

A bijection between loopless maps and triangulations

Éric Fusy

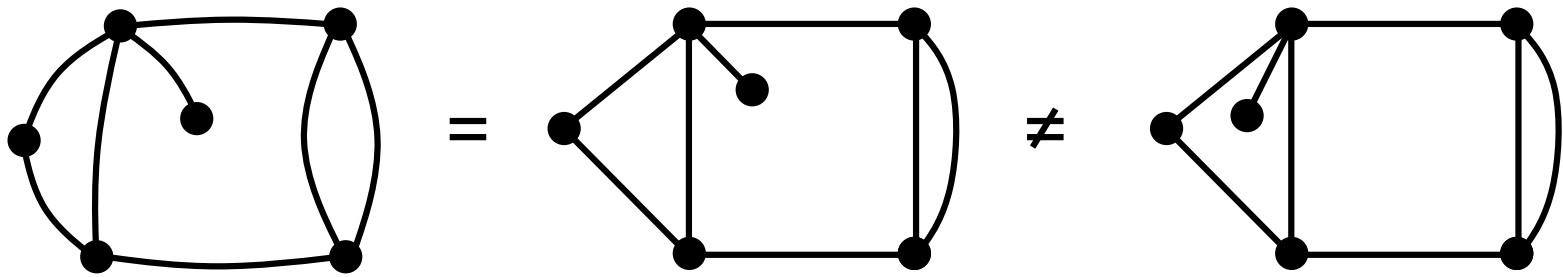
Projet ALGO, INRIA Rocquencourt

and

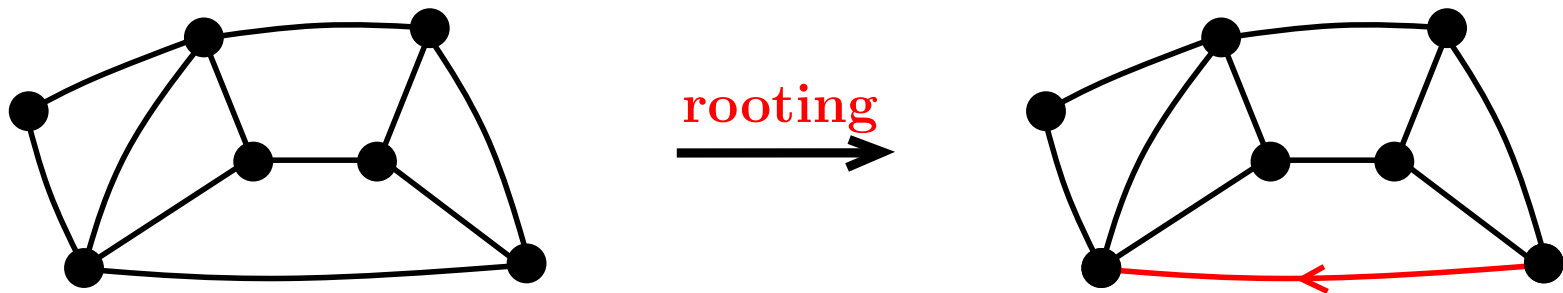
LIX, École Polytechnique (Paris)

Planar maps

- **Planar map** = graph drawn in the plane **without edge-crossing**, taken **up to isotopy** (continuous structure-preserving transformation)

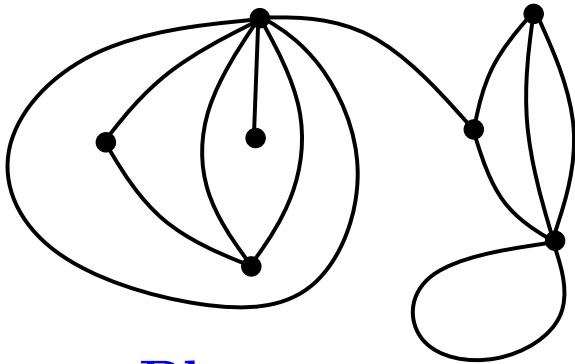


- **Rooted map** = map + root edge

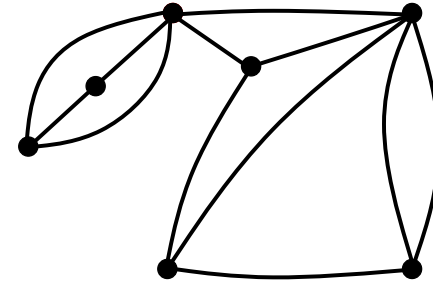


- **Motivations**: mesh compression, graph drawing + nice combinatorial properties

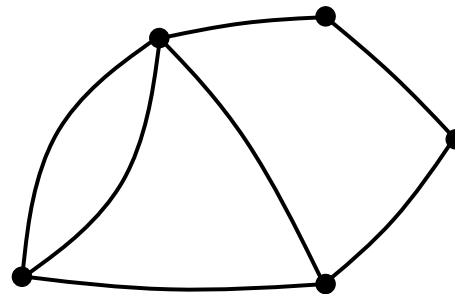
Families of planar maps



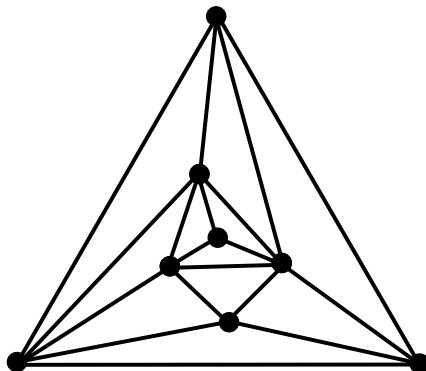
Planar map



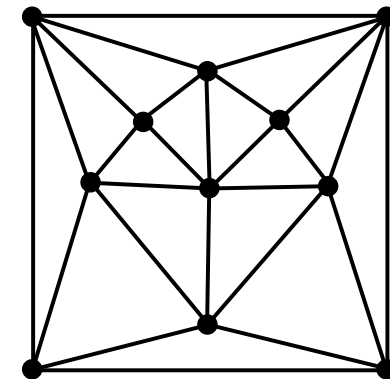
Loopless map



Nonseparable map
(no separating vertex)

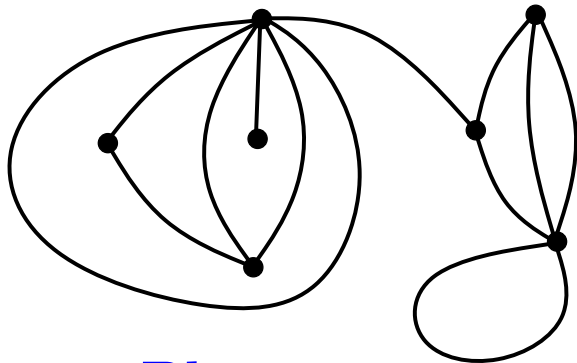


Triangulation



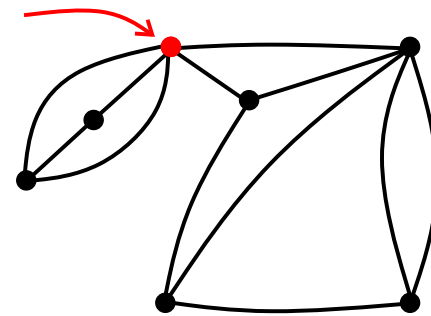
Irreducible triangulation
(no separating triangle) . - p.3/26

Families of planar maps

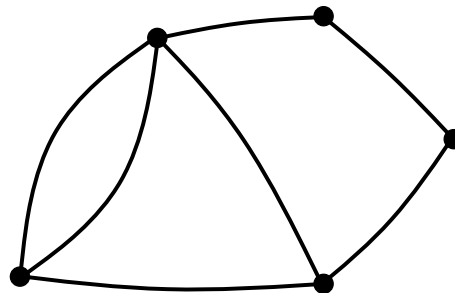


Planar map

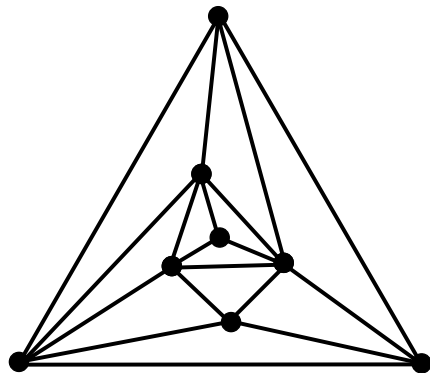
separating
vertex



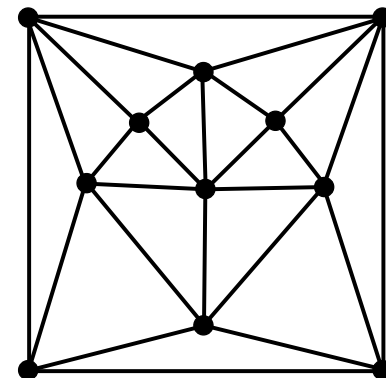
Loopless map



Nonseparable map
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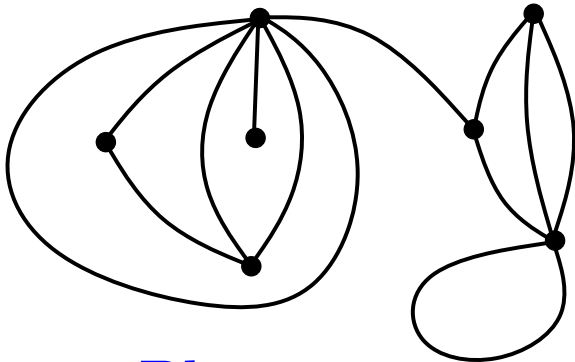


Triangulation



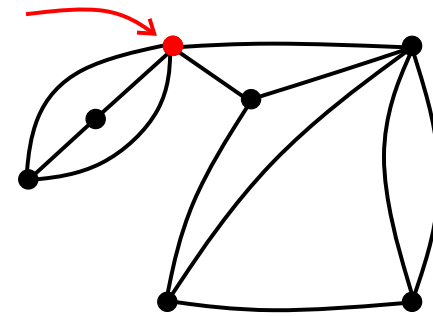
Irreducible triangulation
(no separating triangle) · - p.3/26

Families of planar maps

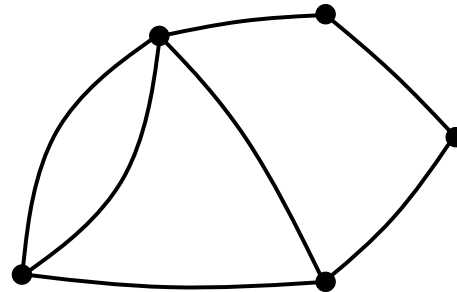


Planar map

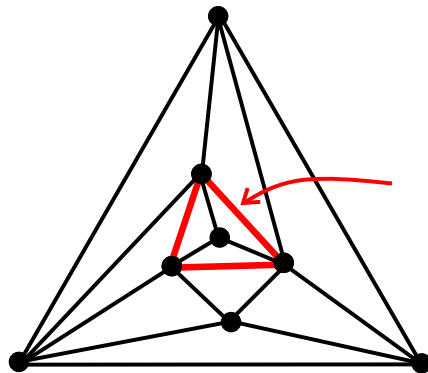
separating
vertex



Loopless map

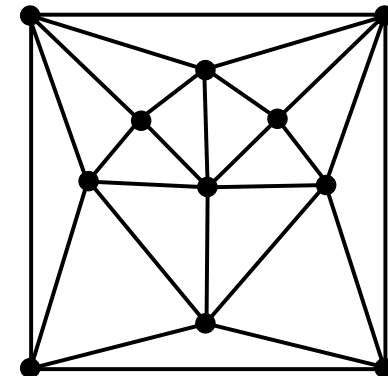


Nonseparable map
(no separating vertex)



Triangulation

separating
triangle

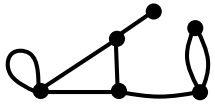


Irreducible triangulation

(no separating triangle) · - p.3/26

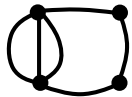
Enumeration of planar maps

- **Symbolic approach:** Tutte, Brown
- **Bijjective approach:** Cori, Schaeffer, Bouttier-Di Francesco-Guitter



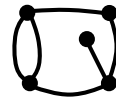
Planar maps

$$\#(\mathbf{n} \text{ edges}) = \frac{2 \cdot 3^n (2n)!}{(n+2)! n!}$$



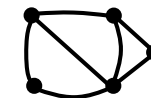
Eulerian

$$\#(\mathbf{n} \text{ edges}) = \frac{3 \cdot 2^{n-1} (2n)!}{(n+2)! n!}$$



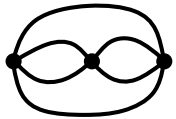
Loopless

$$\#(\mathbf{n} \text{ edges}) = \frac{2(4n+1)!}{(n+1)! (3n+2)!}$$



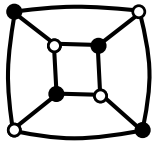
Nonseparable

$$\#(\mathbf{n} \text{ edges}) = \frac{4(3n-3)!}{(n-1)! (2n)!}$$



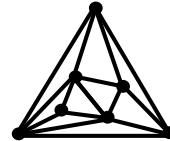
4-regular

$$\#(\mathbf{n} \text{ vert.}) = \frac{2 \cdot 3^n (2n)!}{(n+2)! n!}$$



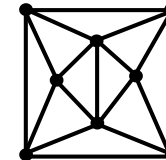
Bicubic

$$\#(\mathbf{2n} \text{ vert.}) = \frac{3 \cdot 2^{n-1} (2n)!}{(n+2)! n!}$$



Triangulations

$$\#(\mathbf{n+3} \text{ vert.}) = \frac{2(4n+1)!}{(n+1)! (3n+2)!}$$

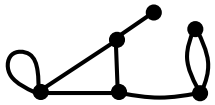


Irreducible

$$\#(\mathbf{n+3} \text{ vert.}) = \frac{4(3n-3)!}{(n-1)! (2n)!}$$

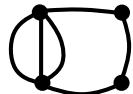
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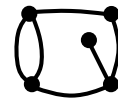
Planar maps

$$\#(\mathbf{n} \text{ edges}) = \frac{2 \cdot 3^n (2n)!}{(n+2)! n!}$$



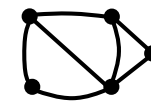
Eulerian

$$\#(\mathbf{n} \text{ edges}) = \frac{3 \cdot 2^{n-1} (2n)!}{(n+2)! n!}$$



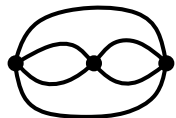
Loopless

$$\#(\mathbf{n} \text{ edges}) = \frac{2(4n+1)!}{(n+1)! (3n+2)!}$$



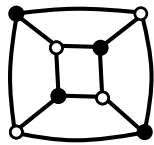
Nonseparable

$$\#(\mathbf{n} \text{ edges}) = \frac{4(3n-3)!}{(n-1)! (2n)!}$$



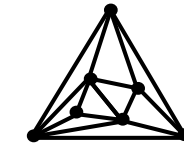
4-regular

$$\#(\mathbf{n} \text{ vert.}) = \frac{2 \cdot 3^n (2n)!}{(n+2)! n!}$$



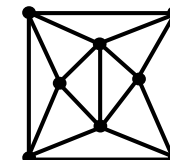
Bicubic

$$\#(\mathbf{2n} \text{ vert.}) = \frac{3 \cdot 2^{n-1} (2n)!}{(n+2)! n!}$$



Triangulations

$$\#(\mathbf{n+3} \text{ vert.}) = \frac{2(4n+1)!}{(n+1)! (3n+2)!}$$



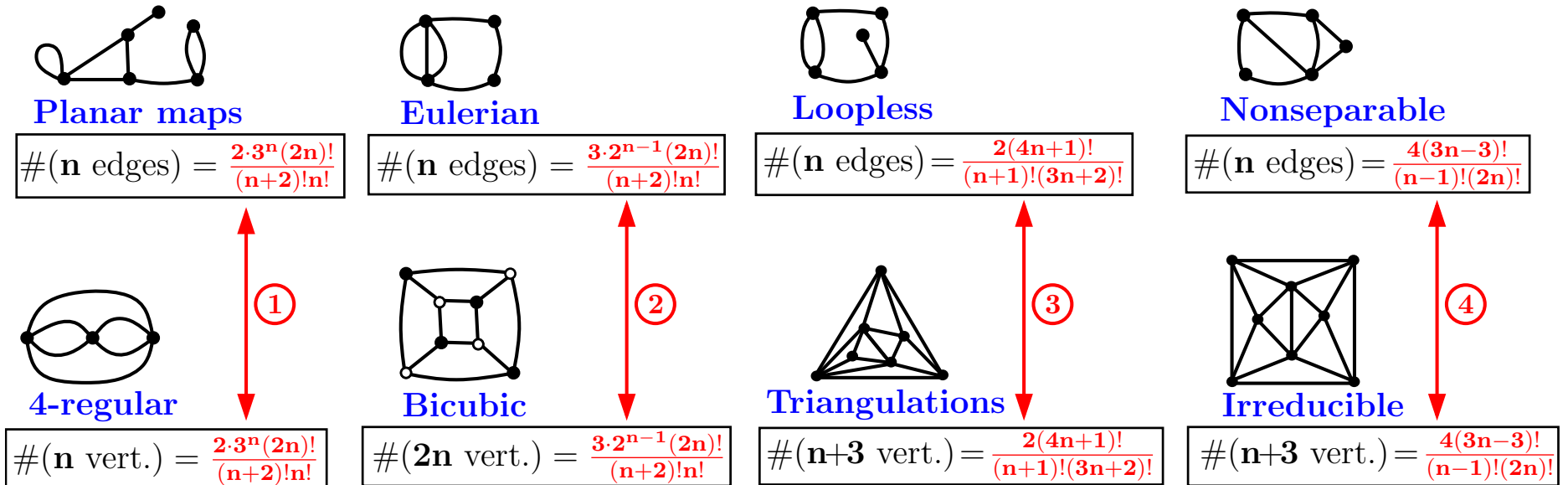
Irreducible

$$\#(\mathbf{n+3} \text{ vert.}) = \frac{4(3n-3)!}{(n-1)! (2n)!}$$



Enumeration of planar maps

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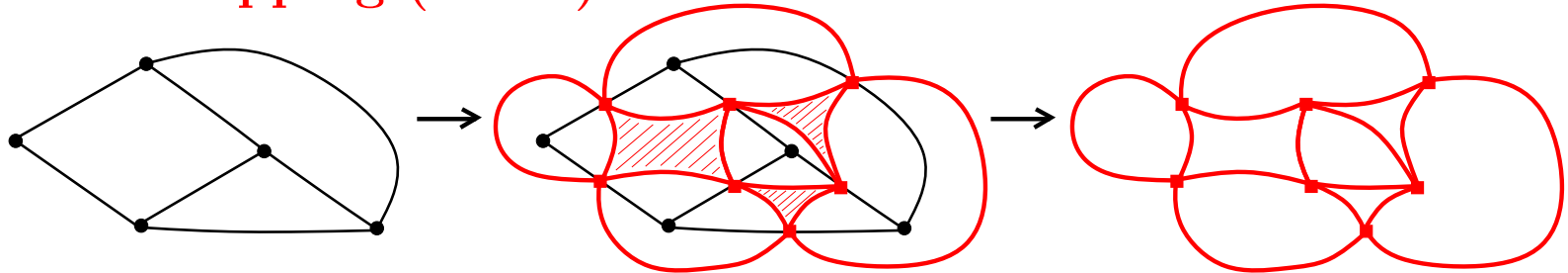


①, ② well known bijections (Tutte) ③ recursive bijection (Wormald)

- This talk:**
- **new** bijective construction for ③
 - **first** bijective construction for ④

Well known bijections

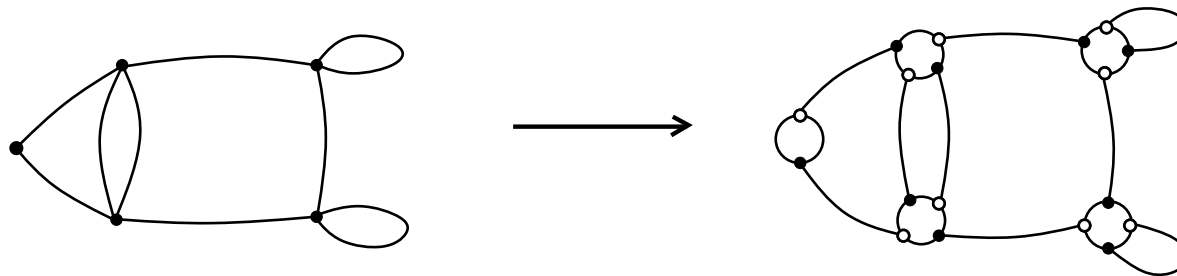
① Radial mapping (Tutte)



Planar maps n edges

4-regular n vertices

② Trinity mapping (Tutte)

