Entropy Compression Method

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Introduction

Entropy compression method

- analyze the performance of randomized algorithms
- prove that the algorithm eventually finds a solution

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$$\varphi : E(G) \to [1, k]$$

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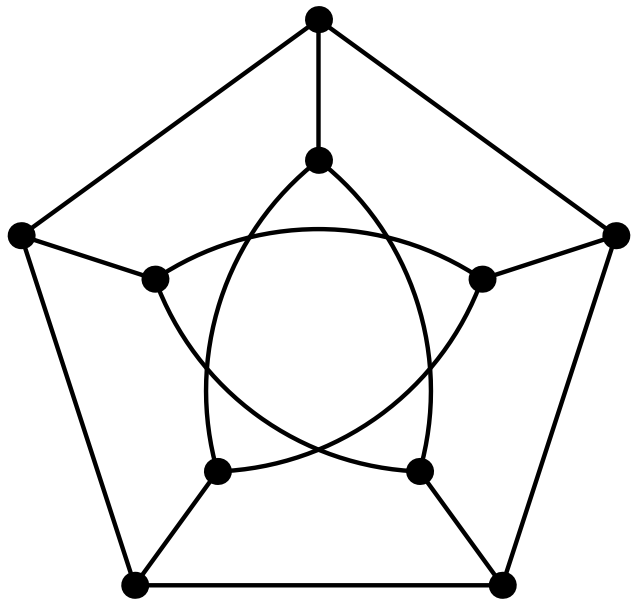

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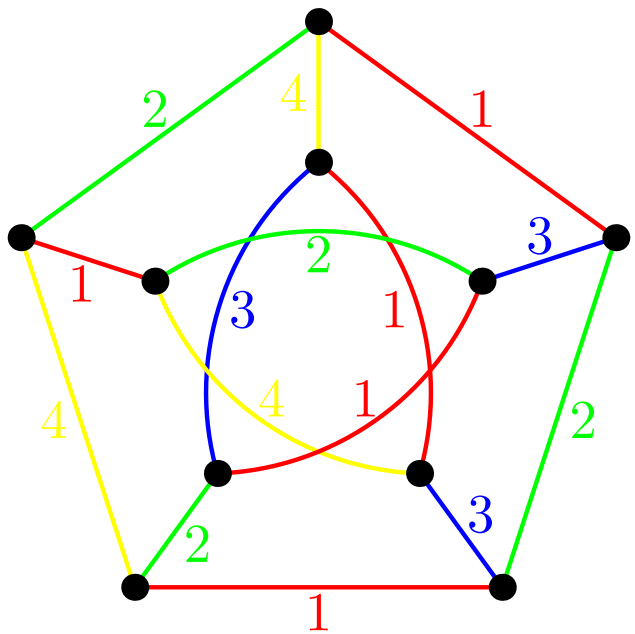

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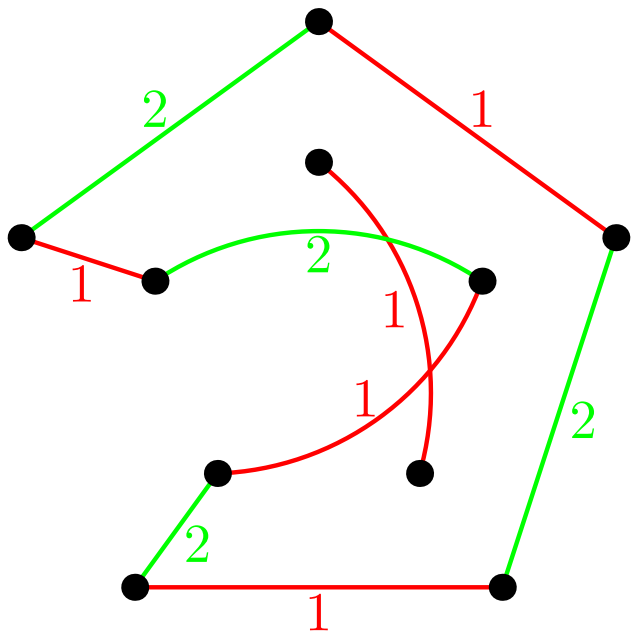

