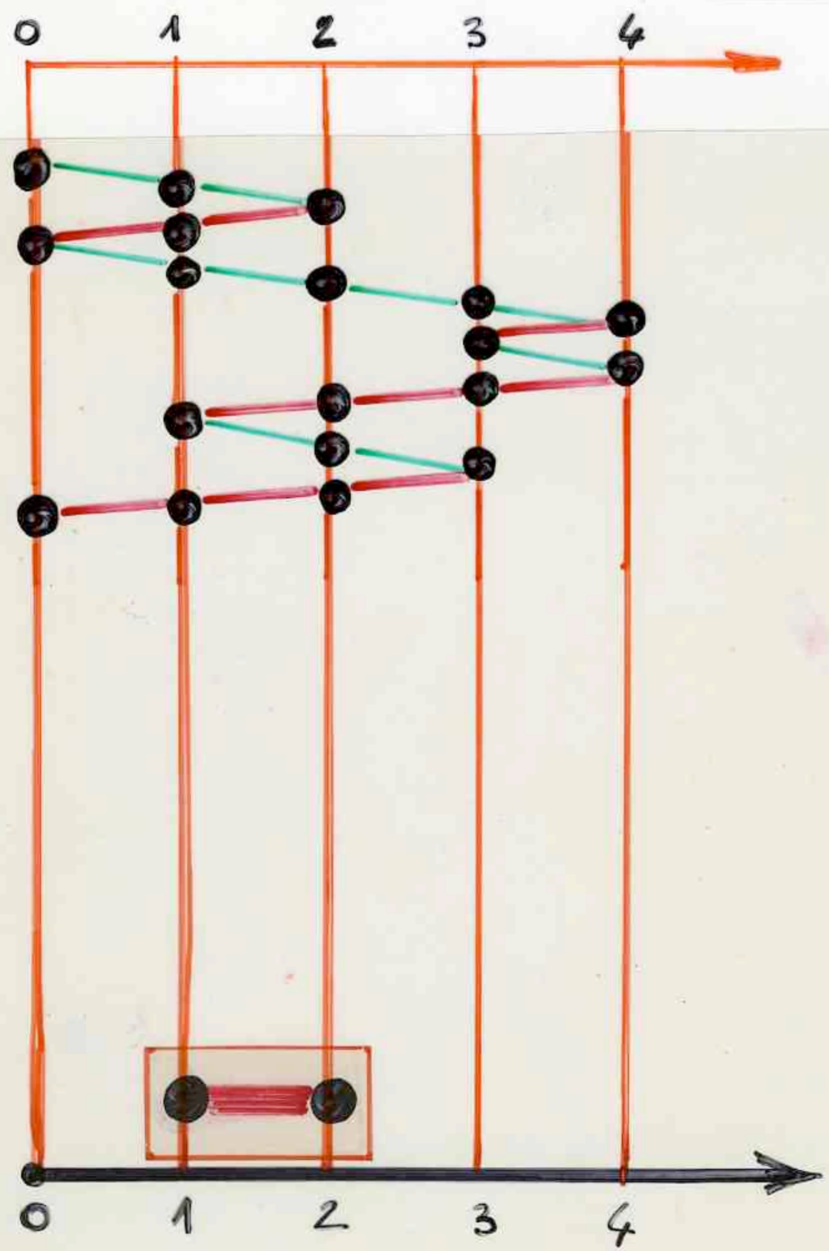
0 1 2 3 4



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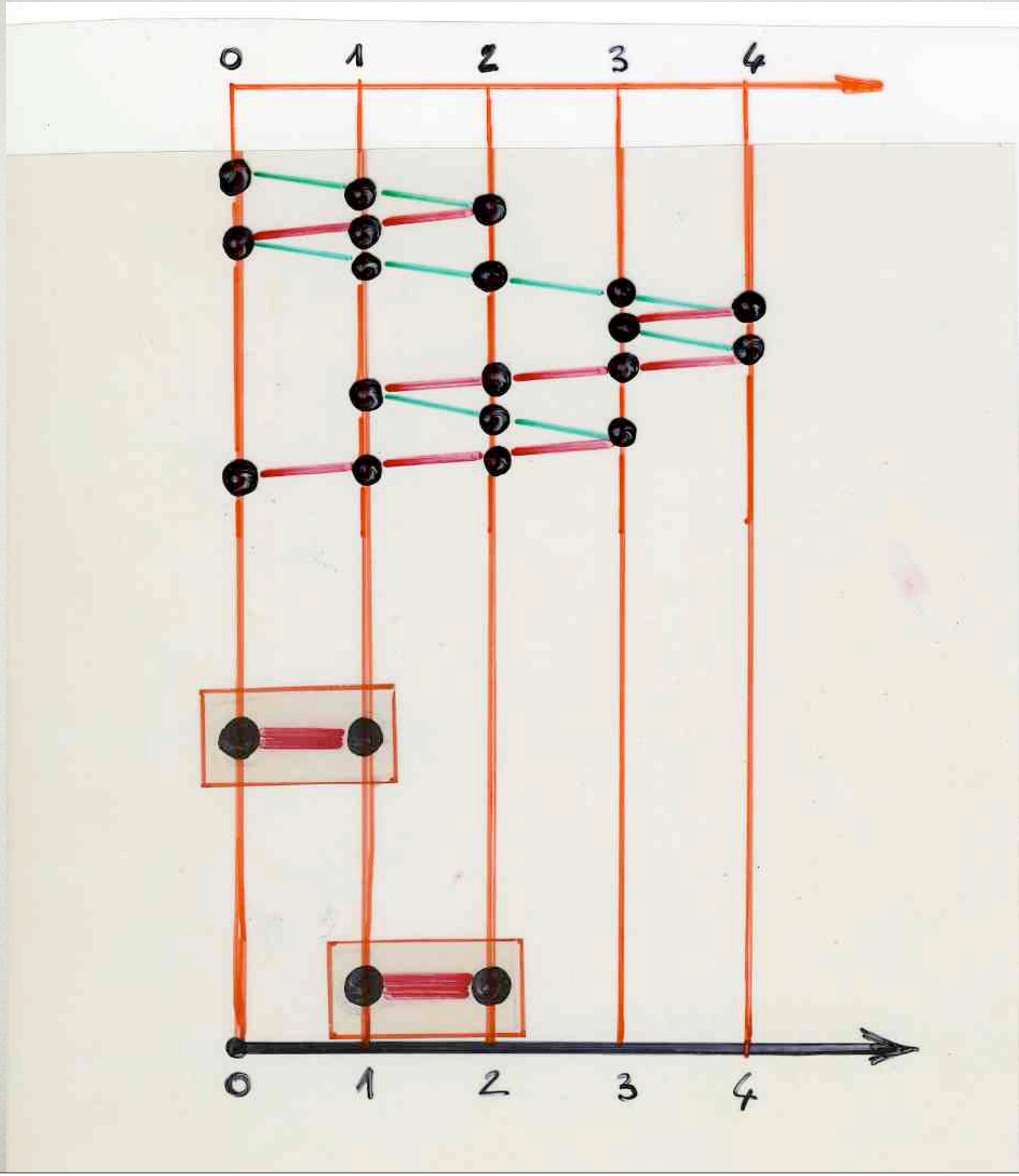

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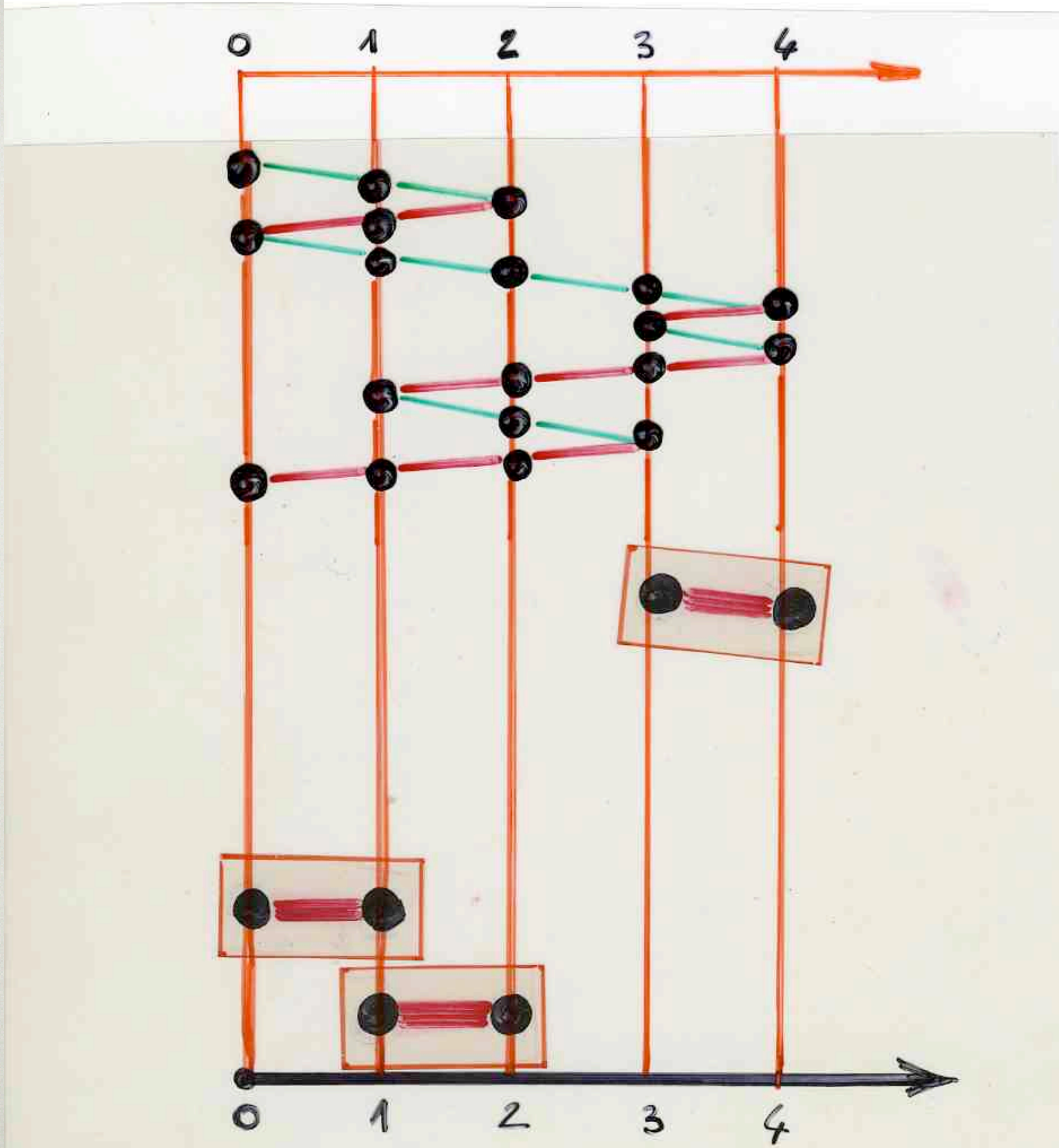

