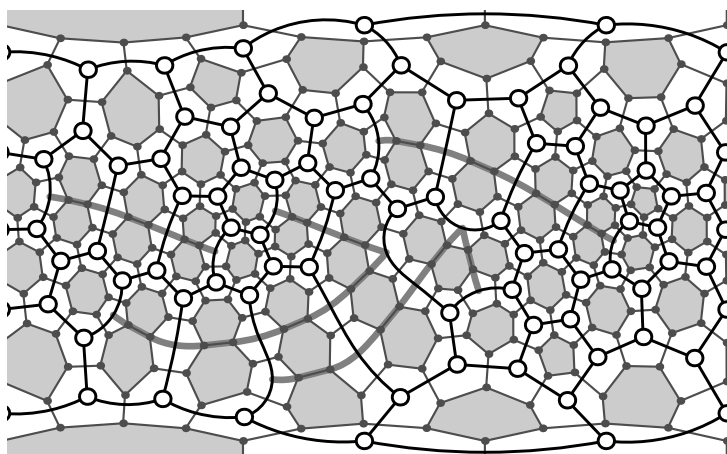


La structure et les invariants combinatoires des graphes cubiques

Structure and combinatorial invariants
of cubic graphs

František Kardoš



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Membres du jury :

Mme. Mireille BOUSQUET-MÉLOU		Examinatrice
Directrice de Recherche, CNRS, Université de Bordeaux		
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M. Matěj STEHLÍK	Professeur, Université Paris Cité	Rapporteur
M. Nicolas TROTIGNON	Directeur de Recherche, CNRS, ENS de Lyon	Examineur

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Introduction

In this thesis, we discuss relationships between structure and certain combinatorial invariants for specific classes of cubic graphs.

In the first chapter, we describe properties of so-called *fullerene graphs*—that is, 3-connected cubic planar graphs whose faces are only of size 5 and 6—and how their specific structure influences their combinatorial properties. Since the number of pentagons is constant (twelve), a sufficiently large fullerene graph locally resembles a cubic planar graph composed only of hexagonal faces—namely, the hexagonal grid. Depending on the type of result or question at hand, various methods and tools can be employed to tackle the problem. For instance, if we only seek to establish the asymptotic behavior of a lower or upper bound, we can often disregard the pentagons and focus instead on the seemingly endless sea of hexagons (see Section 1.3). In other contexts, it may suffice to treat the pentagons approximately (see Section 1.4).

On the other hand, if all the faces must fit together in a precise way—as in the search for a Hamiltonian cycle—the problem becomes significantly more difficult, sometimes requiring computational assistance (see Section 1.5).

In the chapter on fullerene graphs, we present results published in [3, 43, 41, 42], along with complete proofs. We also mention other related results from [44, 48, 24], without providing further details.

In the second chapter, we explore the properties of perfect matchings in cubic graphs. Here, it is not the structure that determines the property, but the property that shapes the structure. In particular, to prove a statement about perfect matchings in cubic graphs by induction, one must find elementary local operations that transform a given graph into one or more smaller graphs in a way that is compatible with perfect matchings. Simply removing a vertex or an edge does not suffice. In this context, we demonstrate how the operation of *path splitting*, combined with a careful analysis of short edge-cuts, can be used to derive new and significant results.

In the chapter on perfect matchings, we present results published in [25, 26, 46, 45], together with complete proofs.

Chapter 1

Fullerene graphs

1.1 Introduction and chemical background

Fullerene graphs, i.e., 3-connected cubic planar graphs with faces of size 5 and 6, are used to capture the structure of *fullerenes*, all-carbon molecules of roughly spherical shape, in which the atoms are arranged in a way that they form pentagonal and hexagonal rings.

The existence of an all-carbon molecule with 60 atoms, realizing one of the Archimedean solids, where each vertex of a polytope is incident with three faces of sizes 5, 6, and 6, respectively, and presenting a third stable form of crystalline carbon (besides the diamond and the graphite) had been discussed and conjectured among chemists since the 60s [63].

The presence of molecules of this particular shape in the soot residue created by vaporising carbon under specific conditions (temperature, pressure) was first proved in 1985 [51]. It was indeed a significant break-through, as the 1995 Nobel prize for chemistry was awarded to Robert F. Curl Jr., Sir Harold W. Kroto and Richard E. Smalley for the discovery of what we now call the fullerene.

The name *buckminsterfullerene* was first chosen for the C_{60} molecule by the discoverers as an homage to American architect Buckminster Fuller for the vague similarity of the structure to the geodesic domes which he popularized [66]. The shortened name *fullerene* eventually came to be applied to the whole family.

Soon after the experimental discovery of the buckminsterfullerene, its existence in the nature was confirmed along with similar structures having 70, 76, 78, 82, 84, 90, 94, or 96 carbon atoms. Each of these all-carbon molecules have polyhedral structure, and all faces of the polyhedron are either pentagons or hexagons. The discovery of the buckminsterfullerene marked the birth of fullerene chemistry and

nanotechnology.

In fact, the C_{60} -fullerene can now be commanded in the standard chemical substances catalogues, for a price similar to gold.

Let us recall some basic properties of fullerene graphs. It can be easily derived from Euler's formula that there are always twelve faces of size 5; the number of faces of size 6 is illimited. A fullerene graph with h hexagonal faces has $h + 12$ faces in total, $2h + 20$ vertices, and $3h + 30$ edges.

The smallest fullerene graph is the dodecahedron, only composed of twelve pentagons. It is easy to check that there is no fullerene having exactly one hexagonal face while all the remaining faces being pentagons. Starting from $h = 2$, there is at least one fullerene graph with h hexagonal faces for every possible value of h [35].

Notice that as the number of vertices (hexagons) grows, the number of fullerenes graphs increases as well. For example the fullerenes graphs on 20, 24 and 26 vertices are unique, but there are 40 non-isomorphic fullerene graphs on 40 vertices, and there are 1812 different fullerene graphs on 60 vertices (including the buckminsterfullerene, which is the only among them without adjacent pentagons). Fowler and Manolopoulos in [30] conjectured the following statement, which was confirmed by Cioslowski [9] :

Theorem 1 (Cioslowski 2014) *The number of fullerene graphs on n vertices is of order $\Theta(n^9)$.*

Although the number of pentagonal faces is constant, and for graphs of large sizem it seems to be negligible compared to the number of hexagonal faces, their layout is crucial for the shape of the corresponding fullerene molecule, as it's the pentagons that bring curvature to the othewise perfectly plane hexagonal grid.

If the pentagons are far from each other, and distributed somehow uniformly, a molecule of a spherical shape is obtained. On the other hand, pentagons might form two clusters of six at large distance from each other, separated by a cylindrical part comprised of hexagons only. In this case we speak about a nanotube (a precise definition will be given below).

1.2 Structure of fullerene graphs

1.2.1 Cyclic edge-connectivity of fullerene graphs

Cyclic edge-connectivity is defined to refine the notion of edge-connectivity for classes of cubic graphs, as the latter cannot exceed three. It will also play a substantial role in the second chapter.

Recall that an *edge-cut* (X, Y) in a graph G is a set of the edges of the form xy with $x \in X$ and $y \in Y$, where $X \subset V(G)$ and $Y = V(G) \setminus X$. An edge-cut is *cyclic* if both X and Y induce a graph containing a cycle. The *cyclic edge-connectivity* is then defined as the smallest possible size of a cyclic edge-cut in G provided G contains at least one. Actually, there are only two cubic graphs without a cyclic edge-cut (three, if parallel edges are allowed): K_4 and $K_{3,3}$ (and K_2^3 , the graph with only two vertices joined by three parallel edges), their cyclic edge-connectivity being defined as 3 and 4 (and 2, respectively).

A cyclic edge-cut (X, Y) is *trivial* if $G[X]$ or $G[Y]$ consists of a single cycle. Cyclic edge-connectivity of fullerene graphs is therefore bounded above by five, since pentagonal faces are always present. Rather surprisingly, fullerene graphs are actually known to be cyclically 5-edge connected, as was proved by Došlić [16].

On the other hand, if faces of size larger than 6 were allowed, the property is no more valid – examples of graphs with faces of sizes from 5 to 7 and with (non-trivial) cyclic 3-edge-cuts are known, see Figure 1.1 for an illustration.

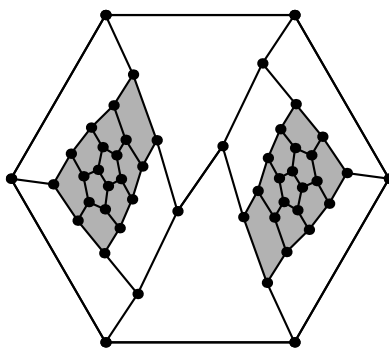


Figure 1.1: An example of a 3-connected cubic planar graph with faces of sizes five, six, and seven, containing a non-trivial cyclic 3-edge-cut. Pentagonal faces are grey.

1.2.2 Goldberg vectors, Coxeter coordinates, and nanotubes

To define a particular fullerene graph, it suffices to specify the mutual position of its twelve pentagonal faces. In order to do this, a specific language was developed.

Let f_1 and f_2 be two faces of an infinite hexagonal grid H . Then there is a (unique) translation ϕ of H that maps f_1 to f_2 . The vector \vec{u} defining ϕ can be expressed as an integer combination of two *unit vectors* – those that define translations mapping

a hexagon to an adjacent one. Out of the six possible unit vectors, we choose a pair \vec{u}_1, \vec{u}_2 making a 60° angle such that f_2 is inside this angle starting from f_1 . Then the coordinates (c_1, c_2) of $\vec{u} = c_1\vec{u}_1 + c_2\vec{u}_2$ are non-negative integers, called the *Coxeter coordinates* of ϕ [10].

We may always assume that $c_1 \geq c_2$. The pair (c_1, c_2) determines the mutual position of a pair of hexagons in a hexagonal grid. It is also called a *Goldberg vector*. Observe that, for example, $(1, 0)$ corresponds to a pair of adjacent faces, $(1, 1)$ corresponds to a pair of non-adjacent faces with an edge connecting them (and thus having two distinct common neighboring faces), whereas $(2, 0)$ corresponds to a pair of non-adjacent faces with two paths of length 2 connecting them (and thus sharing a single common neighboring face), etc.

The Coxeter coordinates are used to define nanotubical graphs in the following way:

Let (c_1, c_2) be a pair of integers with $c_1 \geq c_2$. Fix a pair of unit vectors \vec{u}_1 and \vec{u}_2 making a 60° angle. A graph obtained from an infinite hexagonal grid by identifying objects (vertices, edges, and faces) whose mutual position is (an integer multiple of) the vector $c_1\vec{u}_1 + c_2\vec{u}_2$ is the *infinite nanotube of type (c_1, c_2)* .

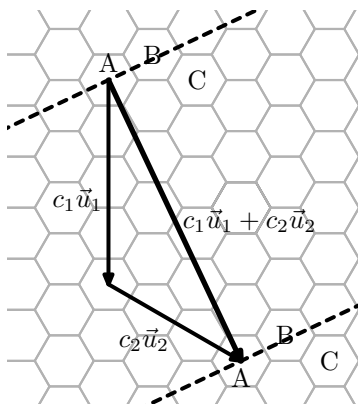


Figure 1.2: Construction of the infinite nanotube of type $(4, 3)$. The pairs of hexagons with the same name overlap.

If $c_1 + c_2 \leq 2$ then the infinite nanotube is not 3-connected. Since nanotubes with $c_1 + c_2 \leq 4$ contain cyclic edge-cuts of size at most four and fullerene graphs are cyclically 5-edge-connected, we will only be interested in nanotubes with $c_1 + c_2 \geq 5$.

Let N be an infinite nanotube of type (c_1, c_2) . Let f_1 and f_2 be two hexagons of the hexagonal grid H at mutual position (c_1, c_2) corresponding to the same hexagon

f of N . Let P be a dual path of length $c_1 + c_2$ connecting the vertices f_1^* and f_2^* in H^* . Then the edges corresponding to the edges of P form a cyclic edge-cut in N of cardinality $c_1 + c_2$. A cyclic sequence of hexagonal faces of N corresponding to the vertices of P is called a *ring* in N .

It is easy to see that if h_1, \dots, h_p is a ring in a nanotube of type (c_1, c_2) , then $p = c_1 + c_2$, moreover, among the p vectors $h_{i+1} - h_i$ (indices modulo p), exactly c_1 are equal to \vec{u}_1 and the remaining c_2 are equal to \vec{u}_2 .

A finite 2-connected subgraph of an infinite nanotube is an *open-ended nanotube* if it contains at least one ring. A fullerene graph is a *nanotube* if it contains an open-ended nanotube of some type as a subgraph. Observe that the same graph may be considered as a nanotube of more than one type.

Let G be a nanotube. We call a *cap* any of the two inclusion-wise minimal 2-connected subgraphs of G that can be obtained as a component of a cyclic edge-cut defined by a set of edges intersecting a line perpendicular to the vector defining the corresponding open-ended nanotube. See Figures 1.17, 1.18, and 1.20 for illustration.

It is known that, for a given type (p_1, p_2) , there is only a finite number of fullerene caps possible [7].

Let G be a fullerene graph and let p_1 and p_2 be two pentagons of G . Suppose that there exists an induced dual path P^* connecting p_1 and p_2 passing only through hexagons. Then if we consider only faces of G corresponding to P^* , and if we replace the two pentagons by hexagons, we obtain a graph with a canonical embedding into an infinite hexagonal grid. The Goldberg vector (c_1, c_2) joining the first and the last hexagon is uniquely determined. We will use this vector to characterize the mutual position of p_1 and p_2 in G . Observe that the vector of two pentagons may depend on the choice of the path joining them, see Figure 1.3 (left) for illustration.

Graver [34] used the Coxeter coordinates to describe the structure of fullerene graphs. A given fullerene graph G is represented by a plane triangulation T , whose vertices represent the pentagons of G , and each edge uv is labelled with a Goldberg vector representing the mutual position of the faces represented by u and v . The angle between face-adjacent edges (incident to the same triangle of T) is well defined and is determined by the labels of the three edges forming the triangle. For a vertex of T representing a pentagon the angles around it sum up to $5\pi/3 = 300^\circ$.

The existence of a triangulation T is guaranteed by a structural theorem of Alexandrov (see e.g. [12], Theorem 23.3.1, or [59], Theorem 37.1), which states (in a more general setting) that any 3-connected cubic planar graph with faces of size at most 6 can be embedded onto the surface of a convex (possibly degenerate) polyhedron so that every face is isometric to a regular polygon with unit edge length; it

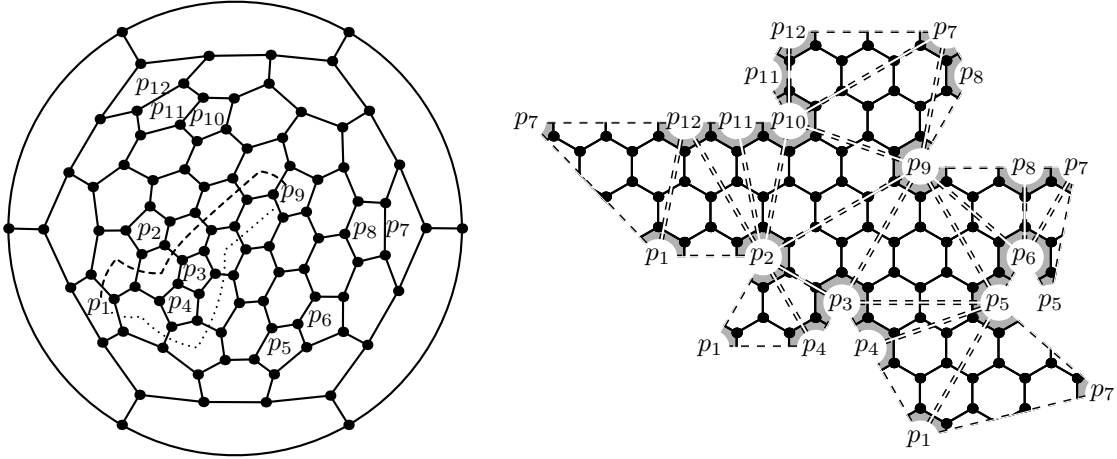


Figure 1.3: An example of a fullerene graph (left). Pentagonal faces are denoted by p_1, \dots, p_{12} . The mutual position of p_1 and p_9 is characterized by vectors $(3, 3)$ (dotted line) or $(4, 2)$ (dashed line). The same graph embedded into a hexagonal grid after being cut along a spanning tree of a triangulation capturing the mutual position of all the small faces (right).

suffices then to triangulate the faces of this polyhedron.

Any spanning tree of T may be used to cut the graph G in order to obtain a graph embeddable into the infinite hexagonal grid, see Figure 1.3 (right) for an illustration.

We say that a Goldberg vector $\vec{u} = (c_1, c_2)$ is *shorter* than $\vec{u}' = (c'_1, c'_2)$ if and only if the Euclidean length of a segment determined by \vec{u} is shorter than the Euclidean length of a segment determined by \vec{u}' when both embedded into the same hexagonal grid.

Observe that the triangulation representing a Barnette graph is not unique: wherever two adjacent triangles form a convex quadrilateral (once embedded into the hexagonal grid), we may choose the other diagonal of the quadrilateral instead of the existing one as an edge of the triangulation. For example, in the graph depicted in Figure 1.3 we could have chosen the edge p_3p_{10} instead of the edge p_2p_9 , etc.

However, for a triangulation T representing a fullerene graph G , the operation switching the diagonals of a convex quadrilateral eventually leads to a triangulation minimal with respect to the sum of lengths of its edges. For example, the triangulation depicted in Figure 1.3 is already minimal.

1.3 Perfect matchings in fullerene graphs

Since all carbon atoms are 4-valent, for every atom precisely one of the three bonds should be doubled. Such a set of double bonds is called a *Kekulé structure* in a fullerene. It corresponds to the notion of perfect matchings in fullerene graphs: a *matching* in a graph G is a set of edges of G such that no two edges in M share an end-vertex. A matching M is *perfect* if any vertex of G is incident with an edge of M .

Let M be a perfect matching in a fullerene graph G . A hexagonal face is *resonant* if it is incident with three edges in M . The maximum size of a set of resonant hexagons in G is called the *Clar number* of G .

From the Four Color Theorem [4, 5, 62] one can easily derive the existence of a proper edge coloring of a fullerene graph G using three colors – it means that the set of edges of G can be decomposed into three pairwise disjoint perfect matchings (see [49]).

In this section we will give a proof of the following result [43]:

Theorem 2 (Kardoš, Král', Miškuf, Sereni 2009) *Let G be a fullerene graph with p vertices that has no non-trivial cyclic 5-edge-cut. The number of perfect matchings of G is at least $2^{\frac{p-380}{61}}$.*

All previously known general lower bounds for the number of perfect matchings in fullerene graphs were linear in the number of vertices [14, 15, 70]. On the other hand, the computation of the number of perfect matchings in typical fullerene graphs with a small number of vertices [17] indicated that this number should grow exponentially with p .

Before, several special classes of fullerene graphs with exponentially many perfect matchings were known but a general result was missing. Such classes of fullerene graphs either have the special structure of nanotubes [53], have high symmetry [17] or are obtained using specific operations [18].

Let us note that Chudnovsky and Seymour [8] proved that every cubic bridgeless planar graph has an exponential number of perfect matchings. In particular, they proved that every such graph has at least $2^{\frac{p}{655978752}}$ perfect matchings. This bound is worse than our bound $2^{\frac{p-380}{61}}$ but it applies to a larger class of graphs. We will discuss perfect matchings in cubic graphs in general in the next chapter.

Proof. (of Theorem 2) The fullerene graphs with non-trivial cyclic 5-edge-cuts have a special structure – they are nanotubes of type $(5, 0)$ [48] – and the number of perfect matchings in them is known to be exponential [53]. Hence, we focus on fullerene graphs with no non-trivial cyclic 5-edge-cuts.

For fullerene graphs with a small number of vertices, the earlier linear bounds [70] provide the desired conclusion. Hence, we consider only fullerene graphs with $p > 380$ vertices.

We find a perfect matching M in G such that there are at least $\frac{p-380}{61}$ disjoint resonant hexagonal faces. Since in each such resonant hexagon we can switch the matching to the other edges of the hexagon independently of the other resonant hexagons, the bound will follow immediately.

The dual graph G^* of the graph G is a plane triangulation with 12 vertices of degree 5 and all other vertices of degree 6. Let $U = \{u_1, \dots, u_{12}\}$ be the set of vertices of degree 5. Our aim is to construct a set W of vertices of G^* of degree 6 and such that:

- the distance between v and v' in G^* is at least 5 for all $v, v' \in W$, $v \neq v'$;
- the distance between v and u in G^* is at least 3 for all $v \in W$ and $u \in U$.

We present a greedy algorithm to construct such a set W . Initially, we set $W_0 = \emptyset$, and we color all the vertices at distance at most 2 from any u_i by the white color. The remaining vertices are colored black. White vertices cannot be chosen as vertices of W . For each $u_i \in U$ there are at most 5 vertices at distance one and at most 10 vertices at distance two. Hence, there are at most $12 \cdot (1 + 5 + 10) = 192$ white vertices initially.

Granted there are some black vertices, we choose a black vertex v_k and add it to the constructed set, i.e. $W_k := W_{k-1} \cup \{v_k\}$. We recolor all vertices at distance at most 4 from v_k (including v_k) white. Since for any vertex v of degree 6 there are at most $6d$ vertices at distance d , there are at most $1 + 6 + 12 + 18 + 24 = 61$ new white vertices. This procedure terminates when there are no black vertices.

Let W be the resulting set W_k . The set W contains at least $\frac{f-192}{61}$ vertices where f is the number of faces of G . By Euler's formula, $f = \frac{p}{2} + 2$ and thus $|W| \geq \frac{p-380}{122}$.