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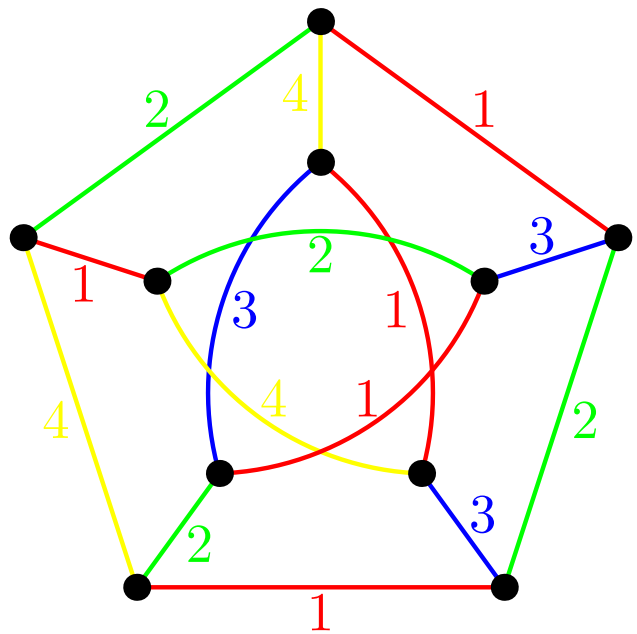

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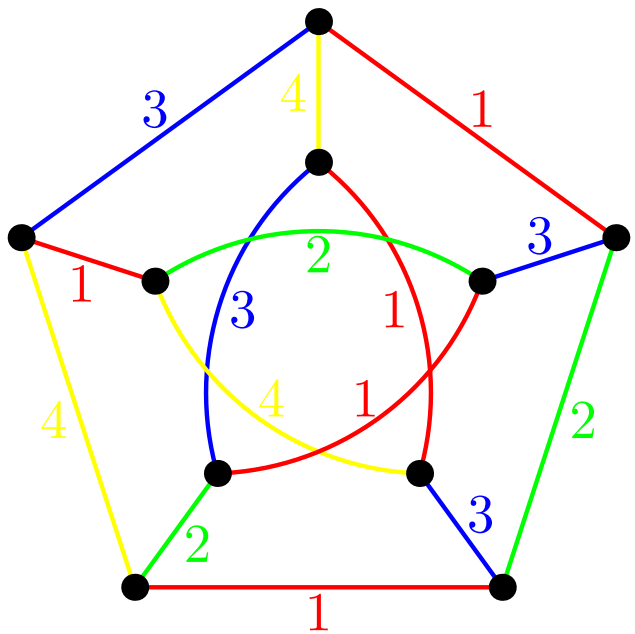

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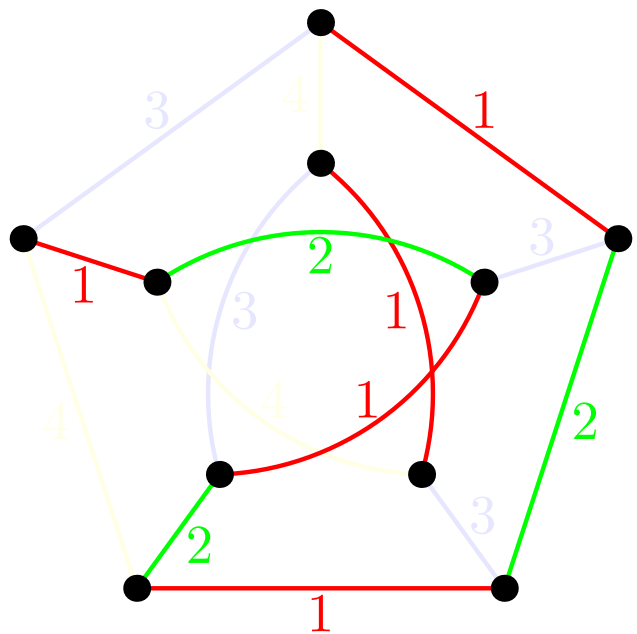