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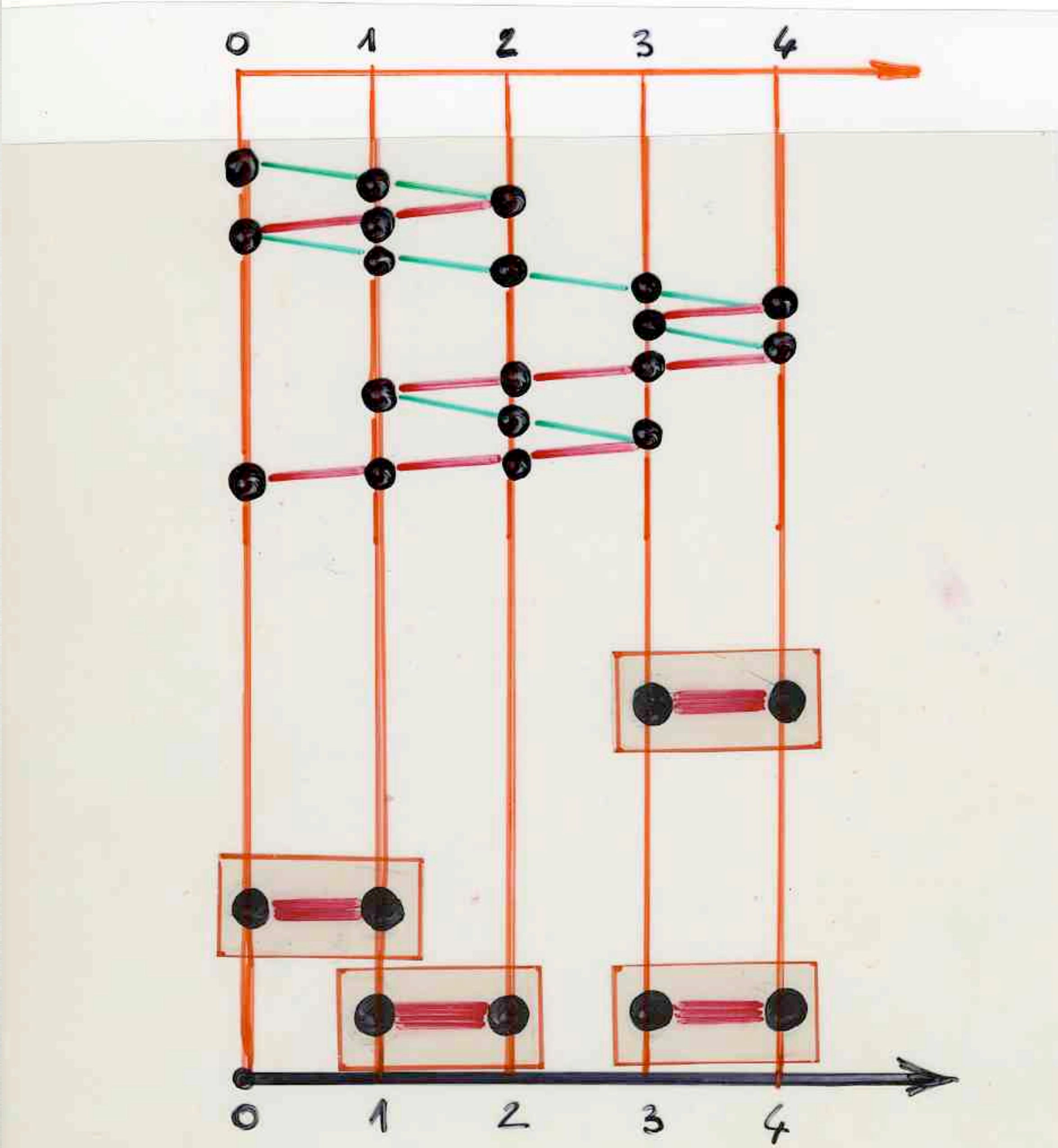
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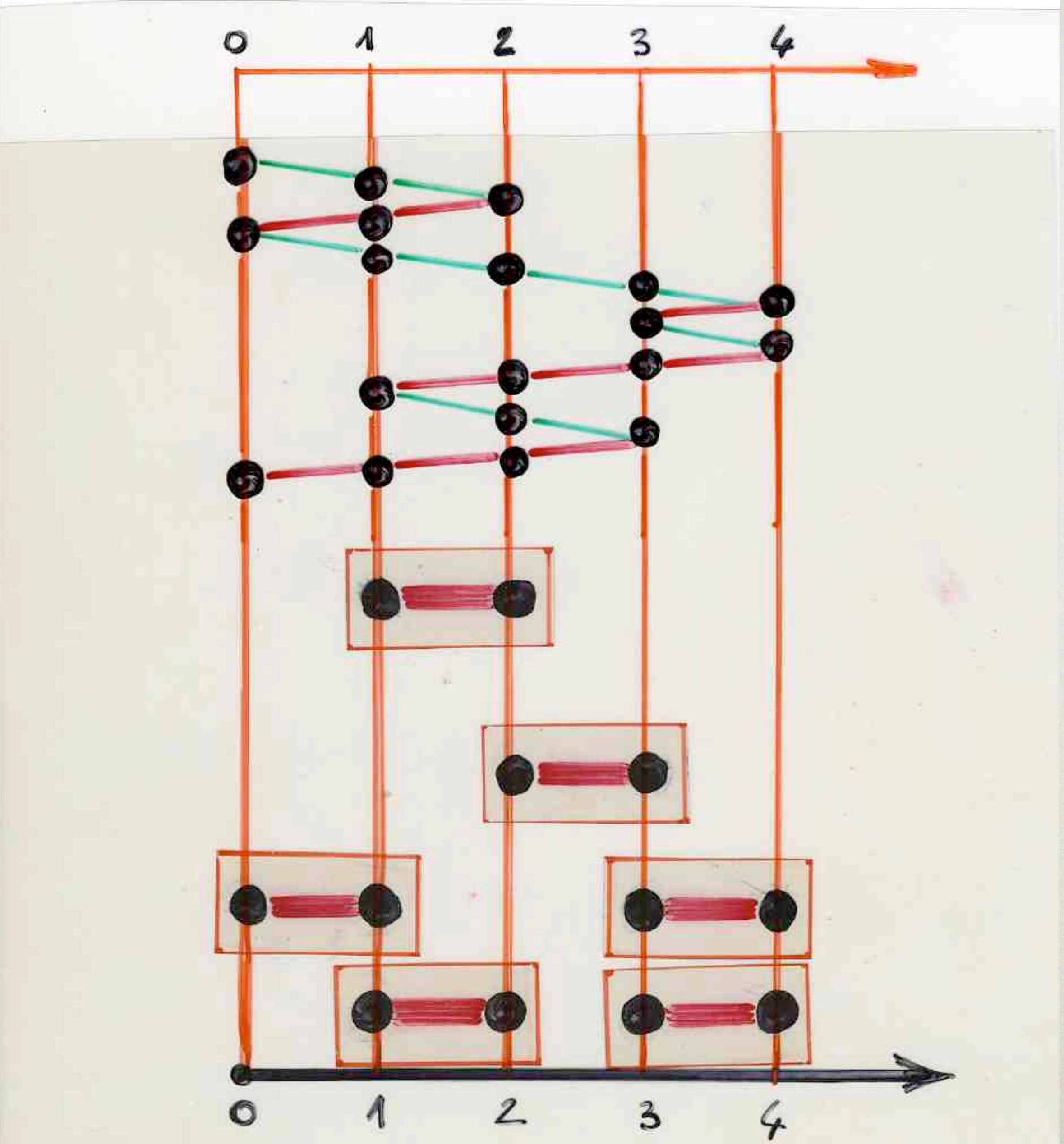
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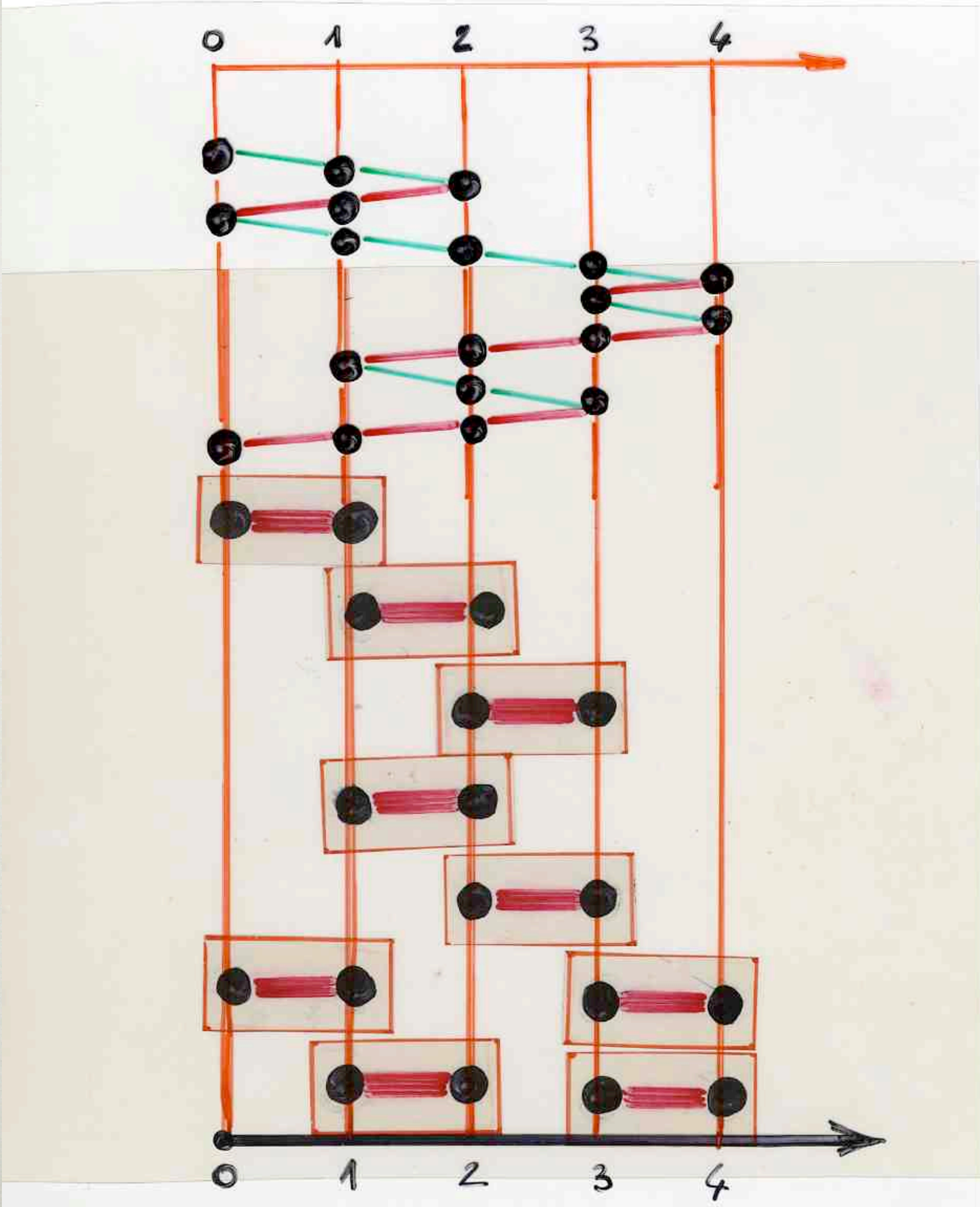
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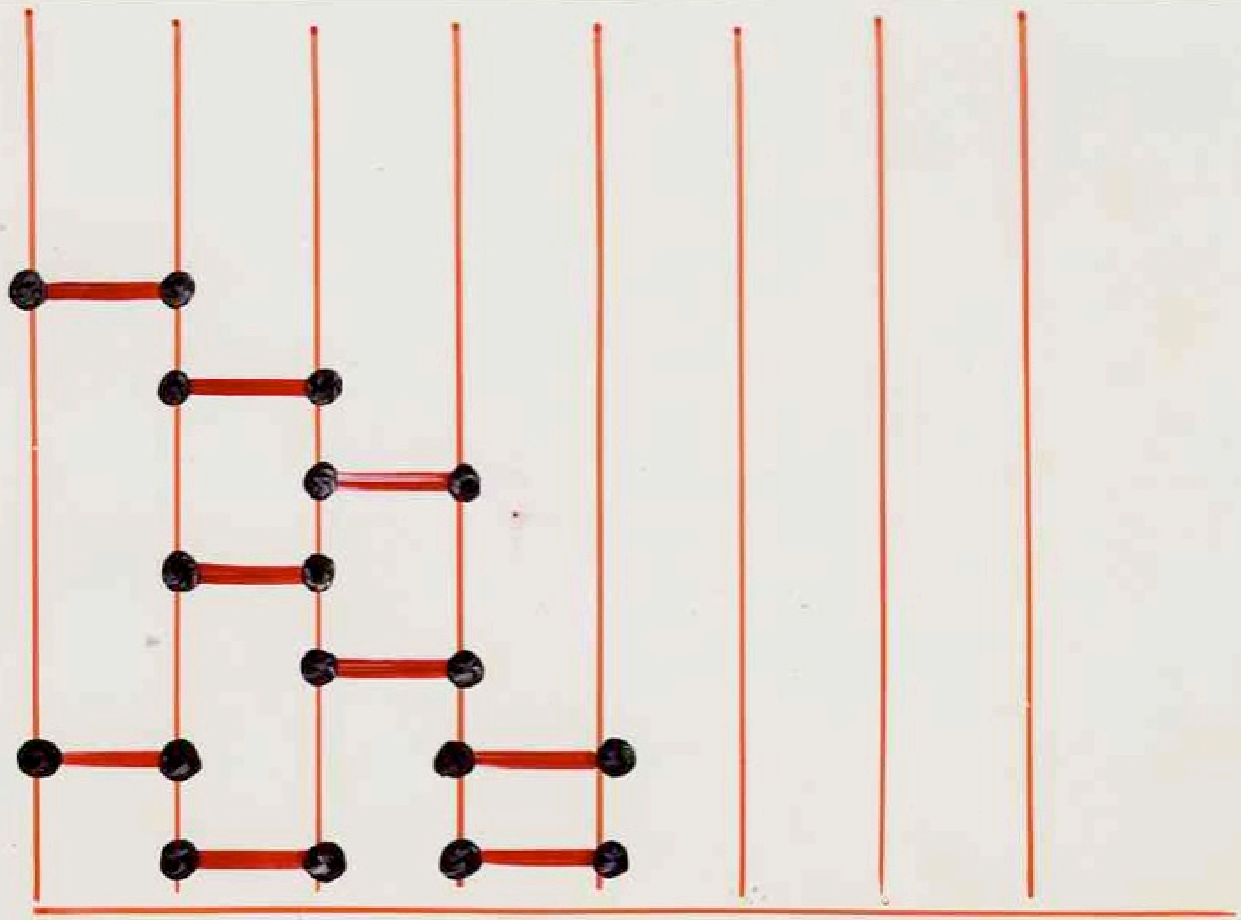
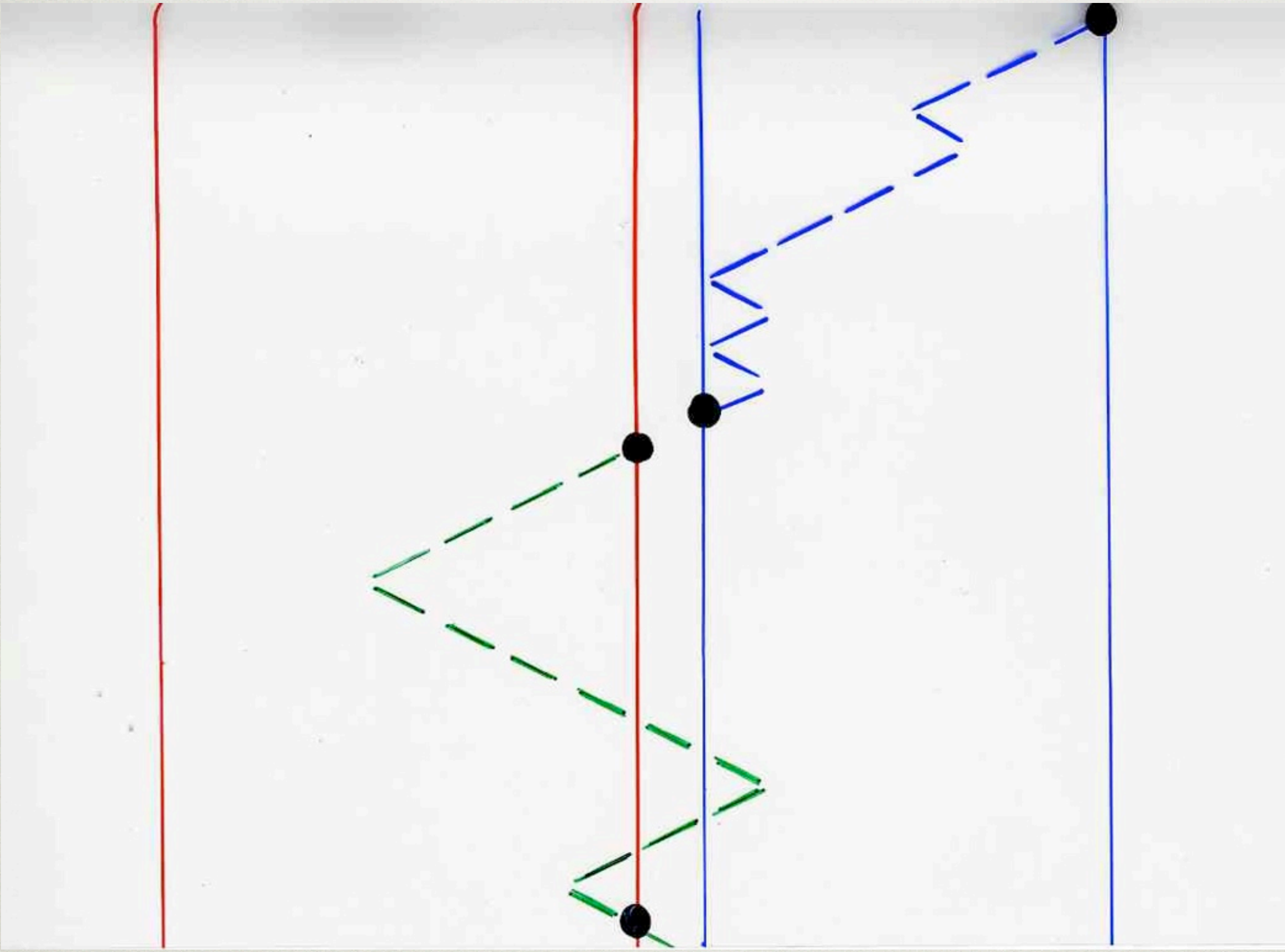
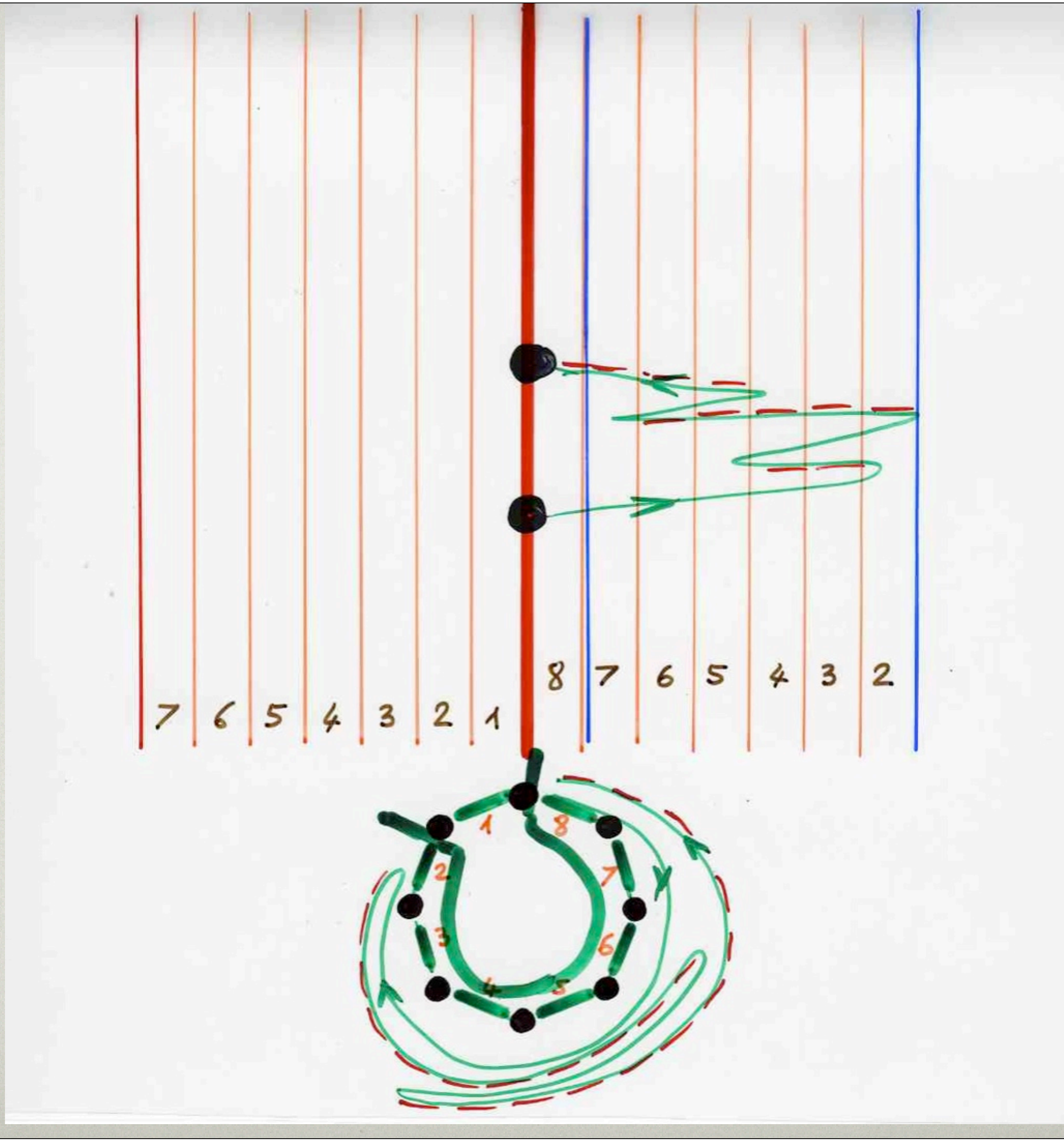


revenons à notre construction





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8 7 6 5 4 3 2
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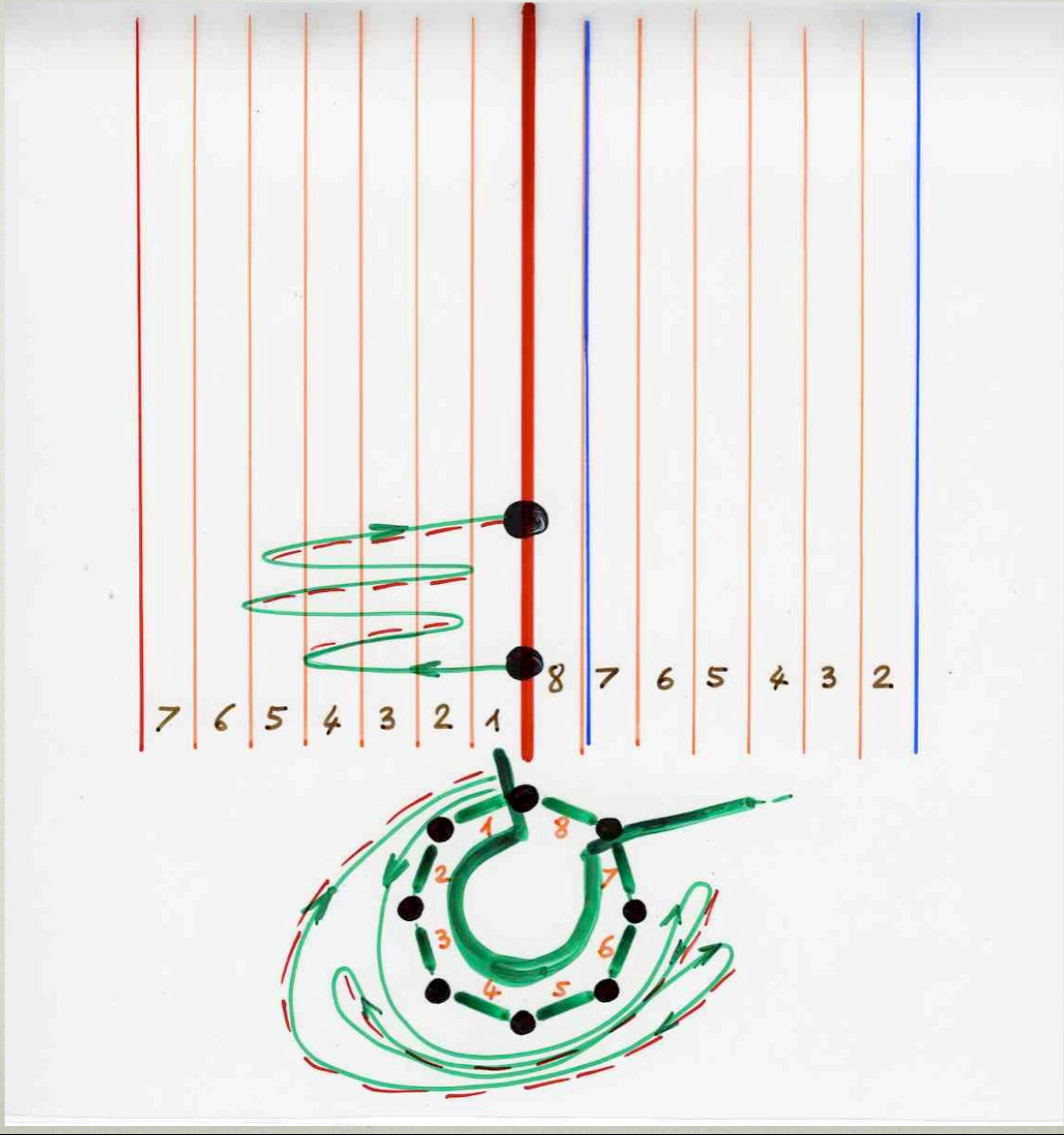

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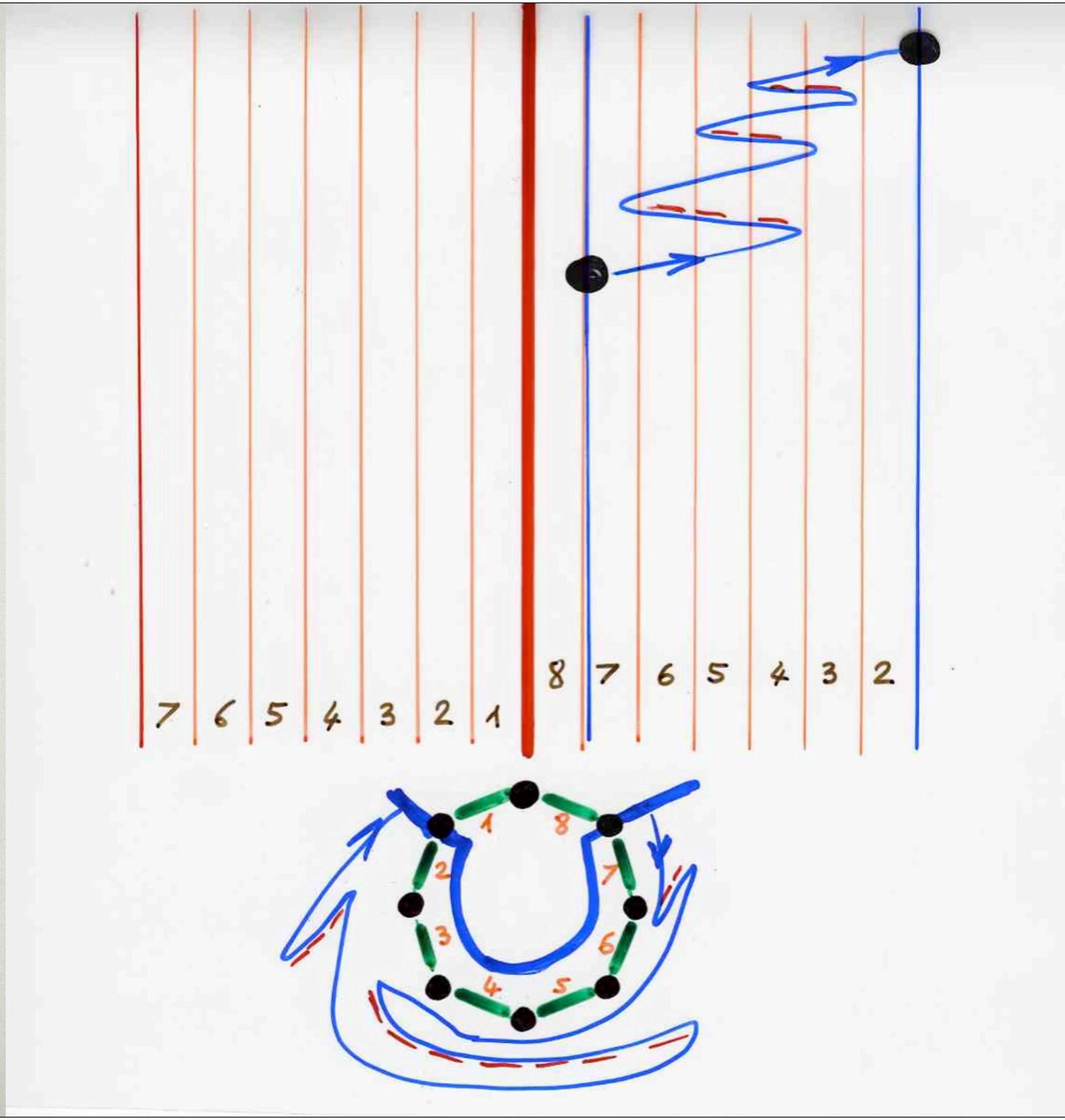