We next describe how to construct a matching in G with a lot of disjoint resonant hexagons. Given a vertex $v \in W$, let $R(v)$ be the set of vertices at distance at most 2 from v (see Figure 1.4). The vertices at distance 2 from v form a cycle of length 12 in G^* . This cycle is an induced cycle of G^* since G^* has no non-trivial cyclic 5-edge-cut. Let $R^*(v)$ be the set formed by the 6 independent vertices of $R(v)$ drawn with full circles in Figure 1.4. Since G has no non-trivial cyclic 5-edge-cut, all the vertices in $R^*(v)$ are different and form an independent set in G^* .

The sets $R^*(v)$ for $v \in W$ are pairwise disjoint since W only contains vertices at distance at least 5. We now modify the graph G^* to planar graphs H_0 and H . For every vertex $v \in W$, delete v and the six neighbors of v . Let H_0 be the resulting

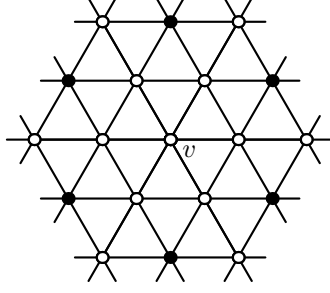


Figure 1.4: The configuration $R(v)$ and the six vertices in $R^*(v)$.

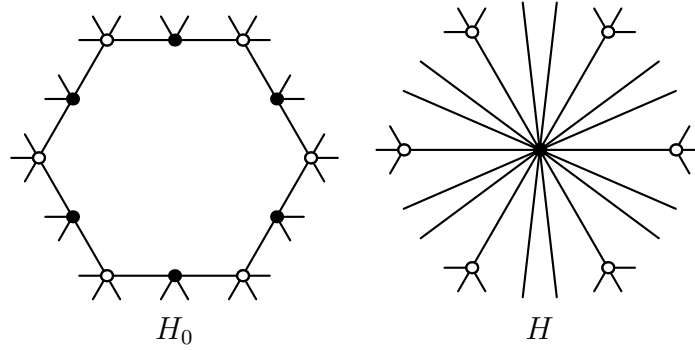


Figure 1.5: The structure of the graphs H_0 and H .

graph. Further identify the six vertices of $R^*(v)$ (see Figure 1.5). The final plane graph is denoted by H .

The Four Color Theorem [4, 5, 62] asserts the existence of a proper vertex coloring of H using four colors. The coloring of H yields a precoloring of H_0 such that the six vertices of each set $R^*(v)$ have the same color. Let $c(v)$ be this color.

We extend the precoloring of H_0 to a proper coloring of G^* . We first color each vertex v by the color $c(v)$. For each $v \in W$, there are only six uncolored vertices inducing a 6-cycle (the vertices adjacent to v), and each such vertex has three neighbours colored with $c(v)$ and one vertex colored with a different color. Therefore, for each such uncolored vertex, there are 2 available colors. Since every cycle of length six is 2-choosable [23, 64], there is an extension of the coloring of H_0 to G^* .

The 4-coloring of G^* corresponds to a proper 3-edge coloring of G . To see this, assume that the vertices of the graph G^* are colored with colors 1, 2, 3, and 4. There are edges of 6 different color types: 12, 13, 14, 23, 24, and 34. Color the edges of G corresponding to the edges of G^* of types 12 and 34 (which are pairwise disjoint)

by the color a , the edges of G corresponding to the edges of G^* of types 13 and 24 by the color b , and the remaining edges, i.e. the edges corresponding to the edges of G^* of types 14 and 23, by the color c . Since the graph G is cubic, each of the color classes a , b , and c forms a perfect matching of G .

Let f be a face corresponding to a vertex w adjacent to $v \in W$ in G^* . Since w has three (non-adjacent) neighbors in G^* colored with the color $c(v)$, the corresponding three non-adjacent edges incident with f are colored with the same color. Hence, the face f is resonant in one of the three matchings formed by the edges of the color a , the edges of the color b , and the edges of the color c .

There are 6 such resonant hexagons for the three matchings for each $v \in W$. Altogether, there are $6|W|$ resonant hexagons. Therefore, one of the matchings has at least $2|W| \geq \frac{p-380}{61}$ resonant hexagons. Observe that the resonant hexagons in one color class are always disjoint: if they were not disjoint, they would correspond to two adjacent neighbors w and w' of some vertex $v \in W$. But the colors assigned to w and w' are different, in particular, the edges corresponding to vw and vw' have different colors. Hence, the hexagons corresponding to w and w' are resonant for different colors a , b , or c . The desired bound on the number of perfect matchings readily follows. \square

Theorem 2 combined with the bound $15 \cdot 2^{\frac{p}{20}}$ by Kutnar and Marušič [53] on the number of perfect matchings in fullerene graphs with non-trivial cyclic 5-edge cuts yields the following.

Corollary 1 *Every fullerene graph with p vertices has at least $2^{\frac{p-380}{61}}$ perfect matchings.*

We do not claim that the constants in our lower bound are best possible. It is clear that the constant 61 could be improved. For example, in the greedy algorithm to construct the set W , if we considered a black vertex adjacent to a white one, we could eliminate a smaller number of new white vertices.

We have included the proof here for two reasons. First, if a fullerene graph is large enough, locally, it just looks as a hexagonal grid, and this property can be properly exploited; Second, the four color theorem really is a powerful tool.

1.4 Saturation number of fullerene graphs

One very important question, a question that receives a lot of attention, is the fullerenes stability. The aim is finding a graph theoretical invariant(s) closely related to the stability of fullerene molecules. Number of different graph invariants

that correlate with the stability were studied. Among those invariant is the *saturation number* $s(G)$ – the cardinality of the smallest maximal matching in a fullerene graph G . Note that it is equal to the cardinality of the minimum independent edge dominating set.

Clearly, the set of vertices that is not covered by a maximal matching is independent [21]. (A set of vertices I is *independent* if no two vertices from I are adjacent.) This observation provides an obvious lower bound on saturation number of the graph G , i.e. $(n - |I|)/2 \leq s(G)$, where G is graph of order n .

The saturation number has another meaning for the chemists: it corresponds to the smallest possible number of large substituents/addends (those that occupy two adjacent atoms) that saturate the molecule. Independent set is another parameter of similar meaning: it is the maximum possible number of addends that cannot be attached to adjacent atoms. Besides in chemistry the saturation number (smallest independent edge dominating set) has a list of interesting applications in engineering, networks, etc.

The saturation number of fullerene graph was studied in [2, 17], where the following bounds were established.

Theorem 3 (Andova et al. 2012) *For any fullerene graph G on n vertices and diameter d , it holds*

$$\frac{3n}{10} \leq s(G) \leq \frac{n}{2} - \frac{1}{4}(d - 2).$$

In particular,

$$s(G) \leq \frac{n}{2} - \frac{\sqrt{24n - 15} - 15}{24}.$$

The saturation number and independent sets on fullerene graphs and triangle-free cubic planar graphs are also studied in [29, 36]. The lower bound in the previous theorem relies only on the 3-regularity of fullerenes. That makes us believe that this bound is not that precise.

Zito [71] provided a probabilistic argument that almost all cubic graphs on n vertices have saturation number at least $0.315812n$. On the other hand there are at least two fullerene graph on n vertices whose saturation number is $3n/10$. Those graphs are dodecahedron and buckminsterfullerene. These two fullerenes are the only fullerenes with icosahedral symmetry whose saturation number satisfies the lower bound of Theorem 3 [17].

In the following subsections we show that the saturation number of fullerenes on n vertices is essentially $n/3$, as published in [3].

1.4.1 Definitions and preliminaries

A *patch* is a 2-connected plane graph with only pentagonal or hexagonal faces, except maybe one face – the outer face; all interior vertices are of degree 3, and all vertices incident to the outer face (on the *boundary* of the patch), are of degree 2 or 3. A patch with no pentagons is called a *hexagonal patch*. Note that by cutting along a cycle in a fullerene graph we always obtain two patches.

Let the number of vertices of degree 2 incident to a face B be denoted by $n_2(B)$. Similarly, let $n_3(B)$ denotes the number of vertices of degree 3 incident to the face B .

In [48], the following lemma is proven.

Lemma 1 (Kardoš and Škrekovski 2008) *Let G be a patch with p pentagons, and an outer face B . Then,*

$$n_2(B) - n_3(B) = 6 - p.$$

It follows that a patch G has 6 pentagons if and only if $n_2(B) = n_3(B)$.

1.4.2 Upper bound on the saturation number

In this subsection we provide an asymptotically tight upper bound on the saturation number in fullerene graphs. We describe a construction to find a maximal matching of size $n/3 + C\sqrt{n}$, where C is a constant.

First we define a maximal matching on an infinite (hexagonal) tube of type (p_1, p_2) , $p_1 \geq p_2$ and $p_1 > 0$.

Proposition 1 *There is a maximal matching M_0 on any infinite nanotube G_0 such that from each hexagon precisely two vertices are not covered by M_0 .*

Proof. We provide a construction of a maximal matching M_0 of the infinite nanotube G_0 of the type (p_1, p_2) , $p_1 \geq p_2$, $p_1 > 0$. We call the edges of M_0 *black* edges; we also call the vertices covered by M_0 *black* vertices. The vertices not covered by M_0 form an independent set; we call them *white* vertices.

For each hexagon h_i we call the common edges with the adjacent hexagons in the direction \vec{a}_1 and $-\vec{a}_1$, an a_1 -edge and $-a_1$ -edge, respectively. Similarly, we name the common edges with the adjacent hexagons in direction \vec{a}_2 and $-\vec{a}_2$ (see Figure 1.6(a)).

We choose a ring $R_1 : h_0, h_1, \dots, h_{p_1+p_2-1}, h_{p_1+p_2} = h_0$ of G_0 . For each hexagon h_i we color the a_1 -edge black; white vertex will be the vertex incident to a a_2 - or $-a_2$ -edge which is not black yet, see Figure 1.6(a). For the hexagons in the next

ring R_2 (the image of R_1 under a translation by the vector $a_2 - a_1$) we propagate similar pattern; for each hexagon we color black the a_2 -edge. White vertices are all the remaining vertices of the ring that are not colored black yet, see Figure 1.6(b).

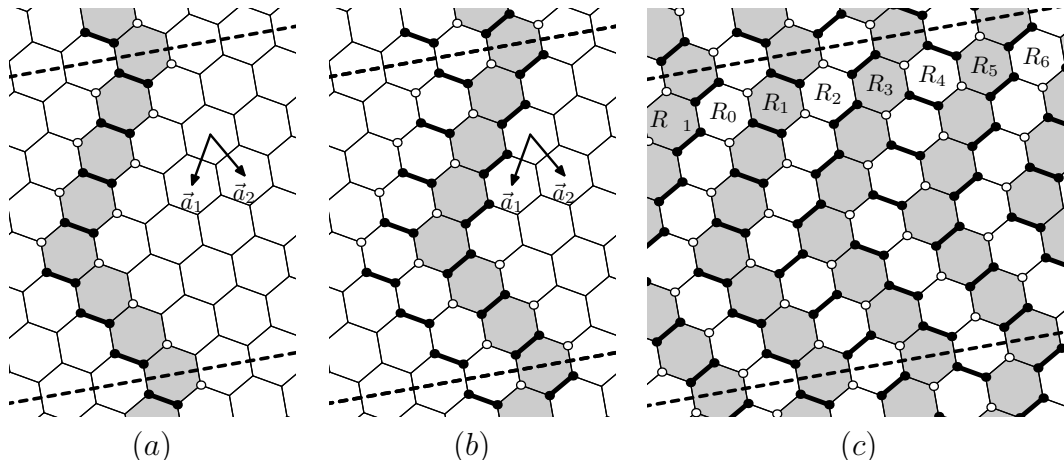


Figure 1.6: Defining a matching on an infinite nanotube of type $(4, 3)$. The dashed lines are overlapping, a_1 and a_2 edges, named after the corresponding vectors. (a) Defining a matching on the first ring (shaded). (b) Defining a matching on the second ring (shaded). (c) Maximal matching covering exactly four vertices from each hexagon. The rings are alternately white and shaded.

For a moment we skip defining a matching on the ring R_3 . Instead we define a matching on the next ring R_4 , in the same fashion as the matching on R_1 . Now, the matching on R_2 and R_4 defines the colors of all vertices incident to the faces of R_3 . We extend this matching to the whole nanotube in the following way: the matching on the ring R_k , $k \in \mathbb{Z}$ is defined in the same way as the matching on the ring R_j , $j = 0, 1, 2$ if $k \equiv j \pmod{3}$.

This way we obtain a desired maximal matching on an infinite nanotube. See Figure 1.6(c) for an illustration. \square

Let us move on from nanotubes to fullerene graphs now.

In [19], Dvořák et al. were investigating how many edges must be removed from a fullerene graph such that the new graph is bipartite. The cardinality of the smallest such a set of edges is called the *bipartite edge frustration* of a graph G , denoted by $\zeta(G)$. It is clear that from each pentagon of a fullerene graph an edge must be removed, but that does not give a bipartite graph since odd cycles might still

exist. They found out that at most $O(\sqrt{n})$ edges must be removed in order to get a bipartite graph.

Theorem 4 (Dvořák, Lidický, Škrekovski 2012) *If G is a fullerene graph with n vertices, then $\zeta(G) = O(\sqrt{n})$.*

Even more, in the same paper they showed that for every pentagon in a fullerene graph, there exist at least five other pentagons at total distance $O(\sqrt{n})$.

Lemma 2 (Six Pentagons Lemma (Dvořák, Lidický, Škrekovski 2012)) *For every pentagonal face f in a fullerene graph G with n vertices, there exist at least five other pentagonal faces whose distance to f in the dual G^* is at most $\sqrt{63n/2} + 14$.*

Using this result we prove the following.

Theorem 5 (Andova, Kardoš, Škrekovski 2015) *Let G be a fullerene graph on n vertices. Then*

$$s(G) \leq \frac{n}{3} + O(\sqrt{n}).$$

Proof. Let G be a fullerene graph. On the set of the twelve pentagonal faces of G we consider the transitive closure \sim of the relation "the distance between f_1 and f_2 in the dual is at most $\sqrt{63n/2} + 14$ ". By Six Pentagons Lemma, for each pentagonal face f there are at least five other pentagonal faces f' such that $f \sim f'$. Therefore, depending on the fullerene structure, two cases are possible:

- (A) *There are two equivalent classes with respect to \sim ;*
- (B) *There is just one class with respect to \sim containing all the twelve pentagons of F .*

We first prove the theorem for case (A), and later we consider the case (B).

- (A) *There are two equivalent classes with respect to \sim .* This means there are two 6-tuples of pentagons "far away" from each other. In this case we find two trees T_1^* and T_2^* in the dual G^* of G , each covering the corresponding six pentagonal faces. Such a tree always exists: it suffices to choose one vertex of degree 5, and to find the shortest paths to the other five vertices of degree 5 using breadth-first search. The union of these is a desired tree in G^* . Let T_i be the set of edges in G corresponding to the edges of G^* with both endvertices in T_i^* , $i = 1, 2$. Observe that T_i may contain (a bounded number of) edges corresponding to edges in G^* which are not edges of the tree T_i^* . Among all the trees possible,

for T_i^* we choose one with as few edges in T_i as possible. The overall number of vertices of $T_1^* \cup T_2^*$ by Lemma 2 is at most $10(\sqrt{63n/2} + 14)$.

Let Q_i be the graph obtained as a union of the boundary cycles for the faces of G corresponding to the vertices of T_i^* . As T_i^* covers exactly 6 vertices of degree 5, Q_i is a fullerene patch containing exactly 6 pentagons. Let C_i be the boundary cycle of Q_i (the binary sum of the boundaries of its faces). Observe that C_i is connected since T_i^* is a tree in G^* .

On the other hand, it is easy to see that each vertex of G is incident to 0, 1, or 3 edges in T_i (otherwise there would be two vertices in T_i^* joined by an edge not corresponding to an edge of T_i). Let R_i be the set of vertices incident to 3 edges in T_i . The graph $P_i = G - (T_i \cup R_i)$ is another fullerene patch containing exactly 6 pentagons. Clearly, $P_i \cap Q_i = C_i$. See Figure 1.7 for an illustration of T_i , C_i , Q_i and P_i . Finally, let $H = G - (T_1 \cup R_1) - (T_2 \cup R_2)$ be the subgraph of G not containing any pentagons. Clearly all the faces of H but two are hexagons.

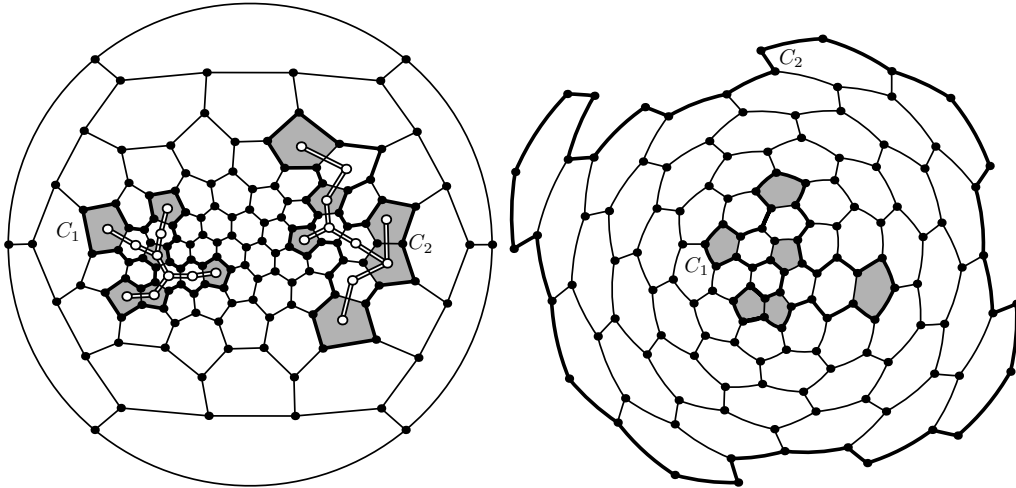


Figure 1.7: An example of a fullerene graph in which the pentagonal faces form two clusters of six. The boundary of the union of faces corresponding to the dual tree T_i is the cycle C_i , $i = 1, 2$ (left). The patch P_2 contains the patch Q_1 (right).

For each vertex v of C_i , either $d_{P_i}(v) = 2$ and $d_{Q_i}(v) = 3$ (if v is incident to an edge of T_i), or $d_{P_i}(v) = 3$ and $d_{Q_i}(v) = 2$ (if v is not incident to any edge of T_i). Let $n_{i,2}$ and $n_{i,3}$ be the numbers of vertices from C_i of degree 2 and 3 in Q_i . Since both P_i and Q_i have exactly six pentagons, by Lemma 1 we have $n_{i,2} = n_{i,3}$.

The patch (with six pentagons) Q_i is a cap of some nanotube. The type of the nanotube having Q_i as a cap can be determined in the following way: Let v_1, v_2, \dots, v_k be the vertices of C_i in a cyclic order ($v_{k+1} = v_1, v_0 = v_k$). If $d_{Q_i}(v_j) = 2$, then the edges $v_{j-1}v_j$ and v_jv_{j+1} are incident to two different added hexagons; towards the tube they form a 240 degree angle. If $d_{Q_i}(v_j) = 3$, then the edges $v_{j-1}v_j$ and v_jv_{j+1} are incident to the same added hexagon; they form a 120 degree angle. Informally speaking, for each vertex, the direction difference between the vectors $v_{j-1}v_j$ and v_jv_{j+1} is either a ‘left turn’ or a ‘right turn’.

If we choose the first edge v_1v_2 on the infinite hexagonal grid, the sequence of degrees of vertices of C_i fully determines the position of all the other vertices. Since there is the same number of vertices of degree 2 and 3 on Q_i , there is the same number of left and right turns, so the edges v_0v_1 and v_kv_{k+1} are equally oriented. Since on the tube $v_0 = v_k$ and $v_{k+1} = v_1$, the difference $v_k - v_0 = v_{k+1} - v_1$ on the infinite hexagonal grid determines the characterizing vector of the tube, see Figure 1.8 for an illustration.

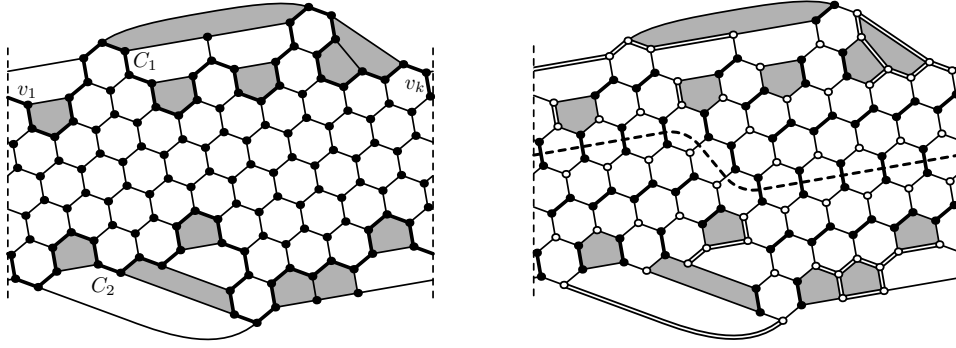


Figure 1.8: The graph from Figure 1.7 drawn in such a way that the part containing only hexagons separating the two patches Q_1 and Q_2 containing pentagons is embedded into a nanotube (left). The difference $v_k - v_0$ on an infinite hexagonal grid determines the type of the nanotube. This graph is a nanotube of type $(9, 2)$. The matching M_1 inherited from the infinite nanotube (right). The dashed line represents a nanotube ring.

Since C_1 is the boundary of both P_1 and Q_1 , and the type of the nanotube is determined solely by C_1 , they can be considered as two different caps of the same nanotube.