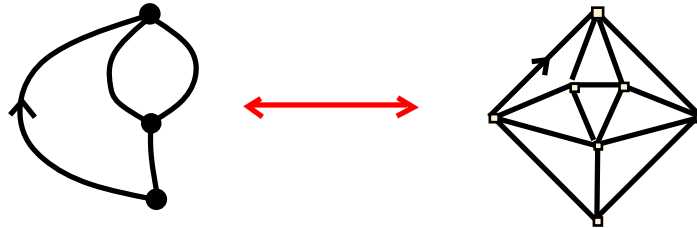


Eulerian maps n edges

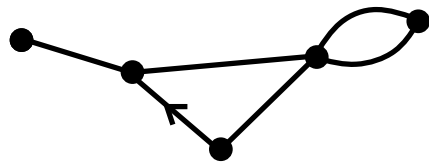
Bicubic maps $2n$ vertices

Overview of the talk

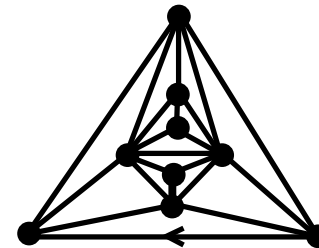
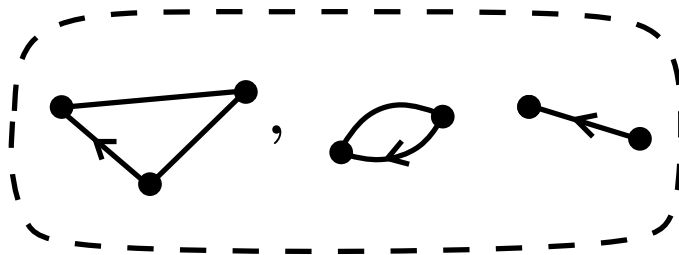
- 1) Bijection **nonsseparable maps** \simeq **irreducible triang**
+ new duality relation for bipolar orientations



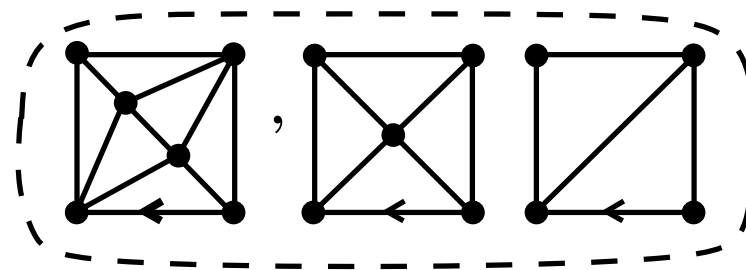
- 2) Bijection **loopless maps** \simeq **triangulations**



nonsseparable components



irreducible components

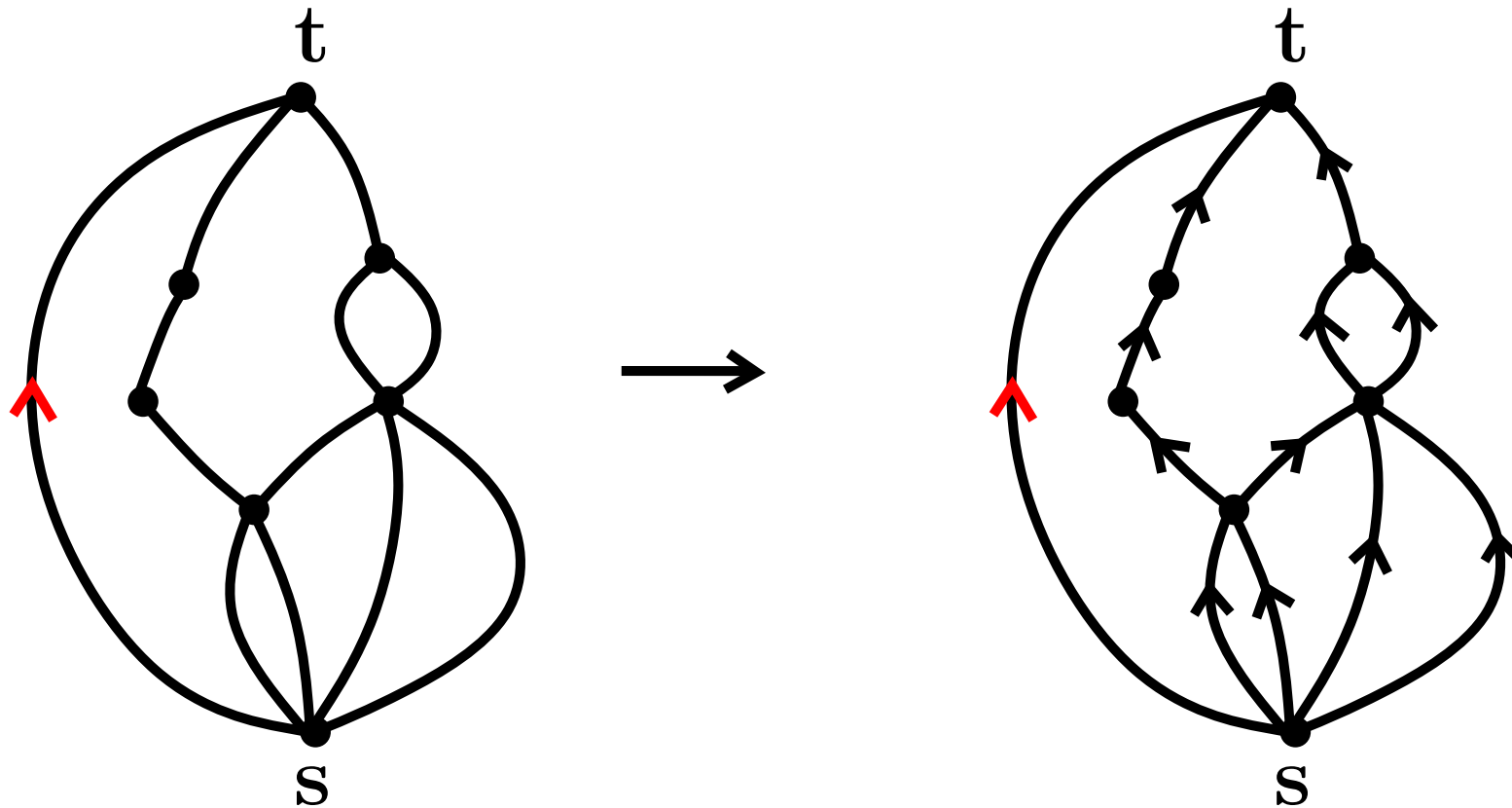


- 3) Applications to random generation and encoding

Bijection between nonseparable maps and irreducible triangulations

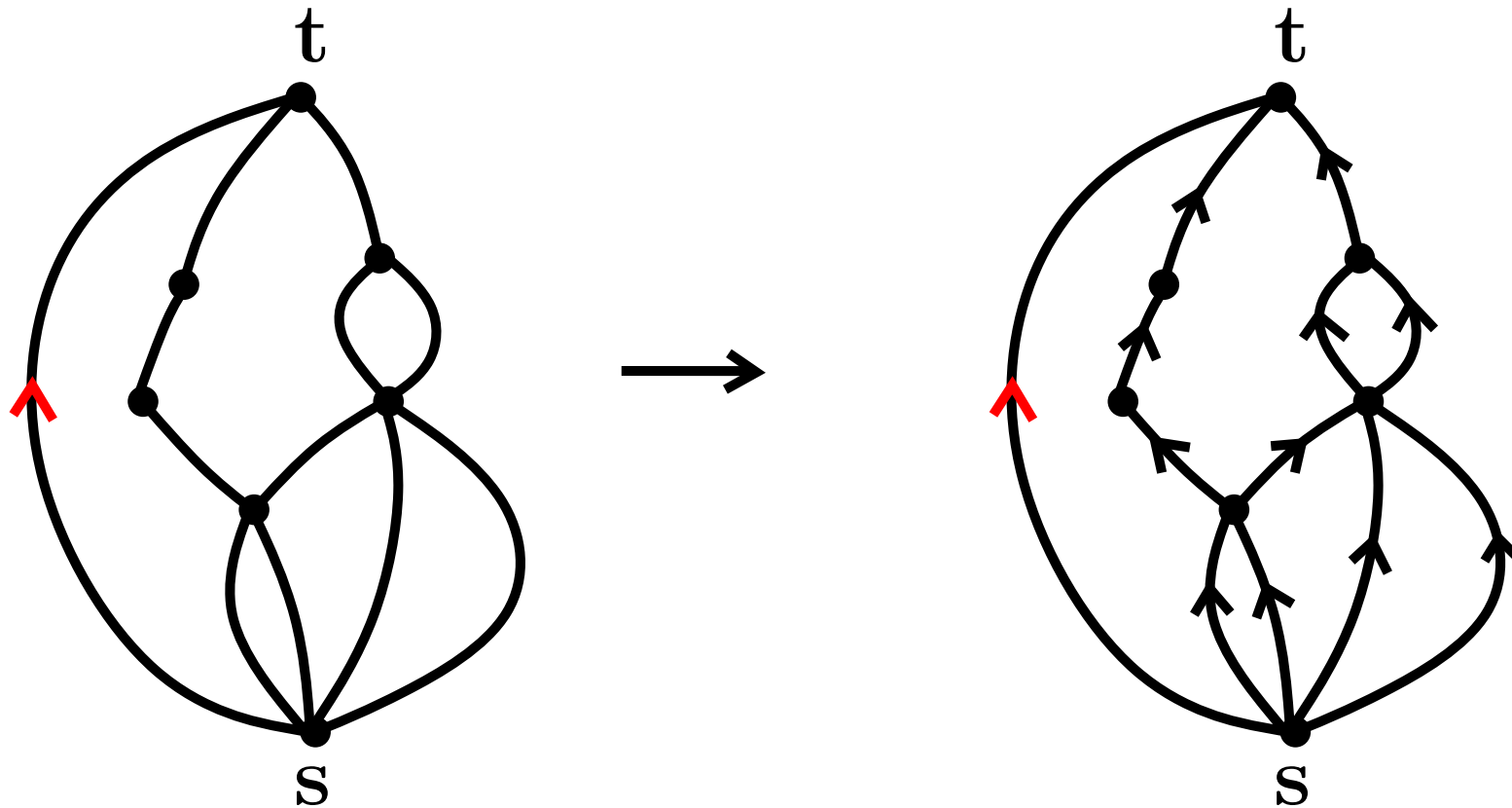
Bipolar orientations

Bipolar orientation = acyclic orientation with a unique source and a unique sink



Bipolar orientations

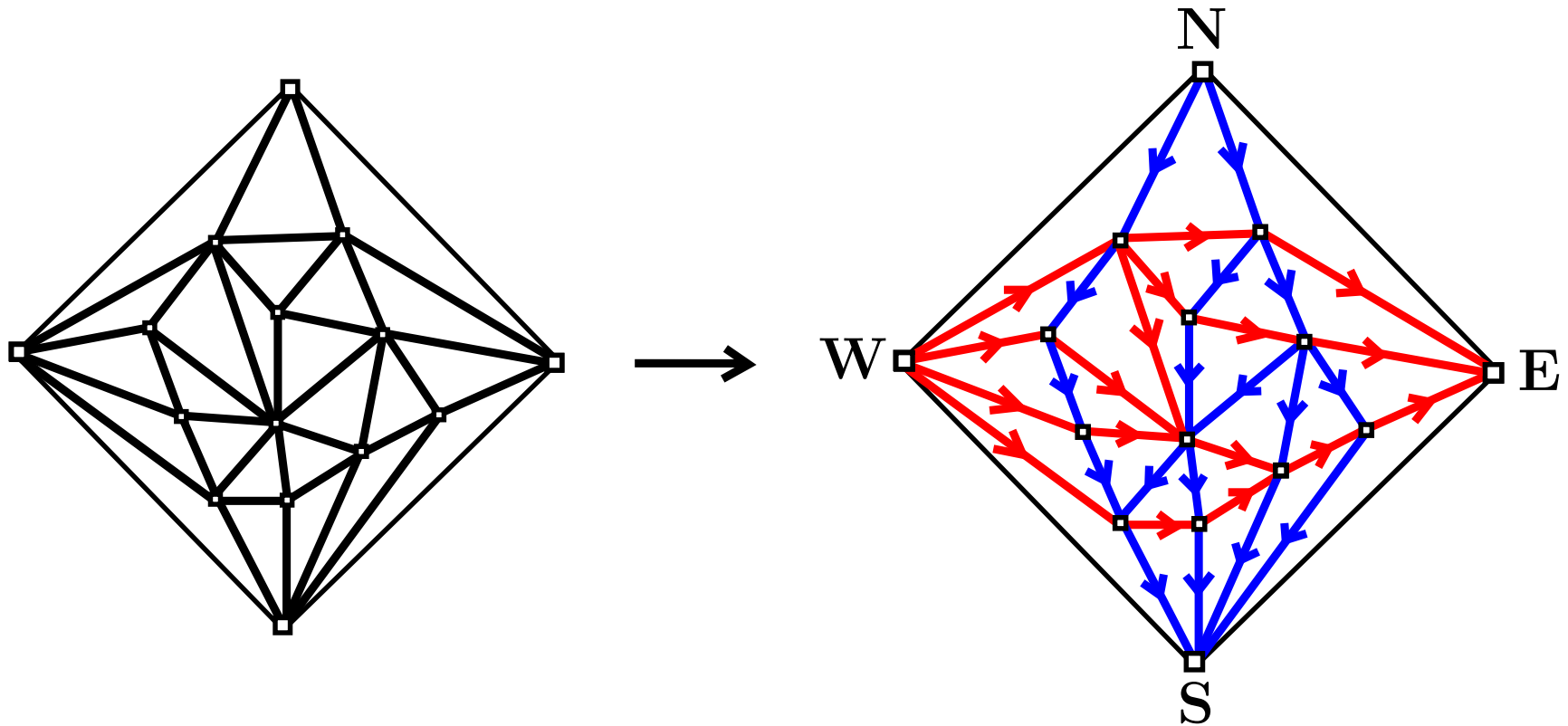
Bipolar orientation = acyclic orientation with a unique source and a unique sink



A map admits a bipolar orientation iff there is no separating vertex (nonseparable)

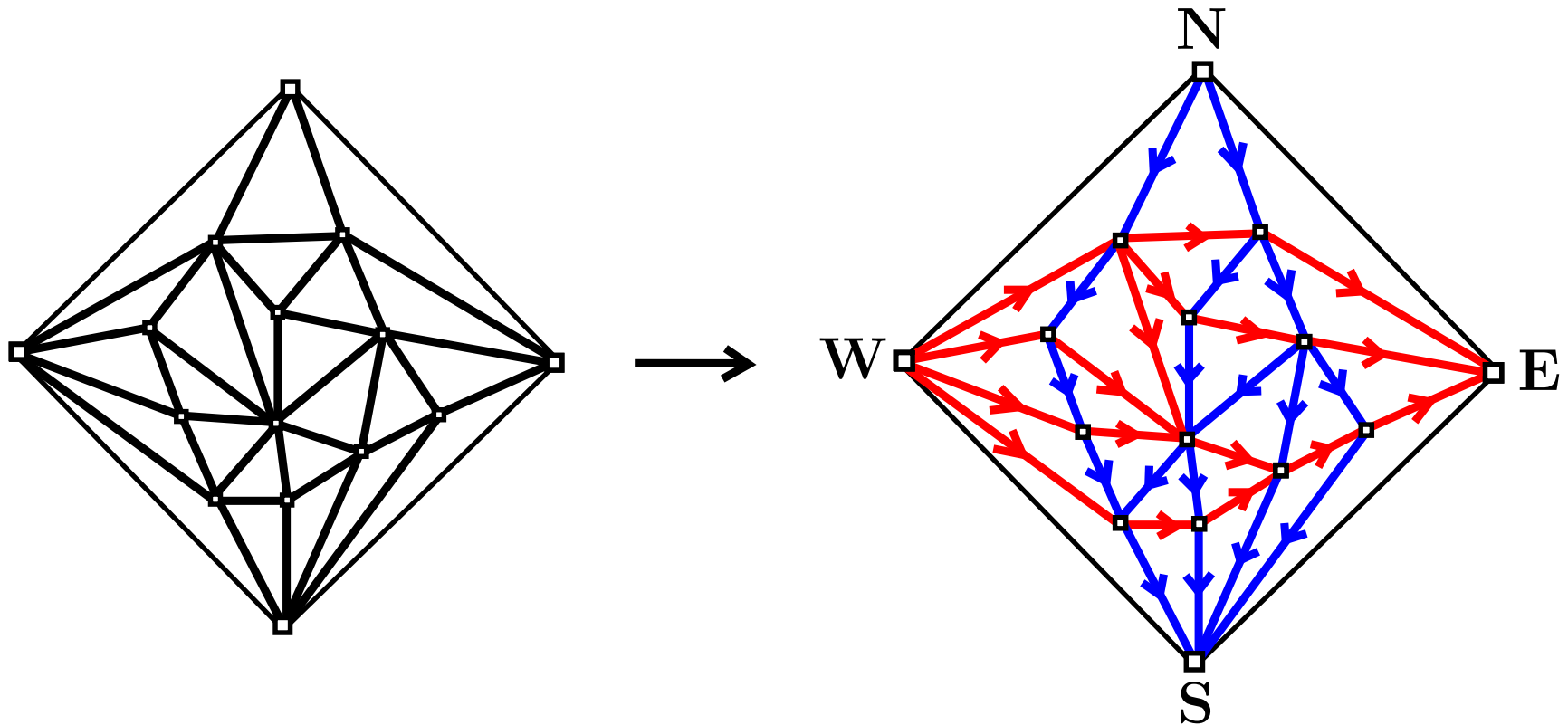
Transversal structures

Transversal structure = partition of inner edges into a red and a blue bipolar orientations that are transversal (introduced by Xin He'93)



Transversal structures

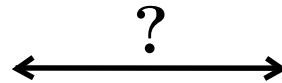
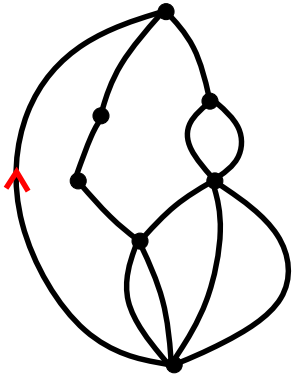
Transversal structure = partition of inner edges into a **red** and a **blue** bipolar orientations that are transversal (introduced by Xin He'93)



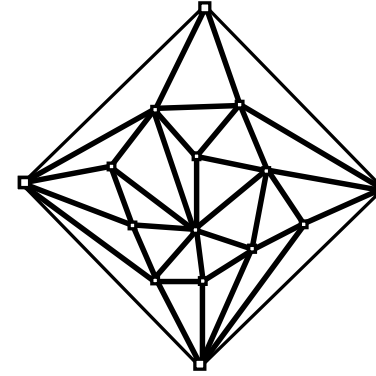
A triangulation of the 4-gon admits a transversal structure **iff** there is no separating triangle (irreducible)

Reformulating the bijection

Nonseparable

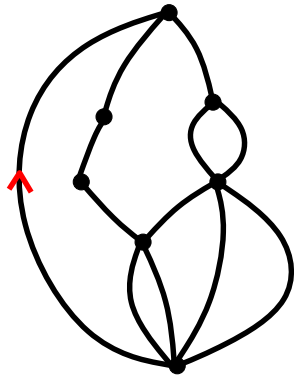


Irreducible



Reformulating the bijection

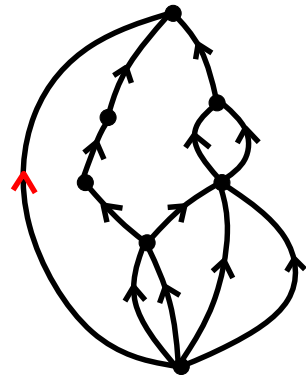
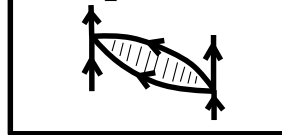
Nonseparable



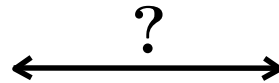
Ossona de Mendez'94



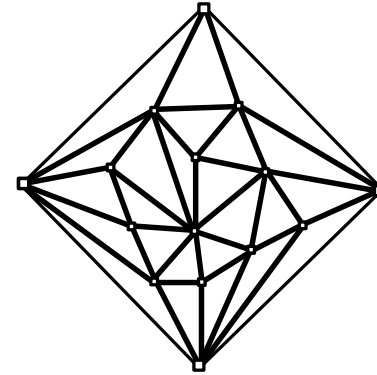
No pattern



Bipolar orientations



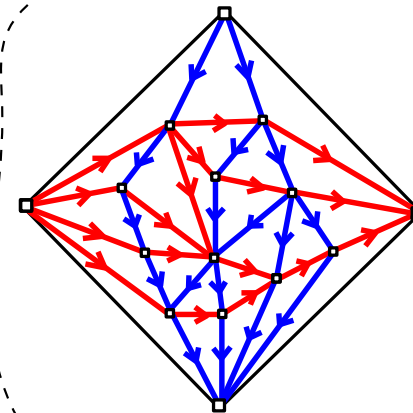
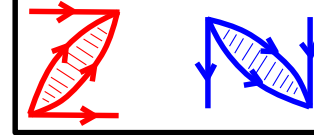
Irreducible



Fusy'05



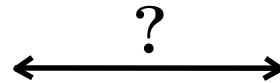
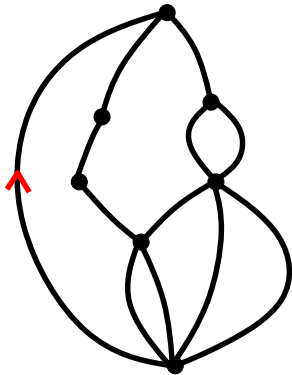
No patterns