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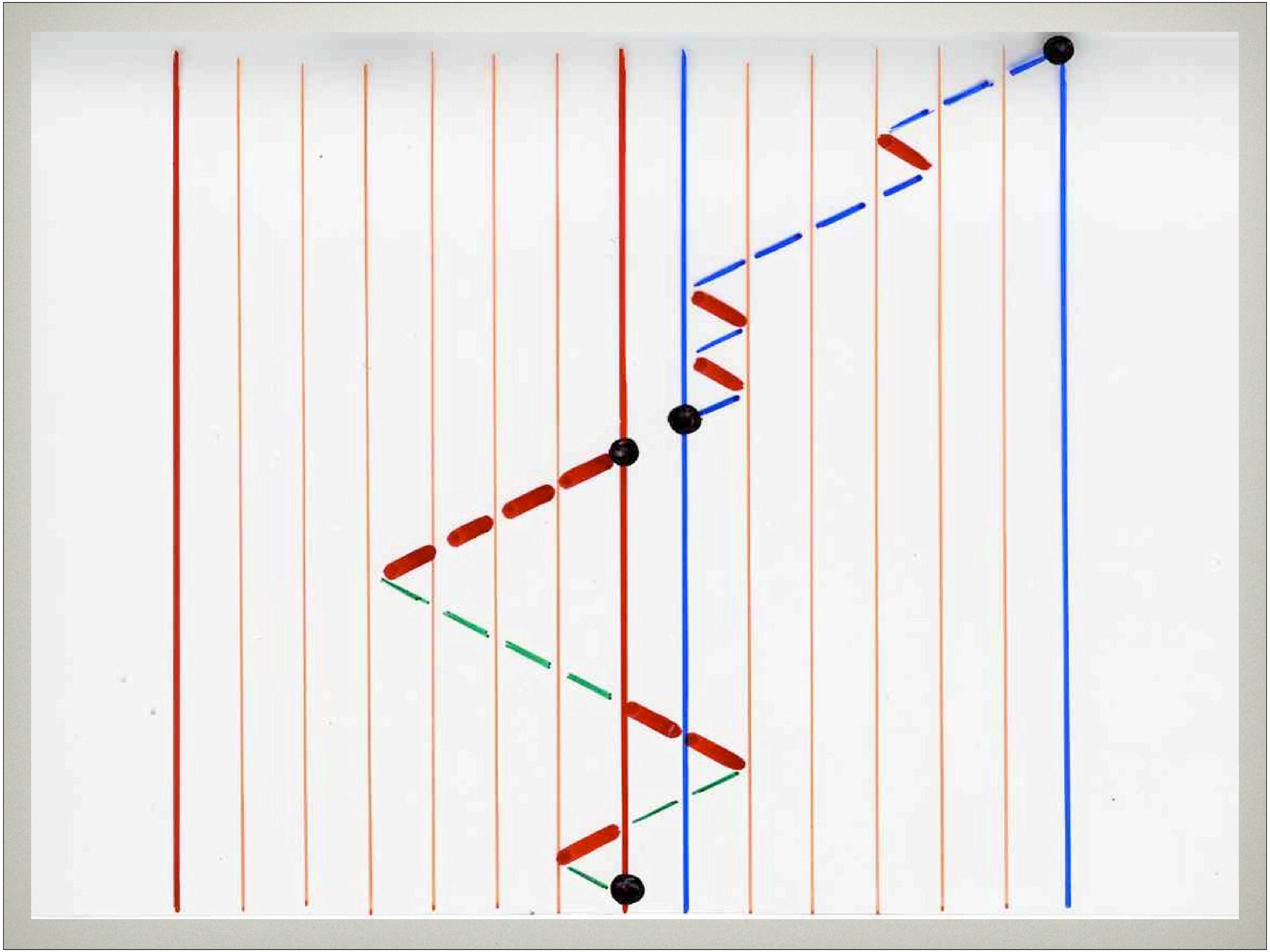

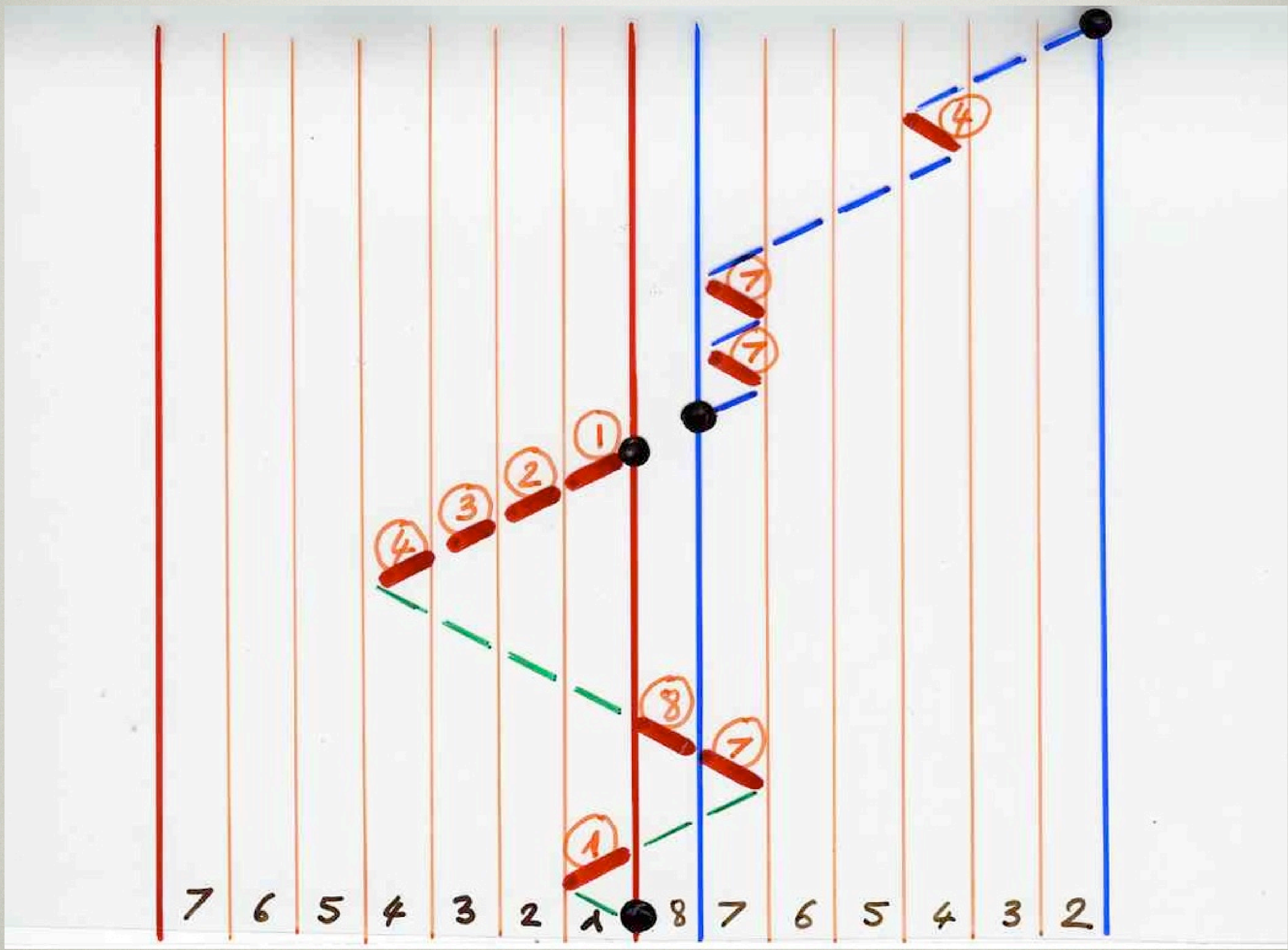
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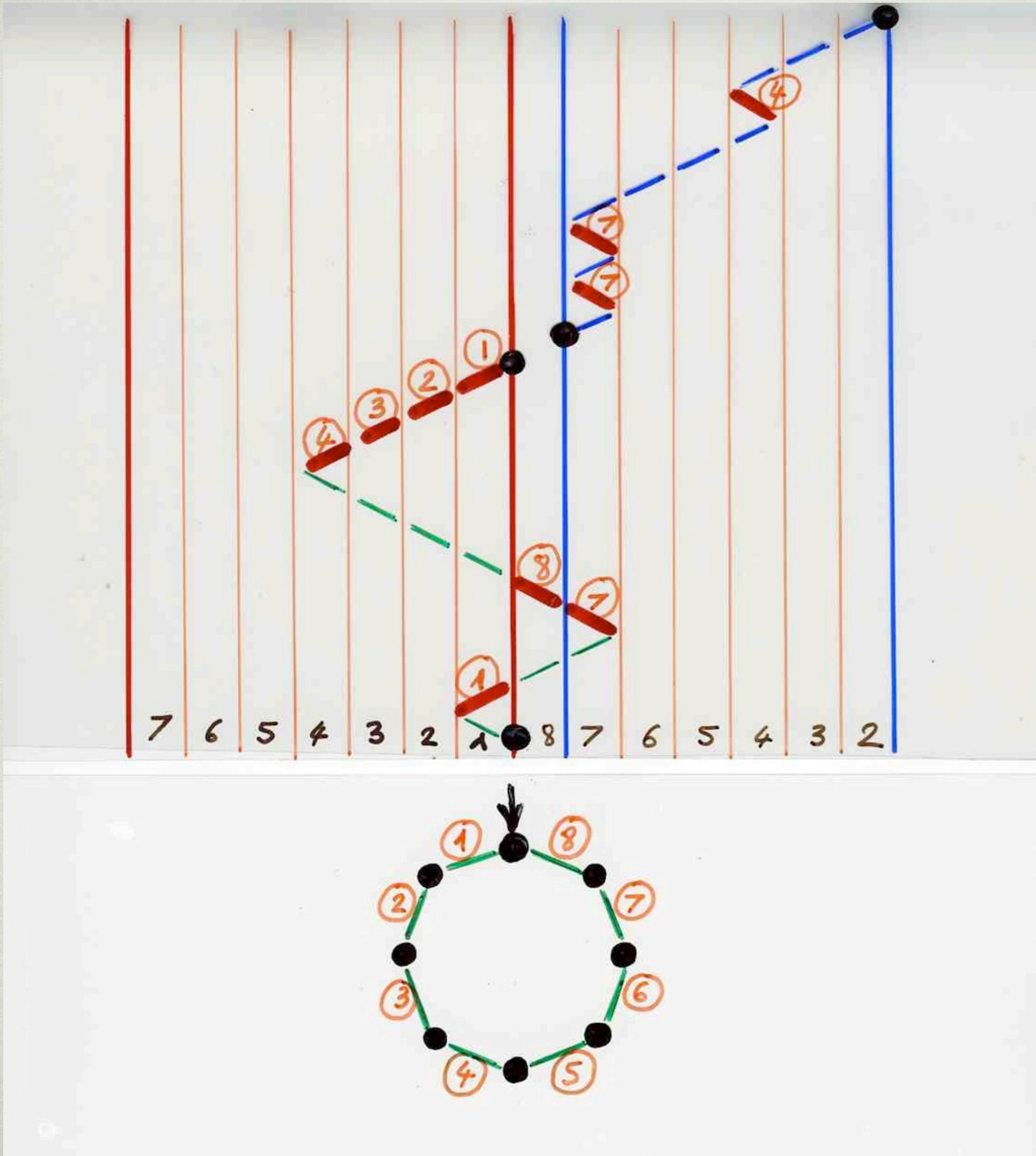

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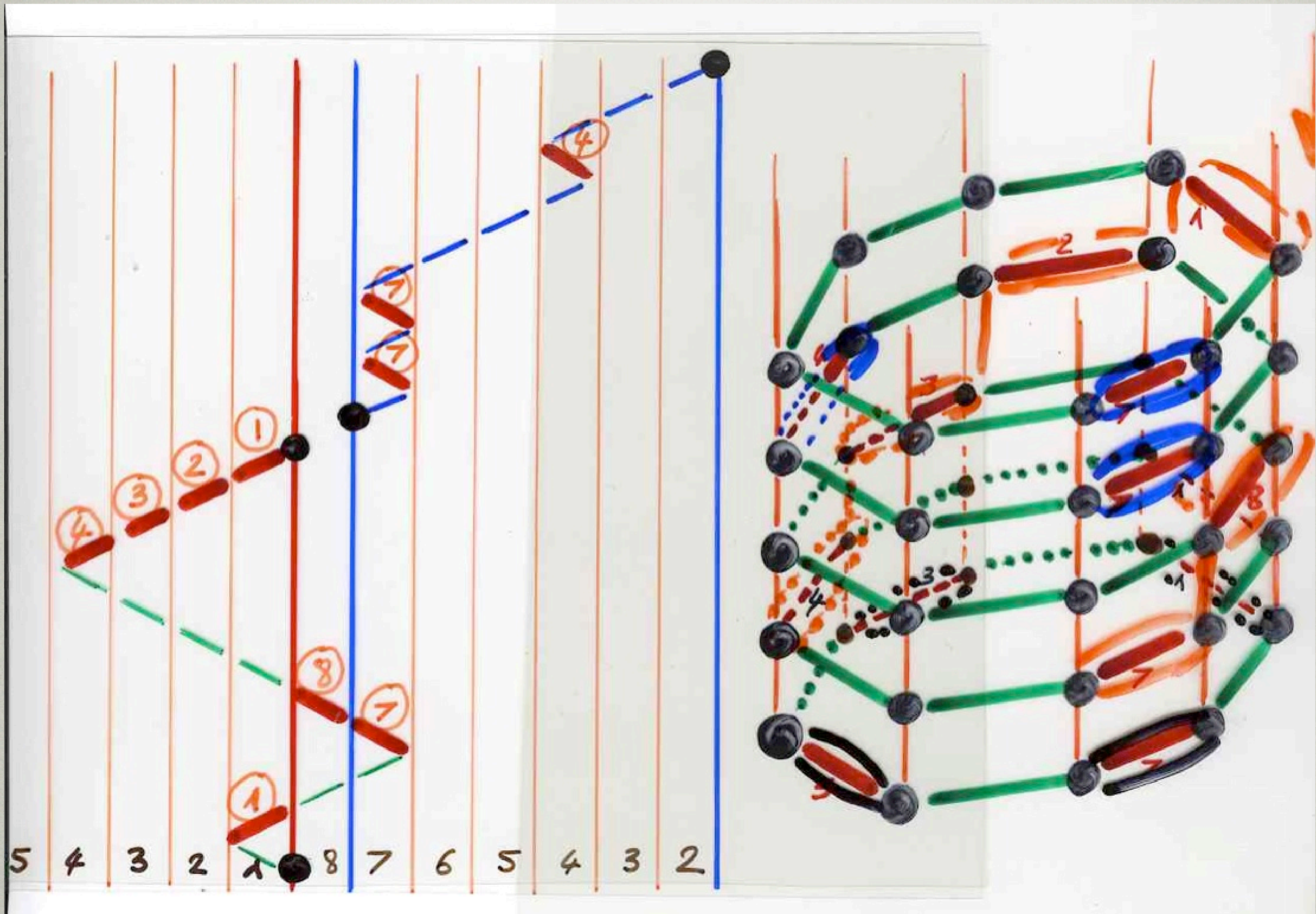

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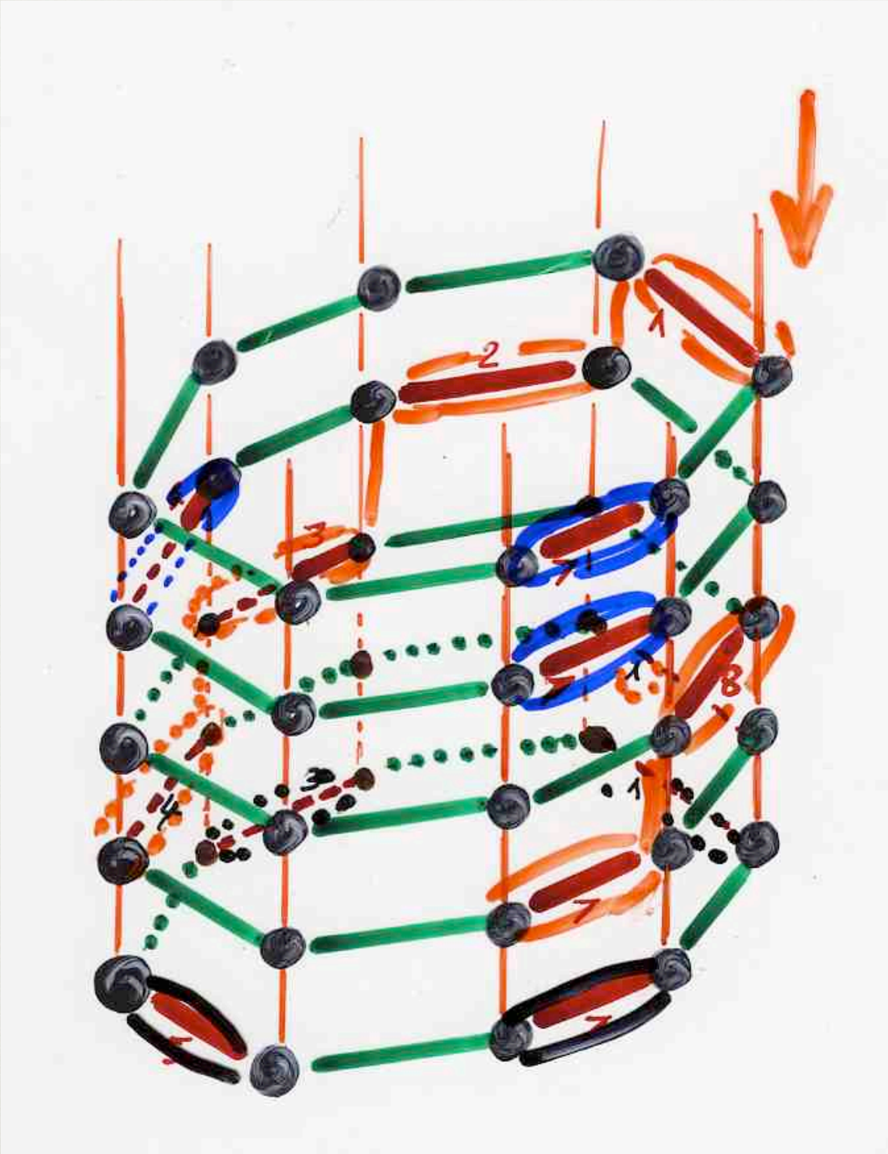