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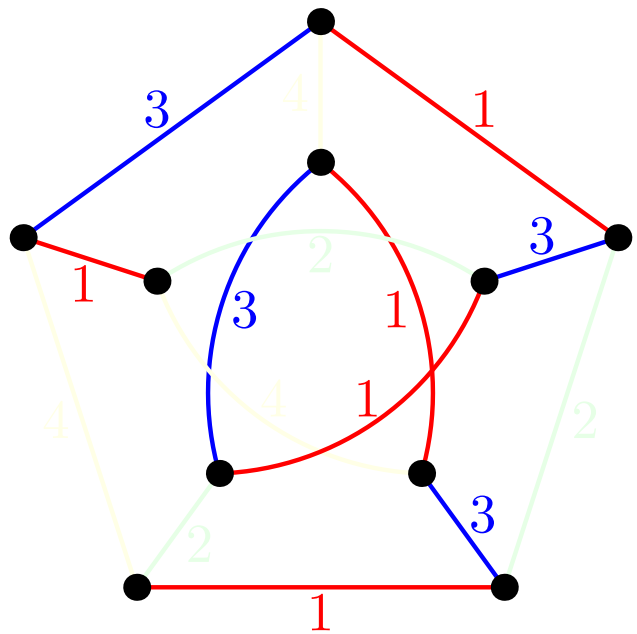

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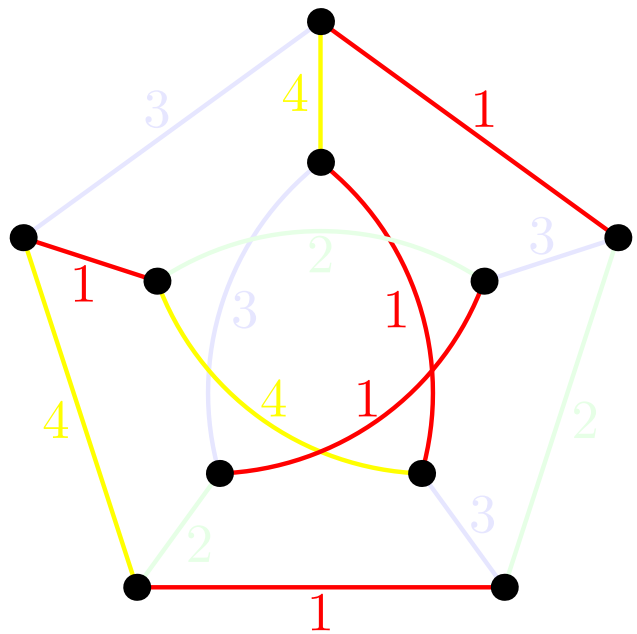

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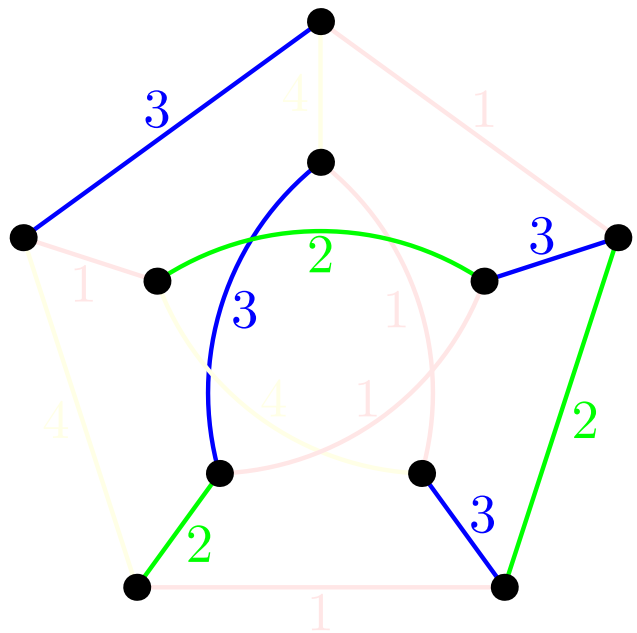

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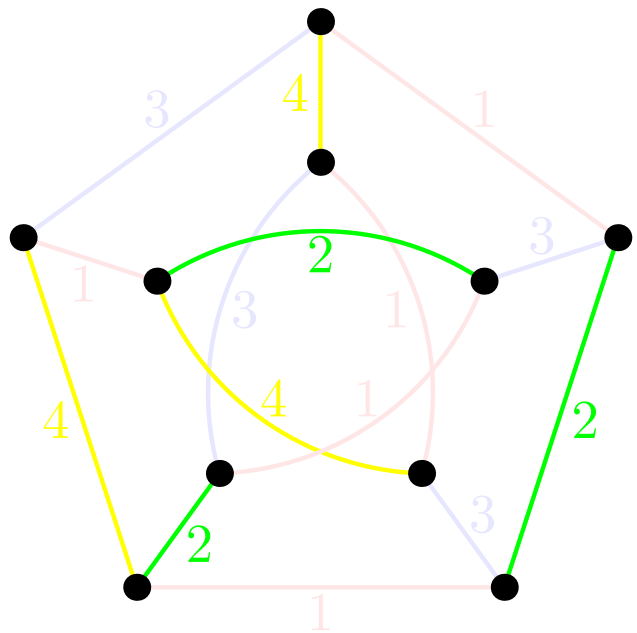