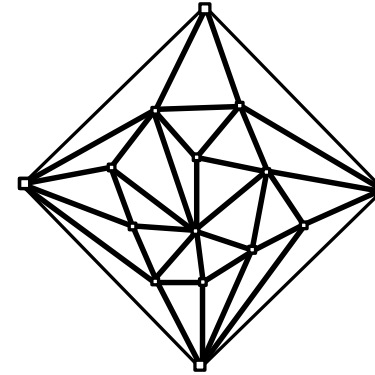
Transversal structures

Reformulating the bijection

Nonseparable



Irreducible



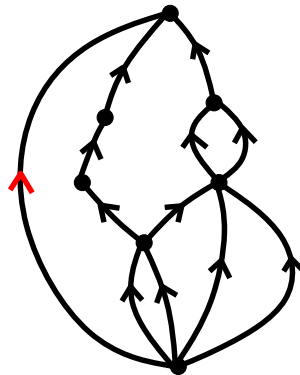
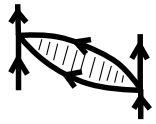
Ossona de Mendez'94



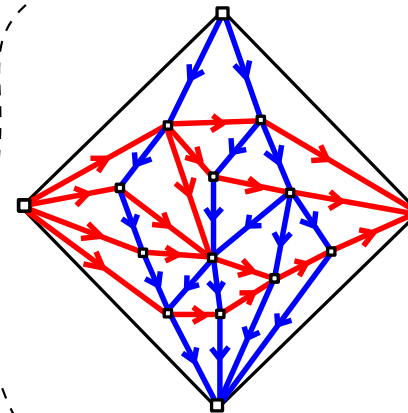
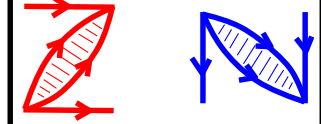
Fusy'05



No pattern



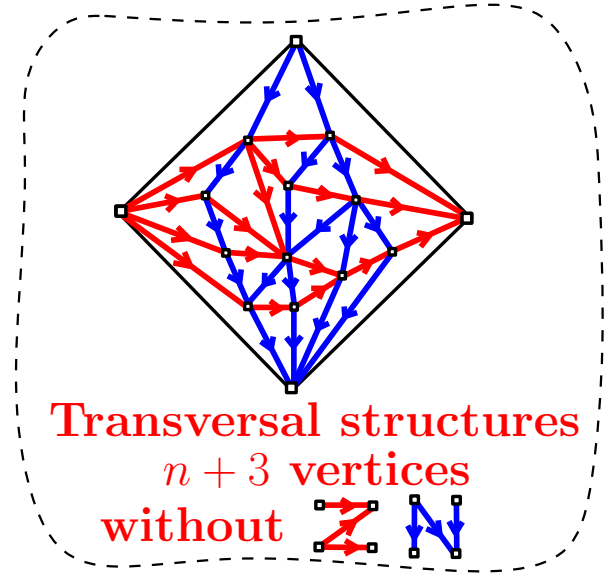
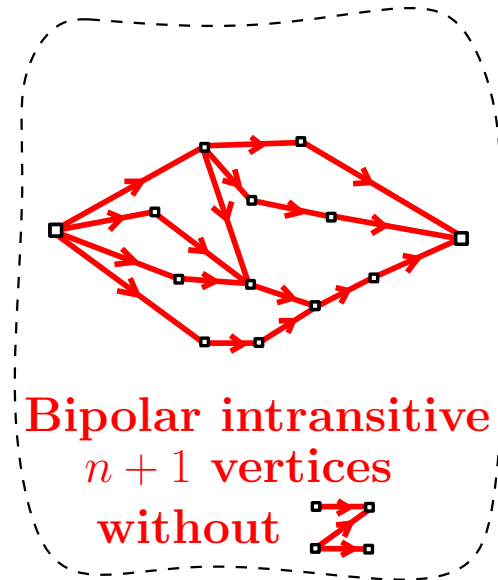
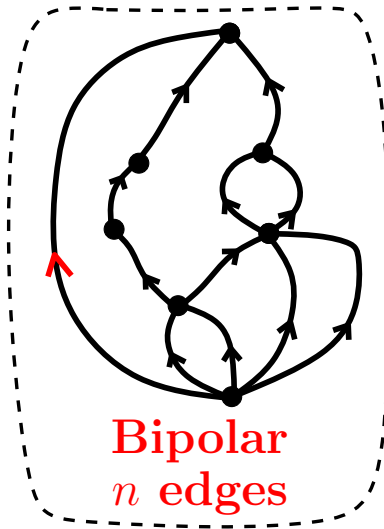
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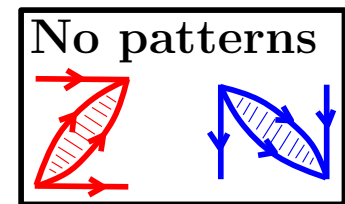
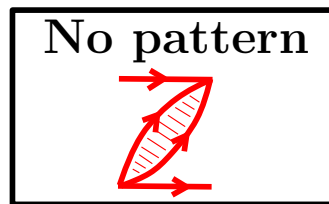
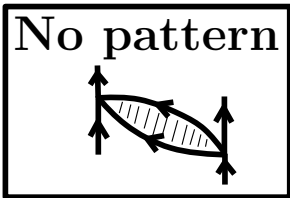
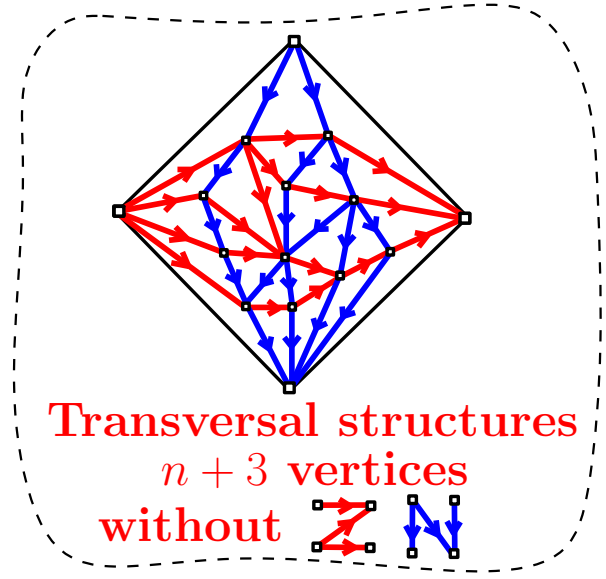
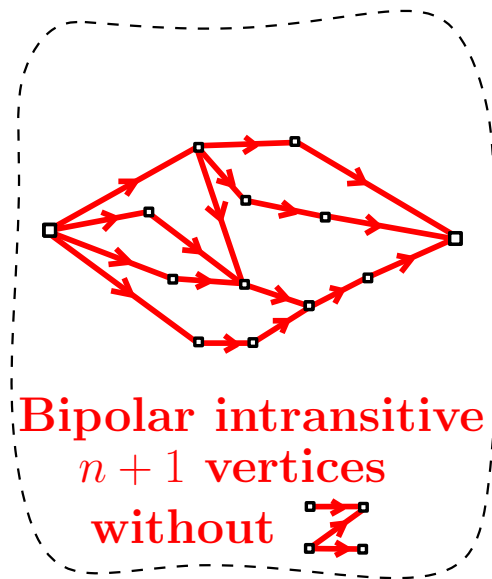
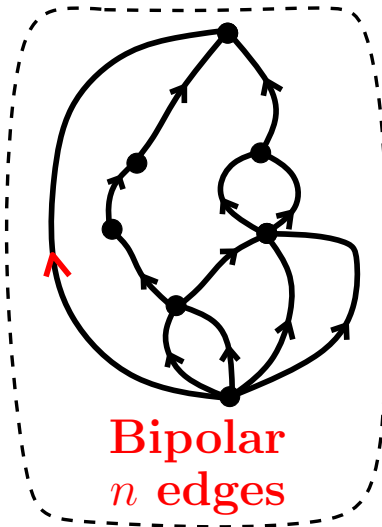
Bipolar orientations

Transversal structures

How the bijection works

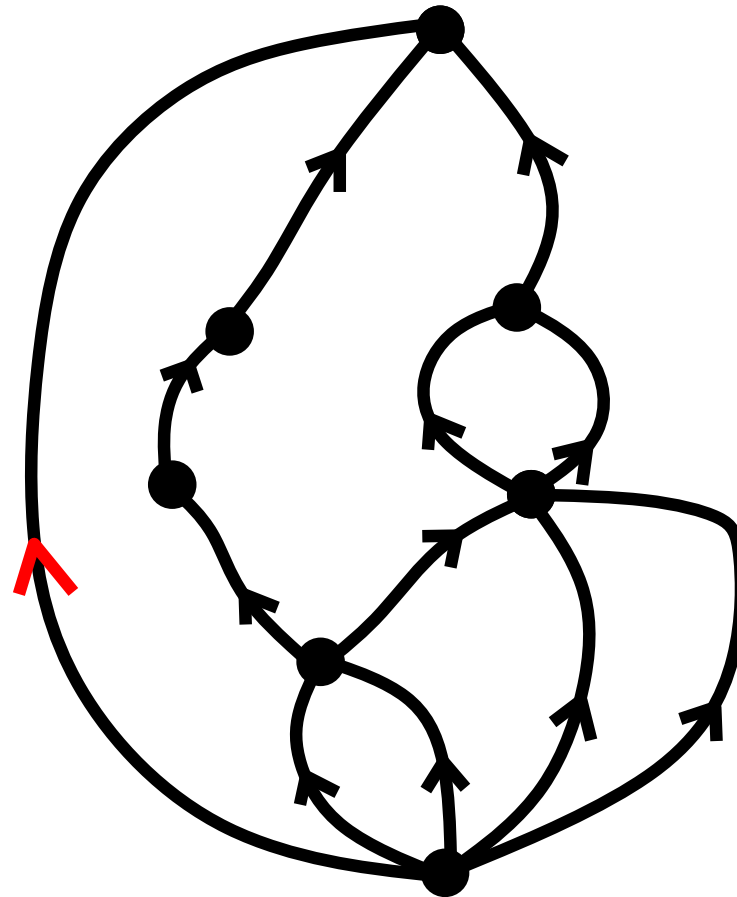


How the bijection works



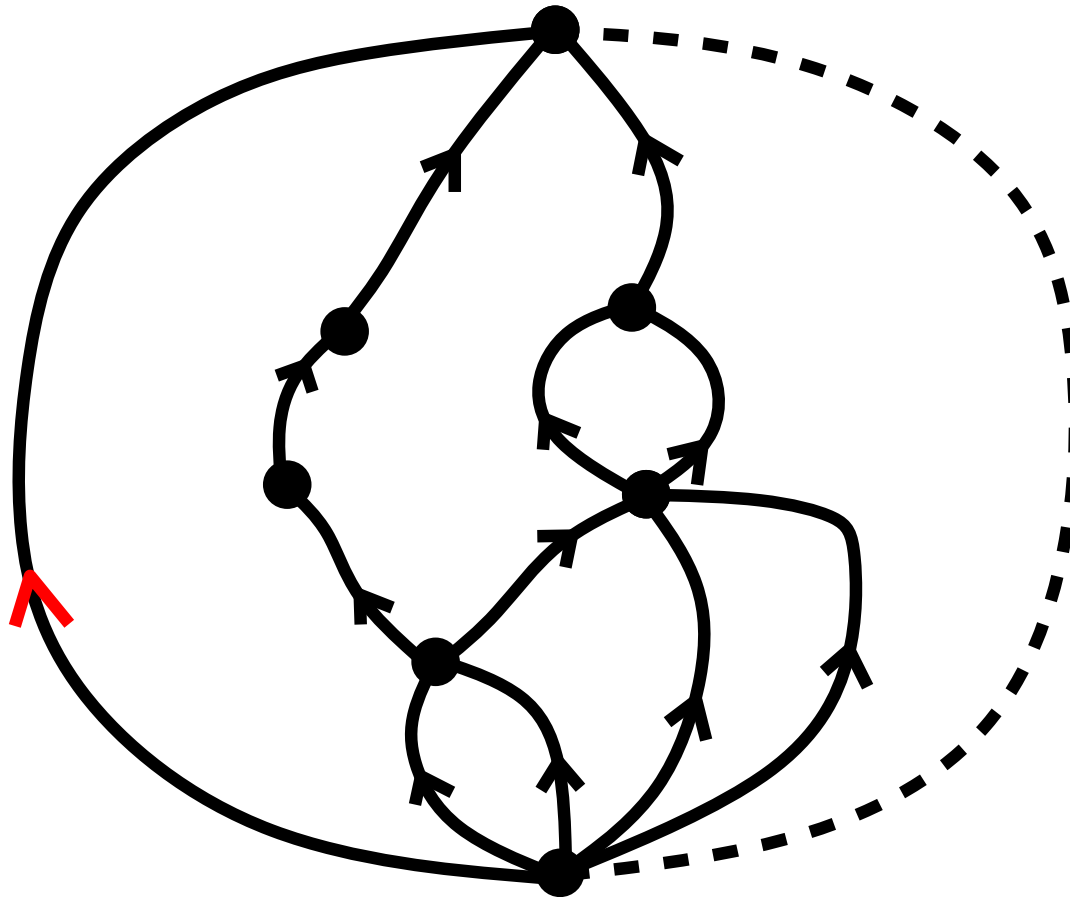
Bipolar \rightarrow Bipolar intransitive

Start with a plane bipolar orientation



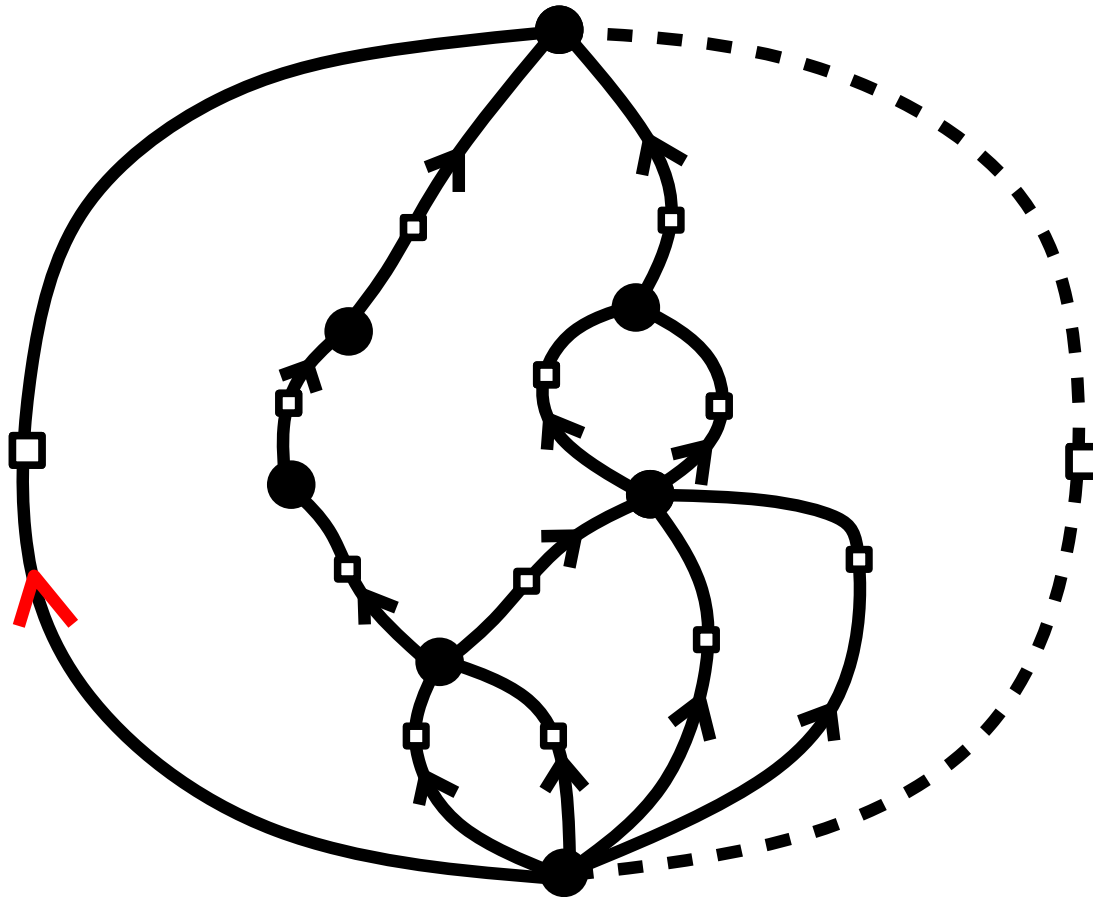
Bipolar \rightarrow Bipolar intransitive

Double the root edge



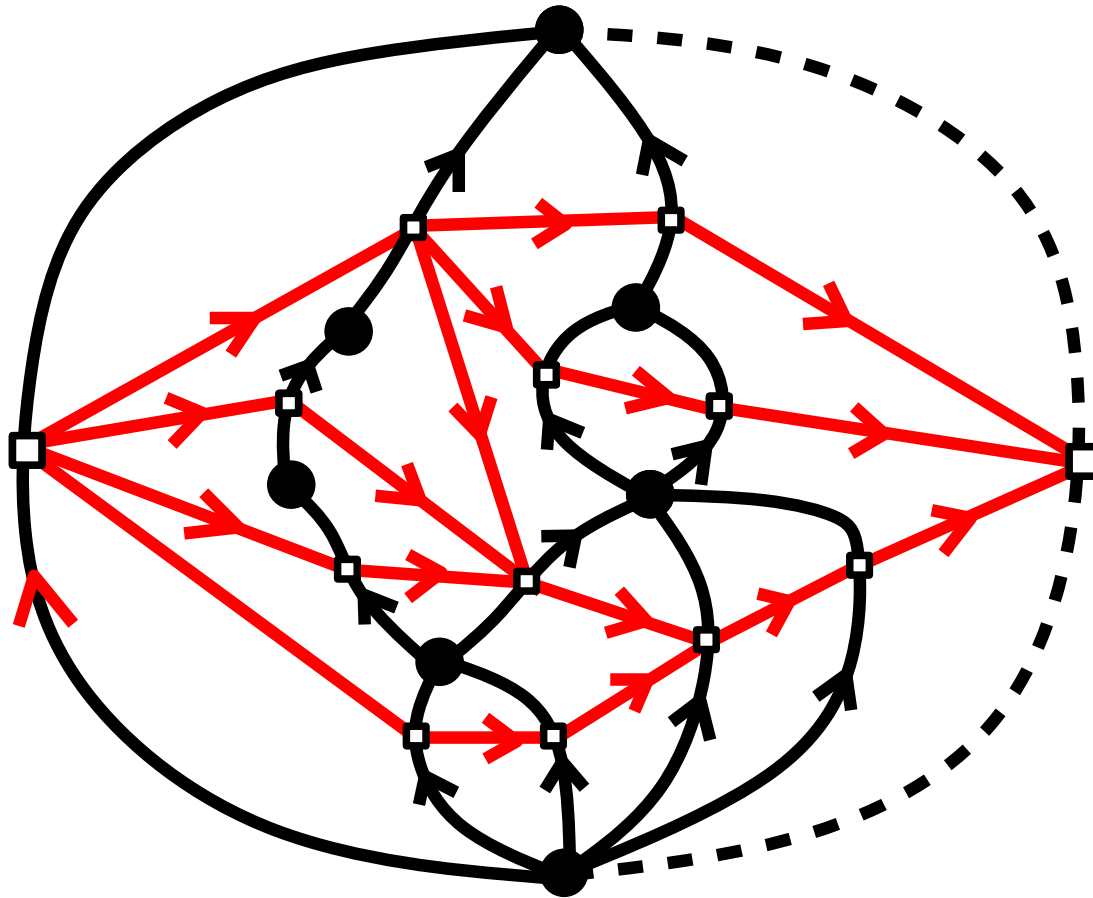
Bipolar \rightarrow Bipolar intransitive