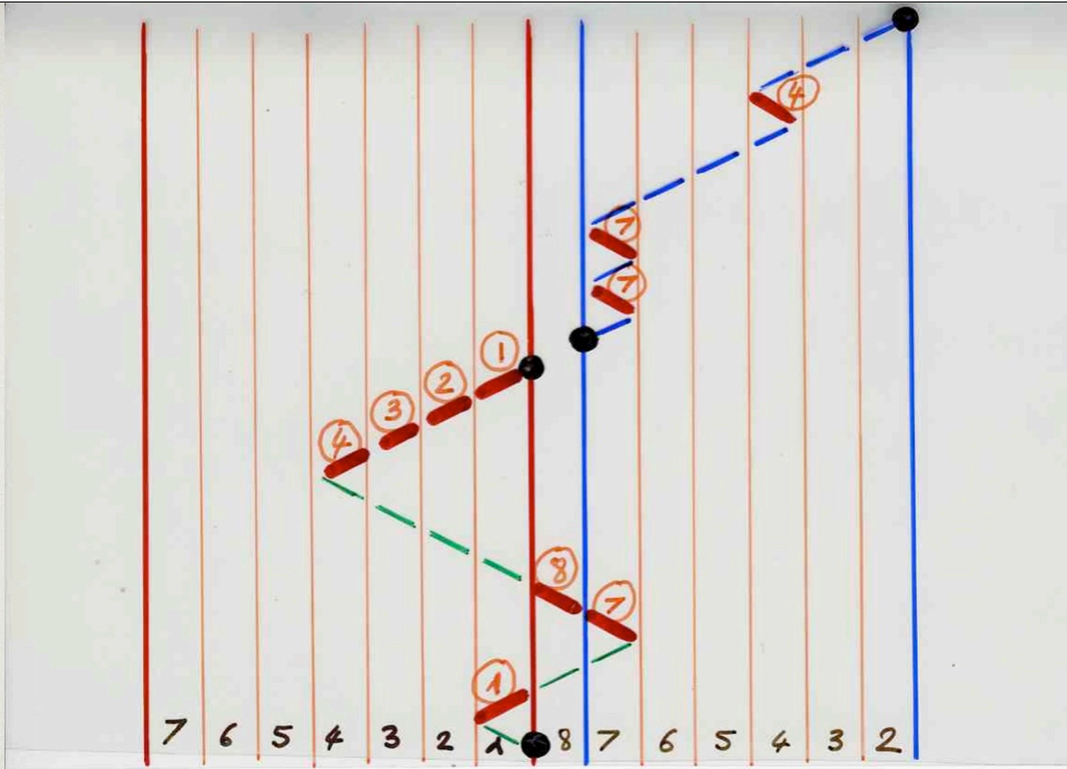

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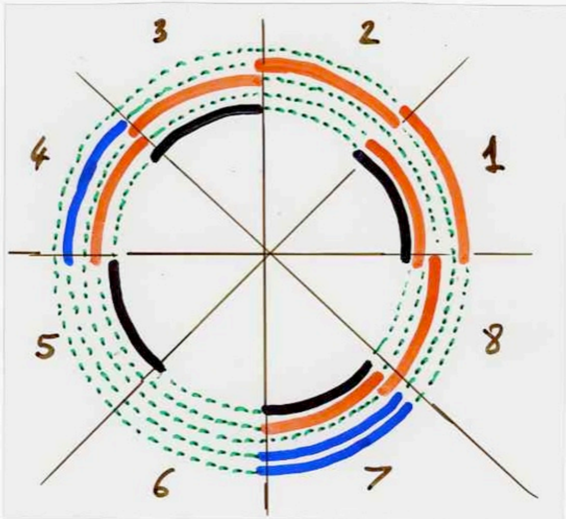

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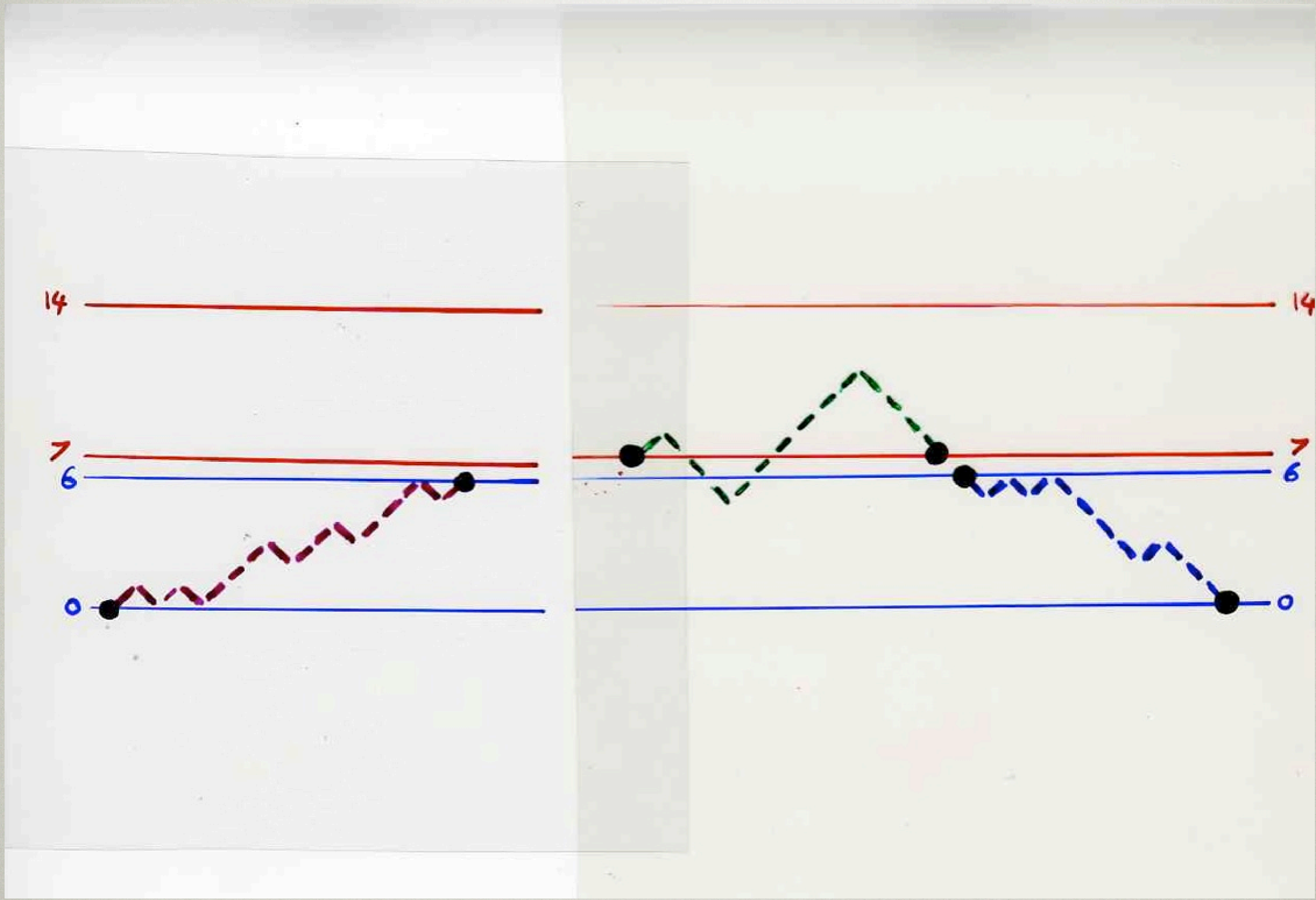

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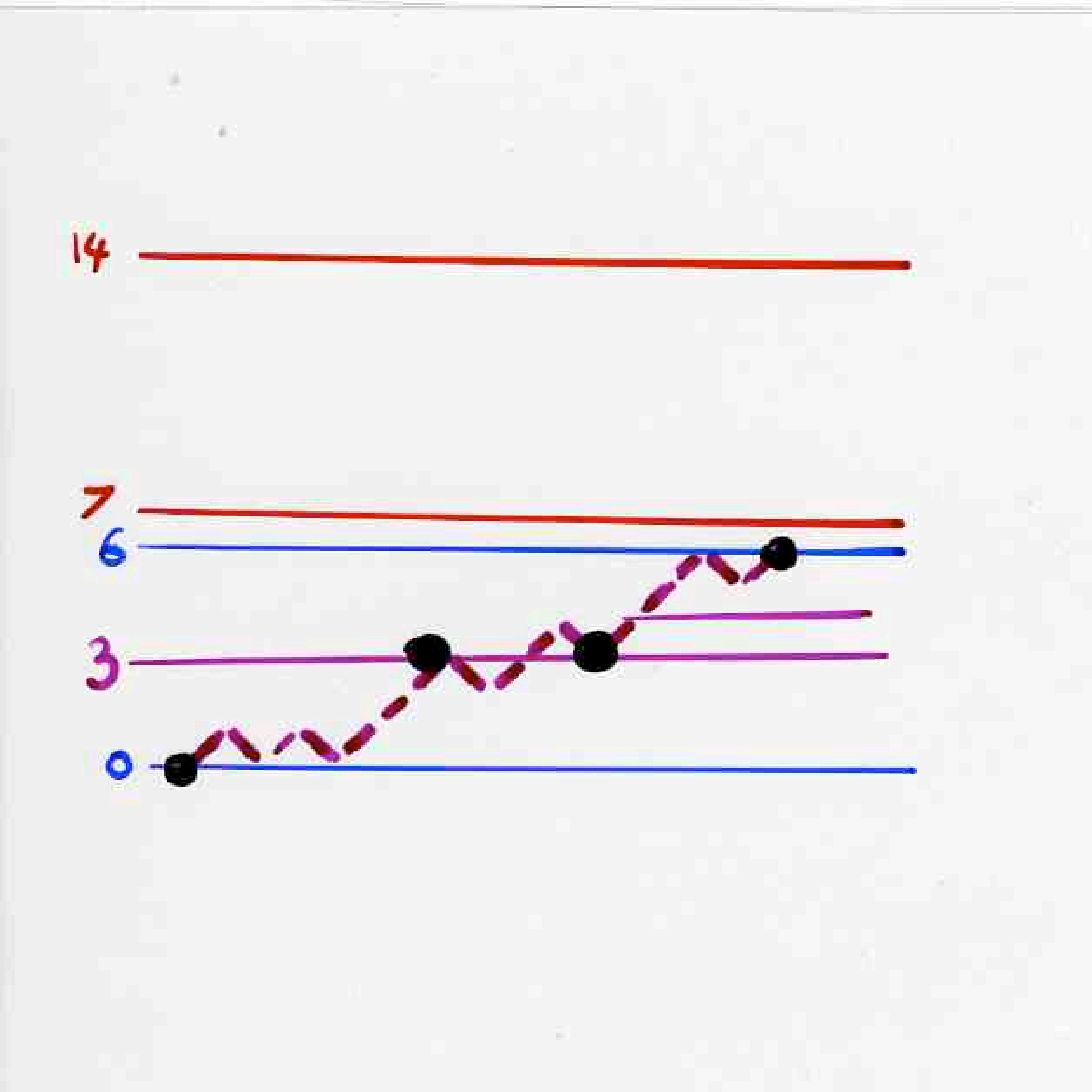