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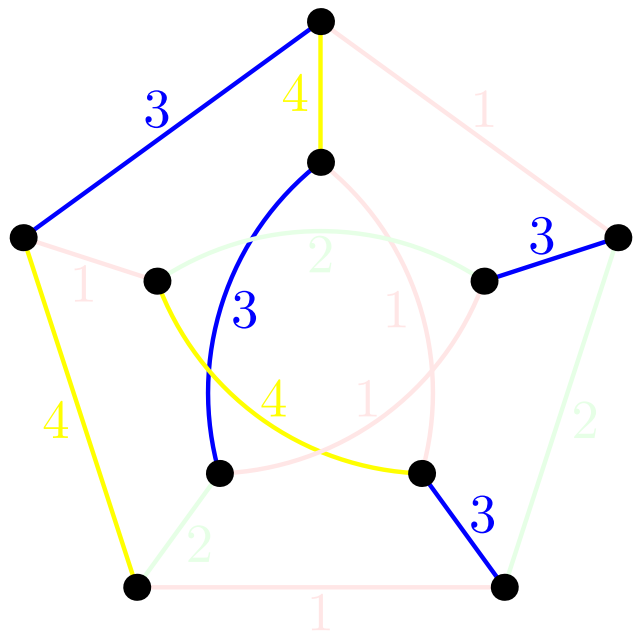

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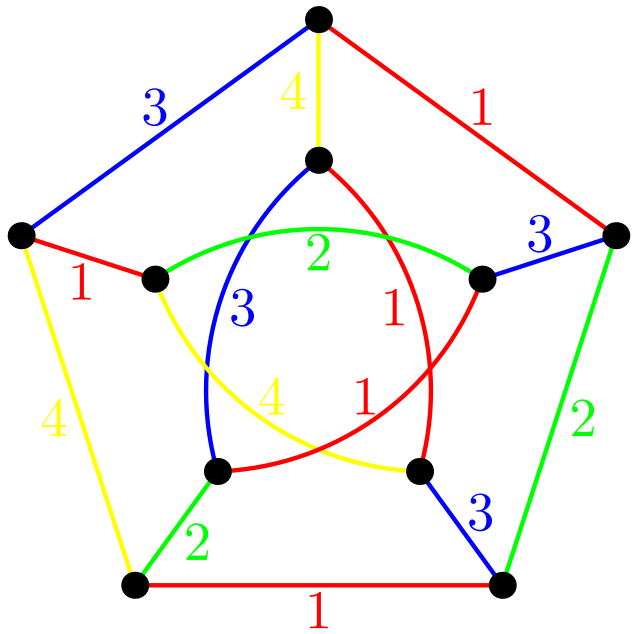

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For Petersen graph P we have

$$\chi'_a(P) = \chi'(P) = 4.$$

Upper bounds: conjectured and known

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Theorem (Esperet and Parreau 2018)

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- pick the first uncolored edge, say e_i
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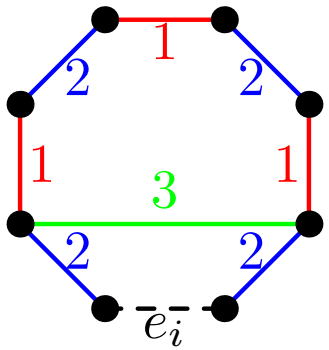

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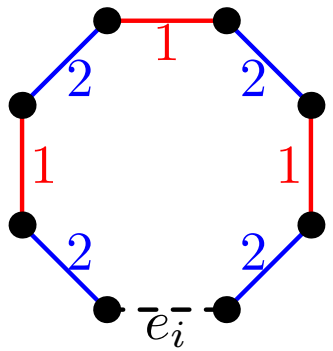

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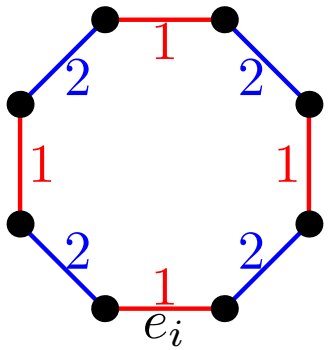

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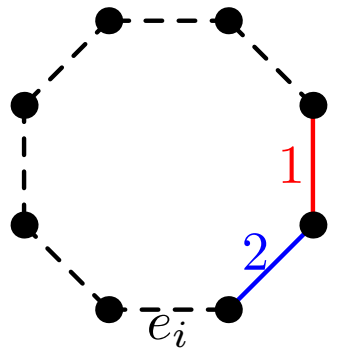

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- the new color for e_i
- (eventually) the path to uncolor.