Since $Q_1 \subset P_2$ (and $Q_2 \subset P_1$), after removing sufficiently large number of

hexagons from P_2 (resp. P_1) one can obtain Q_1 (resp. Q_2). The symmetric difference $H = G - (T_1 \cup R_1) - (T_2 \cup R_2)$ does not contain any pentagons, and therefore Q_1 is a cap for the same nanotube as P_2 (Q_2 as for P_1).

Once the type of the tube is determined, we embed H into infinite tube H_0 with the predefined matching. By Proposition 1, there is a maximal matching M_0 on H_0 such that from each hexagon precisely two vertices are white (not covered by the matching M_0). All the vertices of H inherit the colors from the corresponding vertices of G_0 .

Let n_B and n_W be the numbers of black and white vertices in G , respectively, let $r = |R_1| + |R_2|$. Then $n = n_B + n_W + r$. Let b_2 and b_3 (w_2 and w_3) be the numbers of black (white, respectively) vertices of degree 2 and 3 incident to one of the exceptional faces of H . Since both Q_1 and Q_2 contain exactly 6 pentagons, we have

$$b_2 + w_2 = b_3 + w_3. \quad (1.1)$$

Let h be the number of hexagonal faces of H , let t be the total number of faces of Q_1 and Q_2 . Then by double counting the vertices on the patches Q_1 and Q_2 we get

$$6t - 12 = 3r + 2b_2 + b_3 + 2w_2 + w_3,$$

which combined with (1.1) gives

$$2t - 4 = r + b_2 + w_2. \quad (1.2)$$

Then, by double counting the face-vertex incidences in H we get

$$3(n_B - b_2 - b_3) + b_2 + 2b_3 = 4h \quad \text{and} \quad 3(n_W - w_2 - w_3) + w_2 + 2w_3 = 2h,$$

which together with (1.1) implies

$$3n_B = 2n - 2r + b_2 - w_2 - w_3.$$

Some of the edges of the matching M_0 defined on the infinite tube H_0 can have only one endvertex in H and the other one not. This can only happen for black vertices of degree 2 in H ; let b'_2 be the number of them. We recolor those vertices white temporarily. Observe that for each such vertex, its two neighbors in H are both incident to the same exceptional face. Let M_1 be the matching of G obtained this way. Clearly, $|M_1| = (n_B - b'_2)/2$. See Figure 1.8 for an illustration.

The matching M_1 is not necessarily a maximal matching of G , however, two white vertices can only be adjacent in G if they are both incident to the same exceptional face of H . We make the matching M_1 maximal by adding an arbitrary maximal matching of the subgraph of G induced by the white vertices incident to the two exceptional faces of H and the vertices from $R_1 \cup R_2$. This way we obtain a matching M of size

$$|M| \leq \frac{n_B - b'_2}{2} + \frac{r + w_2 + w_3 + b'_2}{2} = \frac{2n + r + b_2 + 2w_2 + 2w_3}{6}. \quad (1.3)$$

In order to determine the upper bound we used the fact that $w_2 + w_3 \leq b_2 + b_3$, since in H_0 , the white vertices of each cycle C_i induce an independent set and relation (1.1). Now, we have $w_2 + 2w_3 \leq b_2 + b_3 + w_3 \leq 2b_2 + w_2 \leq 2b_2 + 2w_2$, and therefore $r + b_2 + 2w_2 + 2w_3 \leq 3(r + b_2 + w_2)$. Plugging the last relation and relation (1.2) into (1.3), we infer

$$|M| \leq \frac{n}{3} + t - 2.$$

- (B) *There is just one class with respect to \sim containing all the twelve pentagons of G .* In this case we find a subtree T^* of G^* containing all the vertices corresponding to pentagonal faces of G . Let T be the set of edges in G corresponding to the edges of G^* with both endvertices in T^* . The graph $H = G - T$ is a hexagonal patch. The overall number t of vertices of T^* is at most $11(\sqrt{63n/2} + 14)$. We embed H into an infinite tube of the type (p_1, p_2) with $p_1 + p_2$ sufficiently large, and follow the same procedure as in the previous case. Observe that here the patch Q (it is only one) has precisely 12 pentagons, and instead (1.1), now we have $b_2 + w_2 = b_3 + w_3 - 6$. Similarly (1.2) in this case is $2t - 4 = r + b_2 + w_2 + 2$. Applying these changes into (1.3), we can use analogous calculations to prove that this way we find a maximal matching of size at most $n/3 + t - 4$, what concludes the proof of the theorem.

In all the cases we managed to find a maximal matching of size at most $n/3 + 11(\sqrt{63n/2} + 14) - 2$, as desired. \square

1.4.3 Lower bound on the saturation number

In this subsection, we provide a lower bound on the saturation number of fullerene graphs, matching asymptotically the upper bound above. We show that every maximal matching of a fullerene graph contains at least $n/3 - 2$ edges, using a global discharging argument.

Theorem 6 (Andova, Kardoš, Škrekovski 2015) *Let G be a fullerene graph on n vertices. Then,*

$$s(G) \geq \frac{n}{3} - 2.$$

Proof. Let M be a maximal matching in G . Let vertices covered by M be black, edges of M black as well, remaining vertices and edges white. Let B (resp. W) be the set of all black (resp. white) vertices.

In order to prove the theorem, we use the discharging method. We set the charges to vertices and pentagonal faces as follows:

- Let the initial charge of each black vertex be 3;
- Let the initial charge of each white vertex be -6 ; and,
- Let the initial charge of each pentagonal face be 3.

We will prove that the total sum of the charge in the graph $3|B| - 6|W| + 36$ is non-negative. In other words,

$$3|B| \geq 2|B| + 2|W| - 12,$$

and it implies

$$|M| = \frac{|B|}{2} \geq \frac{|B| + |W| - 6}{3} = \frac{n - 6}{3},$$

as desired. The initial charge is redistributed by the following rule:

(R1). *Each white vertex sends -2 of charge to each adjacent black vertex.*

Since M is a maximal matching, W is an independent set in G , i.e., no two white vertices are adjacent. After applying **(R1)**, the white vertices have charge zero.

Let v be a black vertex. It is adjacent to at least one black vertex, hence, it is adjacent to at most two white vertices. Let e_v be the black edge incident with v , and let f_v be the face incident with v , but not with e_v . After having received 0, -2 , or -4 of charge by **(R1)**, according to the number of white neighbors, v has charge 3, 1, or -1 .

(R2). *Each black vertex v sends all its remaining charge to f_v .*

All the charge initially present at vertices of G is now at its faces. The only case when a face was given some negative charge, is when a black vertex v with two white neighbors sends -1 of charge to the face f_v . Therefore, if a face is incident with at

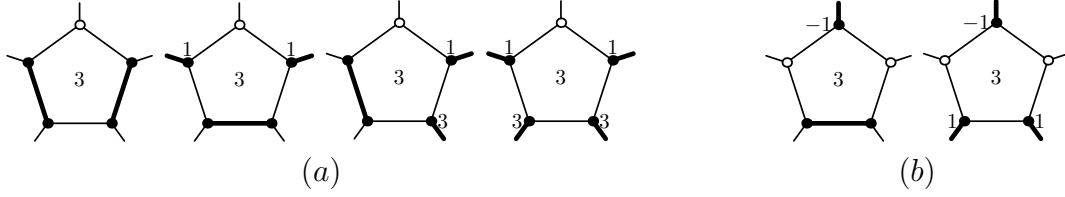


Figure 1.9: (a) All the possible types of pentagonal faces of G incident to one white vertex. The initial charge of the pentagons is 3, and its incident black vertices send charge 0, 1 or 3, as shown in the figure. In all the situations the charge of the pentagon after applying **(R2)** is at least 3. (b) Pentagonal faces of G incident to two white vertices. After applying **(R2)** the charge of the pentagon is at least 2.

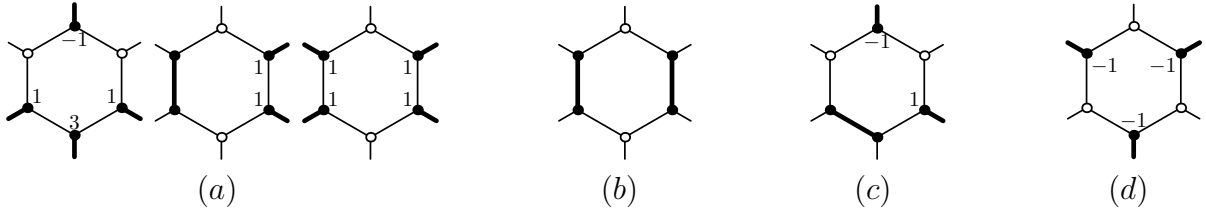


Figure 1.10: All the possible situations of hexagonal faces of G incident to at least two white vertices. The hexagons in (a) have positive charge; the hexagons in (b) and (c) have charge zero; and the hexagon in (d) has charge -3 .

most one white vertex, its charge is non-negative. Moreover, if a pentagon is incident with exactly one white vertex, its charge is at least 3, see Figure 1.9(a). Similarly, if a hexagon is incident with exactly one white vertex its charge is at least 1.

If a pentagonal face is incident with two white vertices, its charge is at least 2, see Figure 1.9(b). If a hexagonal face h is incident with two white vertices as in Figure 1.10(a), it will have a positive charge. If the hexagon h is as in Figure 1.10(b) or (c), then its charge is zero. Let us call these two types of hexagons of charge zero *neutral* and *transition* faces, respectively. The hexagonal face having three white neighbors (Figure 1.10(d)) has charge -3 . Let call this type of hexagons *bad*. All the other hexagons, as well as all the pentagons, have positive charge, and we call them *good*.

Let f be a transition hexagonal face. It is incident to two white vertices, two black vertices forming a black edge, and two other black vertices. Let the white vertex adjacent to the two other black vertices be called *incoming*, let the other one (adjacent to an endpoint of the black edge incident with f) be called *outgoing*.

The next steps of the discharging are given with the following rules:

- (R3). *Each good face sends charge 1 to each incident white vertex;*
- (R4). *Each bad hexagonal face sends charge -1 to each incident white vertex.*
- (R5). *Each transition hexagonal face sends charge -1 to the incoming incident white vertex, and it sends charge 1 to the outgoing incident white vertex.*

It is clear that after applying these three rules there is no negative charge at the faces of G . The only elements of the graph that can contain some negative charge are the white vertices incident to a bad or transition hexagon.

Let v be a white vertex that was sent charge -1 from a hexagon h by (R4) or (R5). Let w_1 and w_2 be the black vertices adjacent to v incident with h . The black edge incident to w_i is not incident to h , $i = 1, 2$. Let u be the neighbor of v not incident with h . Clearly, u is black. Let f_1 and f_2 be the two faces incident with v different from h . Without loss of generality we may assume that the black edge incident with u is incident with f_1 . Then f_1 is good or neutral, so it does not send negative charge to v .

Clearly, f_2 cannot be a bad hexagon, nor a neutral one. If f_2 is not incident to other white vertex but v , it is a good hexagon. If f_2 is incident to another white vertex at distance 3 from v , it is a good hexagon as well. If f_2 is incident to another white vertex at distance 2 from v , then it is a transition hexagon, moreover, v is the outgoing white vertex for f_2 . In all the cases, f_2 has sent charge 1 to v by (R3) or (R5), see Figure 1.11.

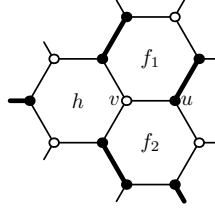


Figure 1.11: For a white vertex v which receives charge -1 from a bad or transition hexagon h , there is always another hexagon, which sends positive charge to v .

Since there is no negative charge in the graph, the total sum of charge is non-negative, as desired. \square

1.4.4 Concluding remarks

We managed to prove a lower and an upper bound on the saturation number of fullerene graphs, which are asymptotically equal.

The bound proved in Section 1.4.3 turns out to be tight. There are infinitely many fullerene graphs with the saturation number equal to $n/3 - 2$: for example, a nanotube of type $(8, 0)$ with $3k + 1$ rings of hexagons and with caps depicted in Figure 1.12 has $48k + 60$ vertices and admits a maximal matching of size $16k + 18$. We are aware of other examples, even without adjacent pentagons.

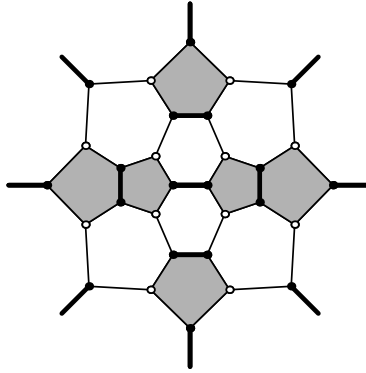


Figure 1.12: A cap of an $(8, 0)$ -nanotube with saturation number $n/3 - 2$.

Comparing the newly established lower bound $n/3 - 2$ with the previous bound $3n/10$ we find that a fullerene graph can only admit a maximal matching of size

exactly $3n/10$ if it has at most 60 vertices. Moreover, this can only occur for fullerene graphs having exactly 20, 30, 40, 50, or 60 vertices.

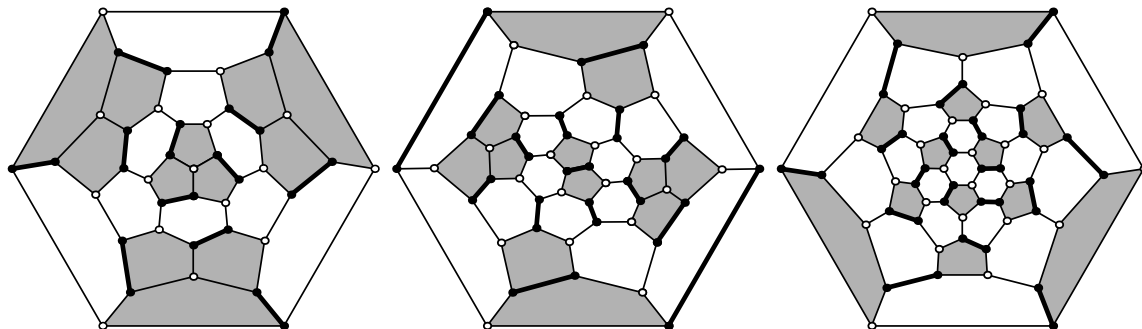


Figure 1.13: Fullerenes on $n = 40, 50, 60$ with saturation number $3n/10$.

Since there are only finitely many such graphs and the lists of those graphs are known, we can find easily those that admit a maximal matching of size $3n/10$ by inspecting each of them by a computer check. The dodecahedron (the only fullerene graph on 20 vertices) admits a maximal matching of size 6; none of the three fullerene graphs on 30 vertices does admit a maximal matching of size 9; there is exactly one fullerene graph on 40 and 50 vertices having a maximal matching of size 12 and 15, respectively; there are 7 fullerene graphs on 60 vertices admitting a maximal matching of size 18, including the buckminsterfullerene (the only fullerene graph on 60 vertices without adjacent pentagons).

The question to determine the exact value of the saturation number remains still open. Here we pose a conjecture concerning the problem.

Conjecture 1 *There is a constant C such that*

$$s(F) \leq \frac{n}{3} + C$$

for any fullerene graph F on n vertices.

The problem of finding a minimum independent dominating set is NP-complete [67]. This problem is NP-complete even when restricted to planar or bipartite graphs of maximal degree three [67], and remains NP-complete for planar cubic graphs [37]. These results imply the next question.

Problem 1 *Is the problem to determine the saturation number for the class of fullerene graphs NP-complete?*

1.5 Hamiltonicity of fullerene graphs

Tait conjectured in 1880 that all cubic polyhedral graphs (i.e., 3-connected planar cubic graphs) are Hamiltonian (i.e., contain a cycle passing through every vertex exactly once). The first counterexample to Tait's conjecture was found by Tutte in 1946; later, many others were found, see Figure 1.14. Had the conjecture been true, it would have implied the Four-Color Theorem.

However, each known non-Hamiltonian cubic polyhedral graph has at least one face of size 7 or more [1, 68]. It was conjectured that all cubic polyhedral graphs with maximum face size at most 6 are Hamiltonian. In the literature, the conjecture is usually attributed to Barnette (see, e.g., [54]), however, Goodey [33] stated it in an informal way as well.

All cubic polyhedral graphs having only faces of sizes 3 and 6 or 4 and 6 are known to be Hamiltonian [33, 32]. Exhaustive generation has shown that

Lemma 3 (Brinkmann, Goedgebeur, McKay 2012 [6]) *Every cubic polyhedral graph with faces of size at most 6 on at most 316 vertices is Hamiltonian.*

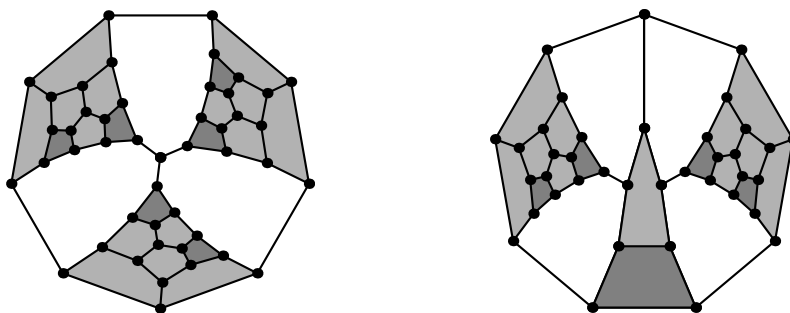


Figure 1.14: Tutte's first example of a non-Hamiltonian cubic polyhedral graph (left); one of the minimal examples on 38 vertices (right).

The conjecture of Barnette and Goodey covers in particular the class of fullerene graphs. Jendroľ and Owens proved that the longest cycle of a fullerene graph of order n covers at least $4n/5$ vertices [39], the bound was later improved to $5n/6 - 2/3$ by Král' et al. [52] and to $6n/7 + 2/7$ by Erman et al. [24]. Marušič [55] proved that the fullerene graph obtained from another fullerene graph with an odd number of faces by the so-called leapfrog operation (truncation of the dual; replacing each vertex by a hexagonal face) is Hamiltonian. In fact, Hamiltonian cycle in the derived graph corresponds to a decomposition of the original graph into an induced forest and a stable set. We will use similar technique to prove the conjecture in the general case.

In this section we prove

Theorem 7 *Let G be a 3-connected planar cubic graph with faces of size at most 6. Then G is Hamiltonian.*

In the next subsections, we reduce the main theorem to Theorem 8 and further to Theorem 10 and we introduce terminology and techniques used in the proof of Theorem 10.

1.5.1 Preliminaries

A *Barnette graph* is a 3-connected planar cubic graph with faces of size at most 6, having no triangles and no two adjacent quadrangles. If we denote f_4 and f_5 the numbers of quadrangular and pentagonal faces, then

$$2f_4 + f_5 = 12,$$

generalizing the observation on the number of pentagonal faces for fullerene graphs. We will call those faces of size smaller than 6 *small*.

1.5.1.1 Cyclic edge-connectivity of Barnette graphs

Lemma 4 *Let G be a Barnette graph. Then G is cyclically 4-edge-connected.*

Proof. Suppose that G contains a cyclic 3-edge-cut (X, Y) . Choose X inclusion-wise minimal. It is easy to see that the cut-edges are pairwise non-adjacent. Let x_1, x_2, x_3 be the vertices of X incident to the cut-edges. We prove that they are pairwise non-adjacent: Suppose that two of them, say x_1 and x_2 , are adjacent. Then, by minimality of X , $X' = X \setminus \{x_1, x_2\}$ is acyclic with $(X', V(G) \setminus X')$ being a 3-edge-cut, and hence, $|X'| = 1$, $X' = \{x_3\}$, so thus $G[X]$ is a triangle, which is impossible in a Barnette graph.

Let y_i be the other endvertex of the cut-edge incident to x_i , $i = 1, 2, 3$. We prove that these three vertices are also pairwise non-adjacent: Since G has no triangles, $G[\{y_1, y_2, y_3\}]$ has at most two edges. If it had exactly two edges, then G would contain a 2-edge-cut, which is impossible since G is 3-connected and cubic. Suppose now that y_1 and y_2 are adjacent, but y_3 is not adjacent to any of them. Each of the two faces incident to the edge x_3y_3 has at least three incident vertices in both X and Y , therefore, it is a hexagon, and there are exactly three incident vertices in both X and Y . Let z_i be the common neighbor of y_3 and y_i , $i = 1, 2$. Then z_1 and z_2 are

adjacent, otherwise there would be a 2-edge-cut in G . But then $y_3 z_1 z_2$ is a triangle in G , a contradiction.

As y_1, y_2, y_3 are pairwise non-adjacent, for each face incident to any cut-edge, there are at least three incident vertices in both X and Y , therefore, each such face is a hexagon having three incident vertices in both X and Y . Let x_{ij} be the common neighbor of x_i and x_j , $1 \leq i < j \leq 3$. By minimality of X , $X' = X \setminus \{x_1, x_2, x_3, x_{12}, x_{13}, x_{23}\}$ is a single vertex, and so $G[X]$ is the union of three 4-faces pairwise adjacent to each other, which is impossible in a Barnette graph. \square

1.5.1.2 First reduction

We reduce Theorem 7 to the case of Barnette graphs:

Theorem 8 *Let G be a Barnette graph on at least 318 vertices. Then G is Hamiltonian.*

Lemma 5 *Theorem 8 implies Theorem 7.*

Proof. Suppose Theorem 8 is true. Let G be a smallest counterexample to Theorem 7. We know that G has at least 318 vertices, since Theorem 7 has already been verified for all cubic planar graphs with faces of size at most 6 on at most 316 vertices (see Lemma 3; the number of vertices of a cubic graph is always even.)