Insert a white vertex in each edge

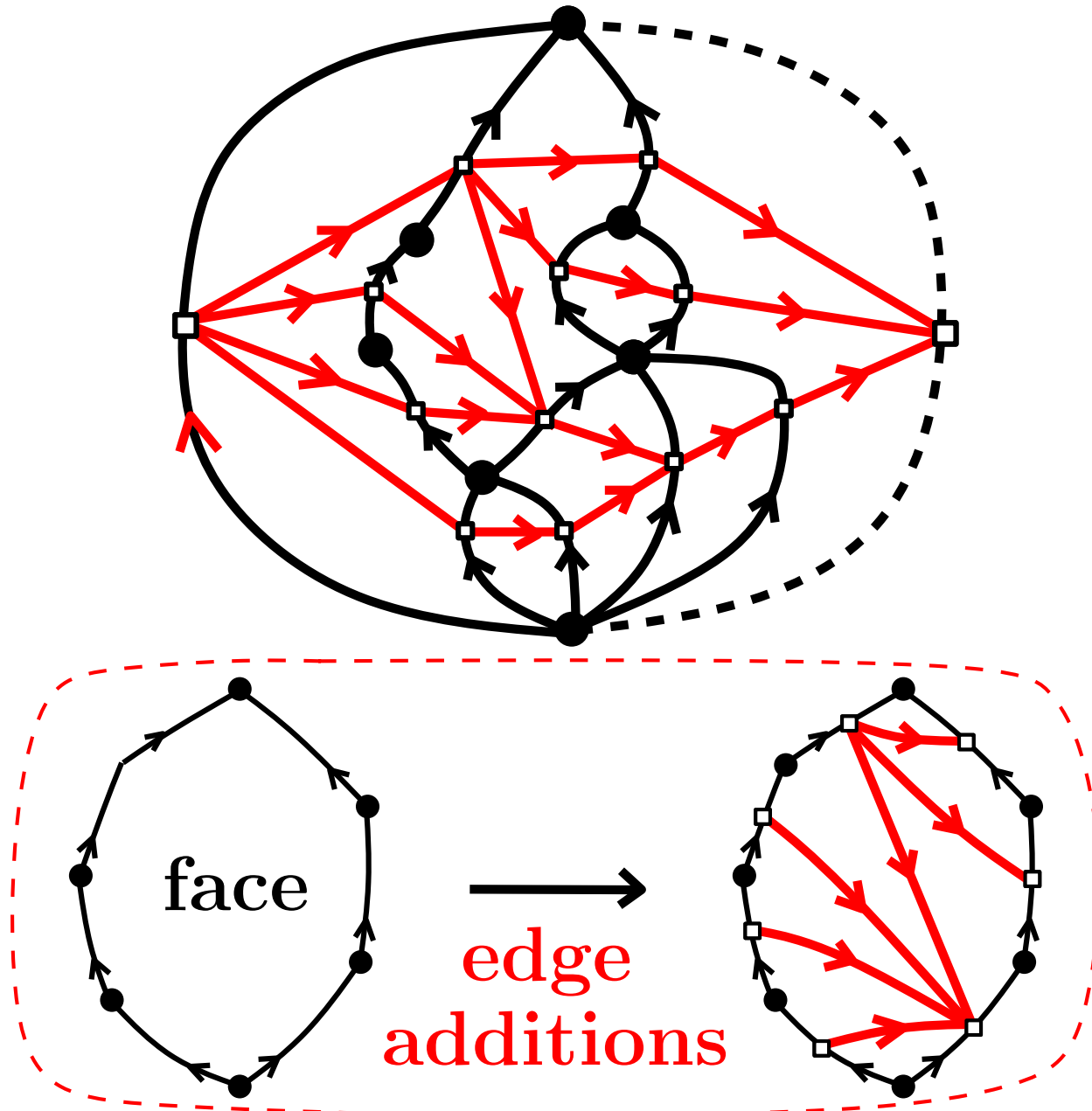


Bipolar \rightarrow Bipolar intransitive

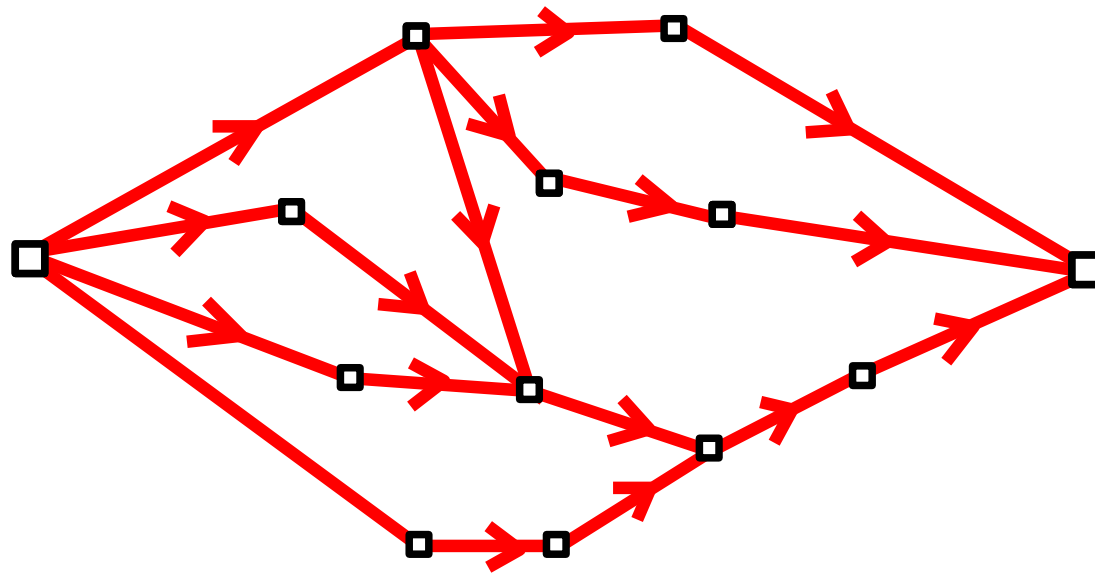
Triangulate the faces by **red edges**



Bipolar \rightarrow Bipolar intransitive

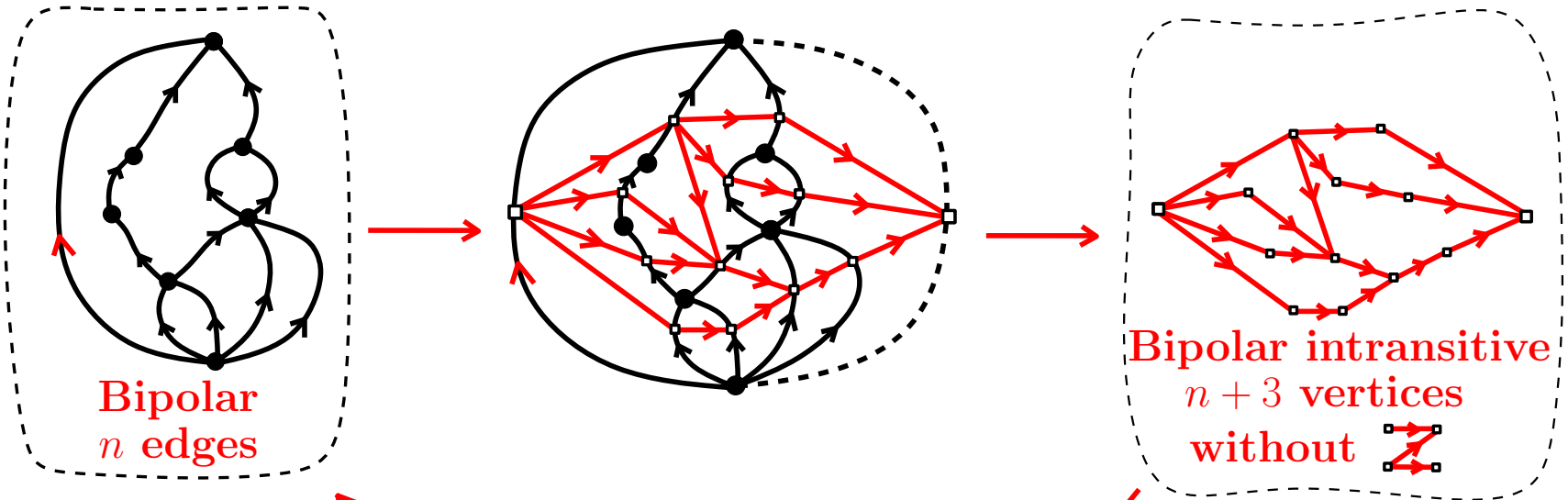


Bipolar \rightarrow Bipolar intransitive

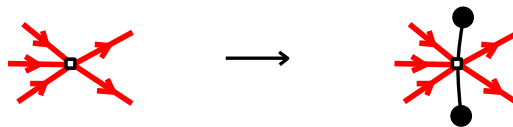


Bipolar \rightarrow Bipolar intransitive

A first bijection:

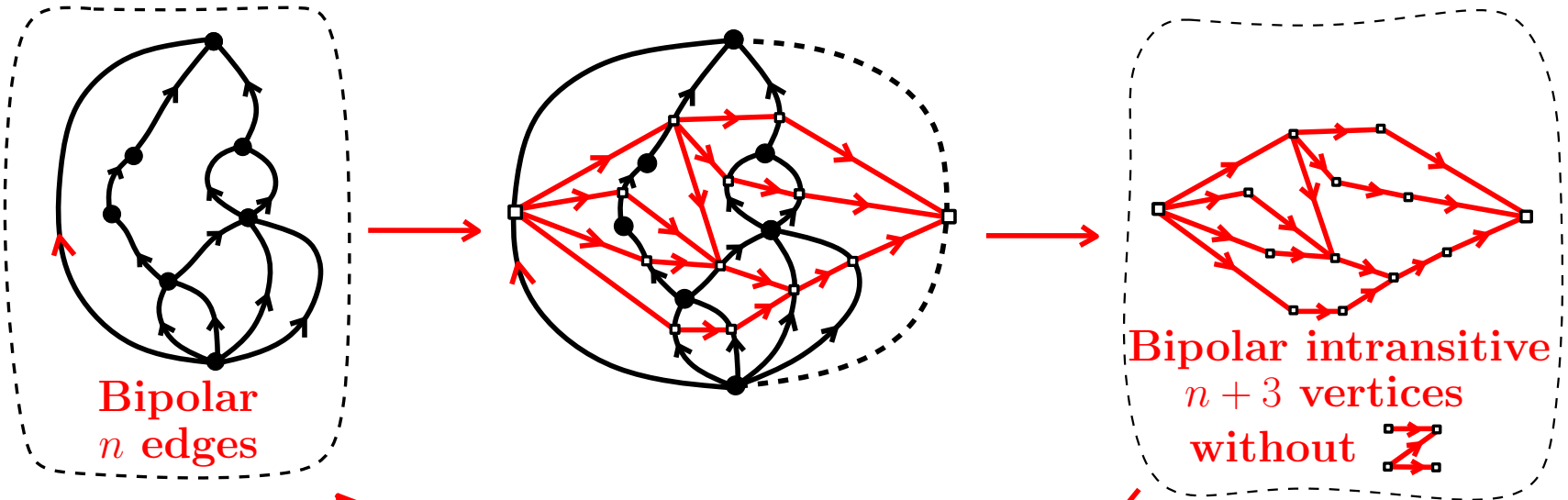


- 1) place a black vertex in each face
- 2) insert one black edge for each white vertex

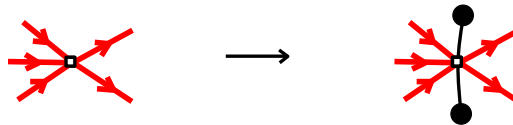


Bipolar \rightarrow Bipolar intransitive

A first bijection:



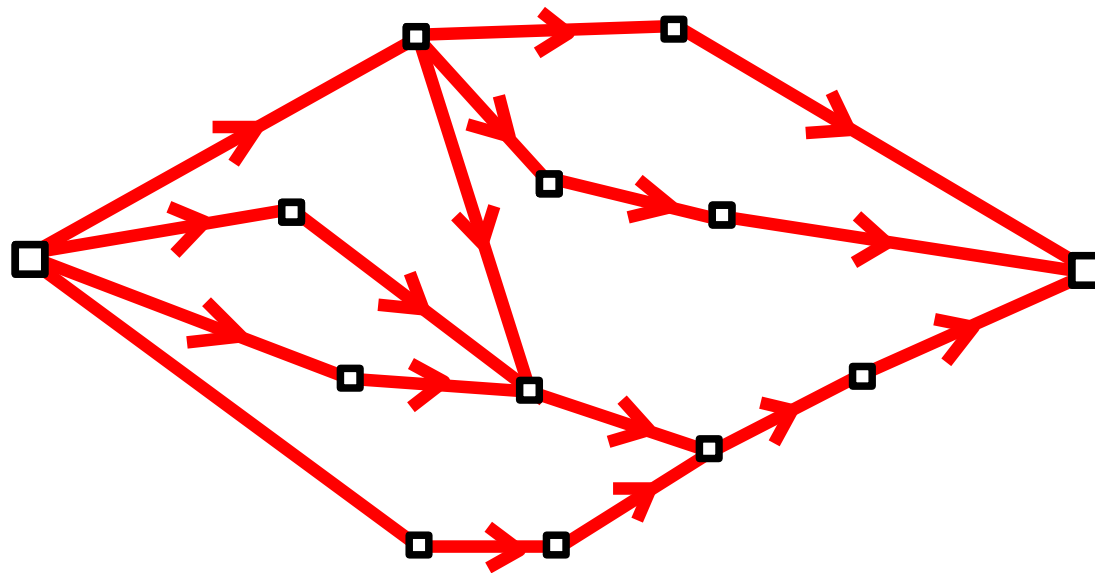
- 1) place a black vertex in each face
- 2) insert one black edge for each white vertex



Remark: These are counted by the **Baxter number**: $B_n = \frac{2}{n(n+1)^2} \sum_{k=0}^{n-1} \binom{n+1}{k} \binom{n+1}{k+1} \binom{n+1}{k+2}$

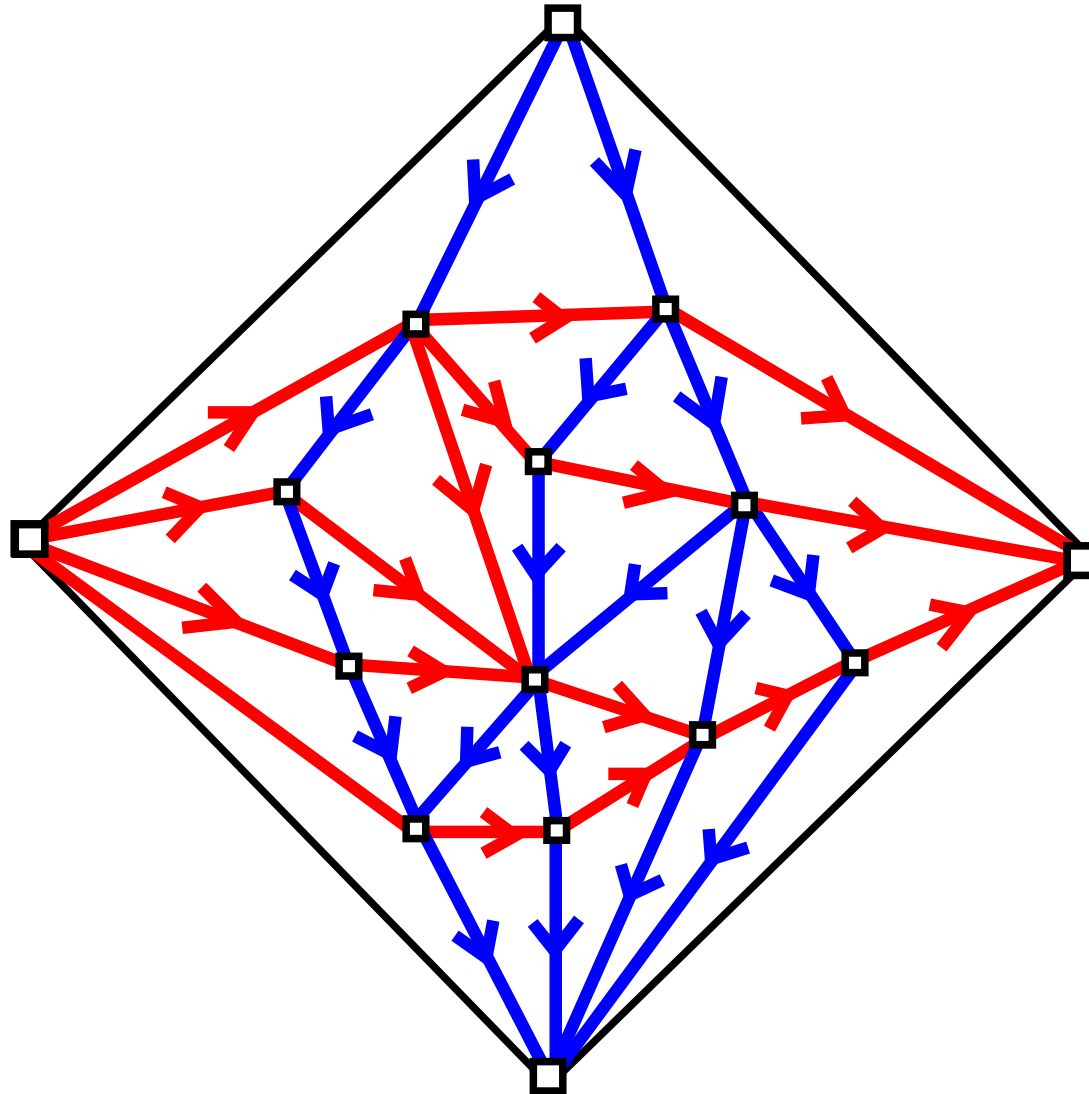
Bipolar intransitive \rightarrow Trans. struct.

Start with an **intransitive** bipolar orientation

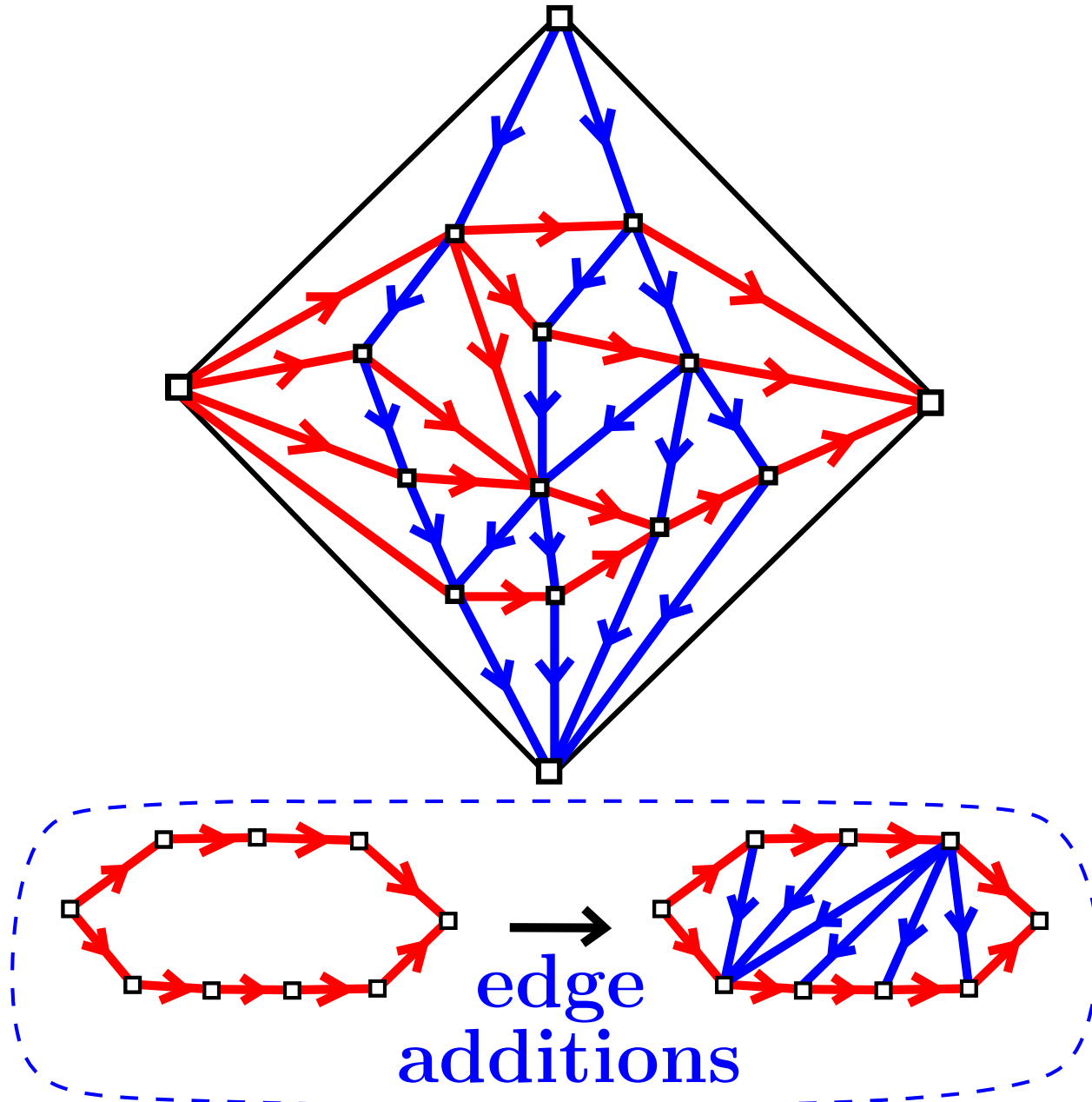


Bipolar intransitive \rightarrow Trans. struct.