Entropy compression principle

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If we can prove that the number of possible combinations of $\{\text{final coloring} \times \text{log file}\}$ is in $o(k^N)$, then we get a contradiction: a run that stops before round N must exist.

Log file: which edges are colored and which are not

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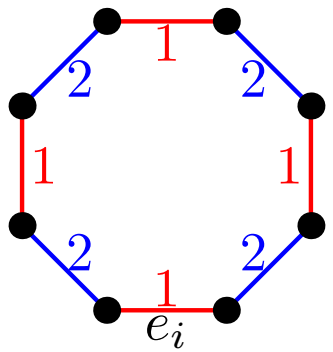

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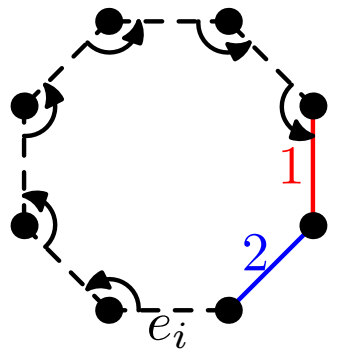

Log file: what else?

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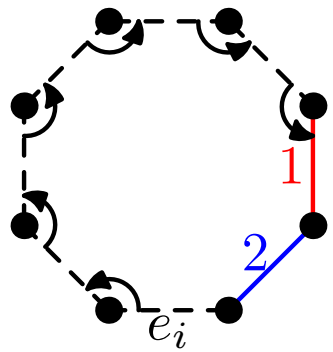

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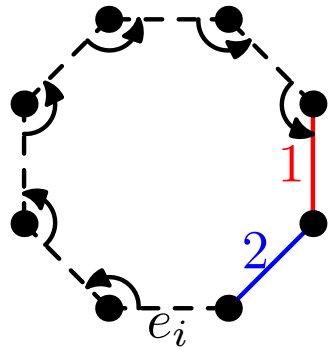

In particular, we can determine the color assigned to e_i.

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Log file contains

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- the number of edges to uncolor, and
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Alternatively, log file can contain

- a series of booleans, indicating whether an edge is colored or uncolored; and
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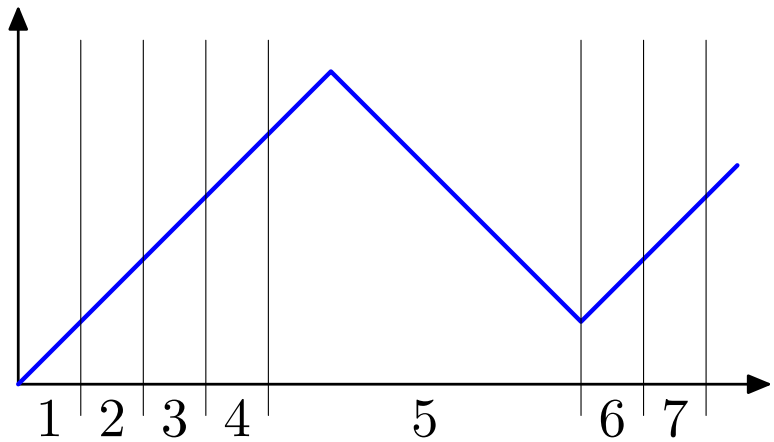