Assume $f = v_1 v_2 v_3$ is a triangle in G . If one of the faces adjacent to f is a triangle, then, by 3-connectivity, G is (isomorphic to) K_4 , a Hamiltonian graph. Therefore, all the three faces adjacent to f are of size at least 4. Let G_1 be a graph obtained from G by replacing $v_1 v_2 v_3$ by a single vertex v . It is easy to see that G_1 is a 3-connected cubic planar graph with faces of size at most 6, moreover, every Hamiltonian cycle of G_1 can be extended to a Hamiltonian cycle of G , see Figure 1.15 for an illustration.

From this point on we may assume that G contains no triangles. Let f_1 and f_2 be two adjacent faces of size 4 in G . Let v_1 and v_2 be the vertices they share; let $f_1 = v_1 v_2 u_3 u_4$, let $f_2 = v_1 v_2 w_3 w_4$. We denote by f_3 (resp. f_4) the face incident to u_3 and w_3 (u_4 and w_4 , respectively). If both f_3 and f_4 are quadrangles, then, by 3-connectivity, G is the graph of a cube, which is Hamiltonian. Suppose $d(f_4) \geq 5$ and $d(f_3) = 4$. Let G_2 be a graph obtained from G by collapsing the faces f_1, f_2, f_3 to a single vertex. Again, G_2 is a 3-connected cubic planar graph with faces of size at most 6, moreover, every Hamiltonian cycle of G_2 can be extended to a Hamiltonian cycle of G , see Figure 1.15.

Finally, suppose that both f_3 and f_4 are of size at least 5. We remove the vertices v_1 and v_2 , identify u_3 with w_3 into z_3 and u_4 with w_4 into z_4 ; in this way we obtain

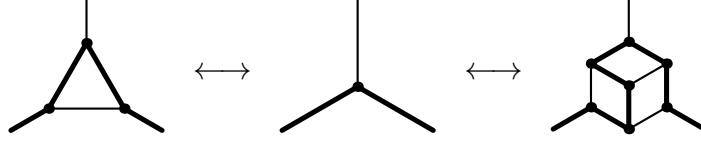


Figure 1.15: A triangle, as well as three quadrangles sharing a vertex, can be reduced to a single vertex.

a graph G_3 . Clearly G_3 is a cubic planar graph with all the faces of size at most 6. Every Hamiltonian cycle of G_3 can be extended to a Hamiltonian cycle of G , as seen on Figure 1.16. It suffices to prove that G_3 is 3-connected, or, equivalently, that G_3 has no 2-edge-cuts.

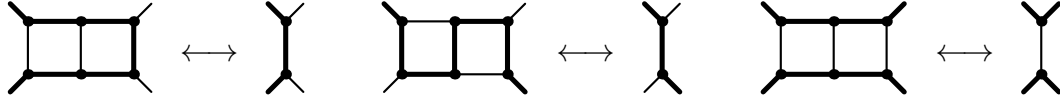


Figure 1.16: A pair of adjacent quadrangles can be reduced to a single edge.

Suppose G_3 has a 2-edge-cut. Then one of the cut-edges is $e = z_3z_4$ (since G is 3-connected); let e' be the other cut-edge. Then $X = \{u_4u_3, v_1v_2, w_4w_3, e'\}$ is a 4-edge-cut in G . Besides f_1 and f_2 , there are exactly two other faces destroyed when removing the four edges from X , say f_u (incident to e' and u_4u_3) and f_w (incident to e' and w_4w_3).

For $i = 1, 2$, let z_i be the endvertex of e' belonging to the same component of $G \setminus X$ as v_i . If z_1 was adjacent to u_4 , then $\{z_1, w_4\}$ would be a 2-cut in G , unless z_1 and w_4 are adjacent, but that would mean the size of f_4 is four, a contradiction. By the same argument we obtain that z_1 is not adjacent to w_4 either; similarly, z_2 is adjacent neither to u_3 nor to w_3 . Since the size of f_u and f_w is at most six, they are both hexagons and z_1 (z_2) is at distance two from both u_4 and w_4 (u_3 and w_3 , respectively). But then the two neighbors of z_1 form a 2-cut in G , unless they are adjacent, in which case there is a triangle in G , a contradiction. \square

1.5.1.3 Barnette nanotubes

Lemma 6 *Let G be a Barnette graph which is a nanotube of type (p_1, p_2) with $p_1 + p_2 = 4$. Then $(p_1, p_2) = (4, 0)$.*

We omit the details of the proof, as it is similar to the proof of Lemma 4: It

suffices to prove that every (potential) cap of a nanotube of type $(3, 1)$ or $(2, 2)$ contains a triangle or a pair of adjacent quadrangles.

Lemma 7 *Let G be a Barnette graph which is a nanotube of type (p_1, p_2) with $(p_1, p_2) \in \{(4, 0), (5, 0), (4, 1), (5, 1), (3, 2), (4, 2), (3, 3), (4, 3)\}$. Then G is Hamiltonian.*

Proof. We may suppose that G has at least 318 vertices (at least 161 faces). For each of the types considered here we have generated all possible caps by hand, and checked the completeness of the list by a computer.

Since the caps of the tube are of bounded size (at most 5, 10, 6, 11, 5, 10, 10, 14 faces, respectively, each), the tubical part of G contains a large number of disjoint rings.

We provide a construction of a Hamilton cycle in such graphs: First, we find a pair of paths covering the vertices of the tubical part of G ; then, we verify that for each possible cap it is always possible to connect the two paths in a way that all the vertices of the cap are covered as well.

In a nanotube of type $(p, 0)$, $p \geq 4$, for each p -edge-cut corresponding to a ring, we construct the two paths transversing the tube in a way that each path contains one cut-edge incident to the same hexagonal face. Let us call this hexagon a *transition face*. For two consecutive rings, the transition faces are adjacent and once the transition face is fixed for one ring, we are free to choose any of the two adjacent hexagons in the next one to be the transition face, see Figure 1.17 for an illustration.

To complete the proof for $(4, 0)$ - and for $(5, 0)$ -nanotubes, it suffices to verify that for every possible cap, there exists a path covering all the vertices of the cap and leaving the cap by two edges adjacent to the transition hexagonal face of the first ring of the tube. Since the tubical part of G is sufficiently long, we can choose a transition face in the first and the last ring of hexagons regardless of the relative position of the two caps.

For nanotubes of type $(3, 3)$, the construction is described in Figure 1.18.

For nanotubes of type (p_1, p_2) with $p_1 > p_2 > 0$, we provide a repetitive pattern to cover the tubical part (see Figure 1.19) and, for every cap and for every position of the cap with respect to the pattern, a path covering the vertices of the cap (see Figure 1.20 for the first two types of nanotubes; we omit the details for the remaining three types). \square

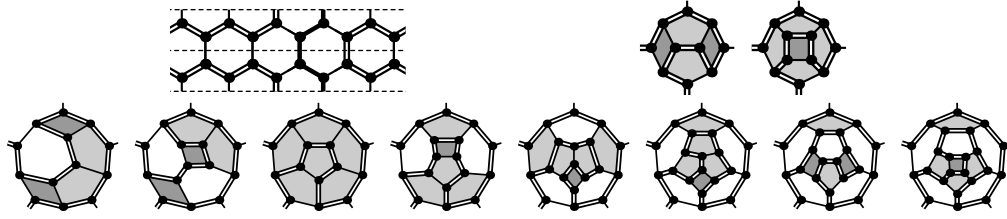


Figure 1.17: Two ways to cover the $2p$ vertices separated by two consecutive cyclic p -edge-cuts in a $(p, 0)$ -nanotube by two paths (top left for $p = 5$, one of the two paths drawn by thick lines, the other by double lines; transition faces are marked with an asterisk). A path joining two consecutive pending edges covering all the vertices, for every possible cap of $(p, 0)$ -nanotubes for $p = 4$ (top right line) and for $p = 5$ (bottom line).

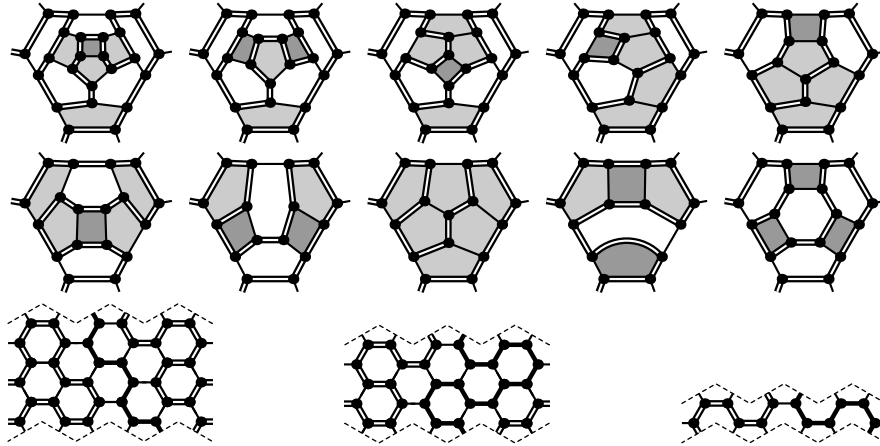


Figure 1.18: For each possible cap of a $(3, 3)$ -nanotube, a path leaving the cap by a prescribed pair of edges is given (first two rows). For the last cap, we added three hexagons of the tube to make the construction work. To connect the two caps and to cover the tubical part of the graph, it suffices to combine an appropriate number of the first two patterns of the last row (and/or their mirror images) and finish by the third one.

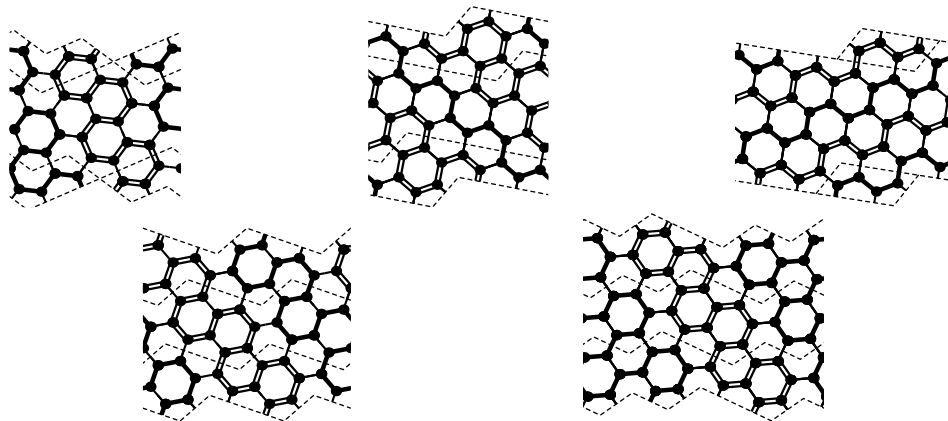


Figure 1.19: Two paths covering all the vertices of a (potentially infinite) open-ended nanotube of type $(3, 2)$, $(4, 1)$, $(5, 1)$, $(4, 2)$, and $(4, 3)$, respectively. For each end of the tube, the two dashed lines separate the smallest period of the covering.

1.5.1.4 Structure of Barnette graphs

Observe that we can use Goldberg vectors to represent the mutual position of small faces in a Barnette graph in the way as it was explained in subsection 1.2.2 for fullerene graphs. The structure of a given Barnette graph G can thus be captured by a plane triangulation T whose vertices represent small faces of G and edges are labelled with the Goldberg vectors representing the mutual position of the corresponding small faces in G .

Let us note that for a vertex of T representing a pentagon the angles around it sum up to $5\pi/3 = 300^\circ$, whereas for a vertex representing a quadrangle it gives $4\pi/3 = 240^\circ$.

Lemma 8 *Let G be a Barnette graph, let T be a minimal triangulation representing G . Then either T is Hamiltonian, or T can be transformed to a Hamiltonian triangulation by a single diagonal switch.*

Proof. T is a planar triangulation on at most 12 vertices. There are exactly three non-Hamiltonian planar triangulations T on at most 12 vertices (one on 11 vertices and two on 12 vertices) [13], with the following structure: T contains a triangular bipyramid B (which has 5 vertices and 6 triangular faces) as a subgraph. $T \setminus B$ then either consists of a set of 6 isolated vertices (each dividing one face of B into three triangles), or is composed of 5 isolated vertices and an isolated edge (i.e., 5 faces of B are divided into three triangles, the remaining one is divided into five triangles;

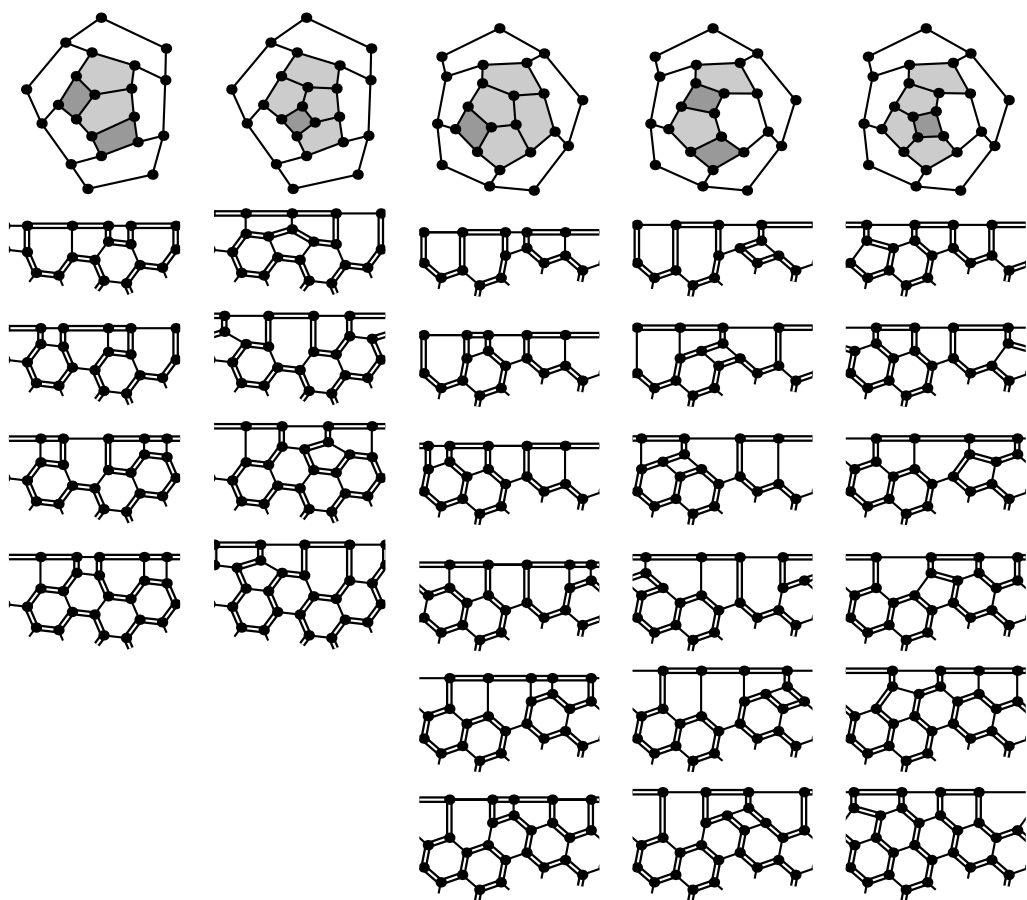


Figure 1.20: For every cap of a nanotube of types $(3, 2)$ (first two columns) and $(4, 1)$ (the rest), and for every position of the cap relative to the two paths covering the tubical part of the graph, a completion of the Hamilton cycle in the cap is given. In the first row, the caps are drawn together with the first ring of the tube.

up to symmetry of B there are only two different ways this can be performed). It means T has five vertices of degree at least 6 (those of B), six vertices of degree 3, and eventually a vertex of degree 4.

Let $e = uv$ be an edge of B . It is incident to two triangles, each incident to a different component of $T \setminus B$. Let x and y be the vertices of $T \setminus B$ such that uvx and uvy are triangles of T . If the quadrilateral $uxvy$ is convex, then the triangulation T' obtained from T by switching uv to xy has at most five vertices of degree 3, so T' has to be Hamiltonian.

It remains to consider the case when for each edge e of B , the union of the two incident triangles is a non-convex quadrilateral, meaning that at one of its endvertices, the sum of the angles in the incident triangles is greater than 180° . Since B has five vertices and nine edges, there is at least one vertex of B with two (disjoint) pairs of incident triangles whose union gives a non-convex angle. But then the sum of the angles around this vertex is greater than 360° , a contradiction. \square

Observe that no matter whether a diagonal switch is necessary or not, for each edge uv of a Hamilton cycle C of a triangulation T given by Lemma 8, the edge uv is the shortest straight line segment joining u and v within the quadrangle obtained as a union of the two triangles incident to the edge uv .

In Figure 1.21, an example of a Barnette graph on 322 vertices is depicted, along with the corresponding triangulation and a shortest Hamilton cycle in it.

1.5.1.5 Second reduction

Let H be a plane cubic graph. We denote H^\parallel the 6-regular multigraph obtained from H by replacing each edge by a pair of parallel edges, equipped with the following black-and-white face-coloring: We color the 2-gons between pairs of parallel edges white and we color the faces of H^\parallel corresponding to the faces of H black. It is easy to see that this is a proper face-coloring of H^\parallel .

Let G be a Barnette graph and let M be a perfect matching of G . Then $F = E(G) \setminus M$ is a 2-factor of G . A hexagonal face of G incident to three edges of M is called *resonant*.

There is a canonical face-coloring of G with two colors, say black and white, such that each edge of F is incident to one black and one white face. Let h be a white resonant hexagon. Since it is incident to three edges from M , the colors of its neighboring faces are alternating black and white.

We transform F into a 6-regular plane pseudograph in the following way: First, inside each white resonant hexagon h of G we introduce a new vertex v_h . We remove the three edges incident to h from F and we replace them by six new edges, joining

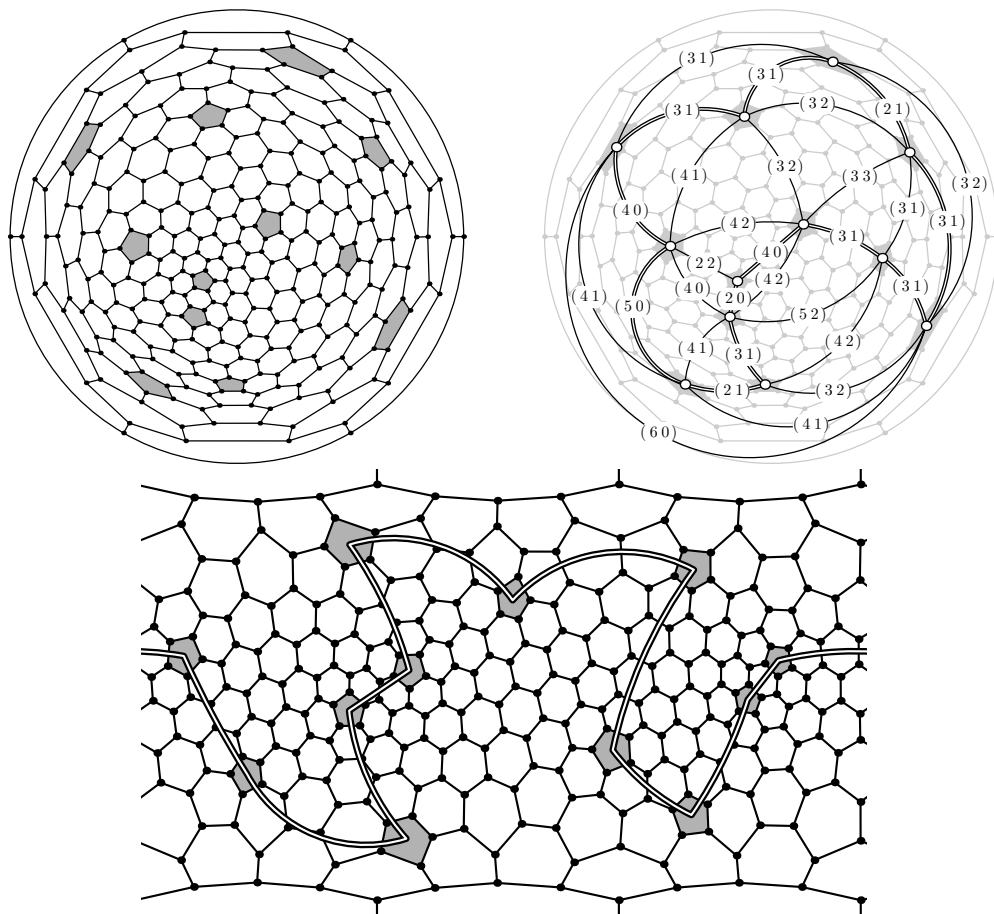


Figure 1.21: An example of a Barnette graph on 322 vertices (top left). A triangulation capturing the mutual position of all the small faces with a Hamilton cycle (top right). Another (tubular) drawing of the same graph (bottom); the three edges sticking to the north (to the south) are incident to an omitted vertex at the north (south) pole.

v_h to all the six vertices incident to h . In G , the hexagon h is partitioned into six triangles; each of them receives the color of the corresponding face adjacent to h in G it is merged with in F . This way we obtain a black-and-white face-colored plane graph with two types of vertices: vertices of degree 2 are the vertices of the underlying Barnette graph, vertices of degree 6 correspond to white resonant hexagons.

Finally, we suppress all vertices of degree 2. This operation may create loops, parallel edges, and even circular edges incident to no vertex, see Figure 1.22 for an illustration. Let G^M be the resulting black-and-white face-colored plane 6-regular pseudograph.