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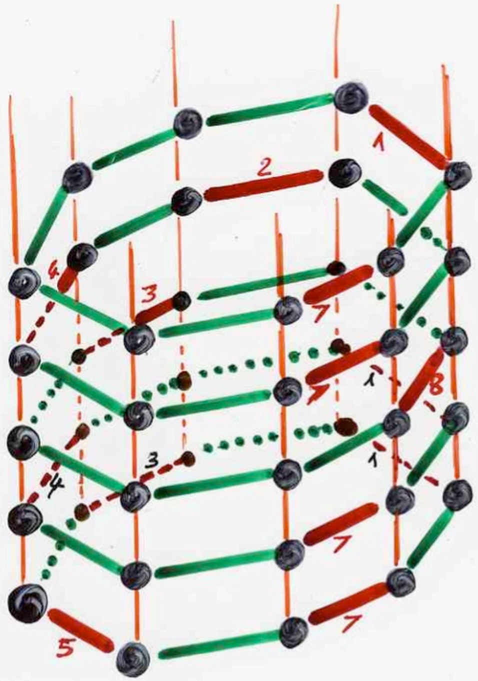
récursivement ....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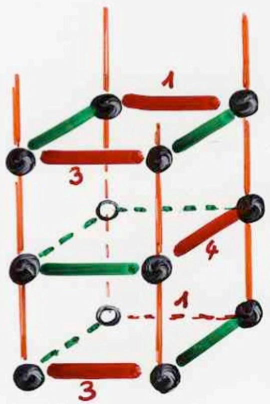
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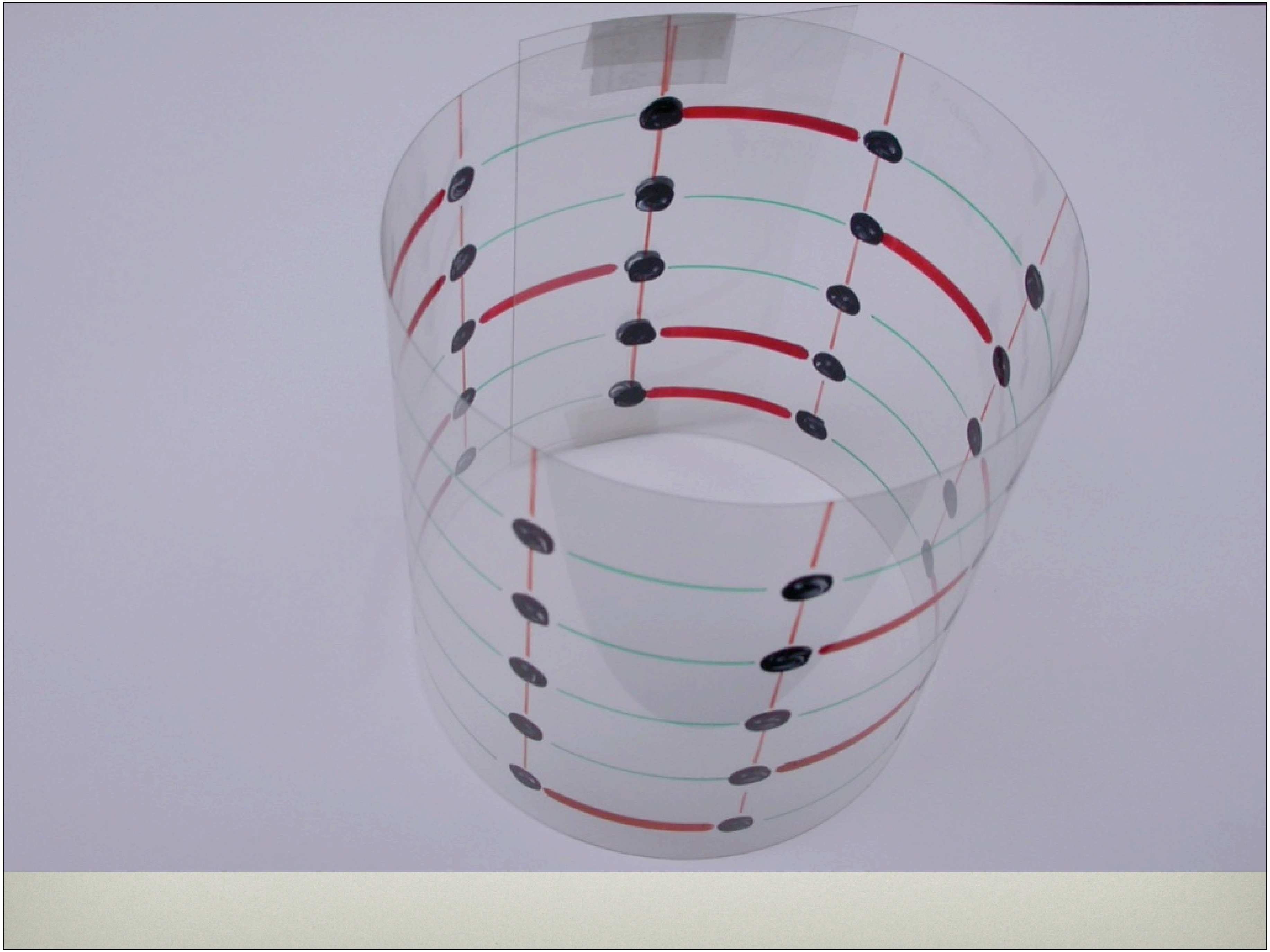
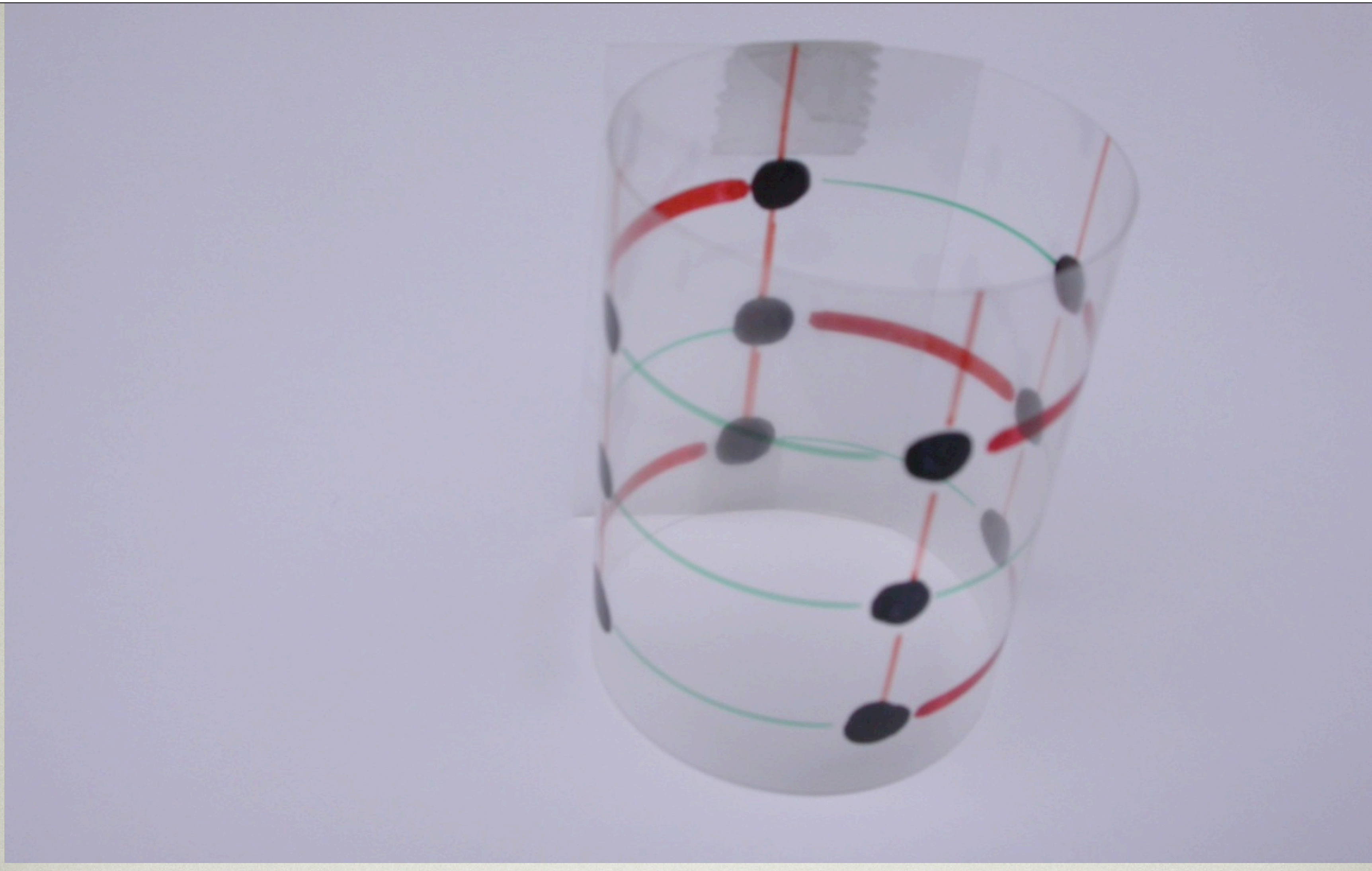
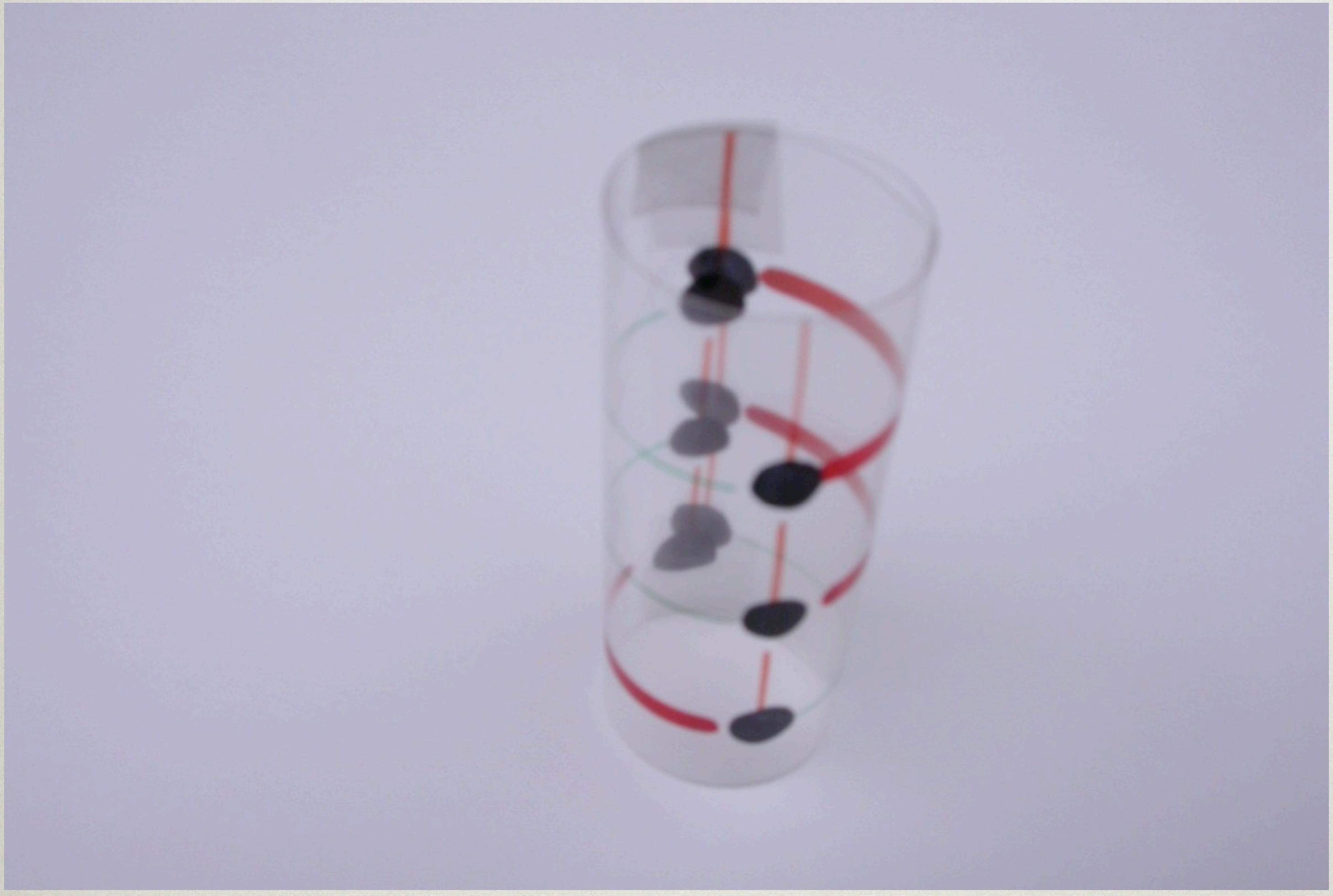
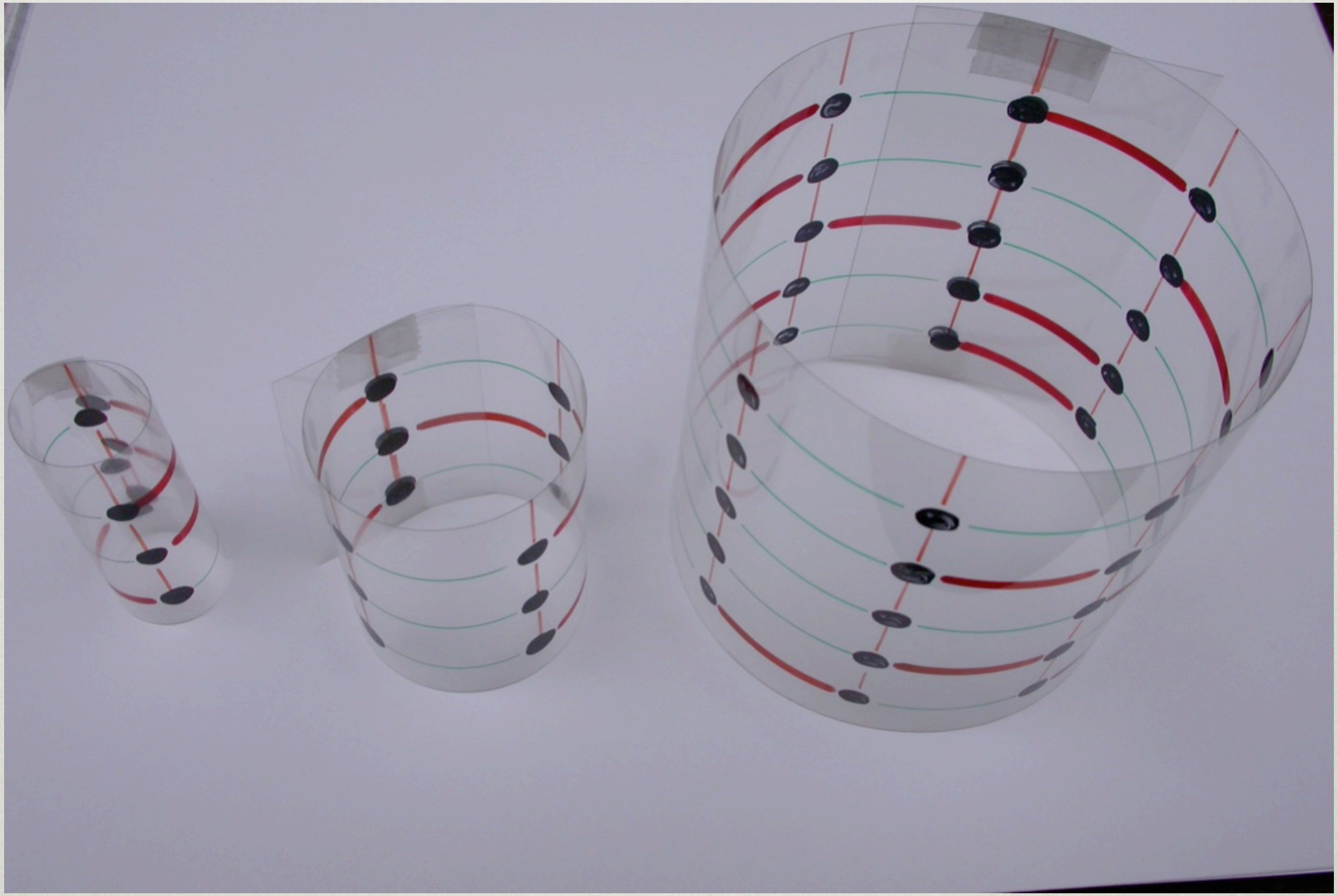
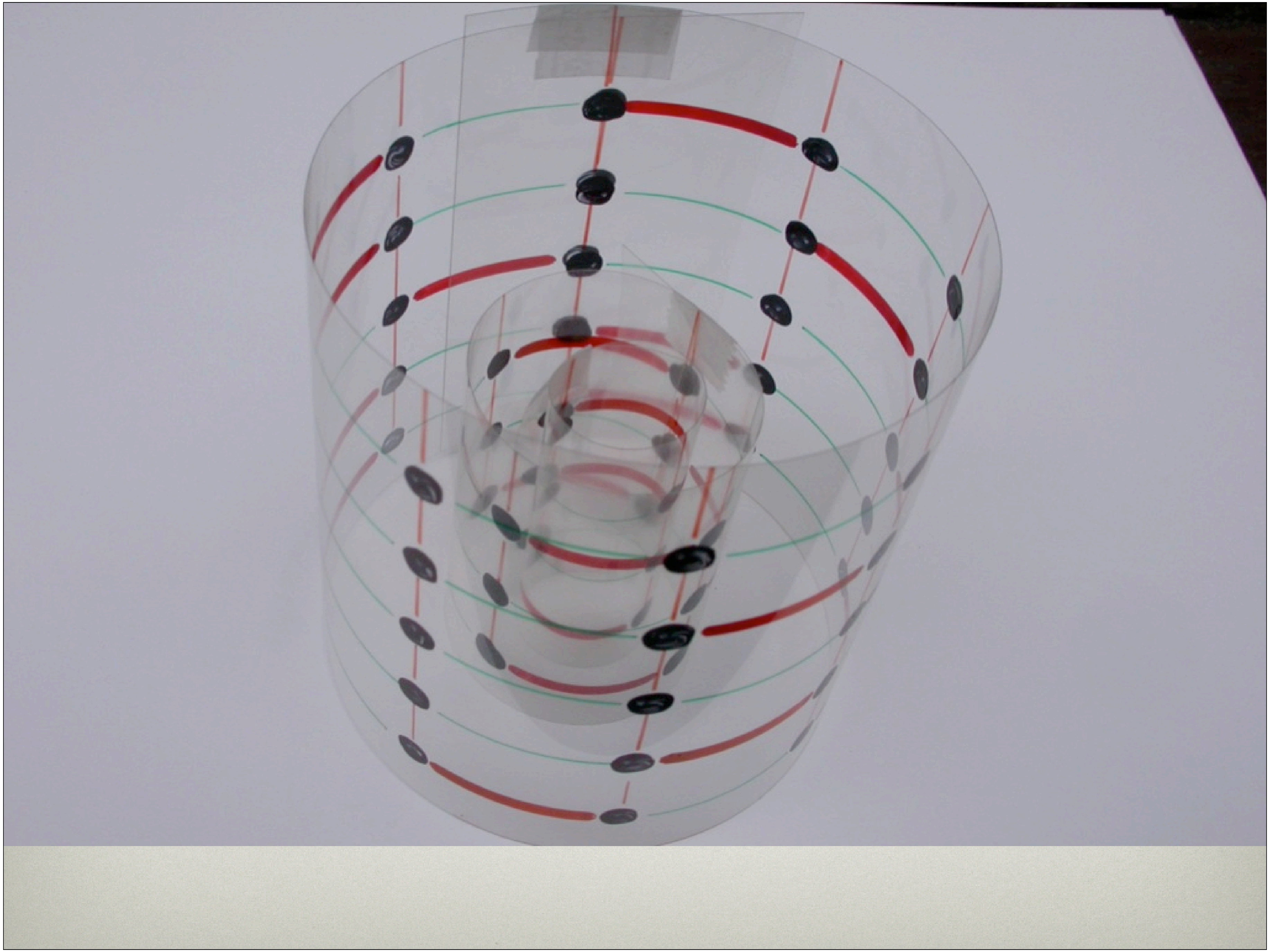


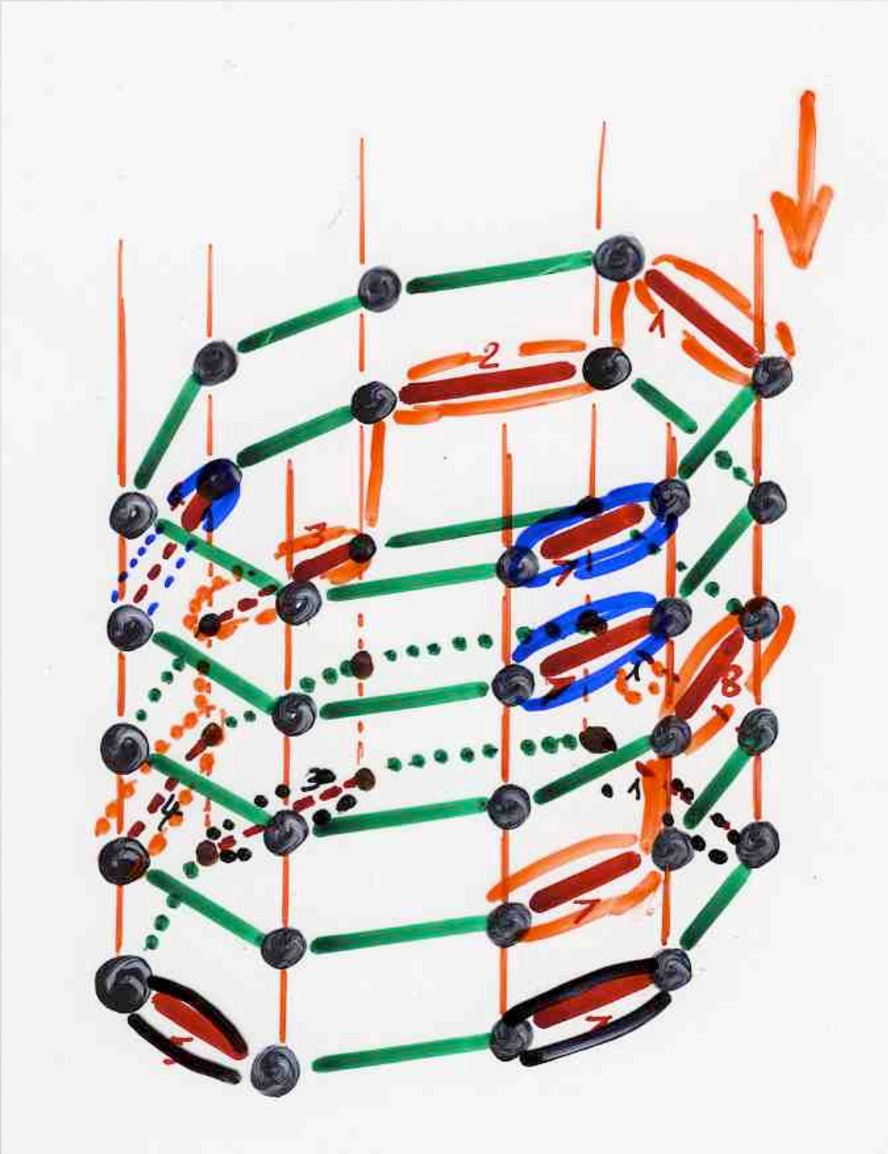






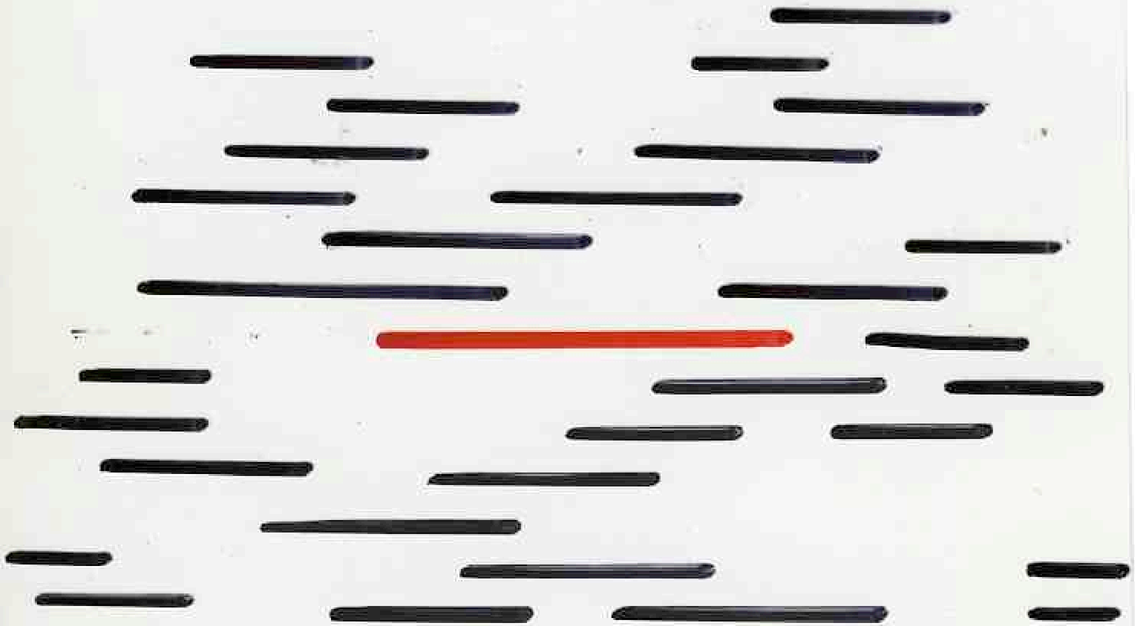


bijection inverse



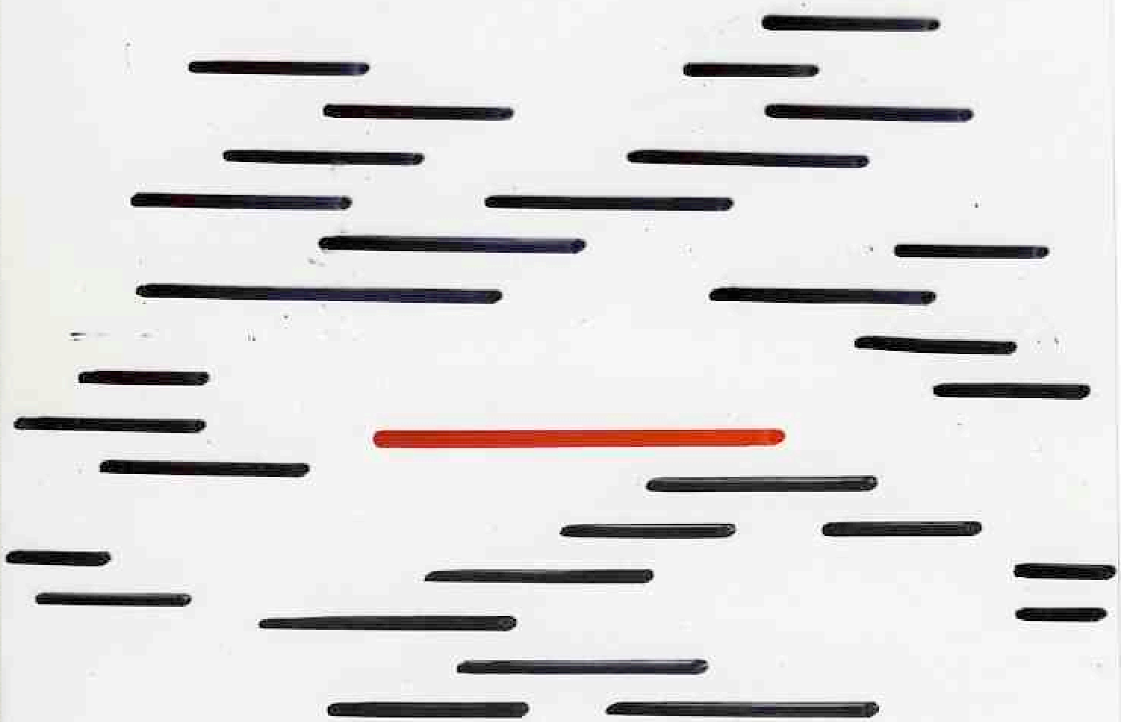
Opérateur

"Poussez" ...