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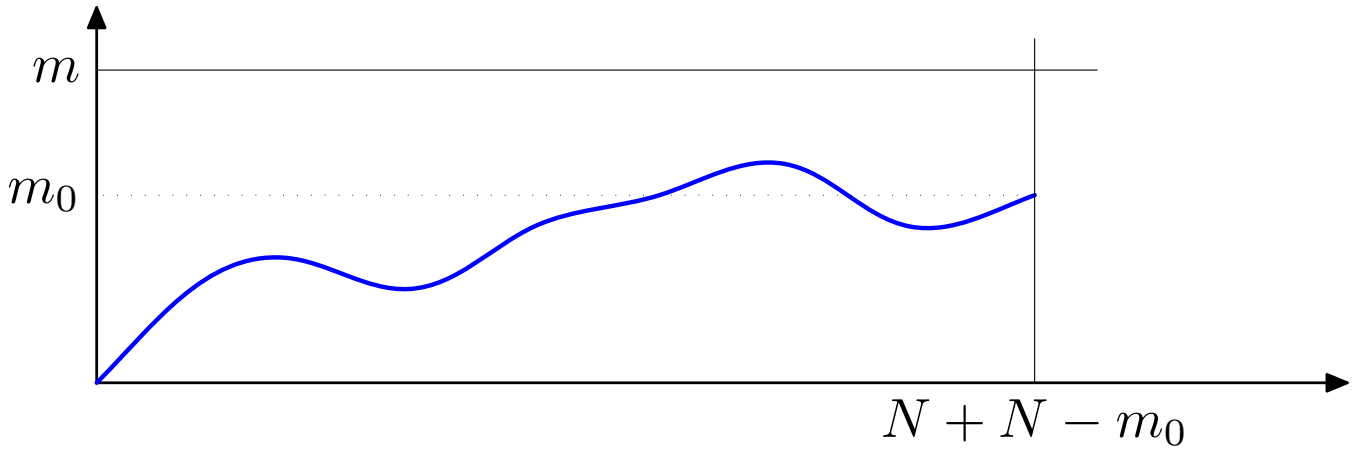

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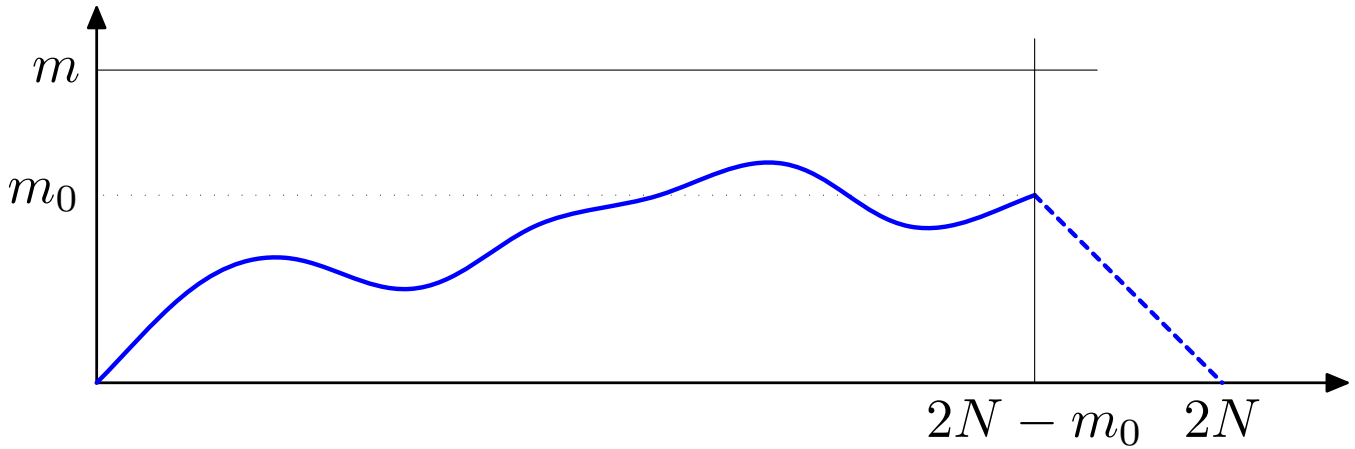

Dyck words

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It is known that the number of Dyck words of length $2N$ is the N-th Catalan number

$$C_N = \frac{1}{N+1}\binom{2N}{N} \sim \frac{4^N}{N^{3/2}\sqrt{\pi}}$$

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$$K^m \cdot \frac{4^N}{N^{3/2}\sqrt{\pi}} \cdot \Delta^N = \frac{K^m}{N^{3/2}\sqrt{\pi}} \cdot (4\Delta)^N = o((4\Delta)^N)$$

How many different colorings?

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until the whole graph is colored.

For an edge e_i, there are at most 2Δ forbidden colors.

How many different colorings?

Repeat

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As long as $K \geq 6\Delta$, the algorithm must find a valid coloring.

Conclusion

Entropy compression: the history of a given process can be recorded in an efficient way – the amount of additional information that is recorded at each step of the process is (on average) less than the amount of new information randomly generated at each step.

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Thank you for your attention!