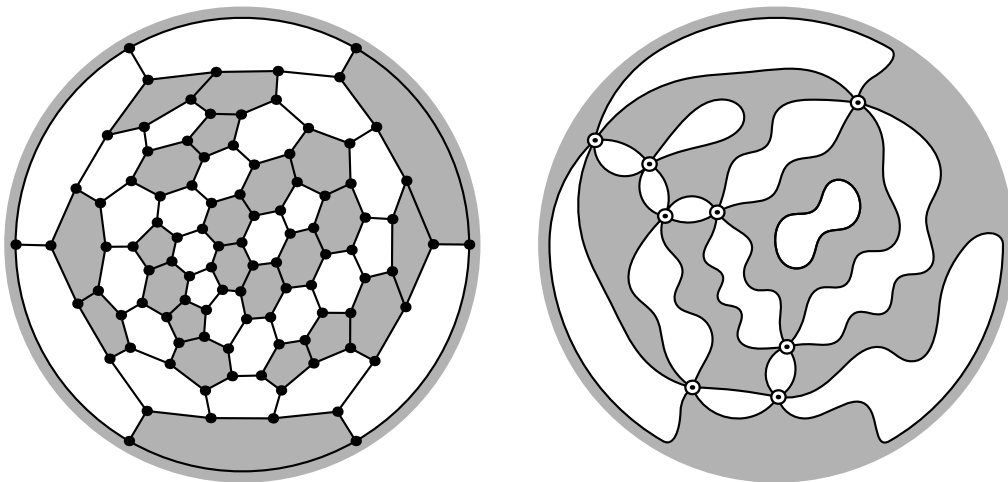


Figure 1.22: An example of a black-and-white face-colored 6-regular pseudograph (right) corresponding to a 2-factor of a Barnette graph (left).

A 2-factor F is called *odd* if it consists of an odd number of (disjoint) cycles; otherwise it is *even*. The same applies to the corresponding perfect matching.

A 2-factor F (as well as the corresponding perfect matching $M = E(G) \setminus F$) is called *simple* if G^M has no circular edges and $G^M \cong H^{\parallel}$ for some cubic planar graph H . If this is the case, H is called the *residual graph*.

Lemma 9 *Let F be a simple 2-factor of a Barnette graph G . Let n be the number of vertices of the corresponding residual graph H . If F is odd, then $n = 4k + 2$ for some $k \geq 1$; otherwise $n = 4k$ for some $k \geq 1$.*

Proof. The number of vertices of a residual graph is always even, since it is a cubic graph. Moreover, the number of cycles in F , say c , is equal to the number of faces

of the residual graph. By Euler's formula,

$$c = 2 + |E(H)| - |V(H)| = 2 + \frac{3n}{2} - n = \frac{n+4}{2},$$

so if $c = 2k + 1$, then $n = 2c - 4 = 4k - 2$; if $c = 2k$, then $n = 4k - 4$. \square

We will make use of the following classical result:

Theorem 9 (Payan and Sakarovitch [60]) *Let H be a cubic graph on $n = 4k + 2$ vertices ($k \geq 1$). If H is cyclically 4-edge-connected, then $V(H)$ admits a partition into two sets, say B and W , such that $H[B]$ is a stable set and $H[W]$ is a tree.*

Observe (by double-counting white-white and black-white edges) that the divisibility condition is a necessary condition for such a partition to exist. That's why we will only be interested in odd 2-factors.

Lemma 10 *Let G be a Barnette graph and let M be an odd simple perfect matching of G . If the residual graph is a cyclically 4-edge-connected, then G is Hamiltonian.*

Proof. Let H be a cyclically 4-edge-connected cubic planar graph on $4k + 2$ vertices ($k \geq 1$) such that $G^M = H^\parallel$. Let $F = E(G) \setminus M$. Recall that vertices of H correspond to white resonant hexagons in G with respect to a fixed canonical face-coloring of F . Let (B, W) be a partition of $V(H)$ into an induced (black) stable set B and an induced (white) tree W given by Theorem 9.

We transform the 2-factor F and the black-and-white face-coloring of G in the following way: For each resonant hexagon h corresponding to a black vertex b of H , replace the three edges from F incident to h in G by the other three edges; recolor the hexagon h black. Since B induces a stable set in H , this operation can be carried out independently for all black vertices of H at once. For each such vertex, the number of edges from F incident to any vertex of G remains unchanged, therefore, F becomes a 2-factor of G , say F' .

We claim that it consists of a single cycle. To prove that, it suffices to observe that the graph $(V(G), F')$ has a single white face (as $H[W]$ is connected) and a single black face (as $H[W]$ is acyclic). See Figure 1.23 for an illustration. \square

It remains to prove that such a situation occurs for at least one perfect matching for any Barnette graph not known to be Hamiltonian yet.

Theorem 10 *Let G be a Barnette graph on at least 318 vertices. Then there exists an odd simple perfect matching M of G such that the residual graph H is cyclically 4-edge-connected, unless G is a nanotube of type $(4, 0)$, $(5, 0)$, $(4, 1)$, $(5, 1)$, $(3, 2)$, $(4, 2)$, $(3, 3)$, or $(4, 3)$.*

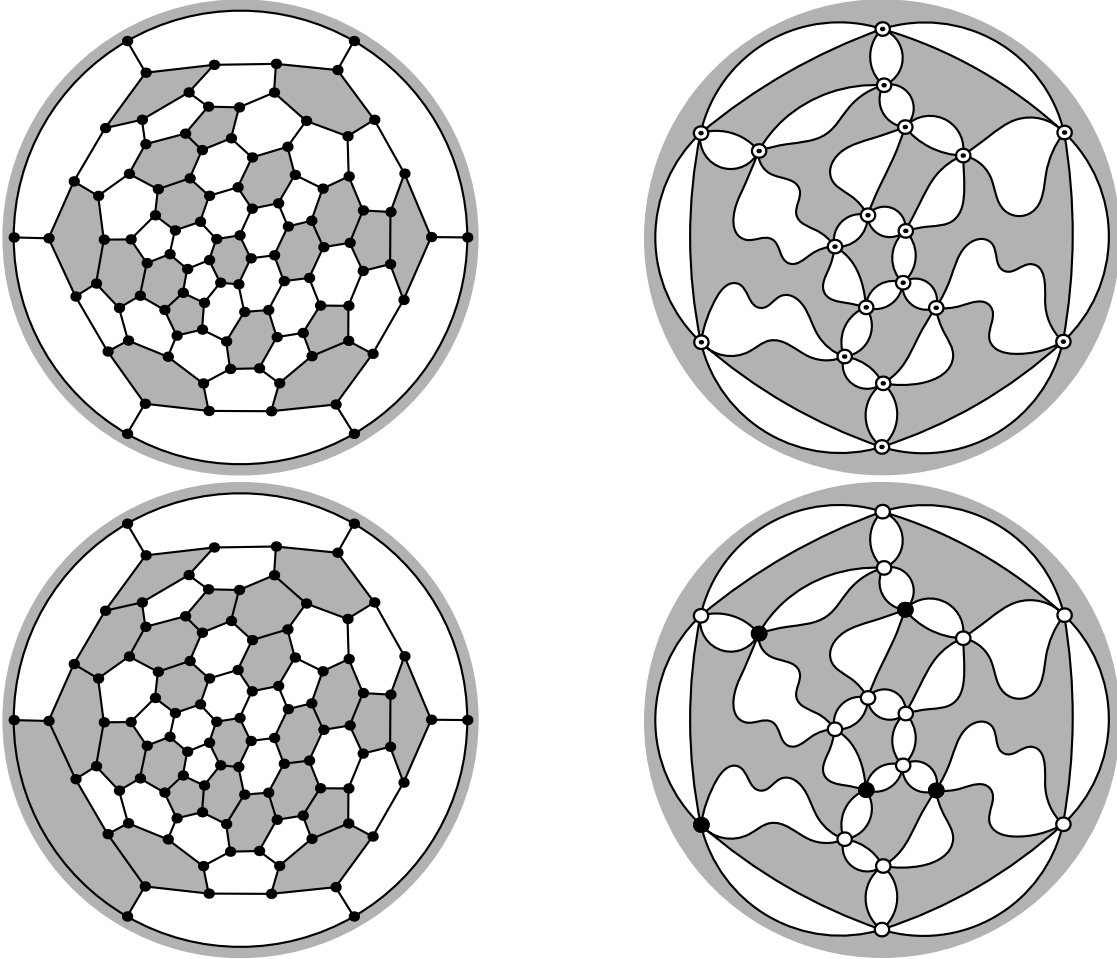


Figure 1.23: Clockwise, starting from upper left: An example of a simple 2-factor F of a Barnette graph G ; the corresponding planar 6-regular pseudograph G^M which is a double of a cyclically 4-edge connected cubic graph H ; A decomposition of H into a (black) stable set and a (white) induced tree; the corresponding Hamilton cycle in G .

In the rest of the chapter, we prove Theorem 10. We describe the general approach in Subsection 1.5.2, and we specify the computer-assisted part in Subsection 1.5.3.

We claim (without proof) that in order to prove Theorem 10 it suffices to consider a simple odd 2-factor maximizing the number of white resonant hexagons.

1.5.1.6 Generalized 2-factors

We will call a 2*-factor of a Barnette graph G any spanning subgraph F of G such that each component of F is a connected regular graph of degree 1 or 2 – an isolated edge or a cycle. For a 2*-factor F of a Barnette graph G , let $F^{(0)}$ be the set of isolated edges of F ; let $G^{(2)}$ be a plane graph obtained from G by replacing each edge of $F^{(0)}$ by a 2-gon; let $F^{(2)}$ be the set of edges of $G^{(2)}$ corresponding to those from F . Then $F^{(2)}$ is a 2-factor of $G^{(2)}$ in the common (strict) sense.

Given a 2*-factor F of a Barnette graph G , there are two canonical black-and-white face-colorings of $G^{(2)}$ (complementary to each other) with the following property: an edge e of $G^{(2)}$ is incident to a white and a black face if and only if e belongs to $F^{(2)}$.

A 2*-factor F of a Barnette graph G is called *quite good* if for each of the two canonical black-and-white face-colorings of $G^{(2)}$ induced by $F^{(2)}$ the 2-gons corresponding to the edges of $F^{(0)}$ have all the same color. Given a quite good 2*-factor of a Barnette graph G , we will always assume that a canonical coloring of $G^{(2)}$ such that all the 2-gons of $G^{(2)}$ are black is given along.

A quite good 2*-factor F of a Barnette graph G is called *good* if, after having fixed a planar embedding of G such that the outer face is a white one, no cycle of F is inside another.

Observe that given a good 2*-factor F of G , for any planar embedding of G with a white outer face, the set of faces inside a fixed cycle C of F is always the same and these faces correspond to a sub-tree of the dual graph G^* (empty if C is a 2-cycle).

Lemma 11 *Let F be a good 2*-factor of a Barnette graph G . Let f be the number of all the faces of G , let q_k be the number of non-resonant white faces of size k in G ($k = 4, 5, 6$), and let c be the number of components of F . Then q_5 is even, moreover, $f + q_4 + q_5/2 + c \equiv 0 \pmod{2}$.*

Proof. Let n be the number of vertices of G , let f_k be the number of all the faces of size k in G , let x_k be the number of black faces of size k in G . Euler's formula yields $n = 8 + f_5 + 2f_6$. If the inside of a cycle contains $c_4 \geq 0$ quadrangles, $c_5 \geq 0$ pentagons, and $c_6 \geq 0$ hexagons, then its length is $2 + 2c_4 + 3c_5 + 4c_6$.

Clearly, each vertex is covered by exactly one cycle, thus we have

$$8 + f_5 + 2f_6 = n = 2c + 2x_4 + 3x_5 + 4x_6 = 2c + 2(f_4 - q_4) + 3(f_5 - q_5) + 4x_6,$$

since only hexagons can be resonant, and thus $f_k = x_k + q_k$ for $k = 4, 5$. Therefore,

$$8 + q_5 + 2f_6 = 2c + 2(f_4 - q_4) + 2(f_5 - q_5) + 4x_6,$$

so q_5 is even. By dividing by two and rearranging the terms we obtain

$$4 + f_4 + f_5 + f_6 + q_4 + q_5/2 + c = 2c + 2f_4 + 2f_5 - q_5 + 2x_6,$$

the claim immediately follows. \square

Let F be a good 2^* -factor in a Barnette graph G . Let us consider the structure of the graph $G^{(2)}$. We introduce an auxiliary graph $\Gamma = \Gamma_G(F)$, defined in the following way: $V(\Gamma)$ is the set of the white non-resonant faces of G (as of $G^{(2)}$). The edges of Γ are defined in the next two paragraphs.

Let C be the facial cycle of a (black) 2-gon f_0 in $G^{(2)}$. Let f_0 be incident to vertices u and v and adjacent to two (white) faces f and f' . Then each of u and v is incident to one more face (which has to be white), say f_u and f_v , respectively. Since f_0 only shares a vertex with f_u and with f_v (but not an edge), the faces f and f' are two consecutive white neighbors of f_u (f_v). Therefore, the faces f_u and f_v cannot be resonant. We add the edge $f_u f_v$ to $E(\Gamma)$; we call this type of edge of Γ *white*.

Let C be a cycle of F (and of $F^{(2)}$) which is not a facial cycle of a face of $G^{(2)}$. It means that C is a boundary of a union of at least two faces of G . We consider every pair of adjacent faces inside C . Let f and f' be such a pair of faces. Let u and v be the endvertices of the edge incident to both f and f' . Then each of u and v is incident to a third face (which has to be white), say f_u and f_v , respectively. The faces f and f' are two consecutive black neighbors of f_u (f_v) in G . Therefore, the faces f_u and f_v cannot be resonant. We add the edge $f_u f_v$ to $E(\Gamma)$; we call this type of edge of Γ *black*.

Observe that for each edge of Γ , its endvertices are two faces of G at mutual position $(1, 1)$. Each edge of Γ covers two vertices of G and these pairs of vertices are pairwise disjoint. Therefore, Γ is a planar graph.

Let f be a white pentagon of $G^{(2)}$. It cannot be resonant, so f is a vertex of Γ . Let f_1, \dots, f_5 be the faces adjacent to f (sharing an edge with f) in $G^{(2)}$. (Observe that some f_i can be a 2-face: if it is the case, then there is another face f'_i adjacent to f in G , and adjacent to f_i in $G^{(2)}$.) Since the size of f is odd, the number of pairs (f_i, f_{i+1}) (with $f_6 = f_1$) of the same color (both black or both white) has to be odd.

If both f_i and f_{i+1} are black, then none of them can be a 2-face, and thus there is a black edge incident to f in Γ . If both f_i and f_{i+1} are white, then again none of them can be a 2-face, and the vertex incident to f , f_i , and f_{i+1} is (in $G^{(2)}$) covered by a 2-cycle adjacent both to f_i and f_{i+1} , and thus there is a white edge incident to f in Γ . Altogether, f is a vertex of odd degree in Γ .

Similarly, for each non-resonant white hexagon f , there is an even number of pairs of consecutive adjacent faces of the same color, hence f is a vertex of non-zero even degree in Γ .

A white quadrangle f is always considered non-resonant. Its degree in Γ is also always even, however, it can be equal to 0 if the neighboring faces are colored alternatively black and white.

As a result of these local observations, the graph Γ can always be edge-decomposed into a set of paths with endvertices at the white pentagons of G , a set of cycles, and, eventually, a set of isolated vertices (corresponding to white quadrangles). The number of paths in the decomposition is equal to $q_5/2$, where q_5 is the number of white pentagons.

1.5.2 Proof of Theorem 10: Finding a 2-factor

In this section we explain the general procedure in the case when the small faces of G are far from each other. We will deal with the case when some small faces of G are close to each other in Section 1.5.3.

1.5.2.1 Phase 1: Cut the graph and fix a coloring

Let G be a Barnette graph, let T be a Hamiltonian triangulation capturing the mutual position of the small faces of G , whose existence is given by Lemma 8. Let C_T be a Hamiltonian cycle in T such that the sum of the lengths of the corresponding Goldberg vectors is minimal. Then there exists a cycle C^* in G^* including all the small vertices of G^* in the same order as the corresponding vertices on C_T .

A cycle in G^* corresponds to an edge-cut in G . We cut the graph G along C^* . We obtain two graphs, say G_1 and G_2 , containing only hexagons as internal faces, and with semi-edges and partial faces on the boundary.

Both G_1 and G_2 are subgraphs of the hexagonal grid, hence, there is a canonical face coloring using three colors for each of them. We will use colors 1, 2, 3 for one and colors A , B , C for the other. We color the partial faces in both graphs too.

We choose one color in each graph, say 1 and A (there are 9 color combinations in total), and recolor black all the faces of G_1 and G_2 colored 1 or A ; we color white

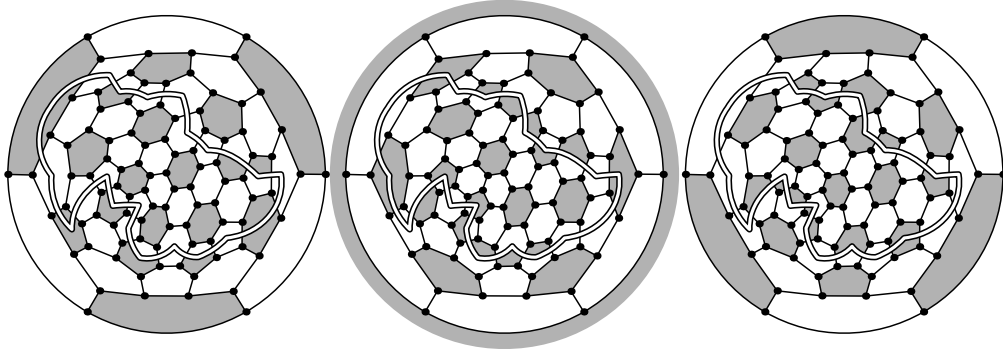


Figure 1.24: Three of the nine black-and-white colorings of the graph in Figure 1.3, (combinations of three different colorings of G_1 and three different colorings of G_2) corresponding to the given order of pentagons.

the other faces. (Later we will inspect all the nine colorings.) This gives a black-and-white face-coloring ϕ_i inducing a 2-factor F_i in G_i , $i = 1, 2$. The 2-factor F_i is composed of cycles and paths. Each cycle is the boundary of an internal face of G_i , hence a 6-cycle; each path has both ends at partial edges leaving G_i and it is a boundary of a partial face of G_i .

Observe that for any choice of a color in G_i ($i = 1, 2$), the edges incident to one face of the other two colors each form a matching M_i such that $G_i^{M_i} = H_i^{\parallel}$, where H_i is the graph whose vertices are the centers of the faces of the other two colors (including the partial faces).

We merge the two black-and-white face-colorings ϕ_1 and ϕ_2 of G_1 and G_2 , respectively, into an intermediate black-and-white (multi-)face-coloring ϕ^{\sim} of G in the natural way: A face not corresponding to a vertex of C^* inherits a color from either G_1 or G_2 ; A face which is cut by the cycle C^* is divided into two partial faces, one inheriting a color from G_1 and the other from G_2 , see Figure 1.24 for an illustration.

Active and inactive segments

The cycle C^* can always be decomposed into a sequence of $\ell \leq 12$ subpaths P_1^*, \dots, P_ℓ^* joining consecutive pairs of small vertices. Let us call these subpaths *segments*.

We may suppose that a segment only contains hexagons with a non-empty intersection with the straight line joining the end-vertices of the segment.

For each segment P_i^* , the two face-colorings of G_1 and G_2 meet along P_i^* , and there is a unique canonical bijection $\varphi_i : \{1, 2, 3\} \rightarrow \{A, B, C\}$ between the two sets of colors.

If $\varphi_i(1) = A$ then the two black-and-white colorings coincide along P_i , we say that the segment P_i is *inactive*; otherwise it is *active*. Out of the nine colorings, each segment is active in precisely six of them. For example, the segments p_6p_7 and $p_{12}p_1$ are inactive in all the three colorings depicted in Figure 1.24, the segment p_4p_5 is active in all the three colorings, whereas the segment p_9p_{10} is inactive in the first coloring and active in the other two.

When switching from P_i^* to P_{i+1}^* , if the i -th small face is a quadrangle, we have $\varphi_i = \varphi_{i+1}$. If the i -th small face is a pentagon, the difference $\varphi_{i+1} \circ \varphi_i^{-1}$ is a permutation of the colors $\{A, B, C\}$ such that the color of the pentagon is stable and the two other colors are switched – a transposition. See Figure 1.25 for an illustration.

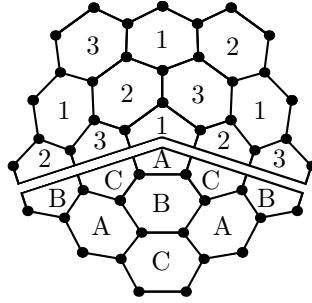


Figure 1.25: A pentagon always causes a single switch of colors – the two colors different from its color are switched.

Let p_i be a pentagonal face of G such that the segments P_{i-1}^* and P_i^* meet at p_i . Then exactly one of the following happens:

- (i) if $\varphi_{i-1}(1) = \varphi_i(1) = A$, then both P_{i-1}^* and P_i^* are inactive, p_i generates a switch of B and C , thus it is colored 1 and A and it is black in both subgraphs;
- (ii.a) if $\varphi_{i-1}(1) = A$ and $\varphi_i(1) \neq A$, then P_{i-1}^* is inactive and P_i^* is active, p_i generates a switch of A and $\varphi_i(1)$, thus it is colored neither A nor 1, so it is white in both subgraphs;
- (ii.b) if $\varphi_{i-1}(1) \neq A$ and $\varphi_i(1) = A$, then P_{i-1}^* is active and P_i^* is inactive, p_i generates a switch of A and $\varphi_{i-1}(1)$, thus it is colored neither A nor 1, so it is white in both subgraphs;
- (iii.a) if $\varphi_{i-1}(1) = \varphi_i(1) \neq A$, then both P_{i-1}^* and P_i^* are active, p_i generates a switch of A and the color in $\{B, C\} \setminus \{\varphi_{i-1}(1)\}$, thus it is colored $\varphi_{i-1}(1)$, so it is black in G_1 and white in G_2 ;

- (iii.b) if $\{\varphi_{i-1}(1), \varphi_i(1)\} = \{B, C\}$, then both P_{i-1}^* and P_i^* are active, p_i generates a switch of B and C , thus it is colored A , so it is white in G_1 and black in G_2 .