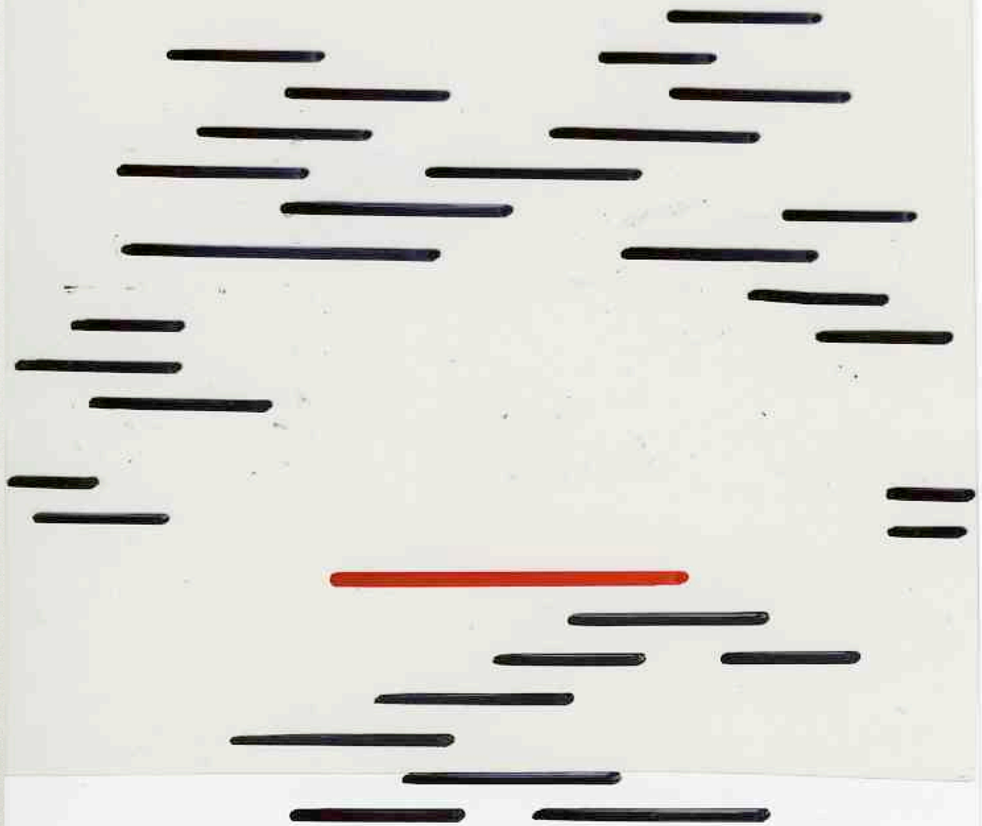
Opérateur

"Poussez" ...



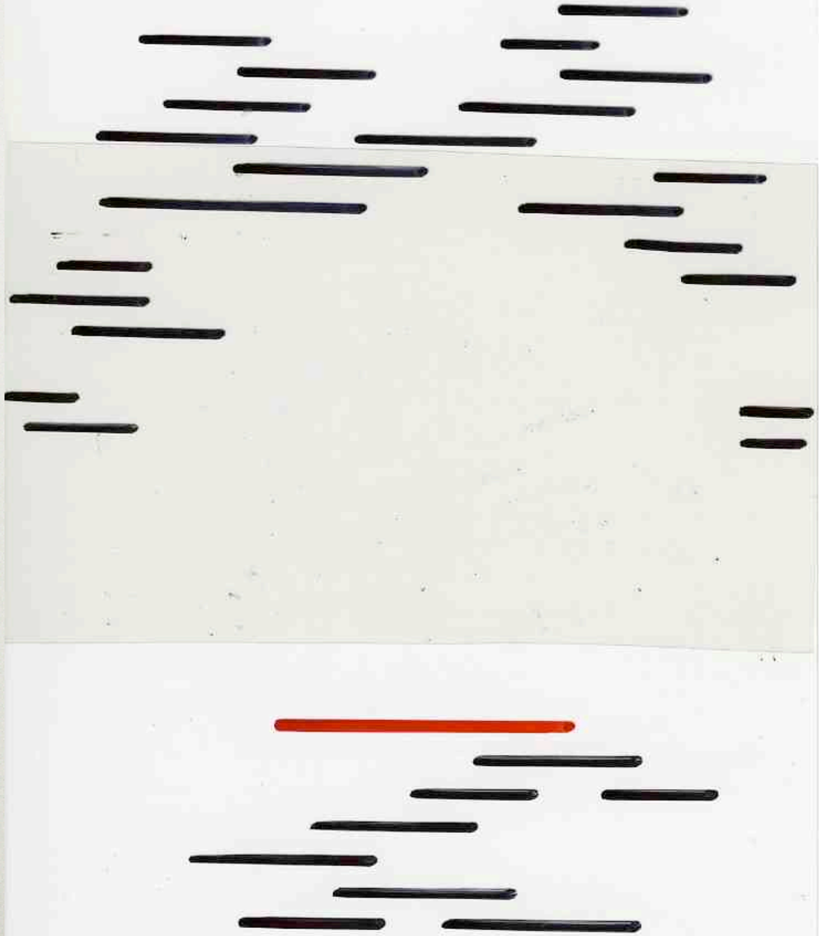
Opérateur

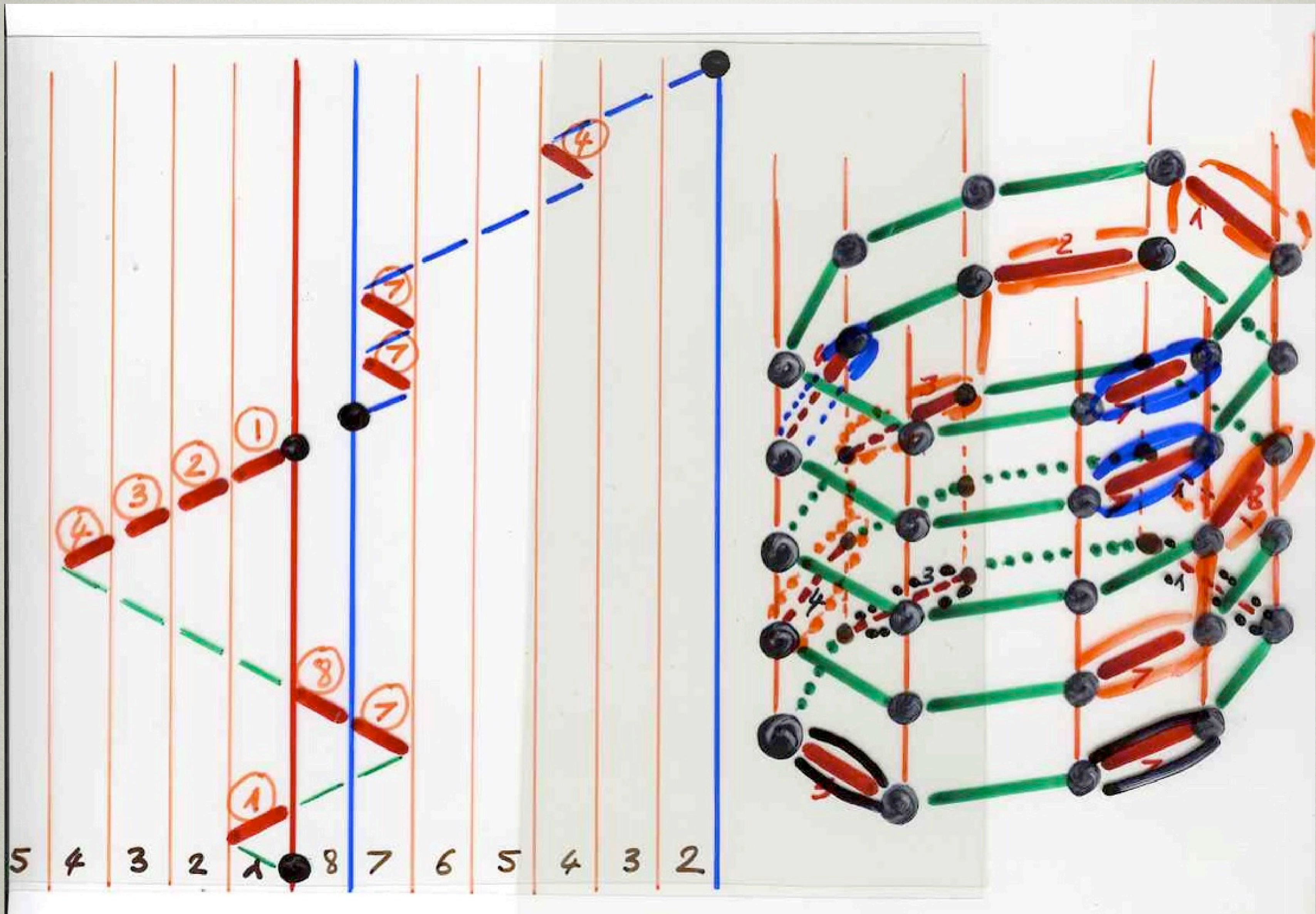
"Poussez" ...



Opérateur

"Poussez" ...

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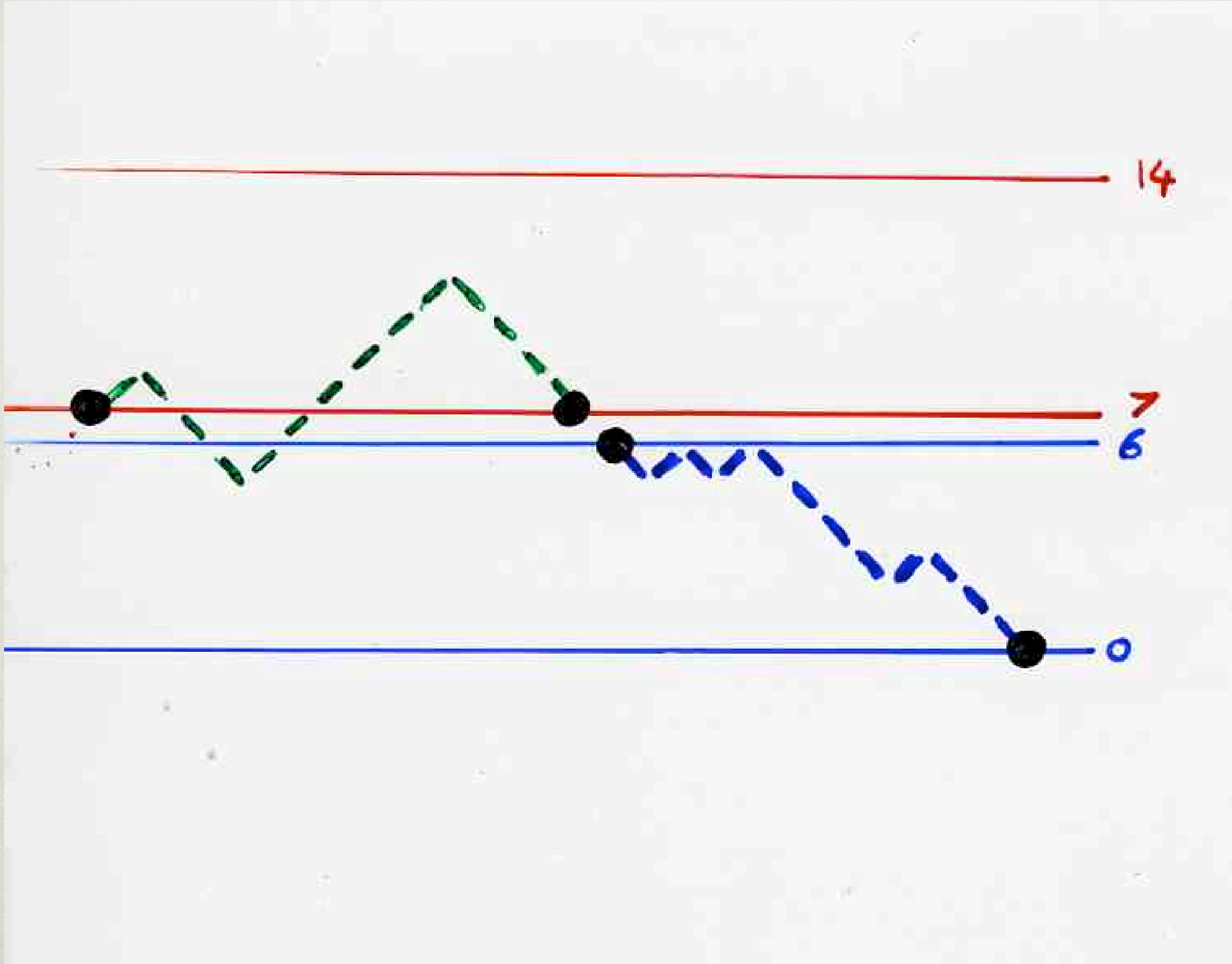
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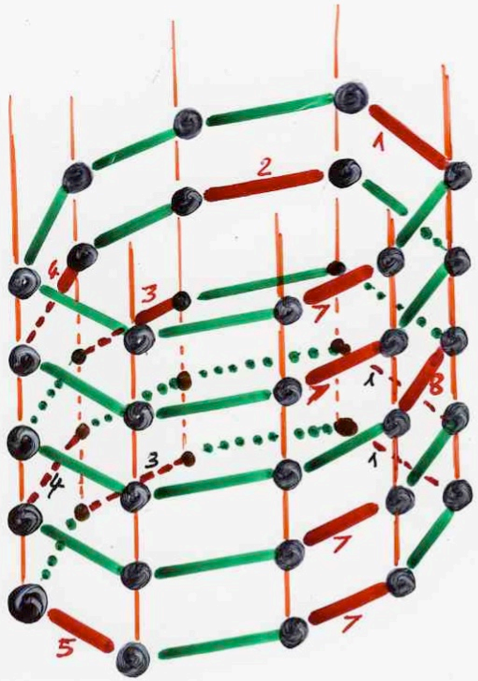
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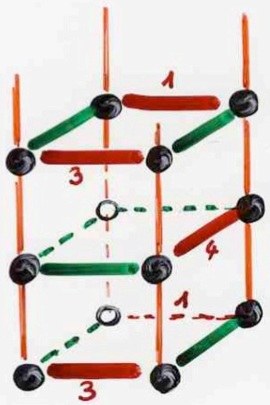
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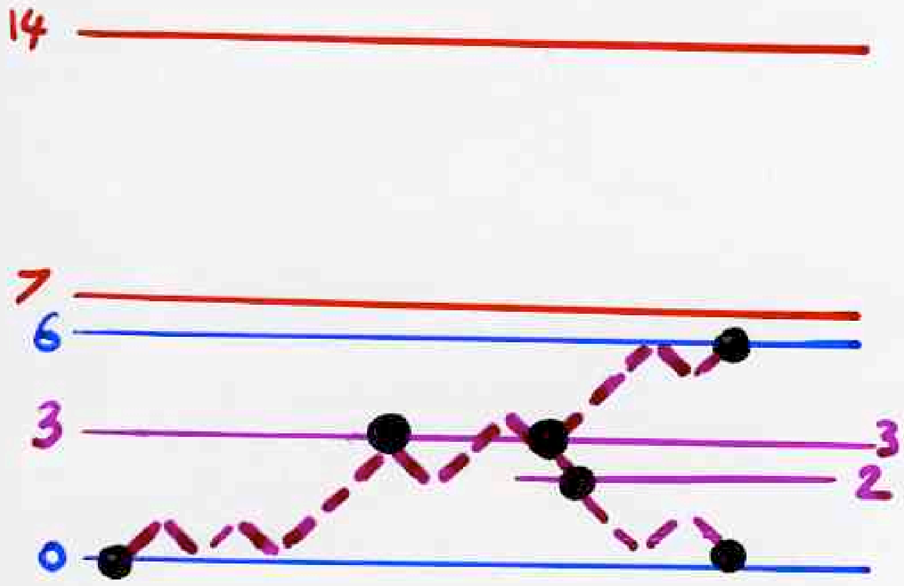
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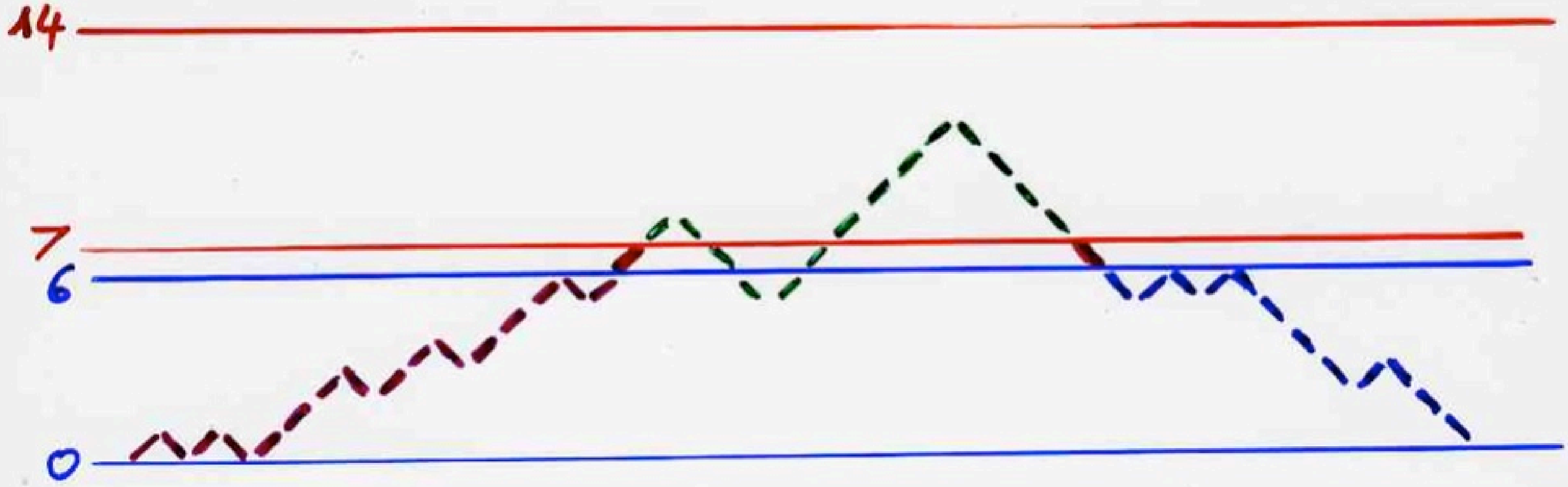
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6. Séries génératrices

generating function

$S_{n,k}$ = nb of (complete) binary trees B
n (internal) vertices
Strahler nb $St(B) = k$

$$S_k(t) = \sum_{k \geq 0} S_{n,k}\, t^n$$

$$S_{\leq k}(t) = \sum_{k \geq 0} S_{n,\leq k}\, t^n$$

$$S_0 = 1$$

$$S_1 = \frac{t}{1-2t}$$

$$S_2 = \frac{t^3}{1-6t+10t^2-4t^3}$$

$$S_3 = \frac{t^7}{1-14t+78t^2-220t^3+330t^4-252t^5+84t^6-8t^7}$$

$$\sin(n+1)\theta = (\sin\theta)\, U_n(\cos\theta)$$

$U_n(t/2)$ reciprocal

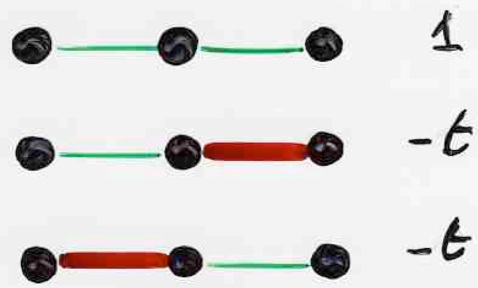
$F_n(t^2)$

Fibonacci polynomial

$$F_{n+1}(t) = F_n(t) - t\,F_{n-1}(t) \qquad F_0 = F_1 = 1$$

alternating generating polynomial for trivial heaps of dimers on the graph segment ~~length~~ n points

