# GEOMETRY OF $q$ AND $q, t$-ANALOGS IN COMBINATORIAL ENUMERATION

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Many familiar quantities from enumerative combinatorics have natural $q$-analogs. Some examples are: (a) two different $q$-analogs of Catalan numbers, (b) a $q$-analog of the number $(n+1)^{n-1}$, which enumerates rooted forests and parking functions, (c) the Kostka-Foulkes polynomials, which are $q$-analogs of the number of semistandard Young tableaux of given shape and content. Examples (a) and (b) are closely connected with the theory of $q$-Lagrange inversion, while (c) is connected with Hall-Littlewood symmetric functions and the Springer correspondence.

The theory of Macdonald polynomials leads to expressions involving two parameters $q$ and $t$, including $(q, t)$-analogs of the aforementioned examples. These $(q, t)$-quantities are defined as rational functions, in such a way that it is far from obvious that they are actually polynomials in $q$ and $t$ with non-negative integer coefficients. Most existing proofs of polynomiality and positivity for $(q, t)$-quantities rely on a geometric interpretation involving the Hilbert scheme of points in the plane, which leads to a quasi-combinatorial interpretation of the coefficients as character multiplicities in a doubly-graded representation of the symmetric group.

Good conjectures are now available giving purely combinatorial interpretations of some of these $(q, t)$-quantities. Some cases, such as that of the $(q, t)$-Catalan polynomials, have been proved using elementary methods. To solve the full conjectures, probably some direct connection between the combinatorial and the geometric/representation-theoretic sides of the question will ultimately be needed.

1. Review of classical $q$-analogs

2. Introduction to Macdonald polynomials and $(q, t)$-analogs

3. Combinatorial conjectures for the $(q, t)$-analogs

4. Symmetric function theory techniques:
    - can compute specializations such as $t = 1$ or $t = 1/q$
    - suffice for elementary proofs of some combinatorial results
5. Geometry/representation-theory techniques
    - main tool for positivity and polynomiality theorems