In order to transform ϕ^\sim into a black-and-white face-coloring ϕ of G corresponding to a good 2-factor of G , we reroute slightly the cut C^* in a way described in the following subsection.

1.5.2.2 Phase 2: Approximate the cut by Γ -paths

Let P_i^* be an active segment, let $\varphi_i(1) = B$. Suppose without loss of generality that $\varphi_i^{-1}(A) = 2$. Then all the faces of P_i colored A (and 2) or 1 (and B) are partially black and partially white; both parts of each face of P_i^* colored C and 3 are white.

In the subgraph of G corresponding to the two triangles of T incident to the edge of T corresponding to P_i^* , the centers of faces colored C and/or 3 form a subgraph H_i^+ of a regular hexagonal grid, the unit edge of which corresponds to a $(1, 1)$ -vector in G .

We approximate the dual path P_i^* by a sequence Q_i of faces colored C and/or 3, each consecutive pair of faces in a mutual position $(1, 1)$, in the following way:

The first (the last) face of Q_i is the pentagon p_i (p_{i+1}) if and only if the segment P_{i-1}^* (P_{i+1}^*) is inactive; otherwise the first (the last) face of Q_i is a hexagon adjacent to p_i (p_{i+1}) and it is the last (the first) face of Q_{i-1} (Q_{i+1} , respectively).

Let the first and the last face of Q_i be f_0 and f'_0 . Then Q_i is defined as the sequence of faces corresponding to such shortest path joining f_0^* and f'_0 in H_i^+ that it approximates the straight line segment joining p_i and p_{i+1} the best. (This is analogous to tracing a diagonal in a pixelized image of rectangular shape). If a face cut by P_i^* is colored C and 3, then it becomes part of Q_i ; if P_i^* passes through a sequence of several A - (and 2-) and B - (and 1-) faces, then some C - or 3-faces adjacent to these faces are chosen into Q_i .

The segment P_i^* is approximated by a series of H_i^+ -edges joining consecutive faces of the sequence Q_i . We approximate the decomposition of G into G_1 and G_2 accordingly. Each face previously cut by P_i^* and having been colored partially black and partially white thus now belongs to just one of the two colorings, and therefore has just one color. The only faces cut by the new cut (and so having a color (white) in both colorings) are the faces of Q_i .

Let f be a white (C - and 3-colored) hexagonal face of Q_i . Then the set of its neighbors is partitioned into a cyclic sub-sequence of A - and B -faces colored alternatively black and white, and another cyclic sub-sequence of 1- and 2-faces colored alternatively black and white, with the coloring being the opposite of the first one. Therefore, there are exactly two pairs (not necessarily disjoint) of adjacent

neighboring faces of the same color: each pair is either a black A -face adjacent to a black 1-face, or a white B -face adjacent to a white 2-face. Therefore, f is a white non-resonant hexagon, corresponding to a vertex of degree 2 in the auxiliary graph Γ being constructed – we will call f a Γ -face.

Let f and f' be two consecutive Γ -faces of Q_i . If the two faces adjacent both to f and f' are black, then the paths belonging to the 2-factors F_1 and F_2 in G_1 and G_2 are merged into a single cycle. If the two faces adjacent both to f and f' are white, then a new 2-cycle of the 2^* -factor is created. In the first case, the Γ -edge ff' is black, in the second case it is a white Γ -edge. In both cases ff' is a Γ -edge for every pair of consecutive Γ -faces of Q_i , therefore, Q_i can be considered as a Γ -path.

Two consecutive Γ -edges of Q_i of the same color always form a 180° angle, otherwise it could be possible to simplify Q_i by removing a face from Q_i . (A shortest path joining two vertices in a hexagonal grid never makes a sharp ($\pm 60^\circ$) turn.) Similarly, two consecutive edges of Q_i of different colors always form an angle of $\pm 120^\circ$.

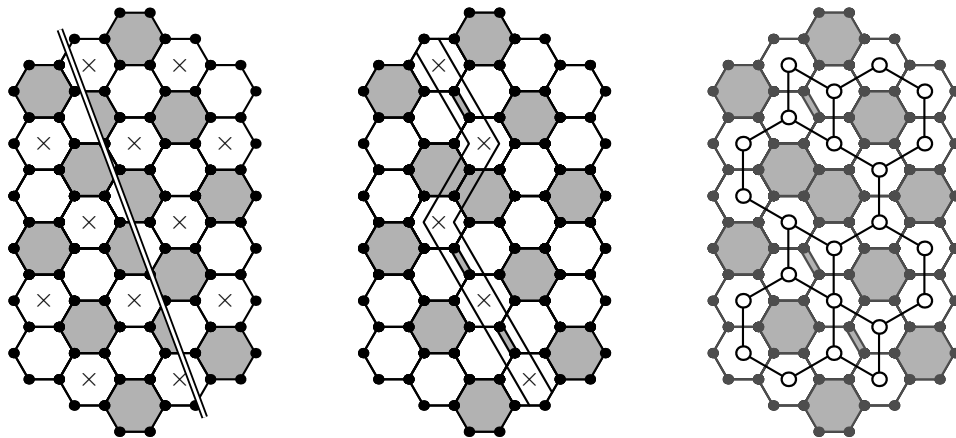


Figure 1.26: Two hexagonal patterns meeting along an active segment P_i^* . Faces labeled \times are the faces colored with the corresponding colors which are chosen to be white in both colorings (left). A Γ -path (a sequence of Γ -faces) Q_i that approximates P_i^* together with the corresponding 2^* -factor (middle). The corresponding auxiliary graph H^\sim (right).

The resulting structure of ϕ^\sim along P_i^* is the following: All vertices are covered by cycles of length 6 (single faces), 10 (two adjacent black hexagons, both incident to a black Γ -edge), or 2 (white Γ -edges). A Γ -path Q_i separates the two subgraphs of regular coloring.

White non-resonant hexagons where two consecutive sequences Q_{i-1} and Q_i meet

are the only occasion where two Γ -edges of the same color might form a 60° angle – if only they are both incident to the same pentagon.

Let us explicit the structure of $H^\sim = H_1 \cup H_2$ and of Γ now: Vertices of H^\sim are all the vertices corresponding to faces of G white in G_1 or in G_2 ; each vertex of Γ where two black edges meet corresponds to a 2-vertex in H^\sim (the corresponding face of G is a non-resonant white hexagon adjacent to four black faces belonging to two different components of the 2^* -factor); each vertex of Γ where two white edges meet corresponds to a 4-vertex in H^\sim (the corresponding face of G is incident to four different compents of the 2^* -factor, including two 2-cycles); each vertex of Γ where a black and a white edge meet at a 120° angle corresponds to a 3-vertex in H^\sim (the corresponding face of G being incident to one, two, and three vertices belonging to three different components of the 2^* -factor: a 2-cycle, a 6-cycle and a 10-cycle, respectively).

If there are q_5 white pentagons, then Γ is composed of $q_5/2$ paths. A white quadrangle is either an isolated vertex of Γ (if both incident segments are inactive) or it is an internal vertex of a path (otherwise).

1.5.2.3 Phase 3: Change the parity of the 2^* -factor

It follows from Lemma 11 that whenever we want to transform an even 2^* -factor into an odd one, it suffices either to increase or decrease the number of black quadrangles by 1 (or, equivalently, increase or decrease the number of non-resonant white quadrangles by 1), or to increase or decrease the number of black pentagons (and white pentagons) by 2. In other words, it suffices either to change the number of isolated vertices in Γ by 1 or change the number of Γ -paths by 1.

Changing the parity using a quadrangle

Let q be a quadrangular face of G . For three of the nine colorings of G_1 and G_2 , both segments incident to q are inactive; moreover, for two out of the three q is a white face. In Phase 1, we choose one of these two.

If the good 2^* -factor obtained in Phase 2 is even, it can be transformed into an odd one by recoloring q black. This way an isolated vertex of Γ is transformed into a cycle of length 2, see Figure 1.27 for an illustration.

Changing the parity using two pentagons

From this point on we may assume that G has no quadrangular faces – it is a (fullerene) graph having 12 pentagonal faces.

Suppose first that some pair of consecutive pentagons p_i and p_{i+1} (consecutive along the cut C) are in the mutual position (c_1, c_2) , $c_1 \geq c_2 \geq 0$, with $3 \mid (c_1 - c_2)$.

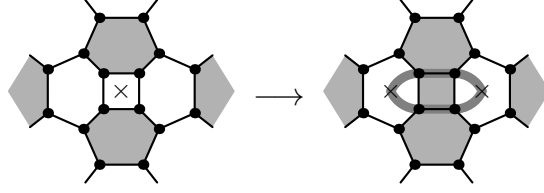


Figure 1.27: We can use a white quadrangle to change the parity of a 2^* -factor. The times sign marks non-resonant faces – vertices of Γ ; edges of Γ are drawn using a thick grey line.

Then in the coloring of G_1 with colors 1, 2, 3 (and of G_2 with A, B, C) the partial faces corresponding to the pentagons p_i and p_{i+1} have the same color. Therefore, for two of the nine colorings the segment P_i joining p_i and p_{i+1} is active whereas the neighboring segments P_{i-1} and P_{i+1} are inactive.

For both such colorings, after Phase 2 there is a Γ -path with endvertices at p_i and p_{i+1} , and the vertex set of this path can be chosen to be the same in both colorings. Each Γ -edge white in one coloring is black in the other and vice versa. Among the two colorings, we may fix the one where the number of white Γ -edges is maximised.

We transform the Γ -path into a Γ -cycle, increasing the number of black pentagons by 2, in the following way: For each black Γ -edge, we recolor both black hexagons forming a black 10-cycle white; then we recolor all faces corresponding to the vertices of the Γ -path black, including the first and the last one (p_i and p_{i+1}). We will denote this operation O_1 . See Figure 1.28 for an illustration.

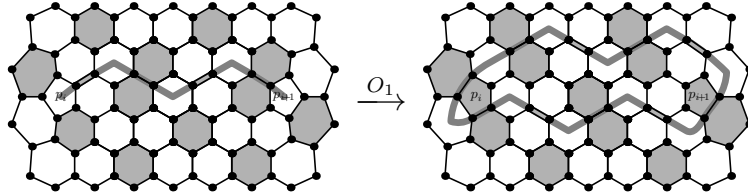


Figure 1.28: Operation O_1 : The parity of a 2^* -factor can be changed by modifying a Γ -path joining two consecutive pentagons into a Γ -cycle.

From this point on we may assume that there is no pair of consecutive pentagons with the same color in G_1 (or in G_2). Then for every pair of consecutive pentagons the nine colorings look like depicted in Figure 1.29.

Let ϕ_i^j be the angle between the two segments meeting at pentagon p_i in G_j , $j = 1, 2$. Clearly, $\phi_i^1 + \phi_i^2 = 300^\circ$. When following the segments composing the cut in an ascending order, say G_1 is to the left and G_2 to the right. If $\phi_i^1 > 150^\circ > \phi_i^2$, then

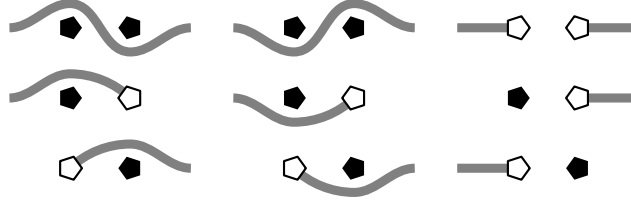


Figure 1.29: A schematic drawing of the position of the Γ -paths in the neighborhood of two consecutive pentagons of different colors.

there is a right turn at p_i when switching from P_{i-1} to P_i . If $\phi_i^1 < 150^\circ < \phi_i^2$, then there is a left turn at p_i when switching from P_{i-1} to P_i . The value $\phi_i^1 = \phi_i^2 = 150^\circ$ means that the segment P_i continues in the same direction as P_{i-1} .

Let $\phi_i = \phi_i^1 - \phi_i^2$ for $i = 1, \dots, 12$. It is easy to see that $\sum_{i=1}^{12} \phi_i = 0$, since $\sum_{i=1}^{12} \phi_i^1 = \sum_{i=1}^{12} \phi_i^2 = 1800^\circ$. Therefore, there exist i such that $\phi_i \cdot \phi_{i+1} \leq 0$ (indices modulo 12). We fix i such that $\phi_i \cdot \phi_{i+1} \leq 0$ and the difference $|\phi_i - \phi_{i+1}|$ is as big as possible.

Without loss of generality we may assume that $\phi_i \geq 0$ and $\phi_{i+1} \leq 0$. In other words, there is a right turn at p_i followed by a left turn at p_{i+1} . There are two colorings in which all the segments P_{i-1} , P_i , and P_{i+1} are active; among them we choose the one where p_i is black in G_2 and p_{i+1} is black in G_1 .

We can now change the parity of the 2^* -factor (if needed) by decreasing the number of black pentagons by 2 in the following way: For each black Γ -edge of Q_i , we recolor both black hexagons forming a black 10-cycle white; then we recolor all faces corresponding to the vertices of the Γ -subpath Q_i black, including the first and the last one (those adjacent to p_i and p_{i+1} , respectively); we recolor p_i and p_{i+1} white. As the last step, we simplify unnecessary 60° turns. We will denote this operation O_2 . See Figures 1.30, 1.31 and 1.32 for an illustration.

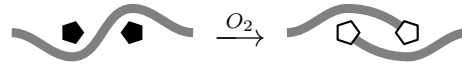


Figure 1.30: A schematic drawing of the operation O_2 .

1.5.2.4 Phase 4: Transform a good odd 2^* -factor into a simple 2-factor

It suffices now, as the last phase, to transform a good odd 2^* -factor into a simple (odd) 2-factor. We do it in the following way:

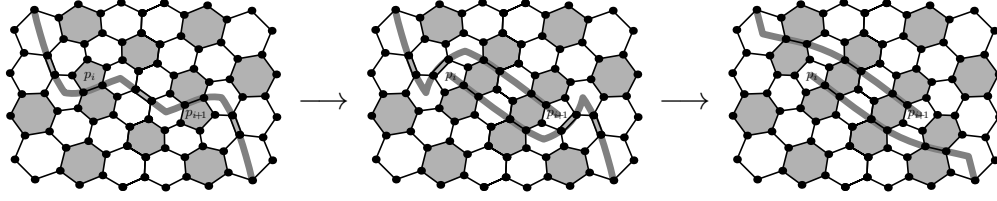


Figure 1.31: Operation O_2 : The parity of a 2^* -factor can be changed by transforming a Γ -path passing by two consecutive black pentagons p_i and p_{i+1} into two different Γ -paths starting at p_i and p_{i+1} , respectively. The modification from the middle to the right corresponds to removing unnecessary sharp turns.

In a good 2^* -factor, each 2-cycle corresponds to a white Γ -edge ff' , incident to two white resonant hexagons h_1 and h_2 (one in each of G_1 and G_2). We can choose either h_1 or h_2 , say h_i , and recolor it black: By doing this, the 2-cycle is merged with two other cycles in G_i ; the other face f_0 incident to both cycles being merged loses its resonantness, it becomes another Γ -face inserted to the Γ -path between f and f' , joint now to f and f' by two black Γ -edges forming a 60° angle and replacing the original white Γ -edge. In H^\sim , a vertex of degree 3 is removed, and thus the degree of three other vertices is decreased by 1: one of them corresponds to f_0 , the other two correspond to f and f' .

Observe that this operation decreases the number of components of the factor by 2, therefore, starting with an odd factor we can only obtain odd factors.

We make a decision for all white Γ -edges sequentially according to their order along Q_i , according to the following rules: If a white Γ -edge e_j forms a 180° angle with e_{j-1} (which has to have been white in this case) and that we have decided to recolor black a hexagon in G_i , $i = 1, 2$, incident to e_{j-1} , then we decide to recolor black a hexagon in G_{3-i} incident to e_j . If a white Γ -edge e_j forms a 120° angle with a black e_{j-1} , we decide to recolor black a hexagon incident to e_j in such a way that one of the new black Γ -edges forms a 180° angle with e_{j-1} .

The resulting structure in G is the following: All the Γ -paths and Γ -cycles are formed of black Γ -edges only. Each vertex of Γ of degree 1 or 2 corresponds to a 2-vertex in H^\sim .

Finally, to obtain H , we suppress all the 2-vertices in H^\sim ; for each Γ -edge we merge the incident partial faces of H^\sim . In other words, every white face is either resonant (and hence corresponds to a 3-vertex of H), or corresponds to a vertex of Γ .

To describe the structure of H , we introduce the following notation: A vertex of Γ is called *direct* if it corresponds to a pentagon or if the two incident (black) Γ -edges

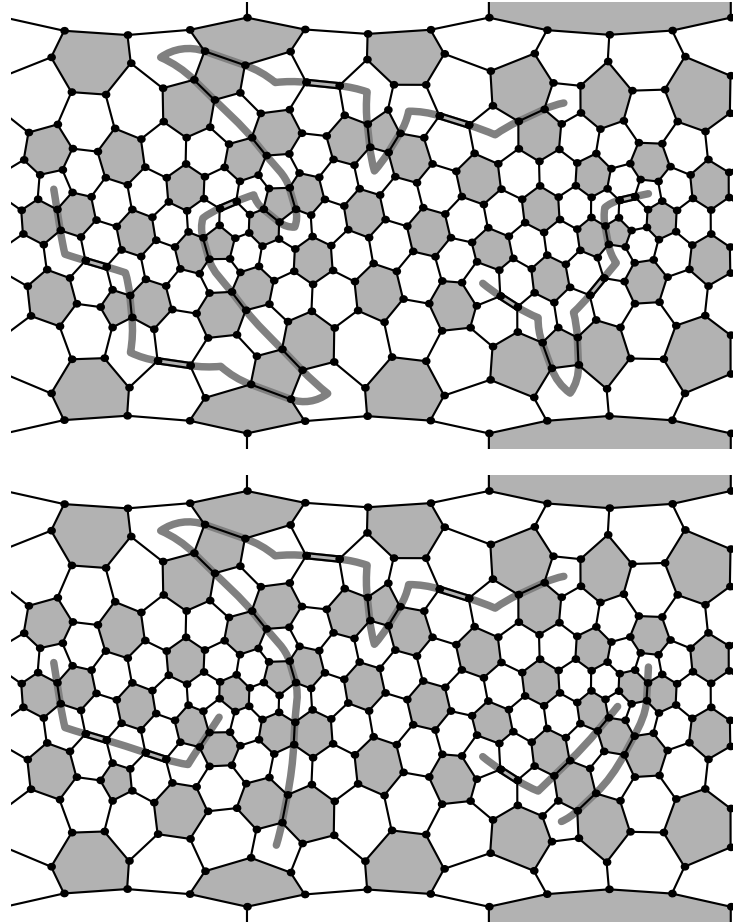


Figure 1.32: The good 2^* -factor induced by one of the nine possible black-and-white face-colorings of the graph in Figure 1.21 (top). It is already an odd 2^* -factor; there are several pairs of pentagons for which the operation O_2 is admissible. Another good 2^* -factor of the same graph obtained by two applications of O_2 at two different pairs of pentagons marked with the same sign (bottom).

form a 180° degree; otherwise it is called *sharp*.

We claim that there cannot be three consecutive sharp Γ -vertices along any Q_i : Suppose some Q_i contains a subpath $f_0 f_1 f_2 f_3 f_4$ with all of f_1 , f_2 , and f_3 sharp and f_0 direct (see Figure 1.33). If $f_1 f_3$ had been a white Γ -edge after Phase 2, we would not have decided to choose f_2 . Therefore, f_2 was a Γ -vertex already after Phase 2, which means that $f_0 f_2$ was a white Γ -edge after Phase 2. If $f_2 f_4$ was also a white Γ -edge after Phase 2, we would have decided one of them in the other way. Therefore, f_3 was a Γ -vertex already after Phase 2, but not f_4 , which means that f_4 is sharp. As f_1 must have been chosen because of the other Γ -edge incident to f_0 , f_4 should never have been chosen, a contradiction.

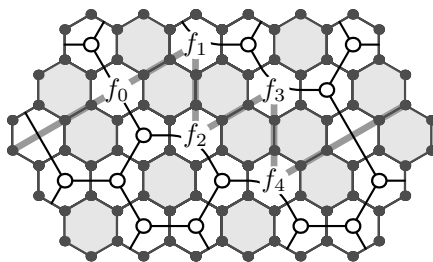


Figure 1.33: An illustration of a sequence of several sharp Γ -vertices along a Γ -path.

A (black) Γ -edge joining two direct Γ -vertices f and f' completes the boundary of two partial faces in H_1 and H_2 , each having three incident 3-vertices. After the suppression of 2-vertices in H^\sim , in H these two partial faces are merged into a hexagon.

The 60° angle at a sharp Γ -vertex f contains a partial face of H^\sim having one 3-vertex, which is to be merged with (at least) two other partial faces.

If both Γ -vertices adjacent to f in Γ are direct, then a face of size 7 is created in H by merging two partial faces each having three incident 3-vertices in H_i with a partial face having one incident 3-vertex in H_{3-i} . On the other hand, opposite to this one, there is a face of H^\sim whose size is decreased by 1 by the suppression of the 2-vertex f – a pentagonal face is created in H .

If one of the vertices adjacent to a sharp vertex in Γ is a sharp one, they are transformed into a face of size 8 and two pentagons in H . See Figures 1.34 and 1.35 for an illustration.