(matching polynomial)



$$F_3(t) = 1 - 2t$$

Fibonacci polynomials

$F_0 = F_1 = 1$

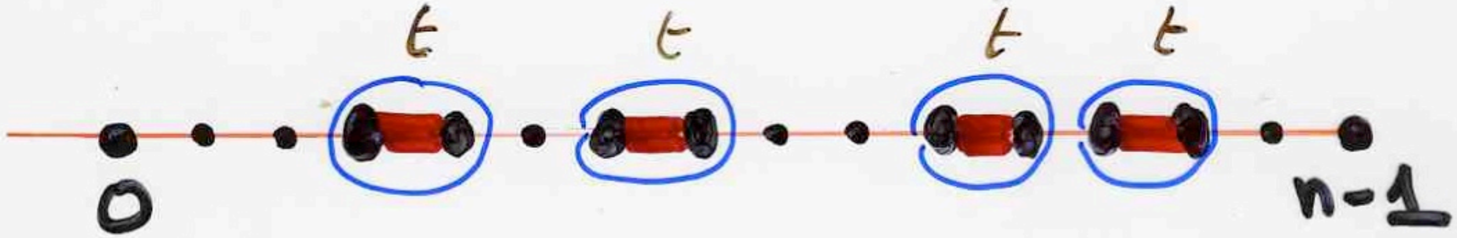
$F_n = F_{n-1} - t F_{n-2}$

1

$1 - t$

$1 - 2t$

$1 - 3t + t^2$



trivial heap of dimers

$$S_{\leq k}(t) = \frac{F_{2^{k+1}-2}(t)}{F_{2^{k+1}-1}(t)}$$

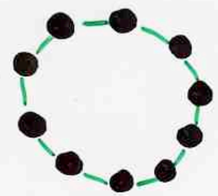
$$S_{k}(t) = \frac{t^{2^{k}-1}}{F_{2^{k+1}-1}(t)}$$

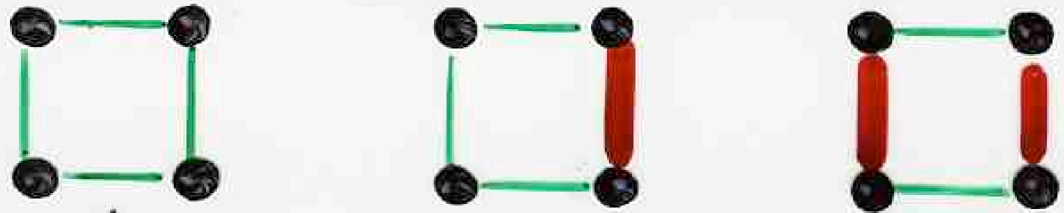
factorisation

$$F_{2n+1}(t) = F_n(t) \times L_{n+1}(t)$$

$L_n(t)$ Lucas polynomial

trivial heaps
(matching polynomial)
graph cycle length n





$1 \qquad -4t \qquad +2t^2$

$$L_4(t) = 1 - 4t + 2t^2$$

$$n = 2^k - 1 \qquad \frac{n+1}{2} = 2^{k-1}$$

$$2n+1 = 2^{k+1} - 1$$

$$F_{2^{k+1}-1}(t) = Z_1(t)\,Z_2(t)\cdots Z_k(t)$$

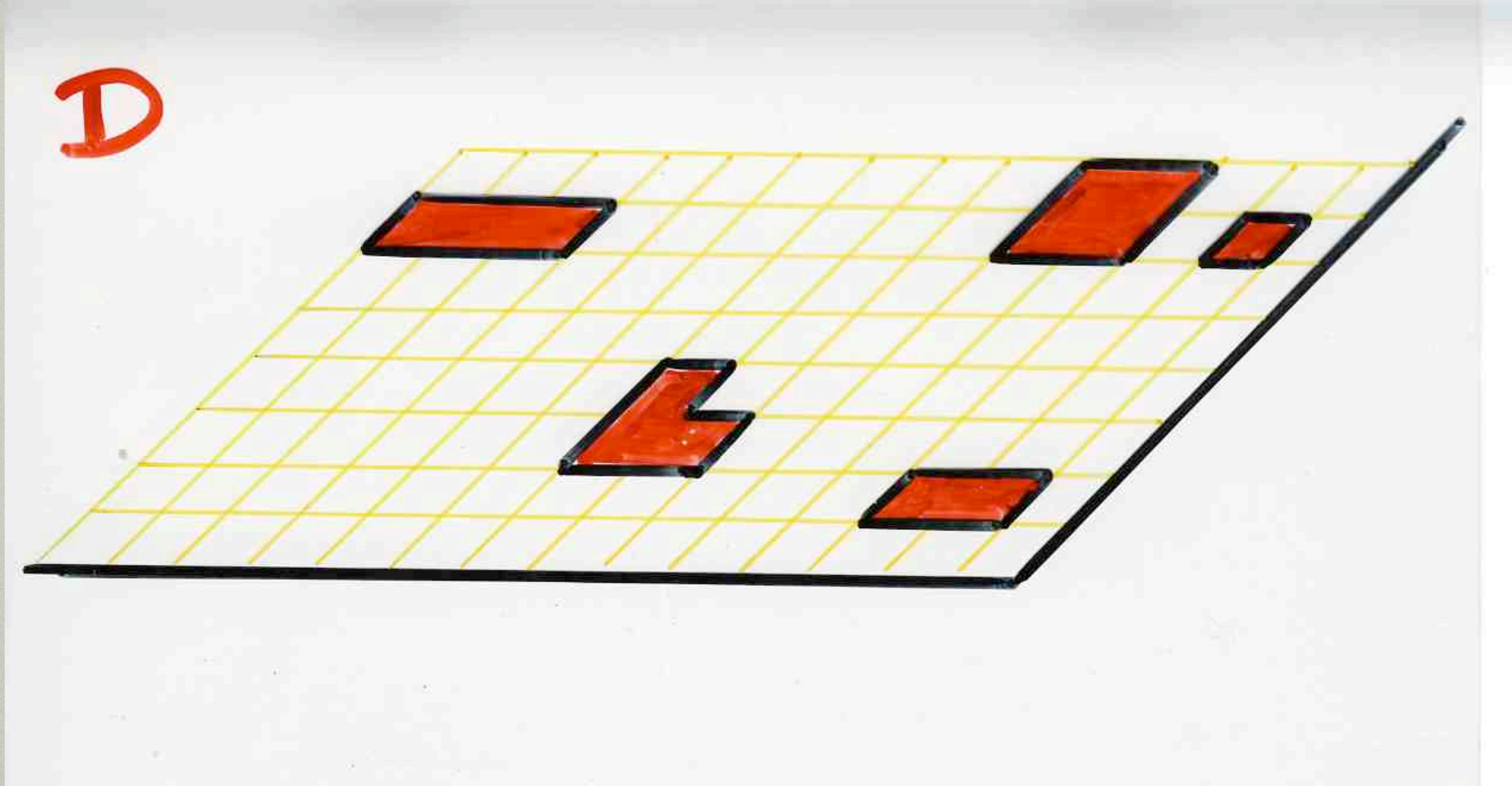
$$Z_i(t) = L_{2^i}(t)$$

$$S_k(t) = S_{k-1}(t) \times \frac{t^{(2^{k-1})}}{L_{2^k}(t)}$$

ex:

$$S_3(t) = \frac{t^7}{F_{15}(t)}$$

$$= \underbrace{\frac{1}{F_1}}_{1} \times \frac{t}{L_2(t)} \times \frac{t^2}{L_4(t)} \times \frac{t^4}{L_8(t)}$$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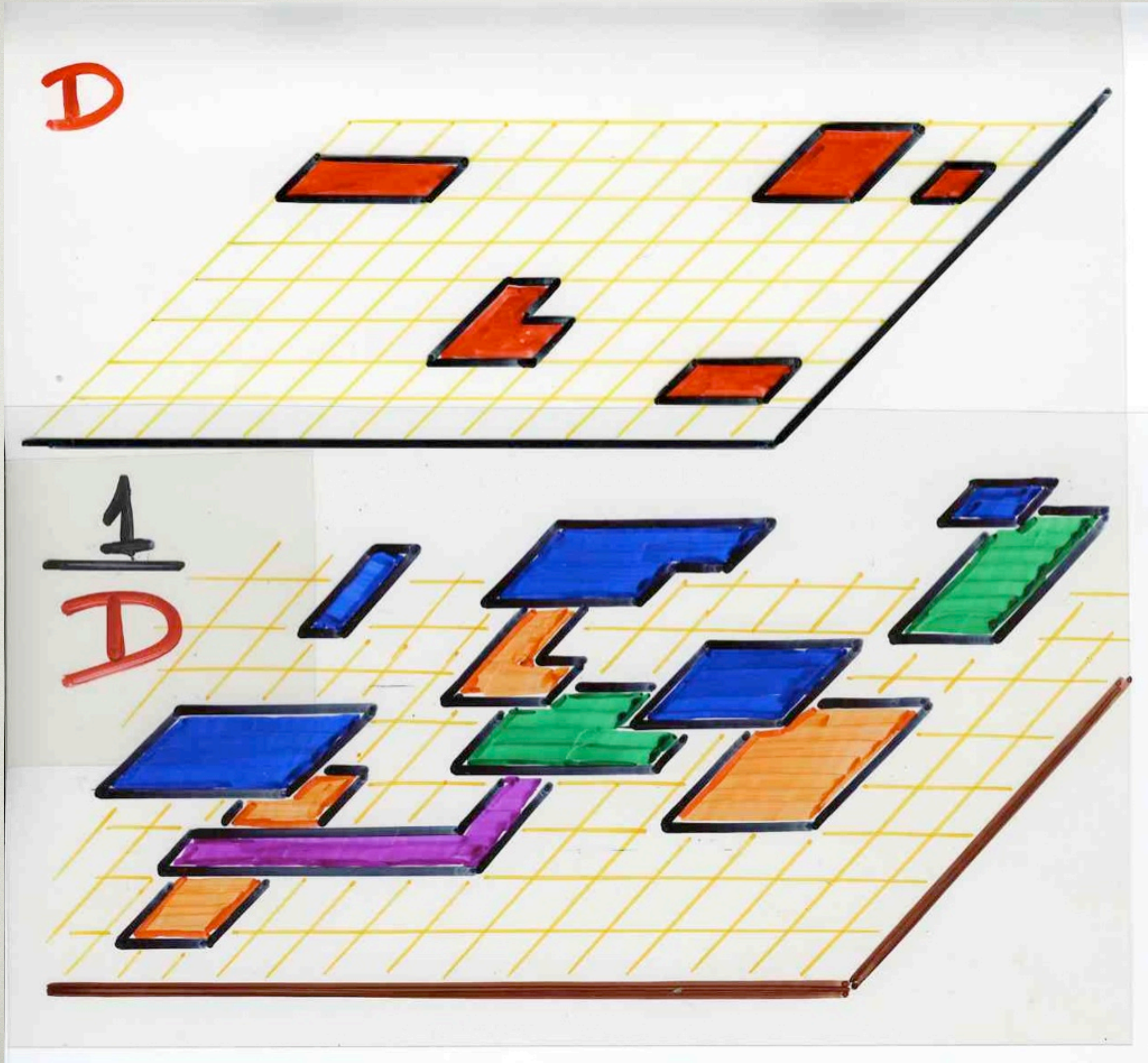
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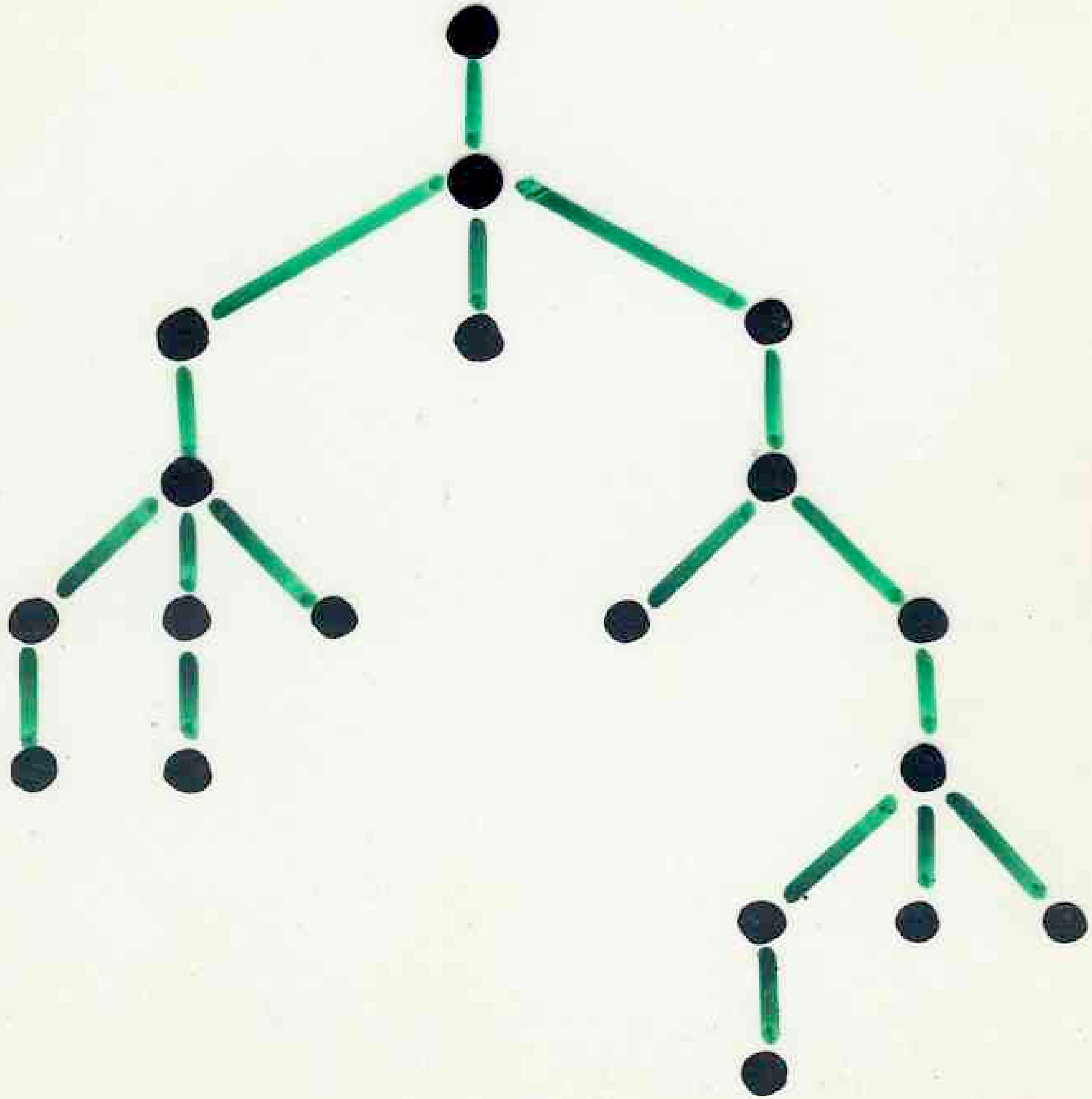
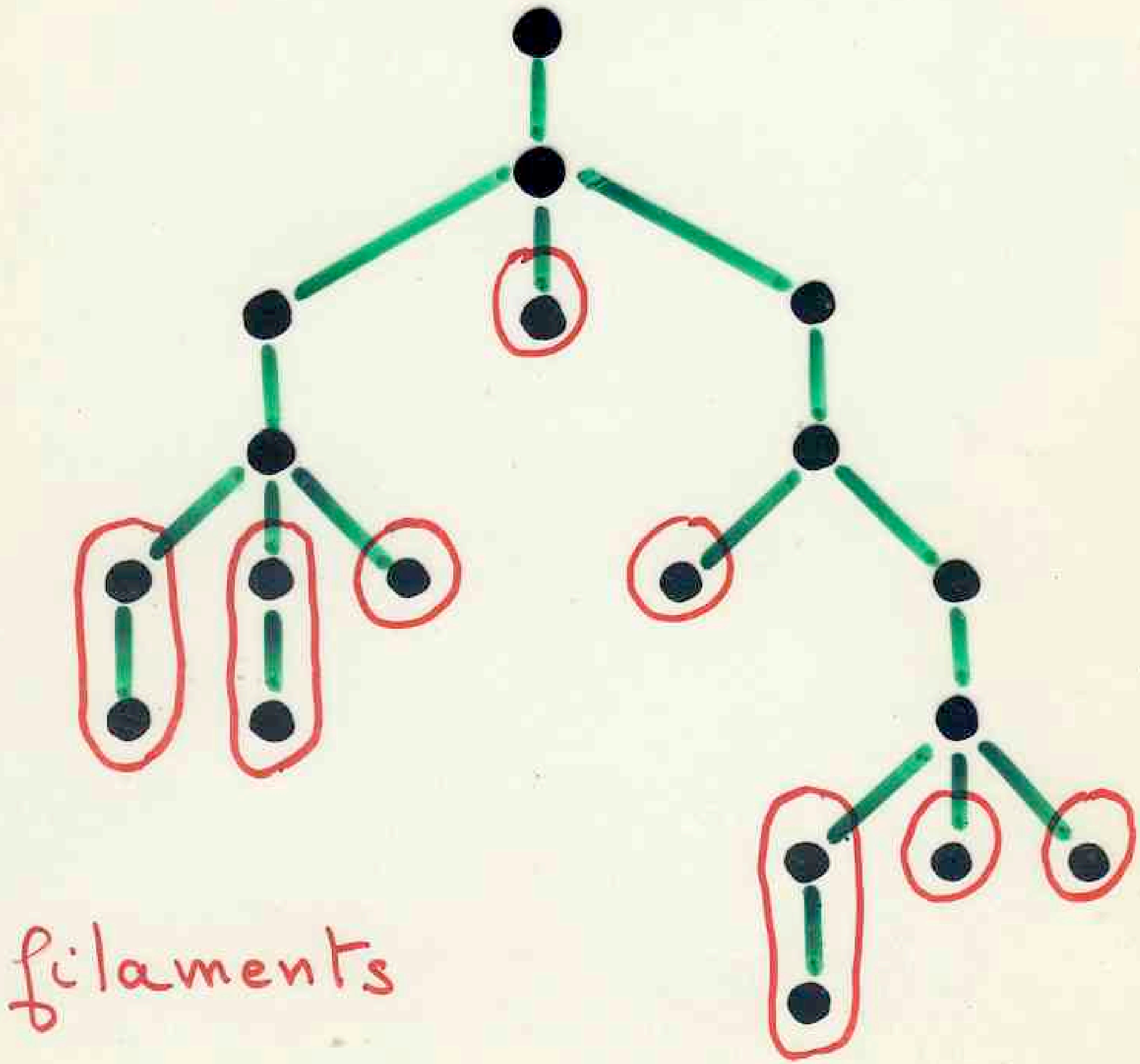
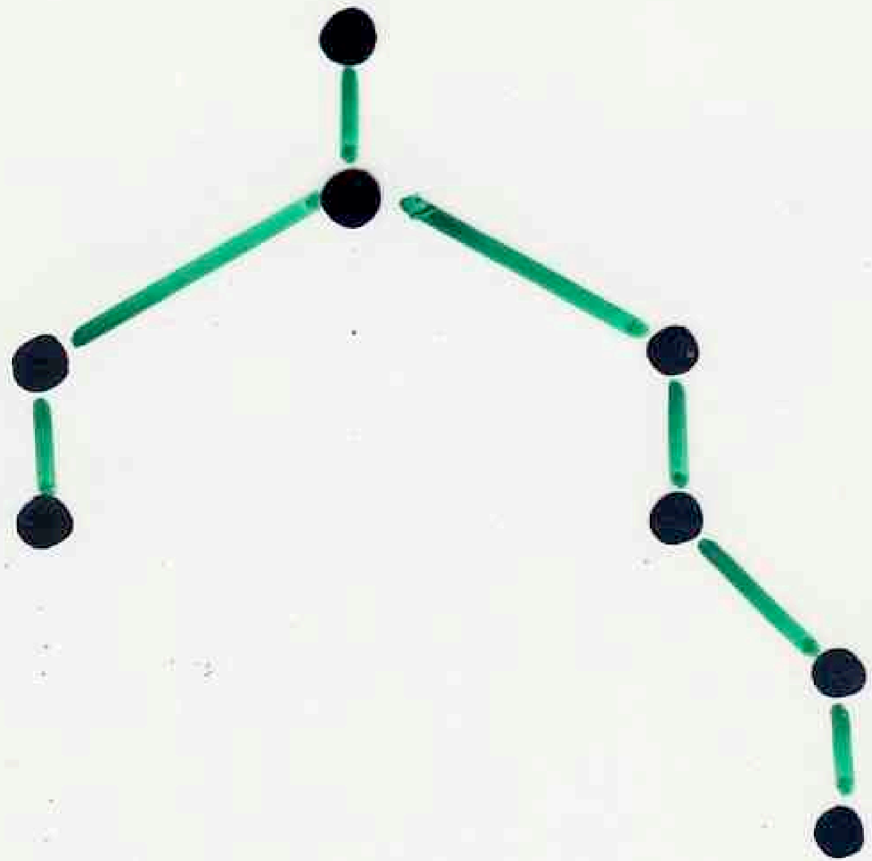
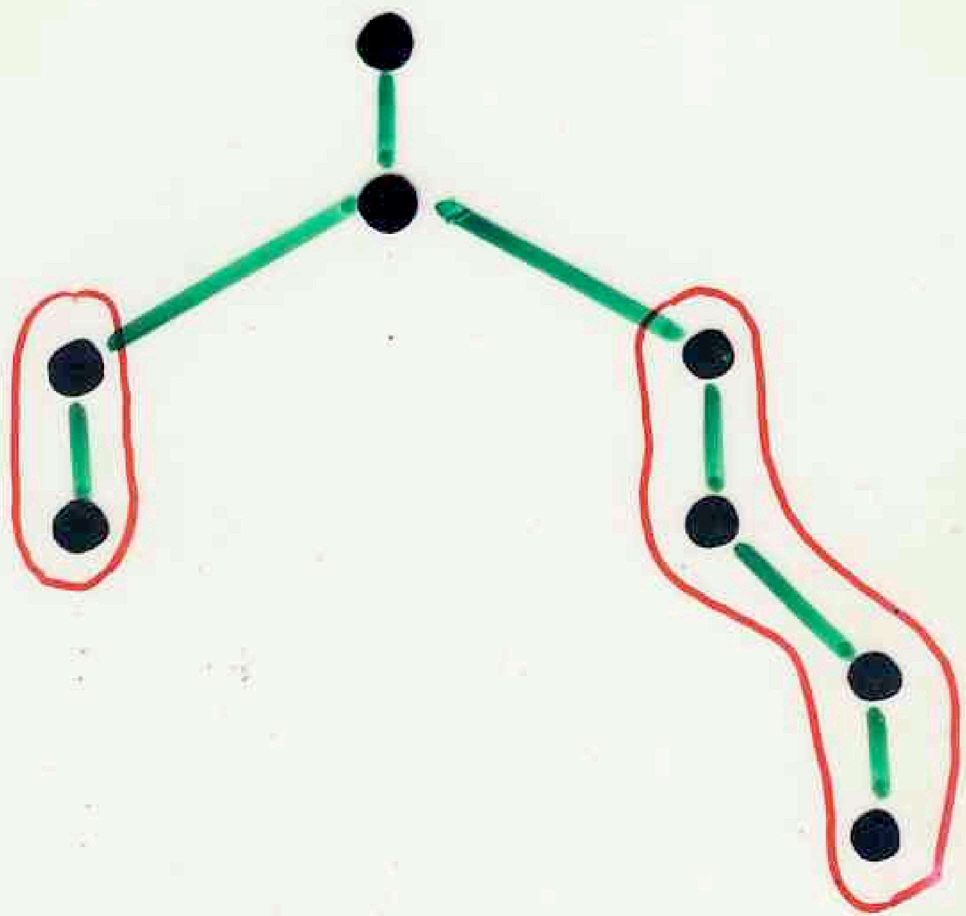
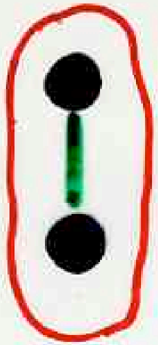
9. Quatre distributions