Triangulate the faces by blue edges

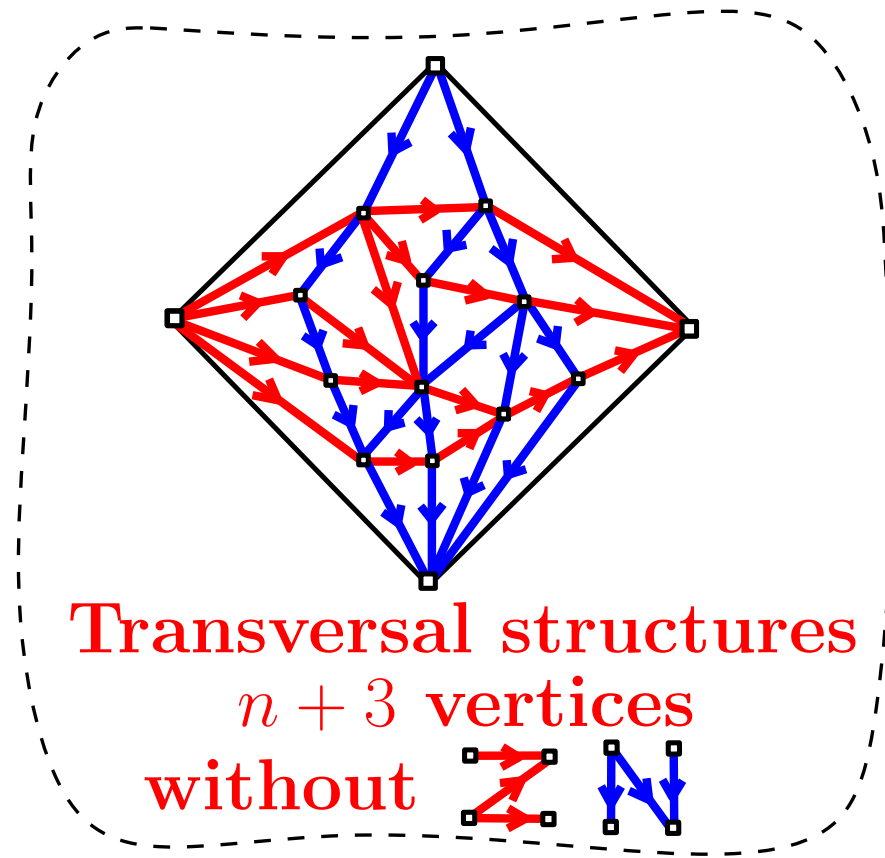
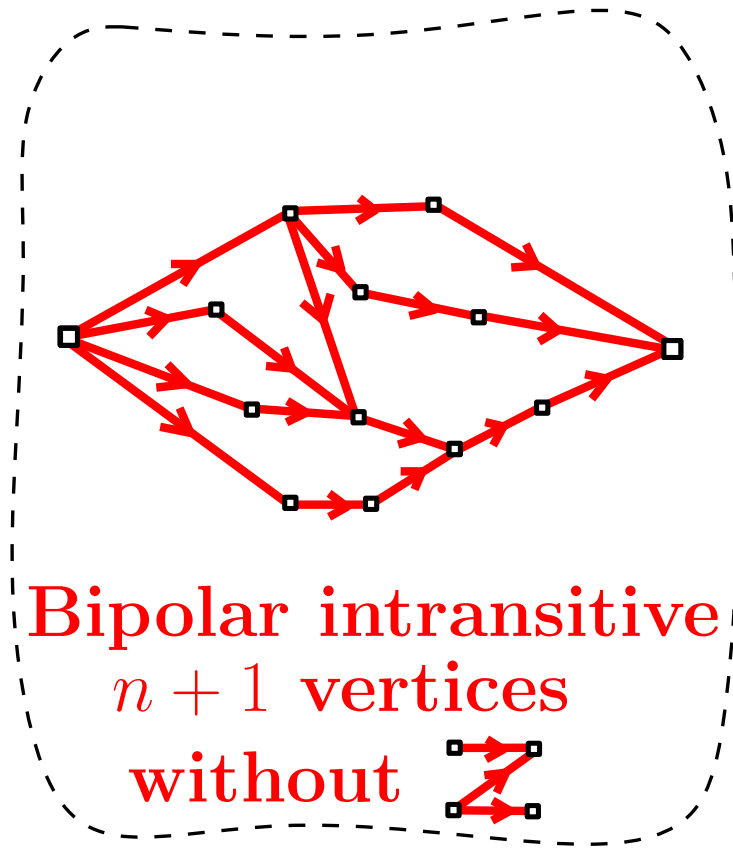


Bipolar intransitive \rightarrow Trans. struct.

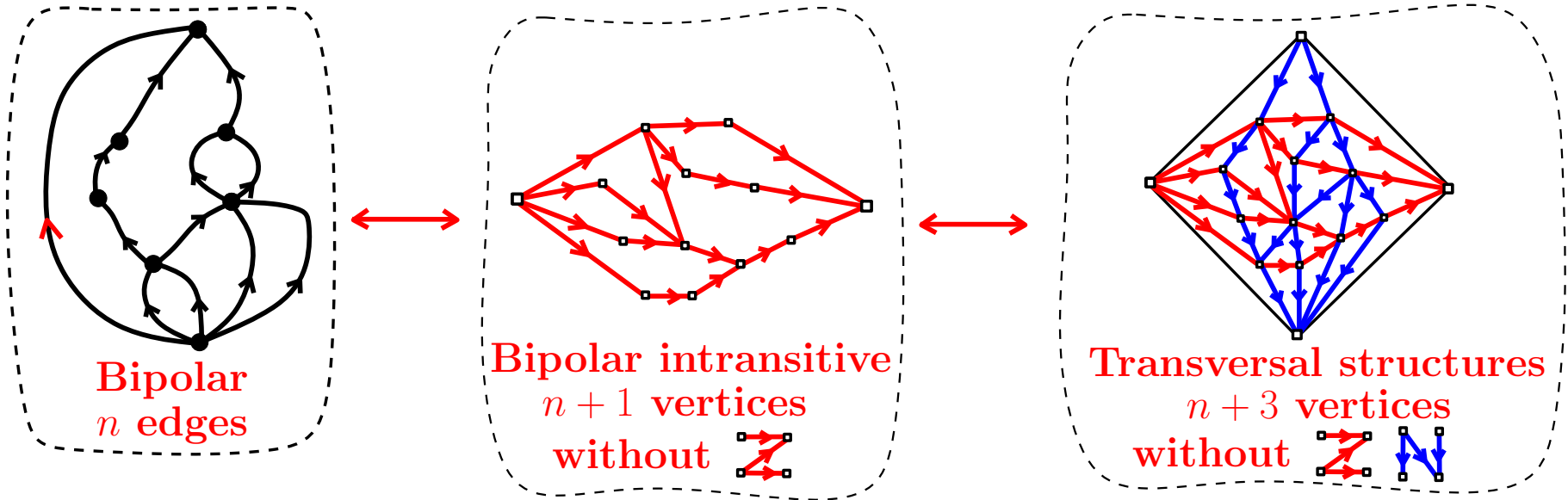


Bipolar intransitive \rightarrow Trans. struct.

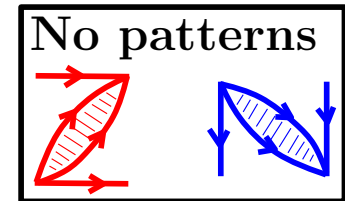
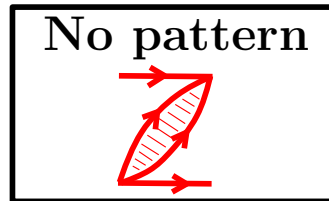
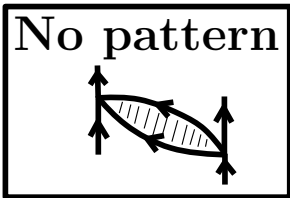
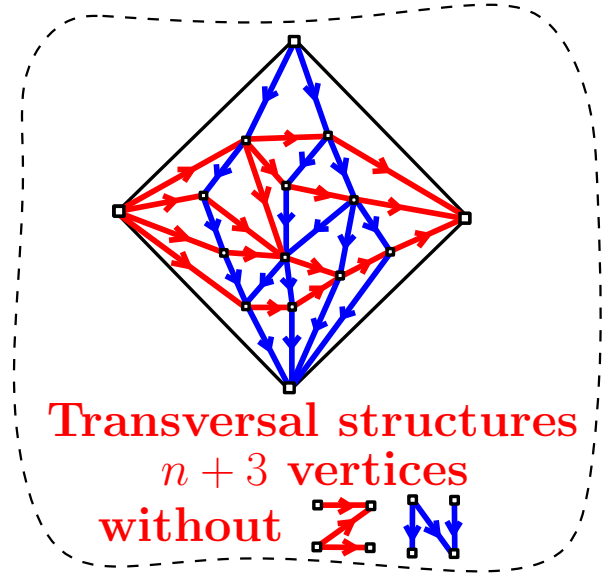
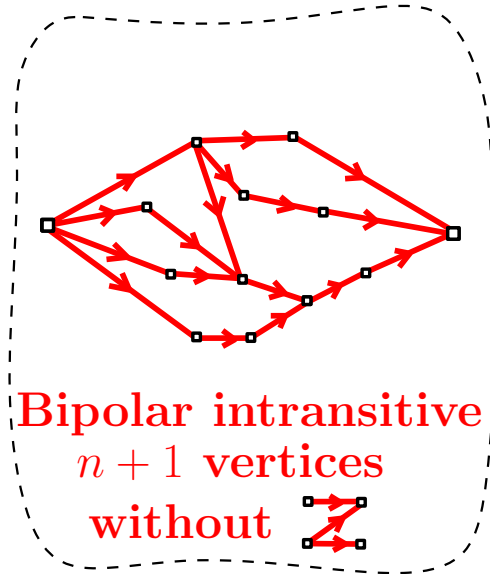
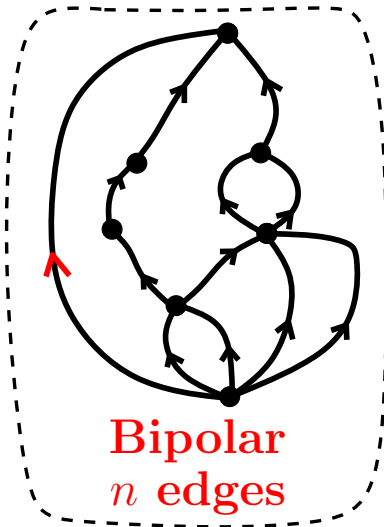
A second bijection:



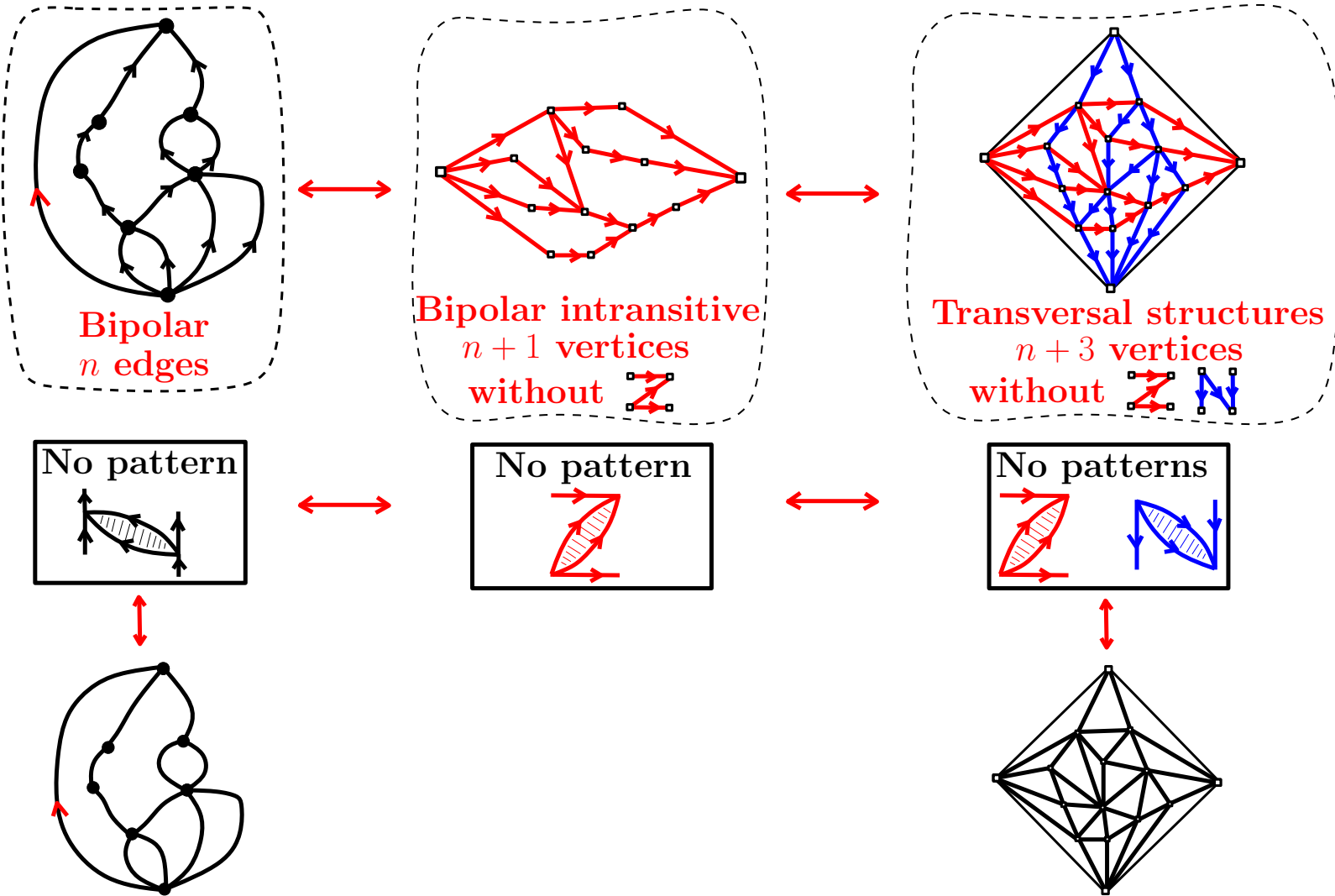
The bijection



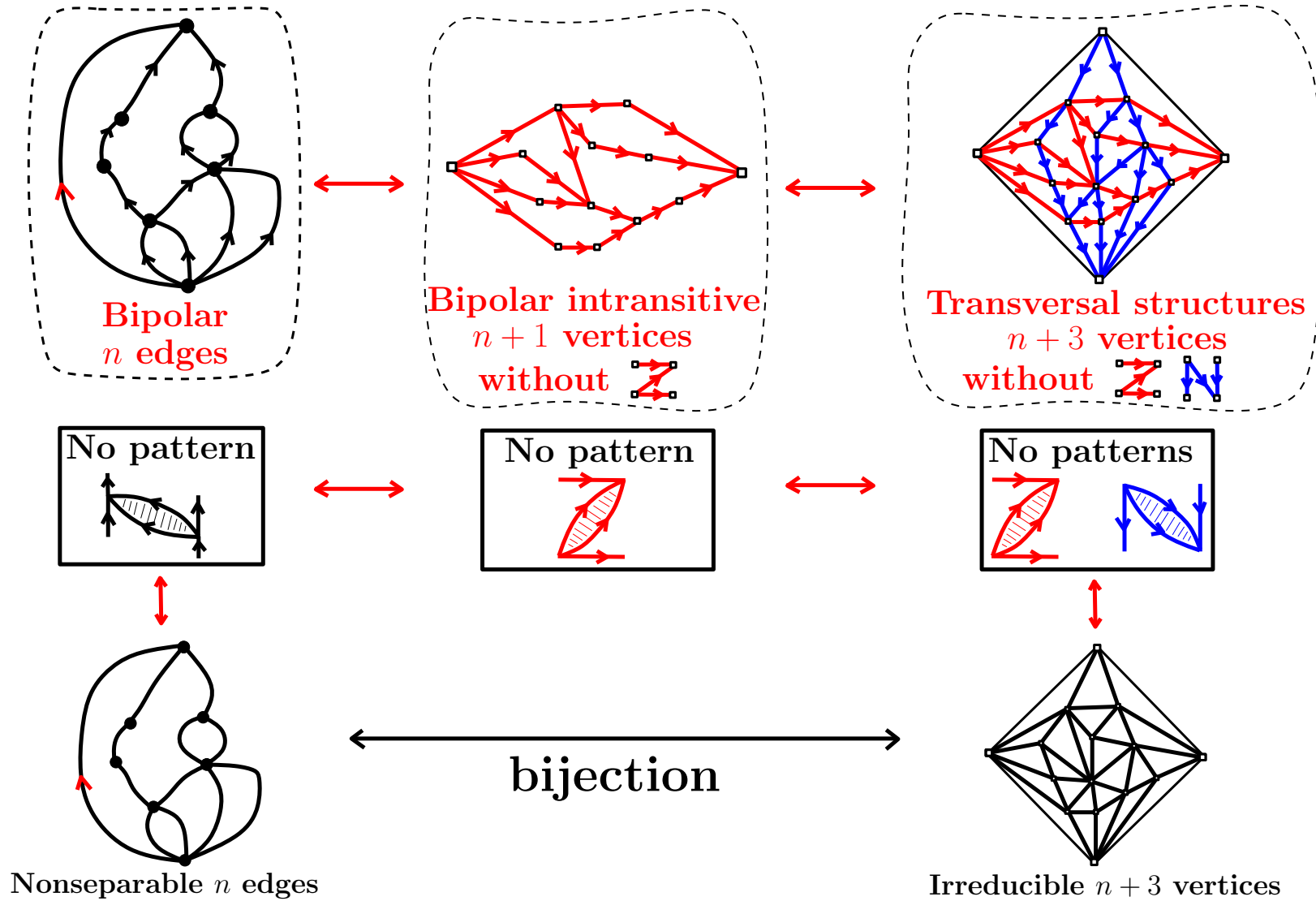
The bijection



The bijection



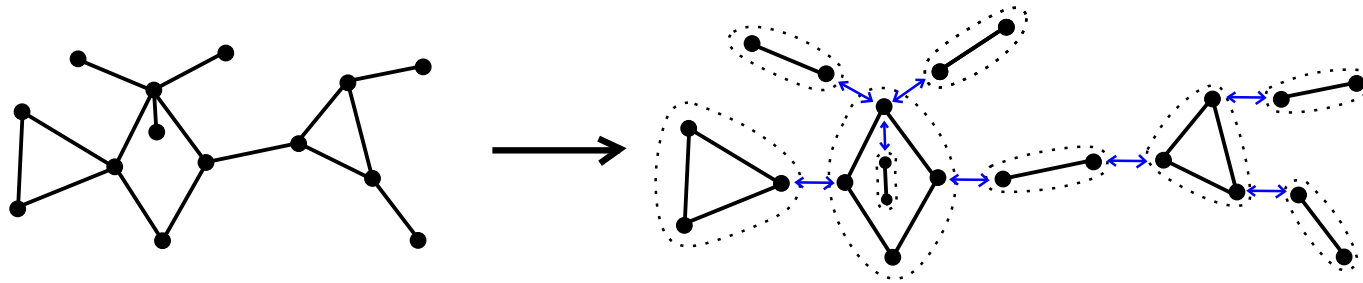
The bijection



Bijection between loopless maps and triangulations

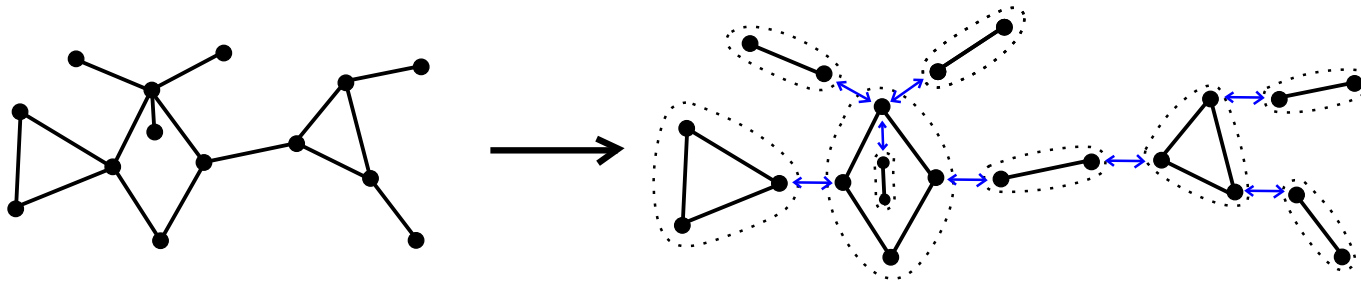
Decomposing a loopless map

- Block decomposition

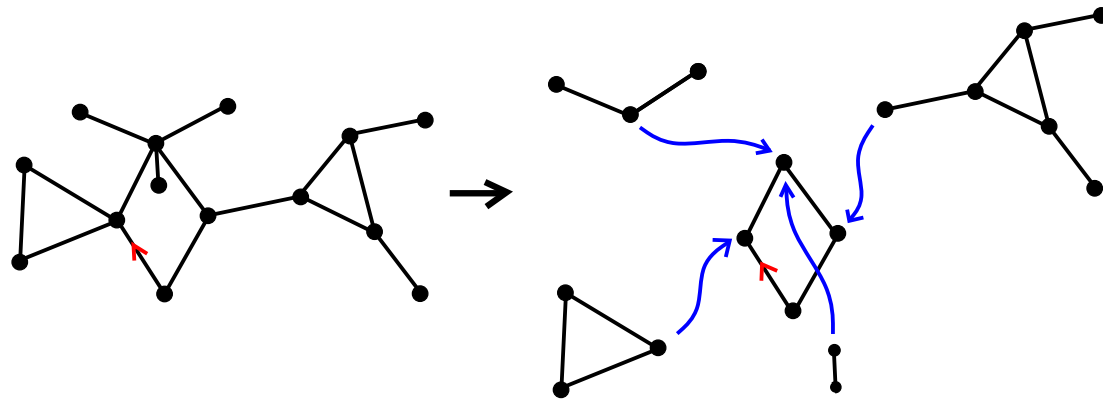


Decomposing a loopless map

- Block decomposition



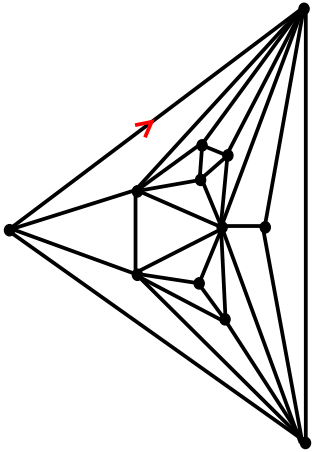
- For rooted loopless maps:



Nonseparable core
where each corner
is possibly occupied
by a loopless map

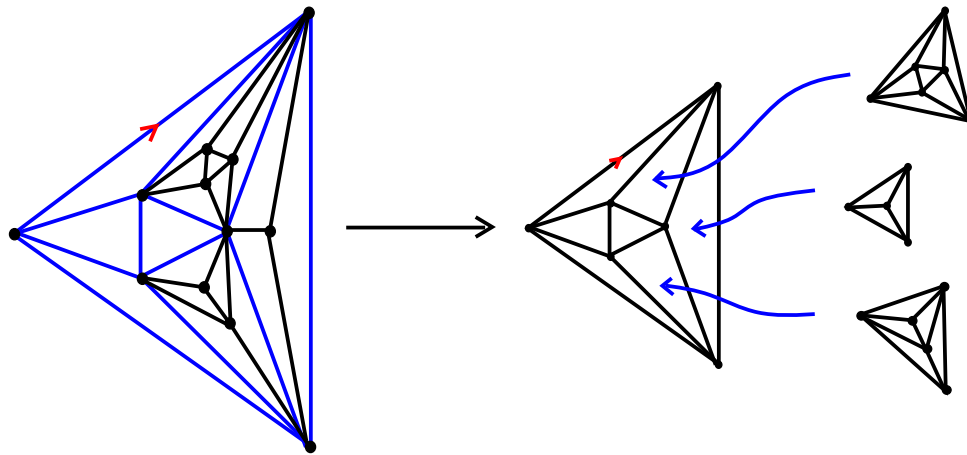
Decomposing a triangulation

- Classical decomposition at separating triangles



Decomposing a triangulation

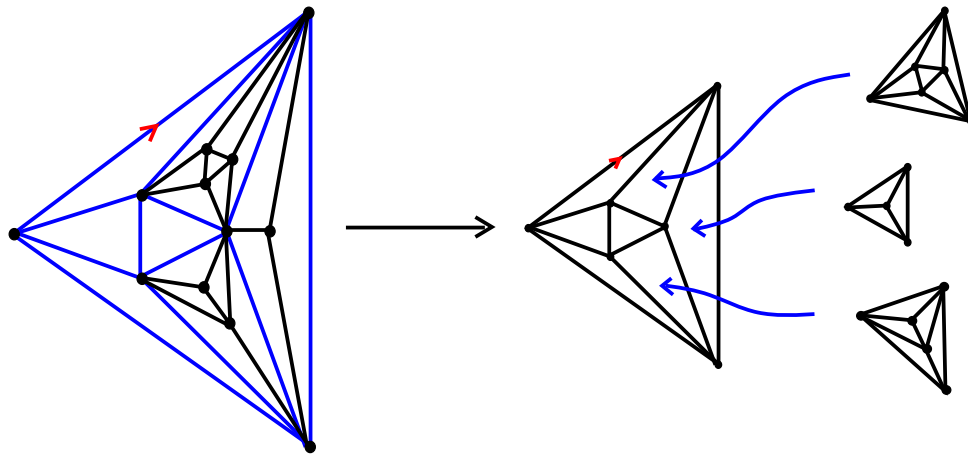
- Classical decomposition at separating triangles



4-connected triangulation
where each face is
possibly occupied
by a triangulation

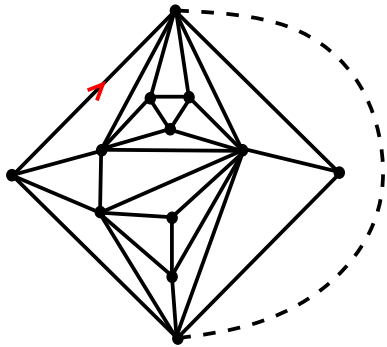
Decomposing a triangulation

- Classical decomposition at separating triangles



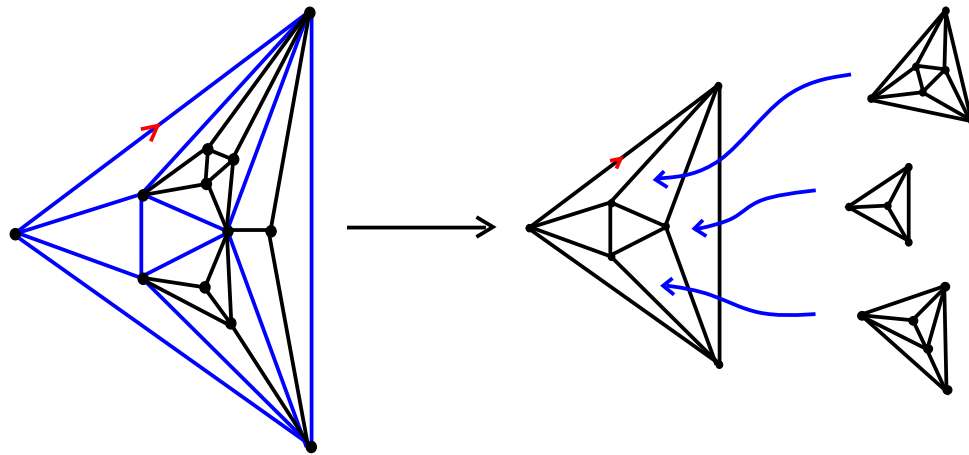
4-connected triangulation
where each face is
possibly occupied
by a triangulation

- **Here:** the same **after deleting an outer edge**



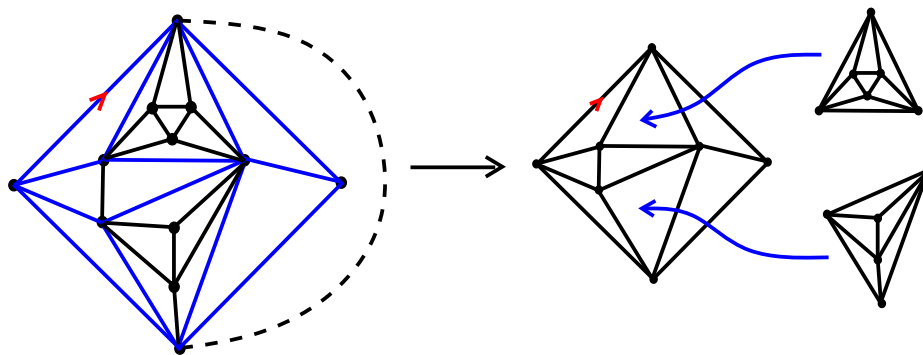
Decomposing a triangulation

- Classical decomposition at separating triangles



4-connected triangulation
where each face is
possibly occupied
by a triangulation

- Here: the same after deleting an outer edge



Irreducible triangulation
where each face is
possibly occupied
by a triangulation

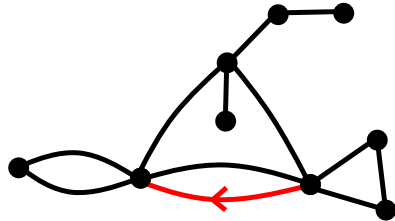
The decompositions are parallel

A **loopless map** M ($|M| = \# \text{ edges}$) is

i) the vertex-map \bullet

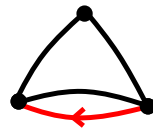
or

ii) $|M| \geq 1$



=

Nonseparable core C



+

$2|C|$ loopless maps $M_1, \dots, M_{2|C|}$



$$|M| = |C| + \sum_{i=1}^{2|C|} |M_i|$$

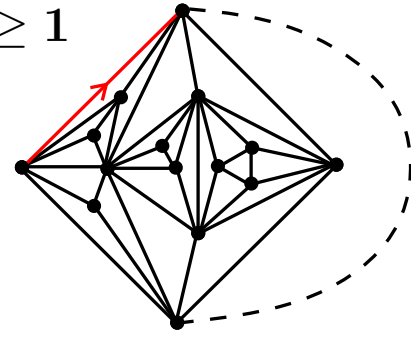
A **triangulation** T ($|T| = \# \text{ vert.} - 3$) is

i) the triangle-map \triangleleft



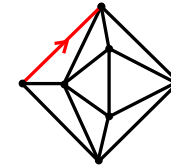
or

ii) $|T| \geq 1$



=

Irreducible core I



+

$2|I|$ triangulations $T_1, \dots, T_{2|I|}$



$$|T| = |I| + \sum_{i=1}^{2|I|} |T_i|$$

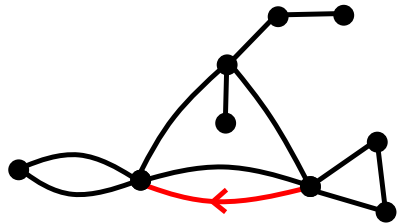
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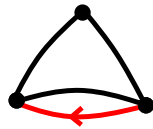
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Nonseparable core C



+

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$$|M| = |C| + \sum_{i=1}^{2|C|} |M_i|$$

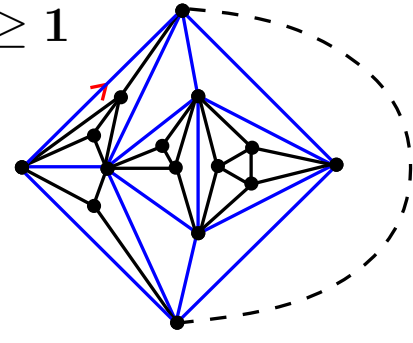
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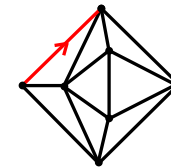
or

ii) $|T| \geq 1$



=

Irreducible core I



+

$2|I|$ triangulations $T_1, \dots, T_{2|I|}$



$$|T| = |I| + \sum_{i=1}^{2|I|} |T_i|$$

Nonseparable maps n edges

\longleftrightarrow *bijection*

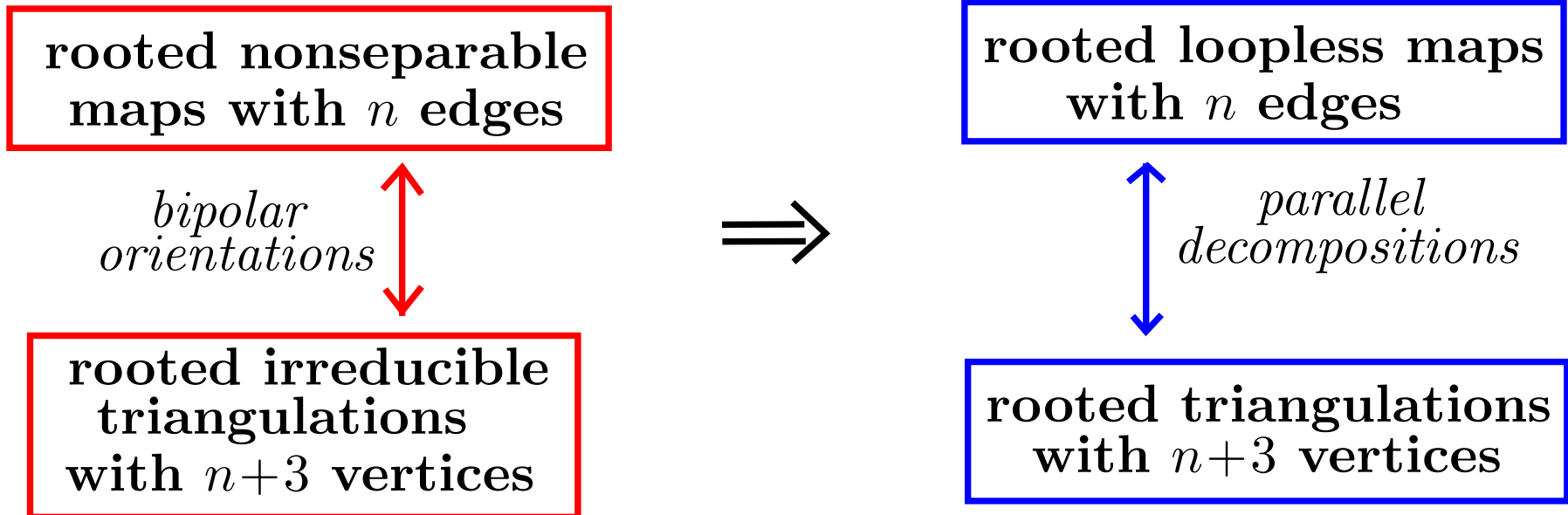
Irreducible triang. $n + 3$ vertices

Loopless maps n edges

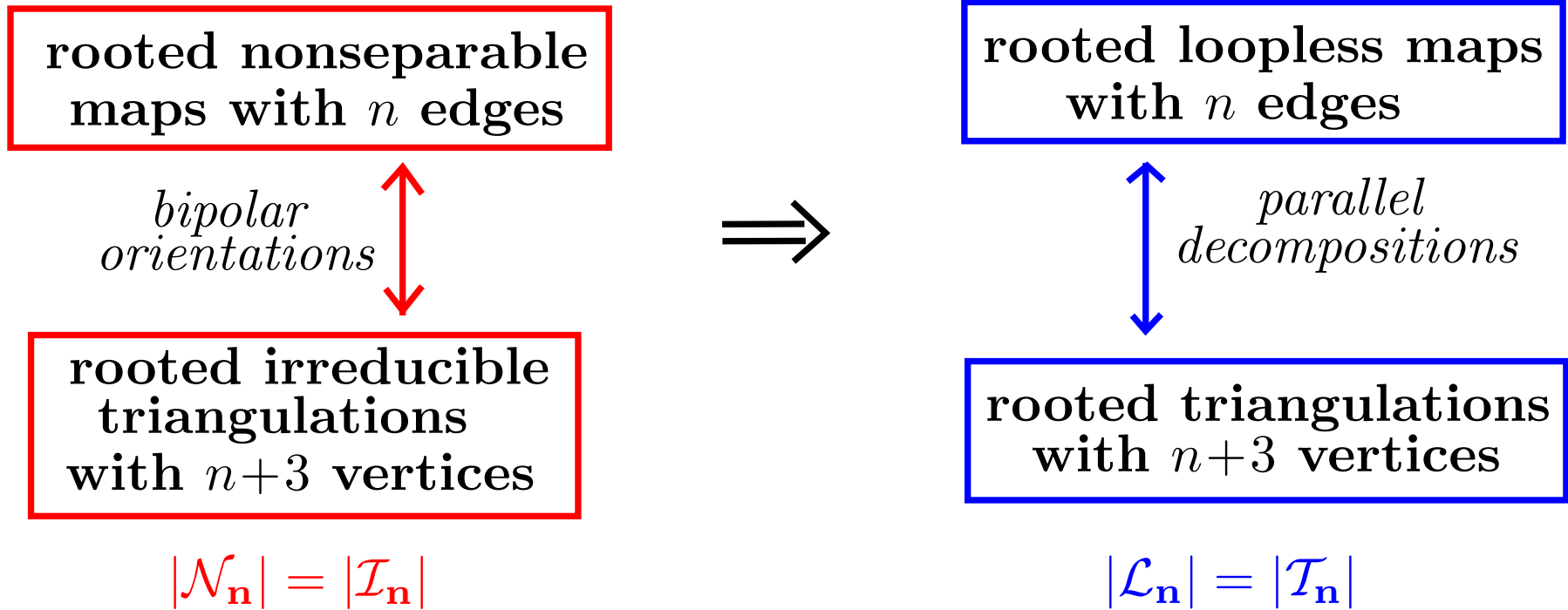
\longleftrightarrow *bijection*

Triangulations $n + 3$ vertices

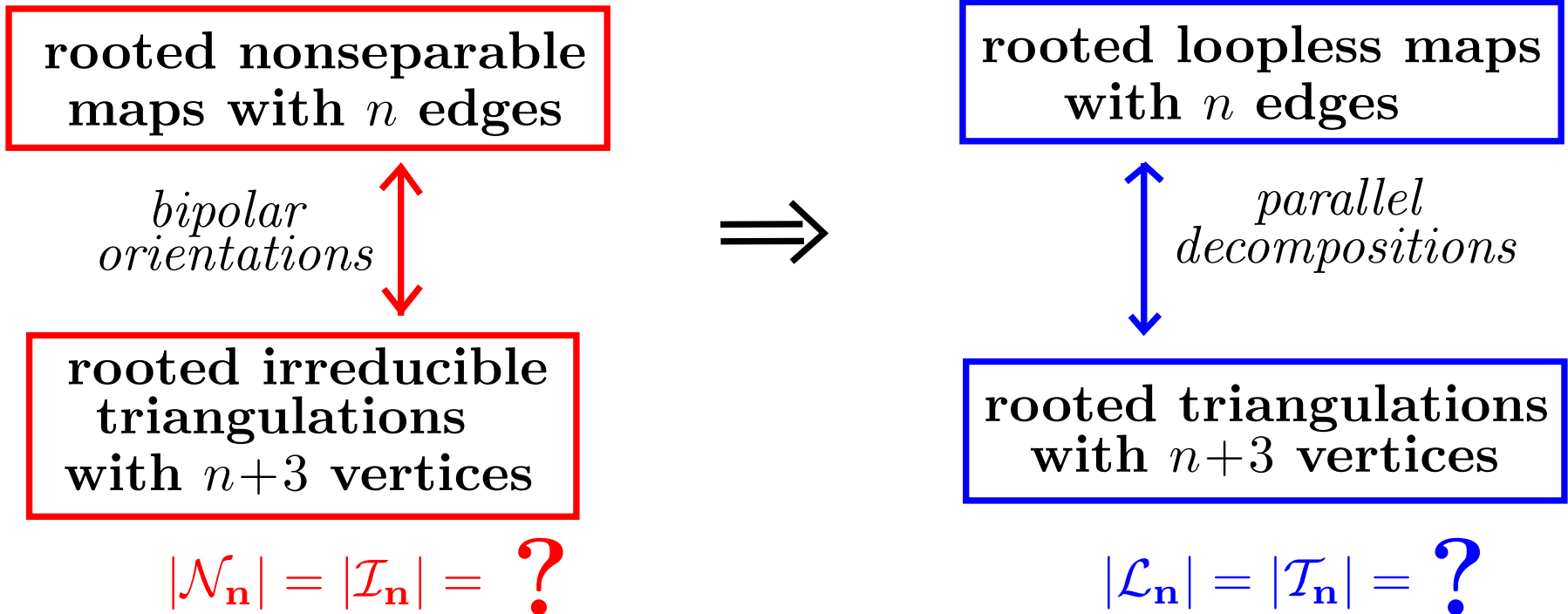
Results obtained so far



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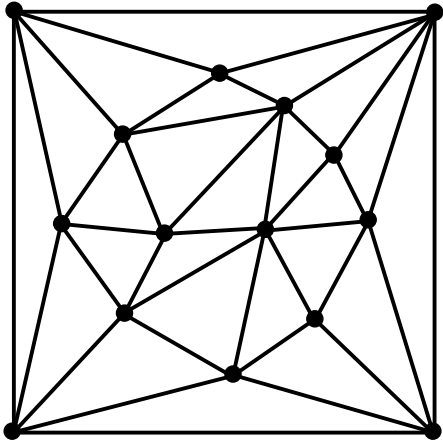


Results obtained so far



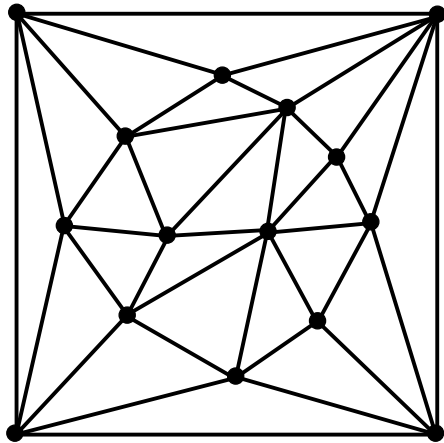
Counting the families

Irreducible triang. \leftrightarrow ternary trees

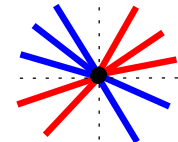
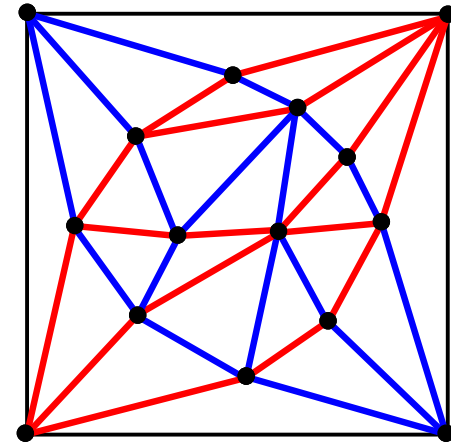