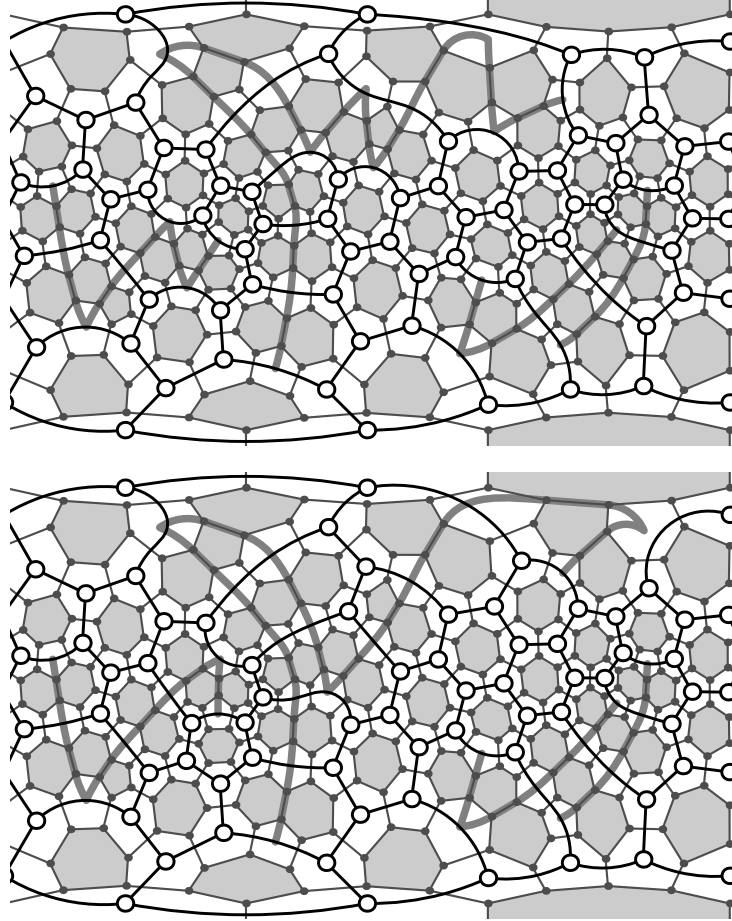


Figure 1.34: The simple 2-factor obtained by the general procedure from the good 2^* -factor in Figure 1.32, depicted together with the residual graph H and the auxilliary graph Γ (top). Another simple 2-factor obtained from the previous one by "flipping out" unnecessary zig-zags of sharp Γ -vertices (bottom). Observe that for the latter, the residual graph H has faces of size 5, 6, and 7 only and it is cyclically 5-edge-connected.

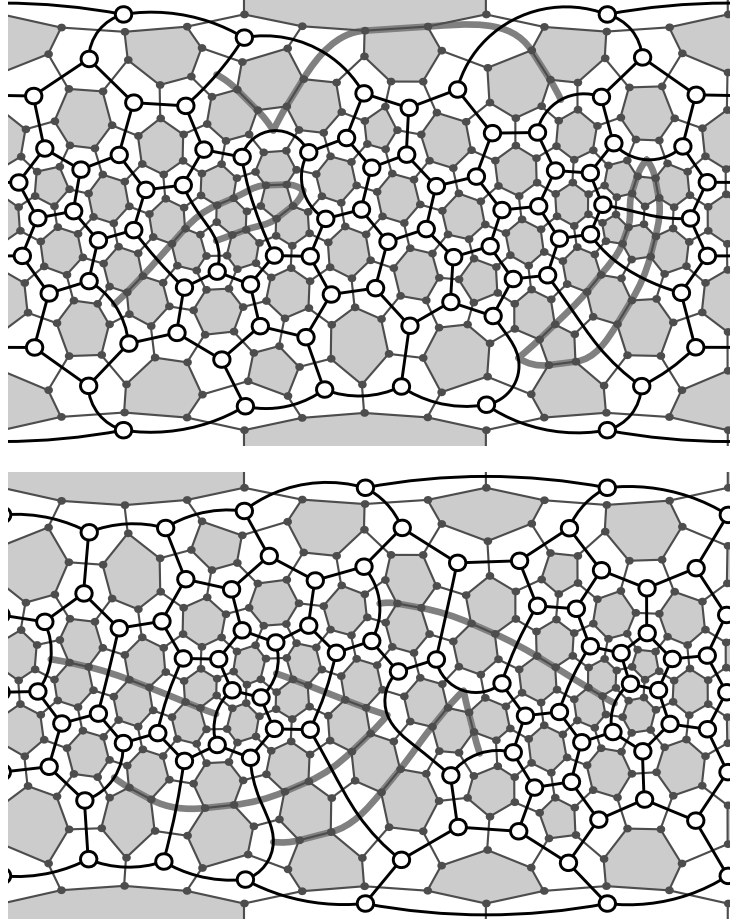


Figure 1.35: Two different simple odd 2-factors of the graph in Figure 1.21 with the largest number of vertices of the residual graph H (82) we were able to find.

1.5.3 The computer-assisted part: Checking the correctness of the algorithm in the neighborhood of small faces close to each other

Let G be a Barnette graph. Let $S(G)$ be the set of the *small* faces (faces of size 4 or 5) of G . Recall that $2f_4 + f_5 = 12$, where f_4 and f_5 are the numbers of quadrangles and pentagons in G , respectively.

1.5.3.1 Patches

A *patch* is a 2-connected subcubic plane graph P , having at most one face of size different from 4, 5 and 6, and such that all vertices of P of degree 2 are incident to this special face, often referred to as the outer face of the patch; moreover, P contains no pair of adjacent 4-faces. When a patch is depicted, there are additional pending half-edges at vertices of degree 2 towards the outer face.

The *curvature* of a patch P , denoted by $\mu(P)$, is equal to $2f_4(P) + f_5(P)$, where $f_4(P)$ and $f_5(P)$ are the numbers of quadrangles and pentagons in P (distinct from the outer face of P), respectively.

We denote $\partial(P)$ the *boundary* of a patch P – the facial cycle of the outer face of P ; we denote $\delta(P)$ the *perimeter* of a patch P , the number of 2-vertices in P .

The *boundary vector* $\sigma(P)$ of a patch P is a cyclic sequence of distances between consecutive 2-vertices on the boundary cycle of P . The length of $\sigma(P)$ is equal to $\delta(P)$ and its sum is equal to the length of $\partial(P)$. When expliciting elements of a cyclic sequence σ , we write x^k as a shortcut for k consecutive occurrences of a value x in σ .

Each vertex of $\partial(P)$ is either a 2-vertex or a 3-vertex in P . The proportion of 2-vertices along $\partial(P)$ is determined by the curvature of P , as is stated explicitly in the following lemma, which is a generalisation of an observation from [48] and can be derived directly from Euler's formula by the same double-counting arguments.

Lemma 12 *Let P be a patch of curvature μ . Then*

$$2\delta(P) - |\partial(P)| = 6 - \mu.$$

Observe that for patches of curvature (greater than, less than) six, the average value of $\sigma(P)$ is (greater than, less than, respectively) two.

Let $X = (x_1 x_2 \dots x_\ell)$ be a cyclic sequence of integers. Let $\mu(X) = 6 + \sum_{i=1}^{\ell} (x_i - 2)$ be the curvature of X . A cyclic sequence X of curvature six is called *convex* if

$1 \leq x_i \leq 3$ for all $i \leq \ell$ and if it contains neither 32^j3 nor 12^j1 for any $j \geq 0$. (However, the absence of one implies the absence of the other.) A cyclic sequence X of curvature $\mu \leq 5$ is called *convex* if it is composed of $6 - \mu$ convex segments. Here a *convex segment* is a sequence of the form $2^a 132^{b_1} 132^{b_2} \dots 132^{b_s} (13)^c 12^{d_1} 312^{d_2} \dots 312^{d_t} 31$ where $s, t, a, b_1, b_2, \dots, b_s, c, d_1, d_2, \dots, d_t$ are non-negative integers, and moreover, $b_1 \geq b_2 \geq \dots \geq b_s$ and $d_1 \leq d_2 \leq \dots \leq d_t$. A cyclic sequence $X = (x_1 x_2 \dots x_\ell)$ of curvature $\mu \geq 7$ is called *convex* if the sequence $Y = (2 - x_i)_{i=1}^\ell$ is convex.

A patch P is called *convex* if its boundary vector $\sigma(P)$ is convex.

Note that, in particular, a patch P of curvature $\mu \leq 5$ with a boundary vector of the form $(12^{k_1} 12^{k_2} \dots 12^{k_t})$ where $t = 6 - \mu$, $k_1, k_2, \dots, k_t \in \mathbb{N}_0$, is convex, and that $k_1 + k_2 + \dots + k_t = \delta(P) - t$.

We denote by $P^{i \leftarrow j}$ a patch obtained from P by adding a face of size j to P along the path corresponding to the i -th element of $\sigma(P)$, if such a patch exists, see Figure 1.36 for an illustration.

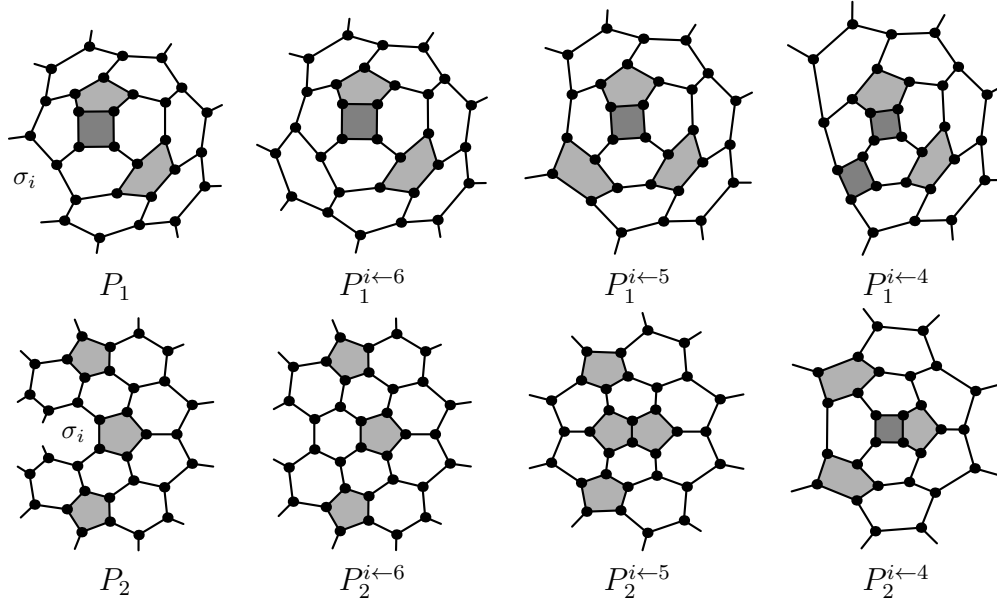


Figure 1.36: Two different examples of three different patches obtained from a given patch (on the left) by inserting a new face at the element σ_i of its boundary vector.

It may happen that while adding a new face to a patch, we have to identify some elements (vertices/edges/faces) of the patch, as in the second row of Figure 1.36. It may even happen that adding a new face of some desired size to a specific place of a patch is not possible, since the faces to be identified are not of the same size.

Patches in Barnette graphs

Let G be a Barnette graph. We say that a patch P is *contained* in G if there is a graph homomorphism $\varphi : P \rightarrow G$ such that all faces of the patch (except for the outer face) are also faces of G . We say that a patch P is *realizable* if it is contained in some Barnette graph. The homomorphism φ restricted to the set of vertices (edges, inner faces) of P does not have to be injective; if it is not, we say that P is *self-overlapping* in G .

Observe that a patch P of perimeter 0 is contained in a Barnette graph G if and only if $P = G$ and the outer face of P is a face of G . Similarly, a patch P of perimeter 2 is contained in a Barnette graph G if and only if $P = G \setminus e$ for some edge e of G and the outer face of P is the union of the two faces incident to e in G . Finally, since Barnette graphs are cyclically 4-edge-connected, a patch P of perimeter 3 is contained in a Barnette graph G if and only if $P = G \setminus v$ for some vertex v of G and the outer face of P is the union of the three faces incident to v in G . On the other hand, no patch of perimeter 1 can be realizable, since it would correspond to a cut-edge in a Barnette graph.

Some (but not all) realizable patches can be obtained in the following way: For any induced cycle C of a Barnette graph G , there are two distinct (but not disjoint) patches P and \bar{P} contained in G such that $\partial(P) = \partial(\bar{P}) = C$. It is easy to see that we have $\mu(P) + \mu(\bar{P}) = 12$ and that $\delta(P)$ is equal to the number of edges of the cut separating P from $G \setminus P$.

Moreover, as each vertex of C is either a 2-vertex in P or a 2-vertex in \bar{P} , $\delta(P) + \delta(\bar{P})$ is equal to the length of $\partial(P)$.

As a direct consequence of Lemma 12 we obtain the following observation.

Lemma 13 *Let C be an induced cycle in a Barnette graph and let P and \bar{P} be the two corresponding patches. Then*

$$\delta(P) - \delta(\bar{P}) = 6 - \mu(P) = 6 + \mu(\bar{P}).$$

However, there are patches contained in Barnette graphs which cannot be obtained this way: it is not always true that the facial cycle of the outer face of a patch corresponds to an induced cycle of the host Barnette graph – a patch can even be self-overlapping. There also exist patches that are not realizable at all, see Figure 1.37 for an example.

Lemma 14 *Let P be a realizable patch of perimeter at least 2. For every element σ_i of its boundary vector there exists $j \in \{4, 5, 6\}$ such that $P^{i \leftarrow j}$ is also a realizable patch, moreover, it is contained (at least) in the same Barnette graph as P .*

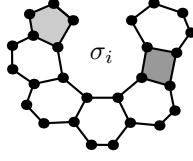


Figure 1.37: An example of a patch that is not realizable at all: Any attempt to insert a new face of size 4, 5 or 6 at σ_i leads always to a contradiction.

Proof. Let P be contained in a Barnette graph G . Each element of σ is a path contained in a facial cycle of some face of G of a certain size $j \in \{4, 5, 6\}$. Therefore, the face added to the patch corresponds to a face of G . \square

Note that in the proof of Lemma 14, even if the embedding of P in G is injective, the embedding of $P^{i \leftarrow j}$ does not have to be injective. Later, we are going to define a way to avoid this situation as much as possible, by carefully choosing the element σ_i to add a new face at.

Primitive patches

A convex patch of curvature $\mu \leq 5$ is *primitive* if the mutual position of its small faces is the same as in one of the patches depicted in Figure 1.38 or it has no small faces at all (for $\mu = 0$).

Observe that each convex patch with at most one small face is primitive.

Lemma 15 *Let P be a convex patch of curvature $\mu \leq 5$ which is not primitive. Then there exists another patch P' with the same curvature and the same boundary vector as P on a bigger number of vertices.*

Proof. Suppose that there exists a convex patch of curvature $\mu \leq 5$ which is not primitive, and all the convex patches of given curvature and boundary vector have at most as many vertices as P .

If P has at most one small face, then it is primitive by definition, a contradiction. Therefore, we may assume that P has at least two small faces.

If all the small faces of P are pairwise adjacent to each other, then P has at most three small faces, moreover, if it has three small faces, at most one of them is a quadrangle. In all the cases the patch is primitive, a contradiction.

We may suppose that P has two small faces f_1 and f_2 which are not adjacent to each other. We claim that f_1 and f_2 are at mutual position $(1, 1)$ and the edge connecting them is incident to a quadrangle:

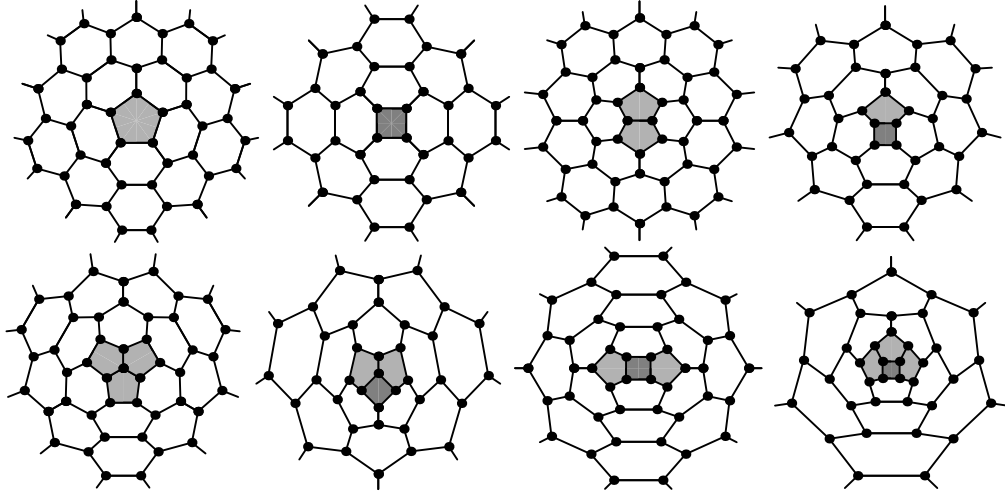


Figure 1.38: Mutual position of small faces in primitive patches for different values of curvature $1 \leq \mu \leq 5$.

Suppose f_1 and f_2 are two small faces in mutual position (c_1, c_2) such that $c_1 \geq c_2 \geq 1$ and $c_1 \geq 2$. Then there exists a new patch P' with the same boundary vector and the same curvature, but with a bigger number of vertices: P' can be found by inserting two pentagons and $c_1 + c_2 - 3$ hexagons along a shortest path joining f_1 and f_2 . (The path is in P due to convexity of P .) By applying this operation, the size of f_1 and f_2 is increased by one; the mutual position of the two new pentagons is $(c_1 - 1, c_2 - 1)$, see Figure 1.39(a) for an illustration. The patch P' is indeed a patch of a Barnette graph, since no pair of adjacent quadrangles can be created this way.

Similarly, if $c_1 \geq 3$ and $c_2 = 0$, then there is a sequence h_1, \dots, h_{c_1-1} of hexagons forming a dual path joining f_1 and f_2 . We subdivide the edge between f_1 and h_1 and the edge between h_{c_1-1} and f_2 once; we subdivide each edge between h_i and h_{i+1} ($1 \leq i \leq c_1 - 2$) twice; we join the new vertices in such a way that h_1 and h_{c_1-1} are split into a pentagon and a hexagon and that all other hexagons in the sequence are split into two new hexagons. Again, the size of f_1 and f_2 is increased by one and a new pair of pentagons at mutual position $(c_1 - 2, 1)$ is created, see Figure 1.39(b) for an illustration.

Analogously, if $(c_1, c_2) = (2, 0)$, then there is a hexagon h adjacent to both f_1 and f_2 . (If the face h adjacent to both f_1 and f_2 was a pentagon, then f_1 and f_2 would rather be in a mutual position $(1, 1)$.) To obtain P' , it suffices to subdivide the two edges h shares with f_1 and f_2 , respectively, and join the two new vertices by a new

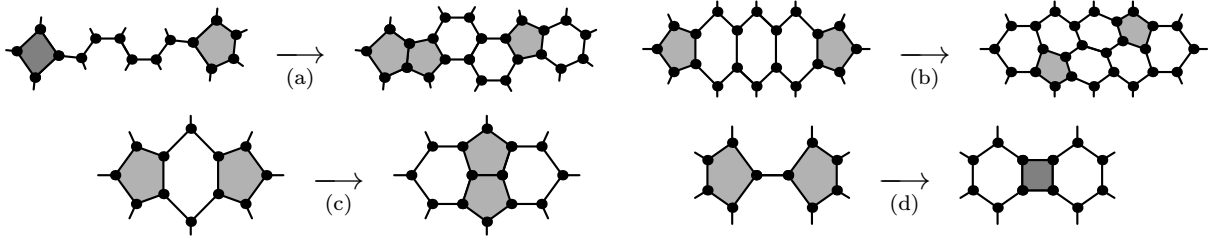


Figure 1.39: If a patch contains at least two non-adjacent small faces, it can be transformed to another one with more vertices, unless the two small faces are in position (1,1) and the edge connecting them is incident to a quadrangle: Two generic cases (a,b) and two special cases (c,d). The size of the two small faces is always increased by one (so if they were pentagons, they are no more small); two new pentagons or one new quadrangle are created.

edge. This way h is split into two pentagons and the size of f_1 and f_2 is increased by one, see Figure 1.39(c) for an illustration.

Finally, let $(c_1, c_2) = (1, 1)$. Then f_1 and f_2 are connected by an edge e . If the edge e is not incident to any quadrangle, then new patch P' can be obtained by replacing e by a quadrangle, see Figure 1.39(d) for an illustration. Since the size of f_1 and f_2 is increased by one, there can not be two adjacent quadrangles in the patch P' .

To conclude, for every pair of non-adjacent small faces of P , there is a quadrangle adjacent to both of them, so both of them are pentagons, and so $\mu \geq 4$ and P contains a quadrangle adjacent to two pentagons (which are not adjacent to each other). If $\mu = 4$, then P has no other small faces, so it is primitive, a contradiction. If $\mu = 5$, then P contains an additional pentagon, which, due to the previous observations, has to be adjacent to (the only) quadrangle – again we obtain a primitive patch, a contradiction. \square

Corollary 2 *For a given curvature $\mu \leq 5$ and a given convex boundary vector, a (convex) patch with maximal number of vertices has to be a primitive one.*

Lemma 16 *Let P be a convex patch of curvature $\mu \leq 5$ and boundary vector σ . Then there exists a unique primitive patch $\bar{P}(\mu, \sigma)$ with the same curvature and the same boundary vector.*

Proof. The existence is given by the previous lemma. The uniqueness can be proven by induction, by adding/removing rows of hexagons from a patch, or, alternatively,

by considering embeddings of patches onto infinite hexagonal cones. We omit the details. \square

Lemma 17 *Let P be a convex patch of perimeter p and curvature $\mu \leq 5$. Then P has at most $\frac{p^2}{6-\mu}$ vertices.*

Proof. It suffices to count the numbers of vertices of primitive convex patches. We omit the details. \square

It is worth mentioning that the bound from Lemma 17 is tight only if $\mu(P) \leq 2$ and the patch contains at most one small face.