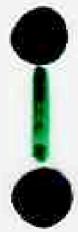



filaments



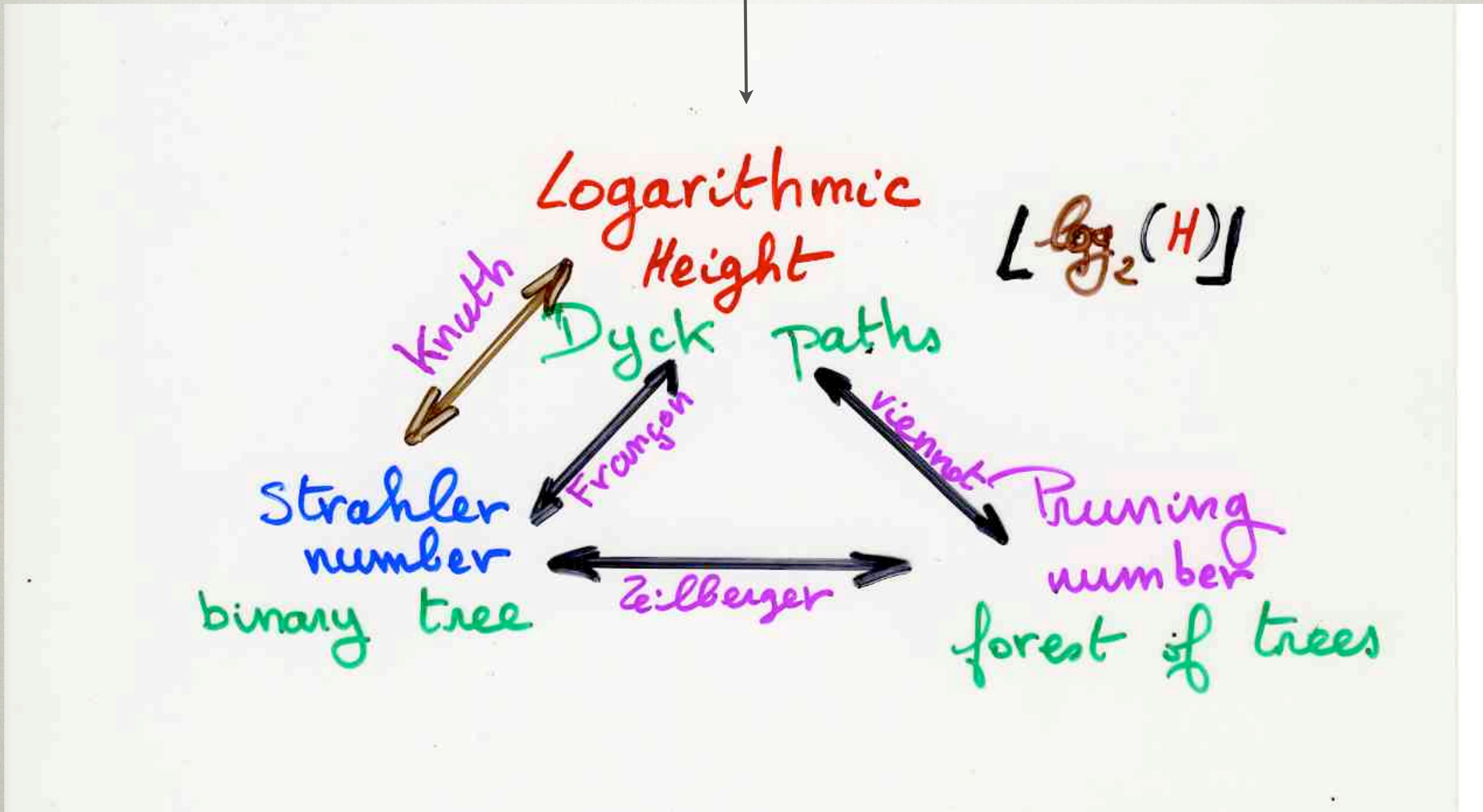






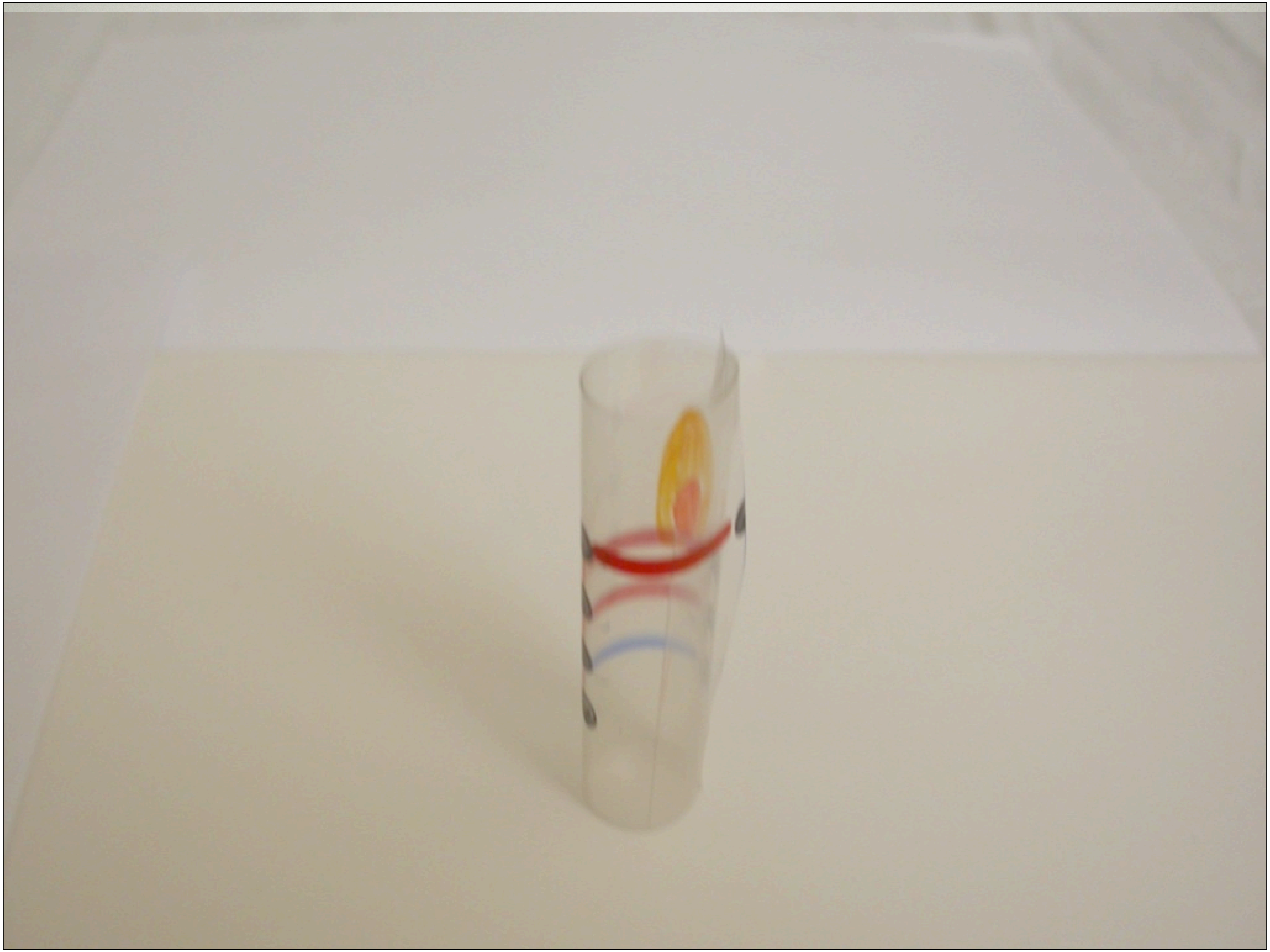
system of Kepler towers

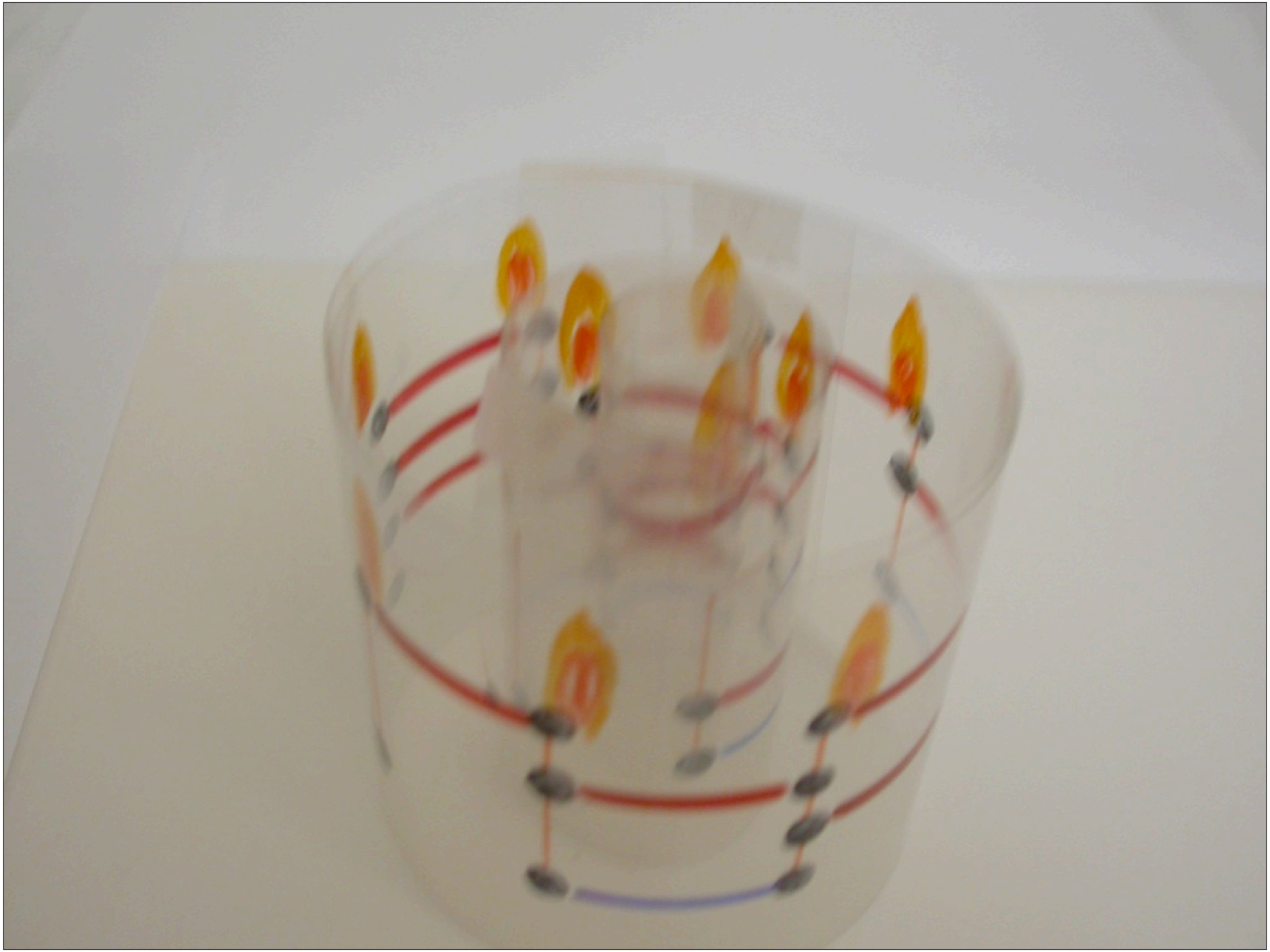
number of towers

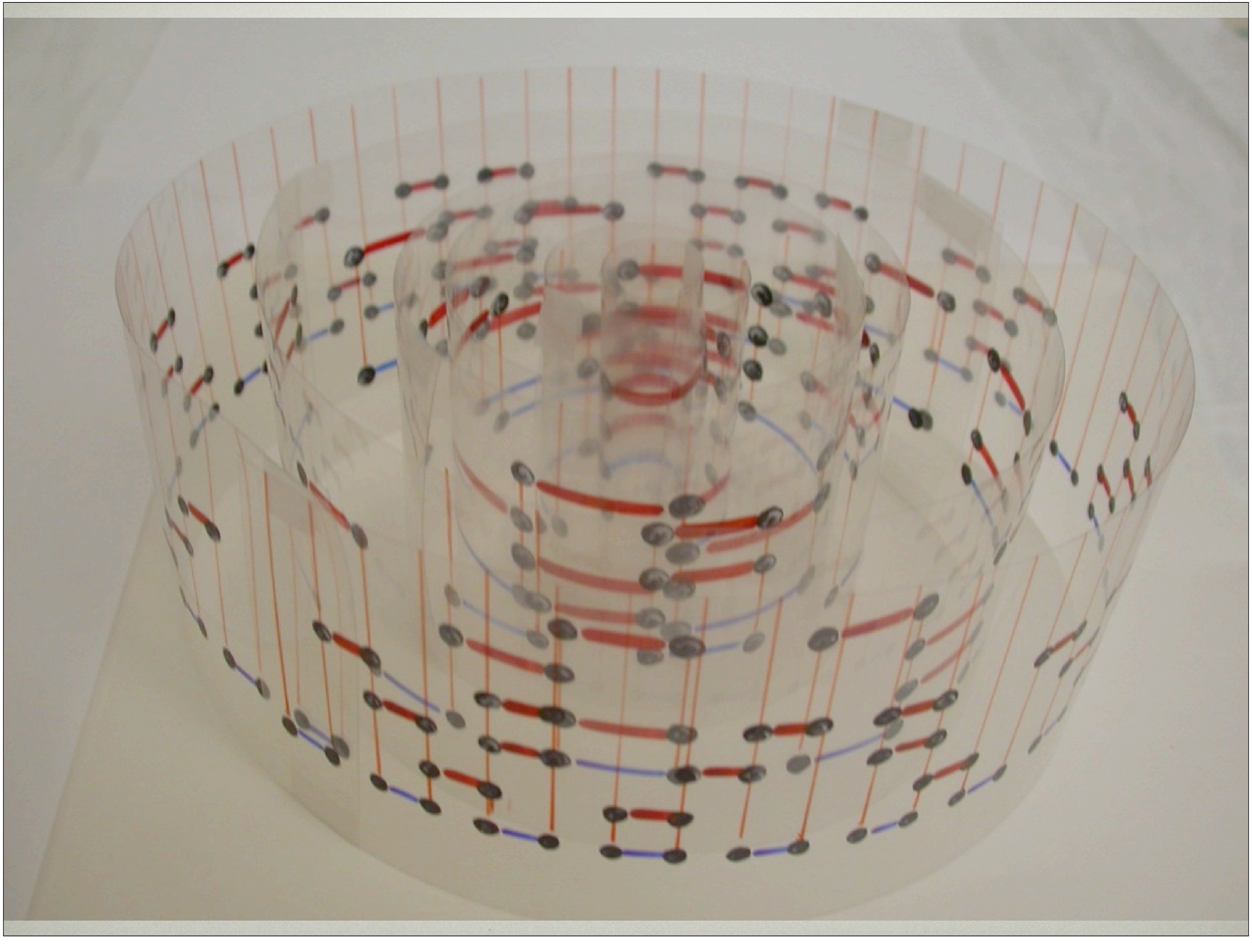


www-cs-faculty.stanford.edu/~knuth/programs/viennot.w

The birthday cake







merci beaucoup

à vous tous