**Fusy'05: irreducible triangulations
are in bijection with ternary trees**

Irreducible triang. \leftrightarrow ternary trees

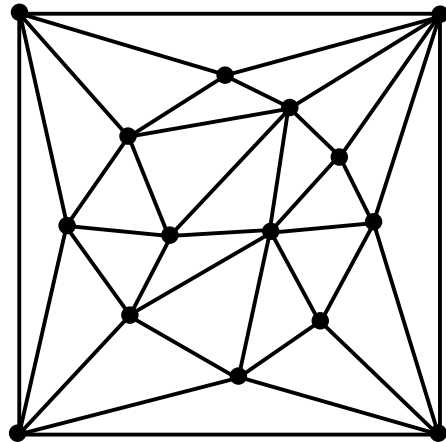


canonical
transversal structure

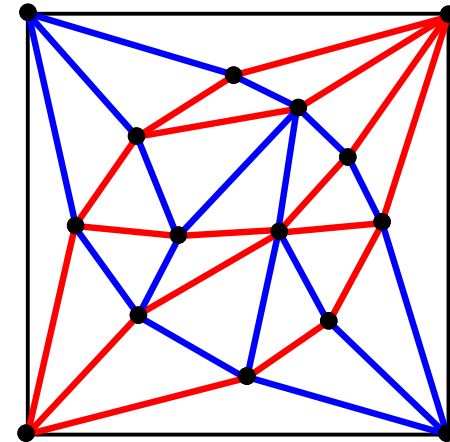


**Fusy'05: irreducible triangulations
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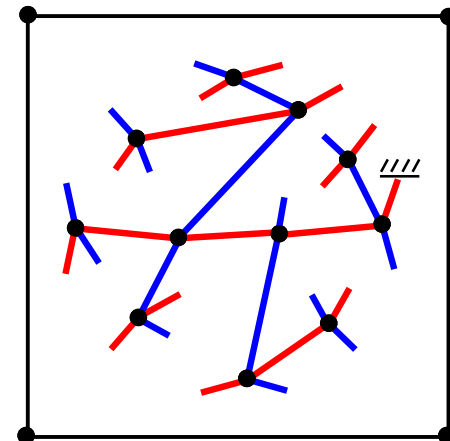
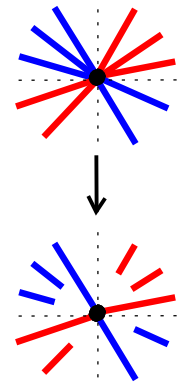
Irreducible triang. \leftrightarrow ternary trees



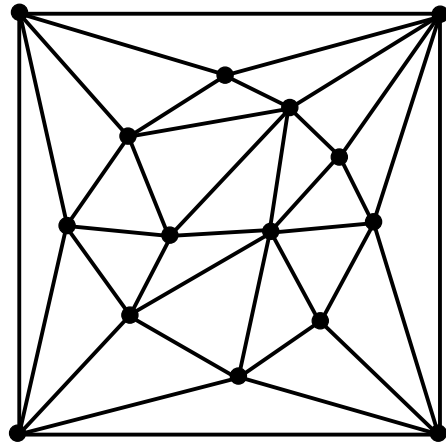
canonical
transversal structure
 \longrightarrow



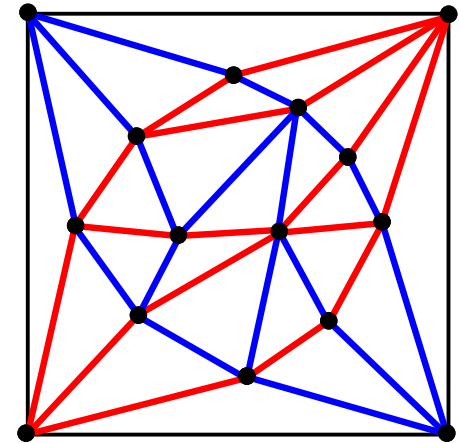
**Fusy'05: irreducible triangulations
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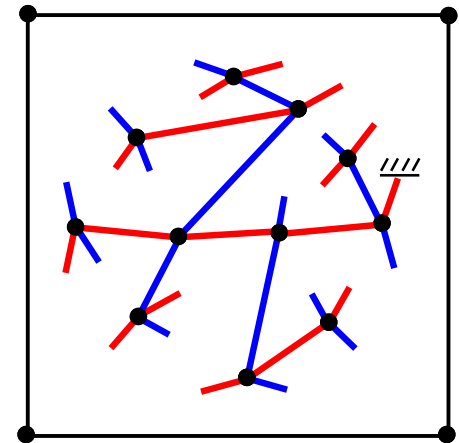
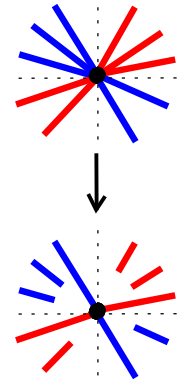
Irreducible triang. \leftrightarrow ternary trees



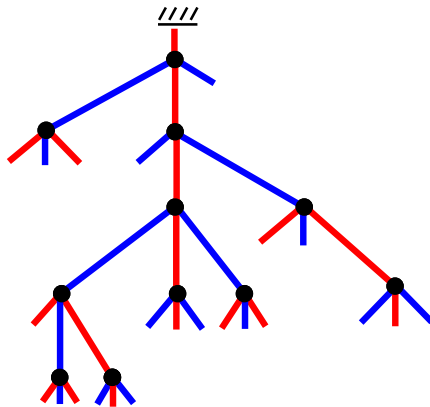
canonical
transversal structure
→



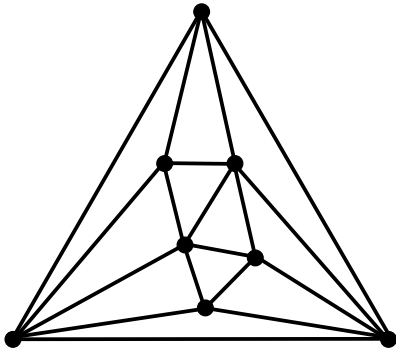
**Fusy'05: irreducible triangulations
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← rooting

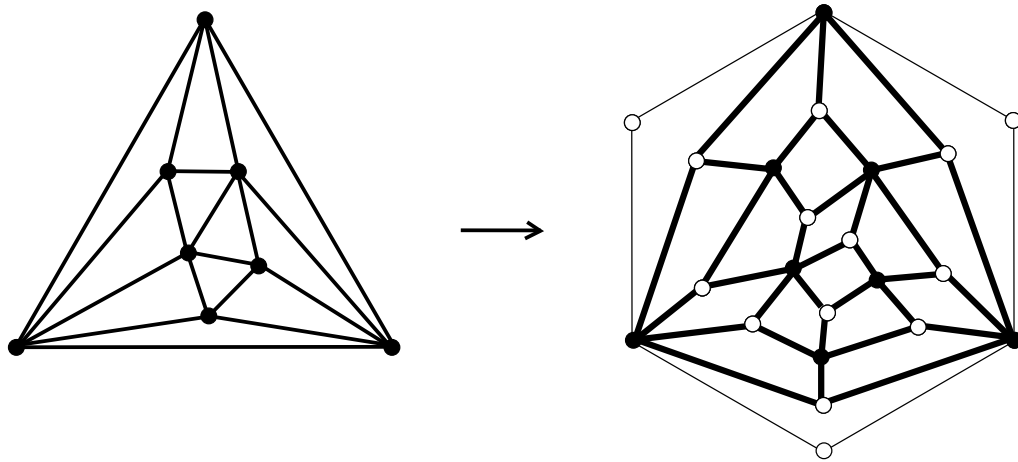


Triangulations \leftrightarrow quaternary trees



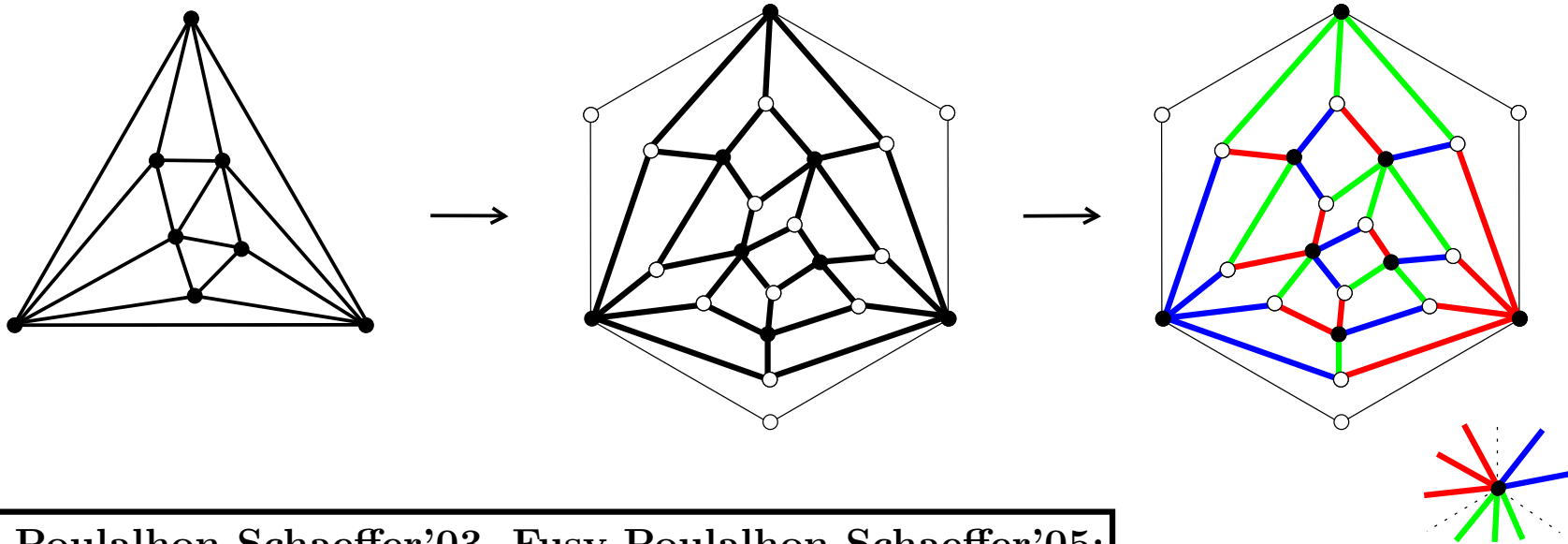
Poulalhon-Schaeffer'03, Fusy-Poulalhon-Schaeffer'05:
triangulations are in bijection with quaternary trees

Triangulations \leftrightarrow quaternary trees



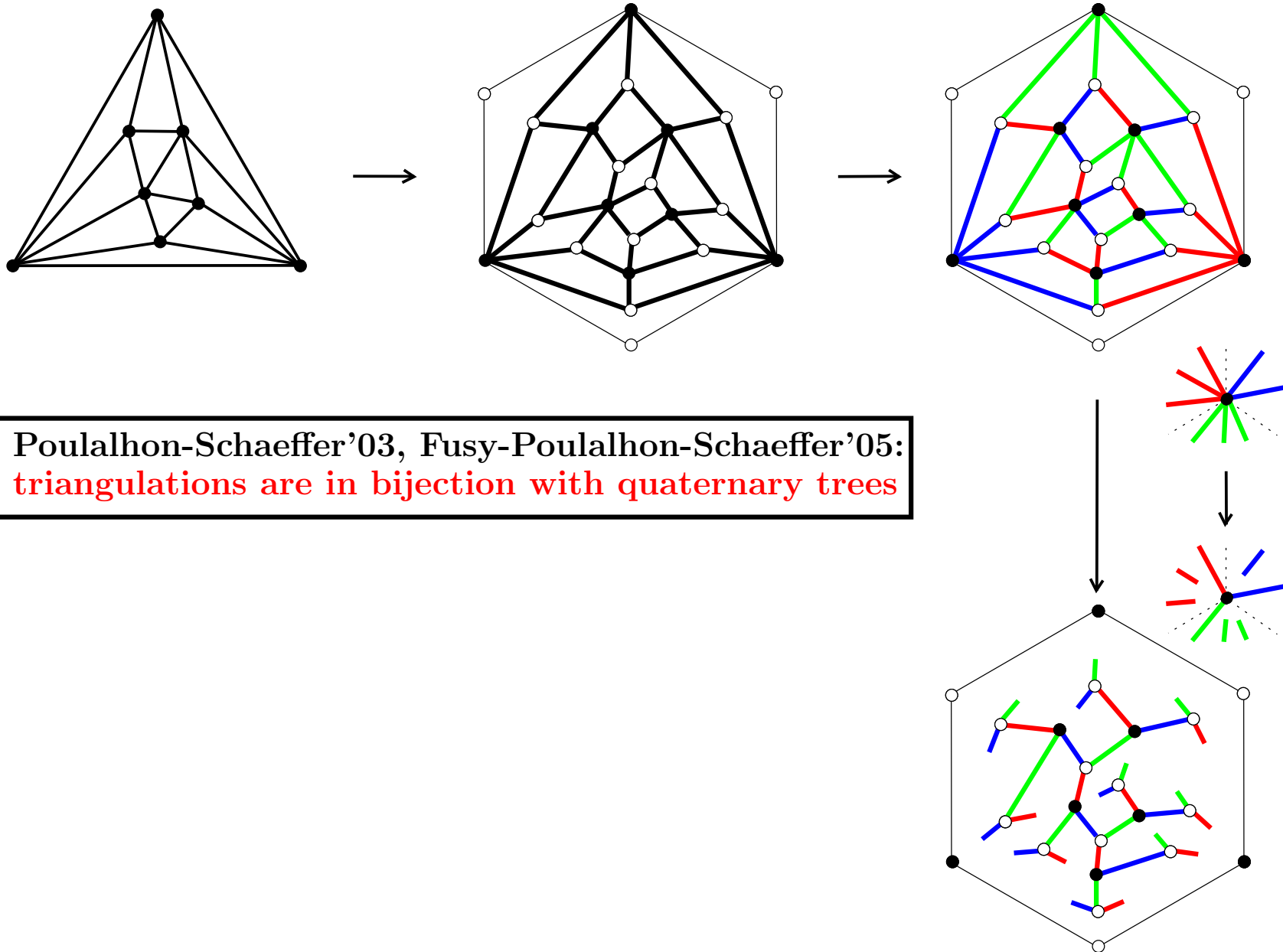
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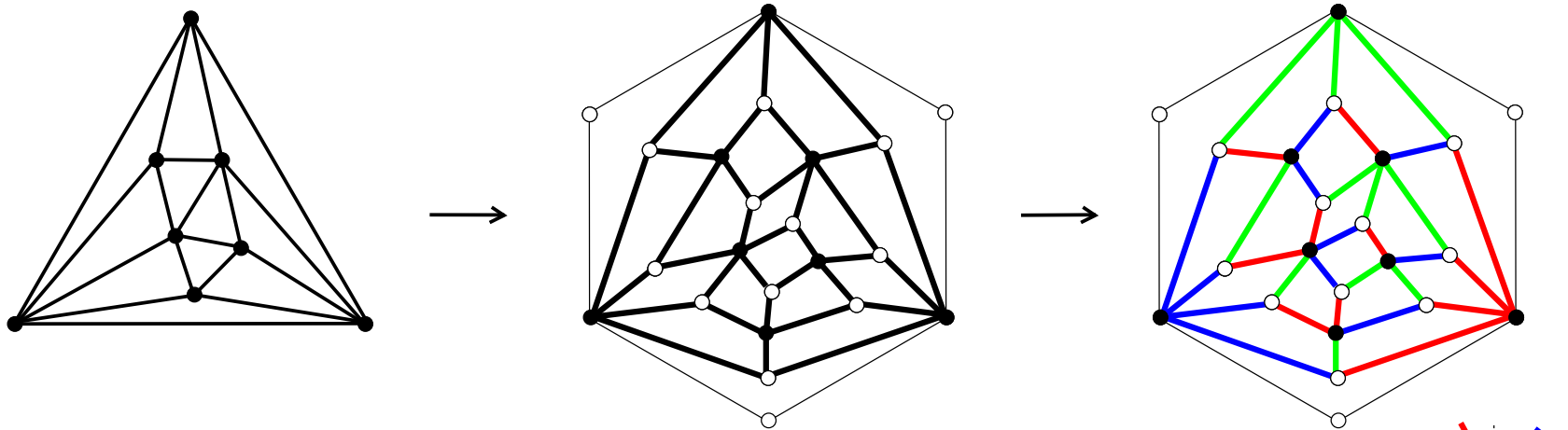
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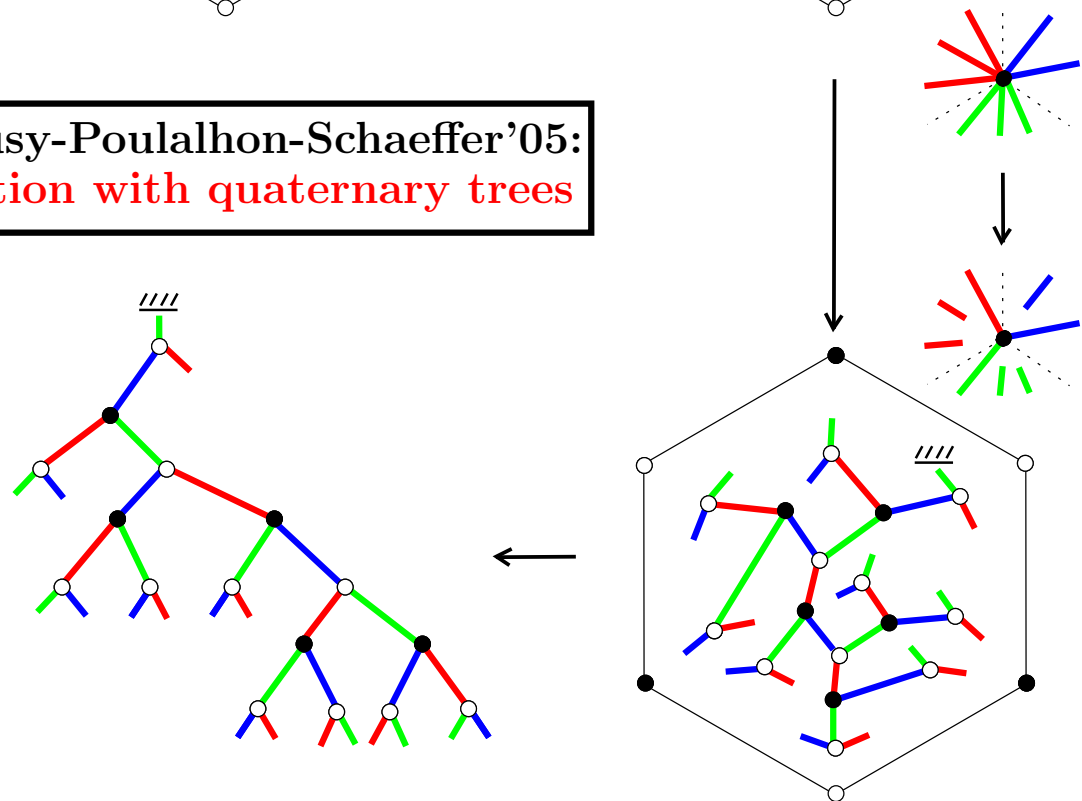


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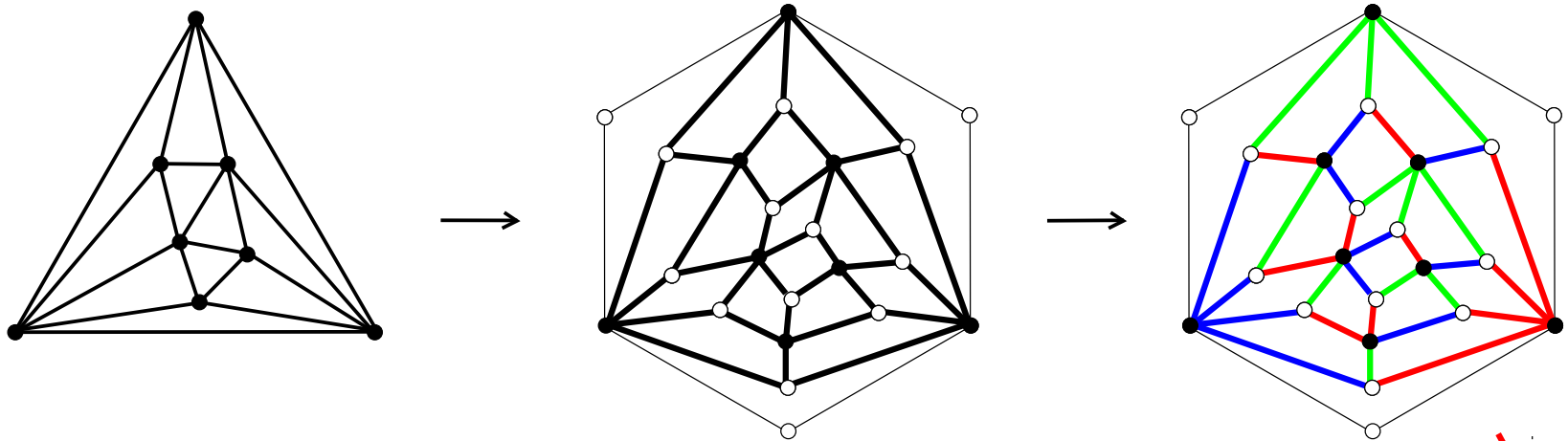
Triangulations \leftrightarrow quaternary trees



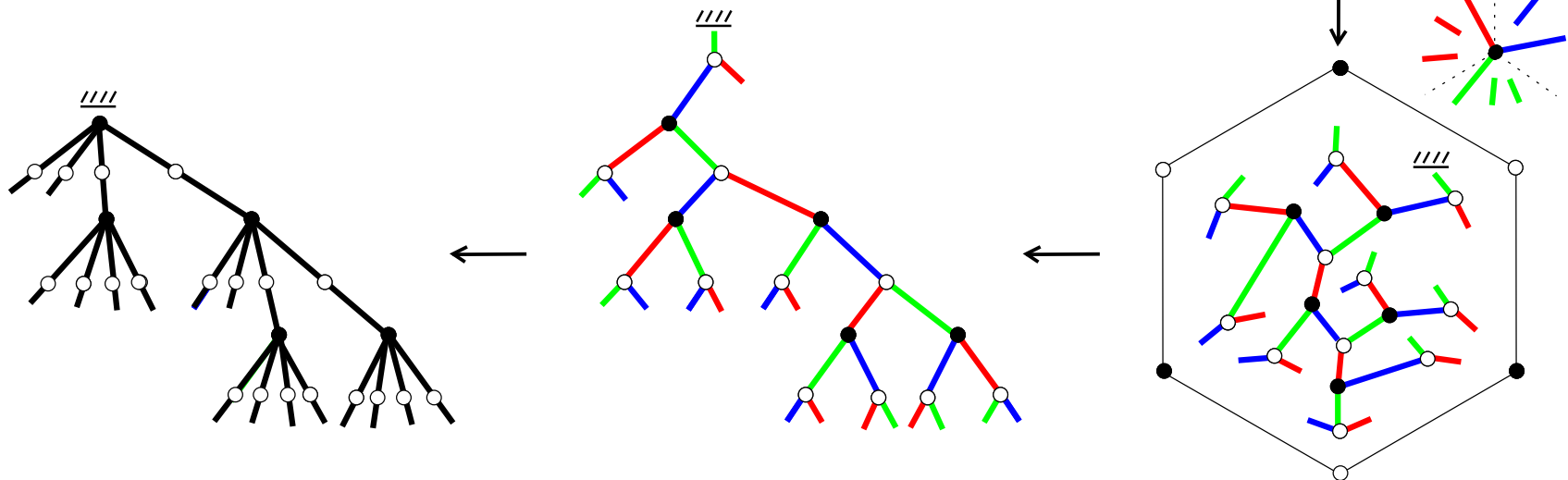
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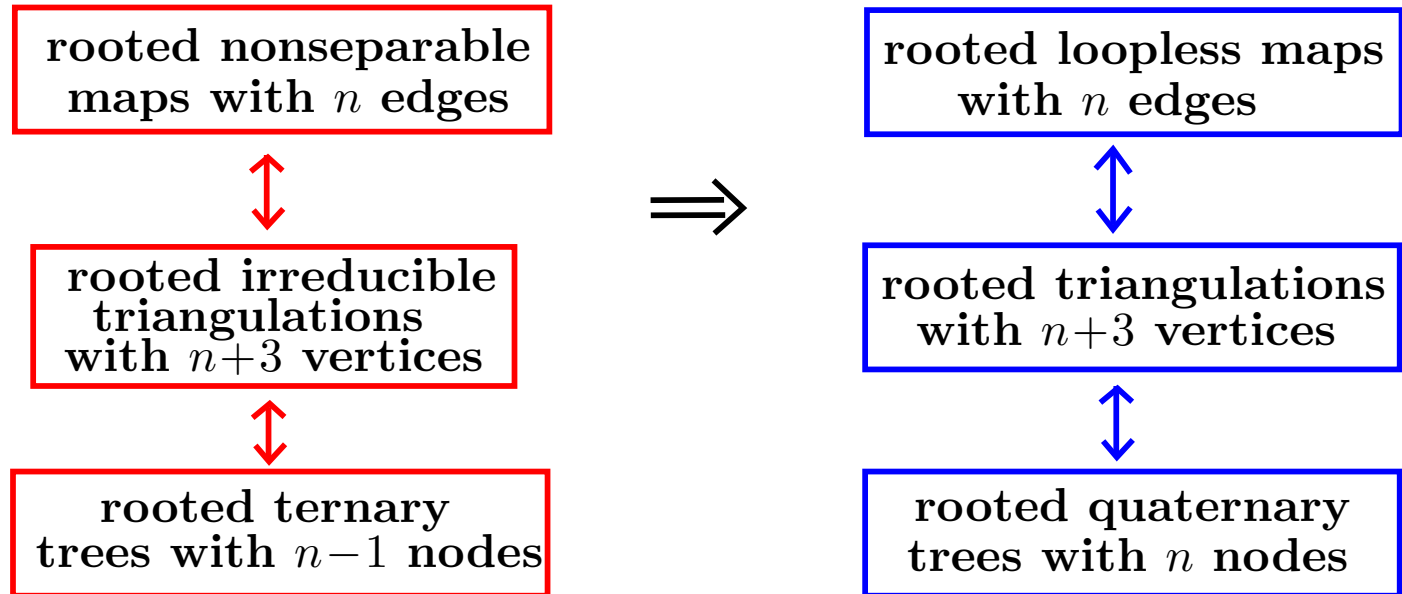
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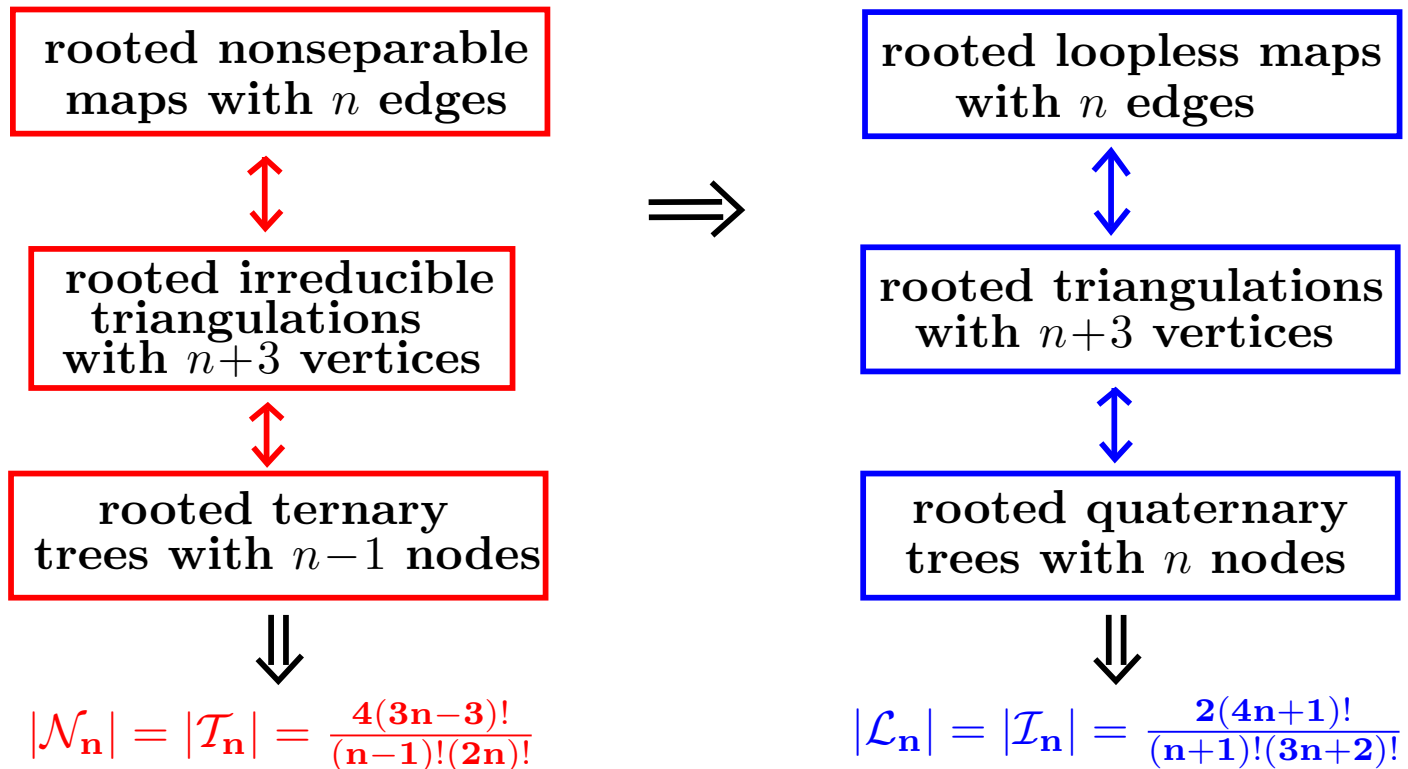
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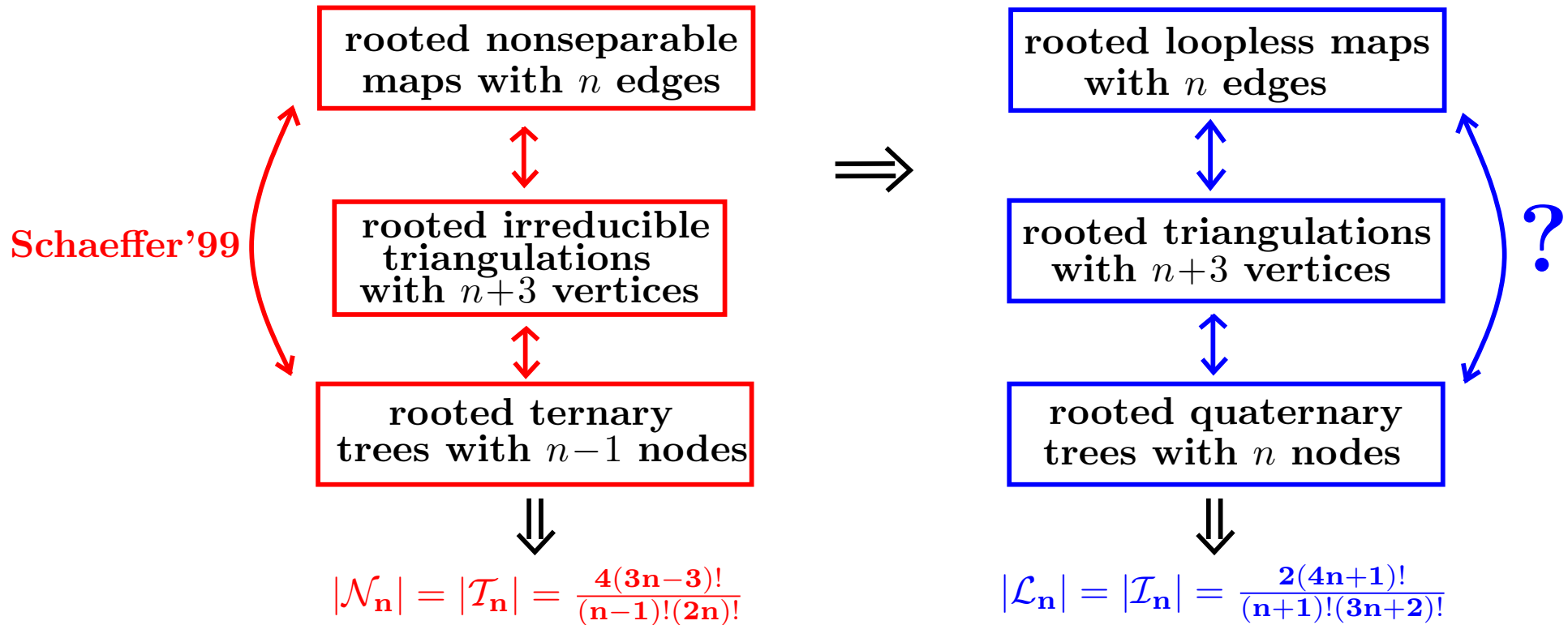
Enumerative Results



Enumerative Results



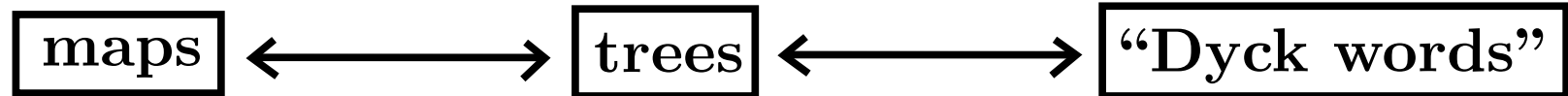
Enumerative Results



Applications

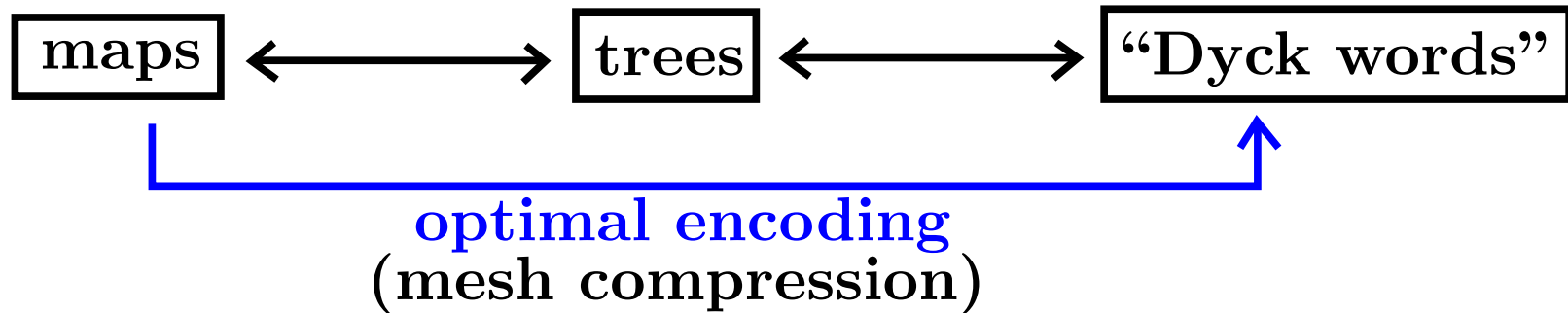
Algorithmic applications

- **General scheme:** (Schaeffer'99, Poulalhon-Schaeffer'03)



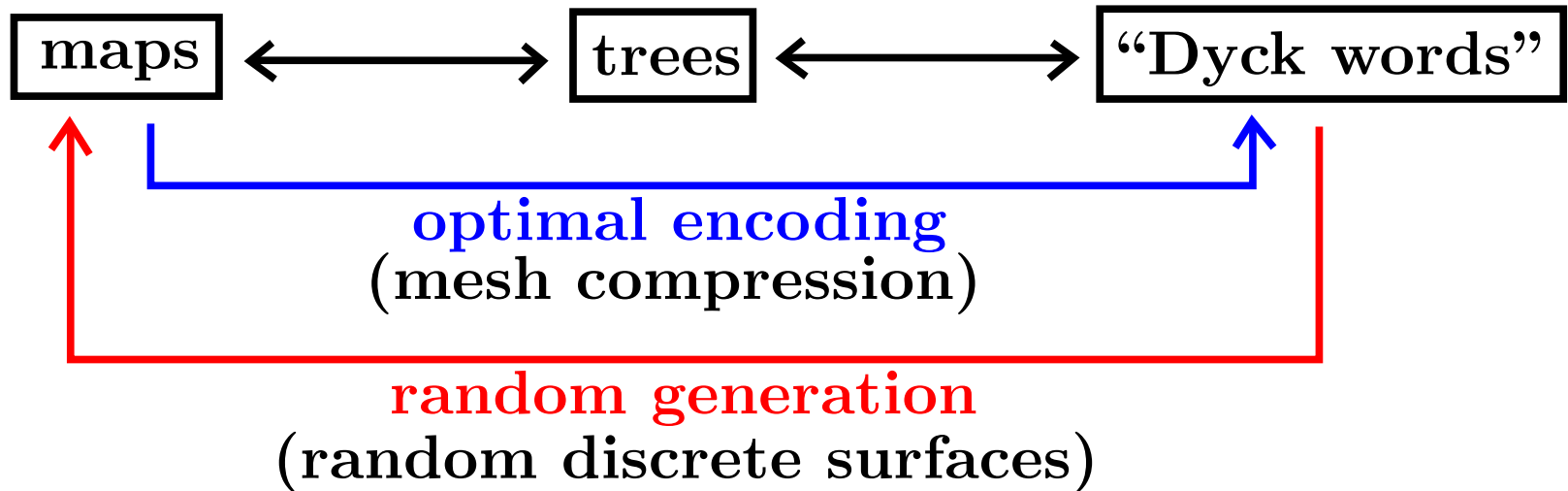
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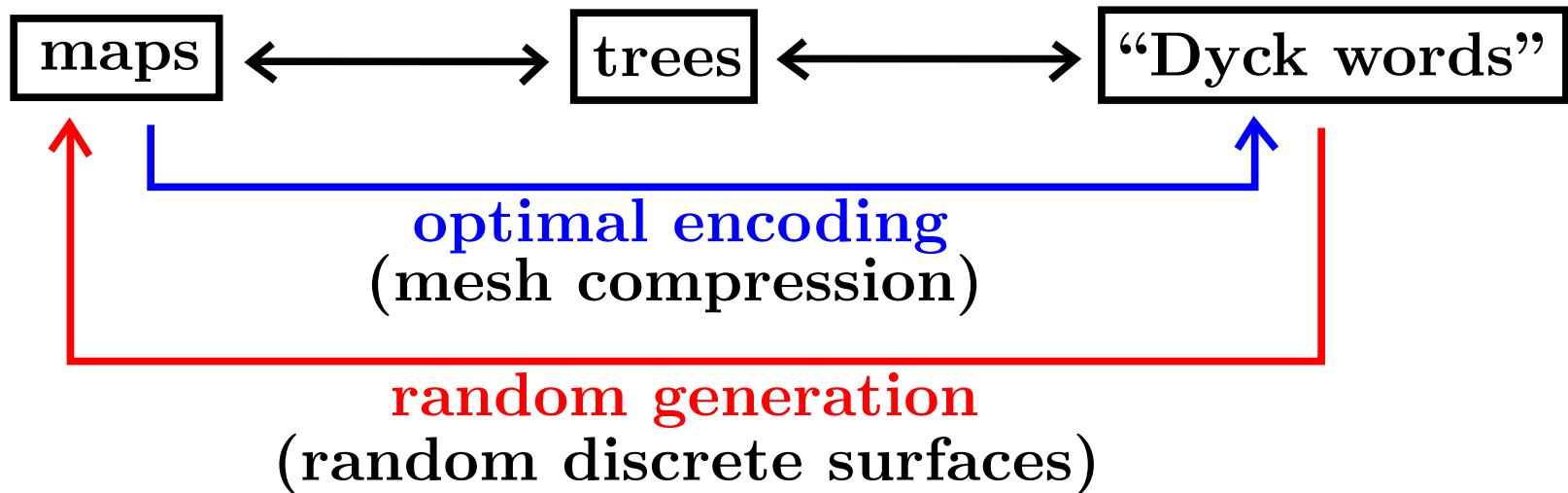
Algorithmic applications

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Algorithmic applications

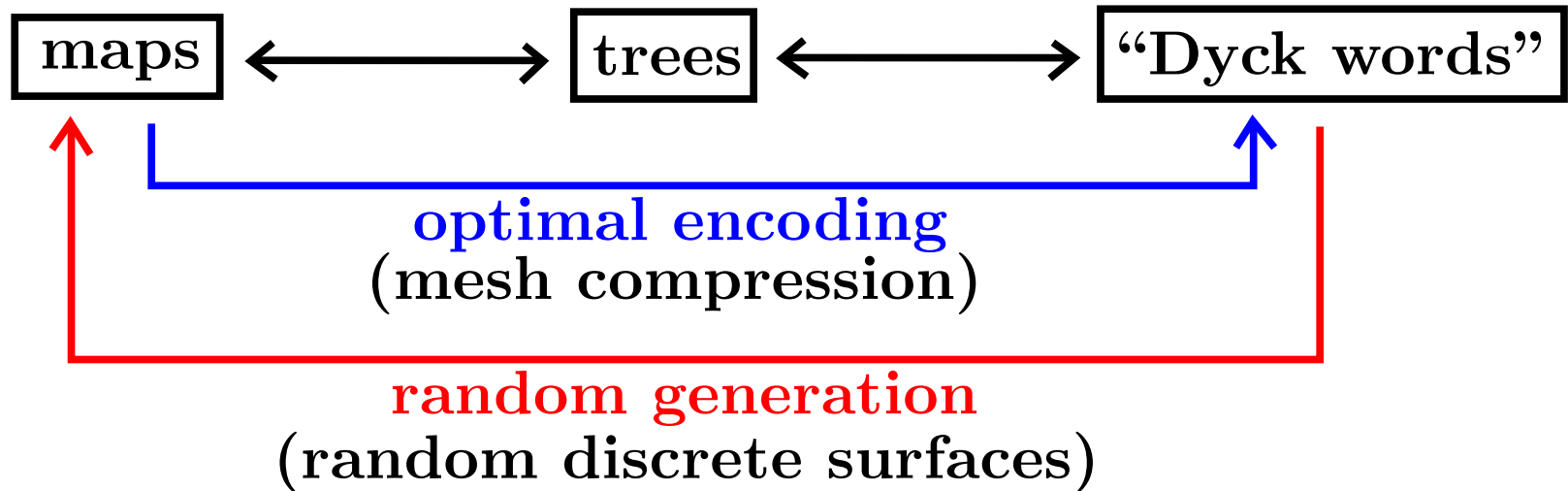
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- Applies here to :
 - irreducible triangulations
 - triangulations

Algorithmic applications

- **General scheme:** (Schaeffer'99, Poulalhon-Schaeffer'03)



- Applies here to :
 - irreducible triangulations
 - triangulationsas well as :
 - loopless maps
 - nonseparable maps