Corollary 3 *Let P be a convex patch of curvature $\mu \geq 7$. Then P can be realized only in finitely many Barnette graphs.*

The largest Barnette graph containing a given realizable convex patch of curvature $\mu \geq 7$ can be found by adding the corresponding (unique) primitive patch of curvature $12 - \mu$.

Patch closure and essential patches

A k -disc centered at a face f of a plane graph G , denoted by $B_k(f)$, is a subgraph of G composed of facial cycles of faces at (dual) distance at most k from the face f . Note that if k is large enough, then $B_k(f) = G$ for any f .

A patch P' is called a *closure* of a patch P , if

1. P is contained in P' ,
2. every small face of P' corresponds to a small face of P ,
3. P' contains the 2-discs centered at the small faces of P , and
4. P' is convex.

A patch P is called *closed* if it is a closure of itself.

Clearly, if P' is a closure of P , then P' can be obtained from P by adding a finite number of hexagons.

Let P be a patch with boundary vector $\sigma(P) = \sigma_1\sigma_2\ldots\sigma_k$. The *small face distance* of a value σ_i is equal to the minimum of the distances $d(f^*, g^*)$, where f is the new face of the patch $P^{i \leftarrow j}$ (for some j sufficiently big), g runs the set of small

faces of P , and the distances are taken in the inner dual (dual without the vertex representing the outer face) of $P^{i \leftarrow j}$.

Let P be a patch which is not convex. Then we set all the values of its boundary vector as *admissible*.

Let P be a convex patch with boundary vector $\sigma(P) = \sigma_1 \sigma_2 \dots \sigma_k$. A value σ_i is called *admissible*, if the small face distance of σ_i is at most 2.

Observe that boundary vectors of closed patches have no admissible values.

Let P be a patch with boundary vector $\sigma(P) = \sigma_1 \sigma_2 \dots \sigma_k$ which is not closed. A *critical element* of the boundary vector of P is an admissible value σ_i such that

- σ_i is maximal, and then
- the sum $\sigma_{i-1} + \sigma_{i+1}$ (incides taken modulo k) is maximal, unless $\mu(P) \leq 6$ and $\max_{i=1}^k \sigma_i = 3$; in which case we choose $\sigma_i = 3$ contained in a subsequence 32^j3 of minimum length, and then
- the small face distance of σ_i is minimal.

The definition of a critical element may seem technical and unnecessarily complicated, however, it was optimised during the developpement of the program that generates the patches in order to avoid situations where existing faces of a patch ought to be identified (as in the lower line of Figure 1.36) and to avoid considering patches that are not realizable at all (cf. Figure 1.37.)

Lemma 18 *Let G be a Barnette graph and let f be a small face of G . Then there exists a finite sequence of patches $\{P_k\}_{k=1}^t$ contained in G such that*

- P_1 is a cycle of length equal to the size of f ;
- $P_{k+1} = P_k^{i \leftarrow j}$, where $j \in \{4, 5, 6\}$ and σ_i is a critical element of $\sigma(P_k)$;
- for each k , in the embedding of P_k into G , the face f corresponds to a face of P_k ;
- either P_t is the first closed patch of the sequence or $P_t = G$.

Proof. The existence of the sequence is guaranteed by Lemma 14. Either adding faces one by one yields a closed patch, or all the faces of G are eventually added. In both cases the sequence is finite. \square

Observe that the sequence $\{P_k\}_{k=1}^t$ of patches contained in a Barnette graph G starting with a fixed small face f of G given by Lemma 18 is not unique – it may depend on the choice of a critical element.

Let f be a small face of a Barnette graph G and let P be a patch. If $P = P_t$ for some sequence described in Lemma 18 starting with f , then we call P an *essential patch* for f in G .

Patches as nanotube caps

Lemma 19 *Let P be a convex closed patch of curvature $\mu = 6$. If P is realized in a Barnette graph G , then G is a nanotube, the type of which is determined by $\sigma(P)$.*

Proof. Let $\sigma = \sigma(P)$. By convexity of P , σ only contains 1s, 2s and 3s. Since $\mu = 6$, the number of 1s and 3s in σ is the same, and, again by convexity of P , σ contains neither 12^k1 nor 32^k3 as a (cyclic) subsequence.

Since P is closed, all the faces of P adjacent to the outer face are hexagons. If $\sigma = 2^p$ for some p , then those faces form a ring of type $(p, 0)$ in P .

If $\sigma = 12^{k_1}32^{l_1} \dots 12^{k_t}32^{l_t}$ (up to a cyclic permutation), then the faces adjacent to the outer face form a ring in P of type (p_1, p_2) , where $\{p_1, p_2\} = \{t + \sum_{i=1}^t k_i, t + \sum_{i=1}^t l_i\}$.

It remains to observe that the ring in P corresponds to a ring in G – that the embedding of P restricted to the set of faces of the ring is injective. \square

Note that the set of faces of the dual distance 2 from the outer face of a convex closed patch of curvature 6 is a ring of a nanotube as well – a convex closed patch of curvature 6 thus contains two disjoint rings of the same type.

1.5.3.2 Patches and the general procedure

Let P be a patch essential for some small face of a Barnette graph G . Then the Hamiltonian cycle C_T of the triangulation T capturing the mutual position of small faces of G enters and leaves P at least once.

We will modify the general procedure in order to ensure that we can choose a cycle C_T entering and leaving P exactly once: For essential patches of curvature at least 6 this is automatically true due to convexity of the patch and minimality of the cycle. For each essential patch P of curvature at most 5 we can temporarily replace P by the corresponding primitive patch \bar{P} ; in the resulting graph we find the cycle C^* visiting each small face exactly once. Since in \bar{P} each small face is adjacent to some other small face, the small faces of \bar{P} are consecutive along C^* by minimality

of C^* . When replacing back the primitive patches by the actual patches, we keep the order in which the (primitive) patches were covered by C^* and we keep the position of the segments joining different patches. We disregard the way how C^* visits the small faces inside each essential patch, since we will inspect that in details later.

From this point on we may assume that for each essential patch P there are exactly two segments leaving P , say P_i^* and P_j^* . For any position of the segments $P_{i+1}^*, \dots, P_{j-1}^*$ inside P , the difference $\varphi_j \circ \varphi_i^{-1}$ is a permutation of three elements which is even if and only if $\mu(P)$ is even (each pentagon of P contributes with a single transposition).

If $\mu(P)$ is odd, then the difference $\varphi_j \circ \varphi_i^{-1}$ is an odd permutation – a transposition. Therefore, among the nine choices of colorings of G_1 and G_2 , for one choice both segments leaving P are inactive, for four choices one of them is active and the other one is inactive, and for the remaining four both segments are active – the patch behaves like a pentagon. We will call these patches type 1.

If $\mu(P)$ is even and the difference $\varphi_j \circ \varphi_i^{-1}$ is the identity, then among the nine choices of colorings of G_1 and G_2 , for three of them both segments are inactive and for the remaining six both segments are active – the patch behaves like a quadrangle. We will call these patches type 0.

If $\mu(P)$ is even and the difference $\varphi_j \circ \varphi_i^{-1}$ is an even permutation different from the identity, then it has to be a cycle of length three. Therefore, among the nine choices of colorings of G_1 and G_2 , for three of them both segments are active and for the remaining six there is one active and one inactive segment – the patch behaves like a pair of pentagons of different colors. We will call these patches type 2.

Let P be a patch essential for some small face of a Barnette graph G . Let the position of two segments leaving P and all the segments inside P be fixed. Let one of the nine colorings of G_1 and G_2 be chosen. Let the procedure described in Section 1.5.2 be applied. We first obtain a 2^* -factor, which is then transformed into at most two 2-factors (depending on the order of decisions at 2-cycles of the 2^* -factor).

Let H_P^0 be the subgraph of the residual graph H induced by the vertices corresponding to the faces of P and faces adjacent to faces of P in G . There can be vertices of degree 1 or 2 in H_P^0 . We add $3 - d$ new vertices adjacent to each vertex of degree d in H_P^0 inside the outer face; we then connect all these new vertices by a new cycle. This way we obtain a plane cubic graph H_P , we call it *partial residual graph*.

Let f^* be a vertex of H_P corresponding to a face f of P . The face f is a white resonant hexagon. If we recolor f black, then three different components of the underlying 2-factor are merged into a single cycle; the vertex f^* is deleted from H_P and the three resulting 2-vertices are suppressed. We call this operation *elimination*

of f^* .

We say that a plane cubic graph is *strongly essentially 4-edge-connected*, if it is cyclically 3-edge-connected, and every cyclic 3-edge-cut separates a triangle adjacent to the outer face from the rest of the graph.

We say that a plane cubic graph is *essentially 4-edge-connected* if it can be transformed into a strongly essentially 4-edge-connected plane graph by a vertex elimination.

We say that a patch P is *regular*, if for every possible position of a pair of segments leaving P and for every choice of the colors of G_1 and G_2 , there is a permutation of small faces of P such that for each of the (at most) two 2-factors obtained by the general procedure the corresponding partial residual graph is essentially 4-edge-connected. See Figures 1.40 and 1.41 for an illustration.

We say that a patch P is *weakly regular*, if for every possible position of a pair of segments leaving P there exists a choice of the colors of G_1 and G_2 such that there is a permutation of small faces of P such that for at least one 2-factor obtained by the general procedure the corresponding partial residual graph is essentially 4-edge-connected.

We say that a patch P is *parity-switching* if for every possible position of a pair of segments leaving P there exists a choice of the colors of G_1 and G_2 such that there exists a permutation of small faces of P such that one of the operations O_1 and O_2 can be applied inside P ; for both 2^* -factors (before and after the operation), for at least one 2-factor the corresponding partial residual graph is essentially 4-edge-connected.

1.5.3.3 Generation of patches

Theorem 11 *There exists a finite set \mathcal{P} of patches such that for every Barnette graph G on at least 318 vertices and every small face f of G , there exists a patch $P \in \mathcal{P}$ essential for f in G .*

Proof. We prove the claim by construction. We used Algorithm 1 to generate all the patches in \mathcal{P} , by two calls of the procedure GENERATE(), passing as a parameter first a 4-cycle and then a 5-cycle, with the database of patches containing initially the closures of the two initial patches. The procedure uses Algorithm 2 as a subroutine to calculate a closure of a given patch.

If the insertion at lines 6, 11, or 16 of Algorithm 1 fails, it means that there is no Barnette graph containing the current patch P such that the element σ_i corresponds to a j -face for $j = 4, 5$, or 6 , respectively. If this is the case, the following lines are ignored until the next insertion. Similarly for the insertion at line 6 of Algorithm 2.

□

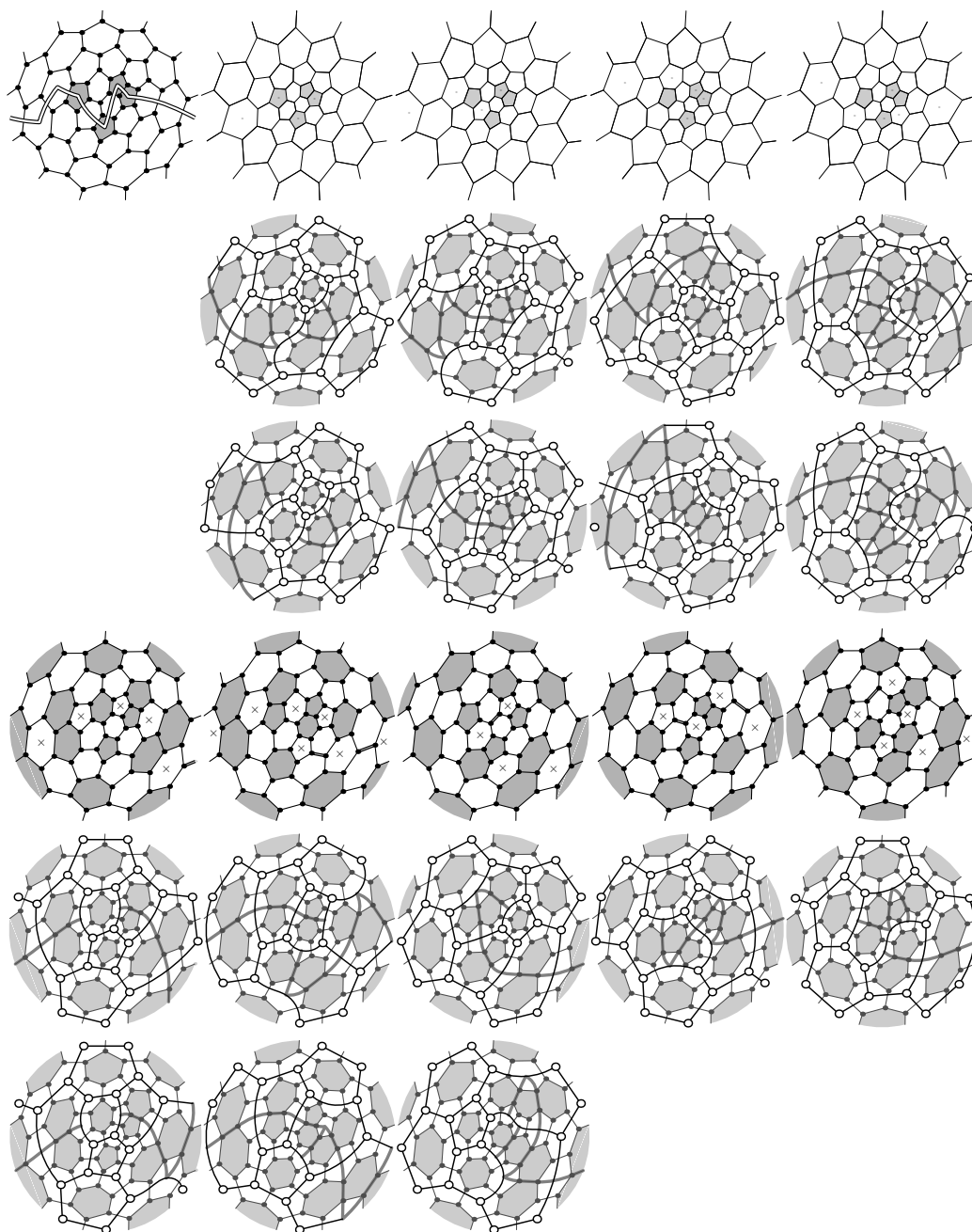


Figure 1.40: For a patch P with four pentagons and a fixed position of two segments leaving P , for each of the nine colorings of G_1 and G_2 the 2^* -factor and (at most) two simple 2-factors obtained by the general procedure are depicted. The third drawing in the third column of the second row proves that for this position of the segments leaving P the patch P is parity-switching.

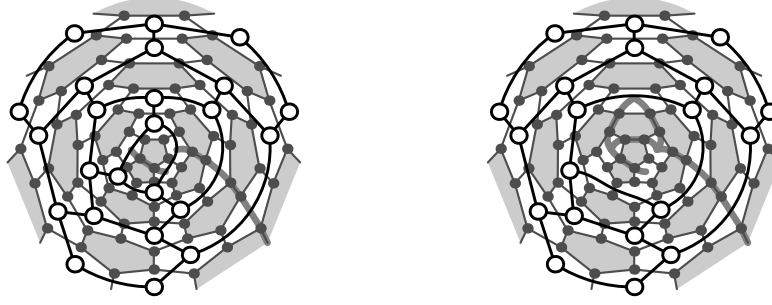


Figure 1.41: For a few patches with many small faces adjacent to each other, the first outcome of the general procedure is a 2-factor such that the corresponding partial residual graph is not strongly essentially 4-edge-connected (left). However, to obtain a strongly essentially 4-edge-connected graph, it suffices to eliminate a vertex incident to a short cycle (right).

Algorithm 1 Generation of all closed patches containing a given patch

```

1: procedure GENERATE(patch  $P$ )
2:   if  $\mu(P) \geq 7$  and the largest graph containing  $P$  has at most 316 vertices then
3:     return
4:   else
5:     let  $\sigma_i$  be a critical element of the boundary of  $P$ 
6:     if the path along  $\sigma_i$  is not adjacent to a 4-face then
7:        $P' \leftarrow P^{i \leftarrow 4}$ 
8:        $P'' \leftarrow \text{CLOSURE}(P')$ 
9:       if  $P''$  is not in the database of patches then
10:        Add  $P''$  to the database of patches
11:        GENERATE( $P'$ )
12:    $P' \leftarrow P^{i \leftarrow 5}$ 
13:    $P'' \leftarrow \text{CLOSURE}(P')$ 
14:   if  $P''$  is not in the database of patches then
15:     Add  $P''$  to the database of patches
16:     GENERATE( $P'$ )
17:    $P' \leftarrow P^{i \leftarrow 6}$ 
18:    $P'' \leftarrow \text{CLOSURE}(P')$ 
19:   if  $P'' \neq P'$  then
20:     GENERATE( $P'$ )

```

Algorithm 2 Computation of a closure of a given patch

```

1: procedure CLOSURE(patch  $P$ )
2:   if  $P$  is closed then
3:     return  $P$ 
4:   else
5:     let  $\sigma_i$  be a critical element of the boundary of  $P$ 
6:      $P' \leftarrow P^{i \leftarrow 6}$ 
7:     return CLOSURE( $P'$ )

```

The counts of patches generated in the proof of Theorem 11 are depicted in Table 1.1.

$f_4 \setminus f_5$	1	2	3	4	5	6	7
0	1	3	11	69	450	4184	1
1	3	26	240	2789			
2	23	423					

Table 1.1: Numbers of essential patches in \mathcal{P} , given number of pentagons and quadrangles. Amongst the patches of curvature greater than 6, only patches contained in at least one graph on at least 318 vertices are counted.

1.5.3.4 Analyse of patches

The following statements were checked by computer:

Theorem 12 *There is no patch $P \in \mathcal{P}$ with $\mu(P) \geq 8$.*

This means that for every patch P of curvature at least 8 considered by the generating algorithm, the largest graph containing P has less than 318 vertices. See line 2 of Algorithm 1 and the remark after Corollary 3.

Theorem 13 *There is a unique patch $P_7 \in \mathcal{P}$ with $\mu(P_7) = 7$.*

Moreover, amongst the Barnette graphs containing P_7 , all but one have less than 318 vertices. The unique graph on (at least) 318 vertices realizing P_7 contains, besides P_7 , the primitive patch of curvature five. See Figure 1.42 for an illustration.

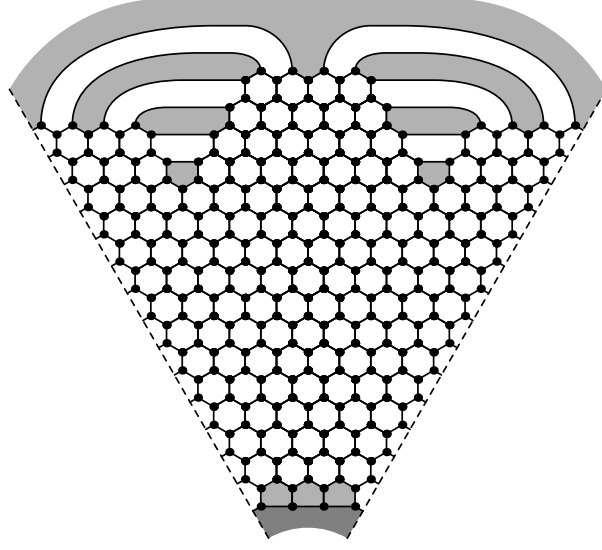


Figure 1.42: The unique graph on at least 318 vertices containing an essential patch of curvature larger than six.

Theorem 14 *Every patch $P \in \mathcal{P}$ with $\mu(P) \leq 5$ is regular. Every patch $P \in \mathcal{P}$ with $\mu(P) \in \{6, 7\}$ is weakly regular, unless $\mu(P) = 6$ and P contains a cap of a nanotube of type (p_1, p_2) with $(p_1, p_2) \in \{(4, 0), (5, 0), (4, 1), (5, 1), (3, 2), (4, 2), (3, 3), (4, 3)\}$.*

Theorem 14 guarantees the existence of a simple 2-factor such that the residual graph is cyclically 4-edge-connected. The only missing part is that we cannot be sure that this 2-factor is odd.

We do not need to check for regularity of patches of curvature 6 and 7, weak regularity suffices instead: If a Barnette graph contains an essential patch of curvature $\mu = 7$, then it only contains one. Therefore, we can choose the coloring of G_1 and G_2 such that no segment leaving P is active.

If a Barnette graph G contains an essential patch P of curvature $\mu = 6$, then, by Lemma 19, G is a nanotube. We can choose the coloring of G_1 and G_2 such that the tubical part of G is traversed by at most one active segment (one if P is type 2, none if P is type 0). Moreover, we may choose the direction in which the active segment traverses the tubical part of G .

If G is a nanotube of type (c_1, c_2) , $c_1 \geq c_2 \geq 0$, and the caps are (contained in) patches of type 0, then $3 \mid (c_1 - c_2)$, so we can write $(c_1, c_2) = (3a + b, b)$ for some integers $a, b \geq 0$. If we choose any of the three colorings of G_1 and G_2 such that

no active segment traverses the tubical part of G , then the residual graph H is a nanotube of type $(a + b, a)$.

If G is a nanotube of type (c_1, c_2) , $c_1 \geq c_2 \geq 0$, and the caps are (contained in) patches of type 2, then $3 \nmid (c_1 - c_2)$, so we can write $(c_1, c_2) = (3a + b, b + 1)$ or $(c_1, c_2) = (3a + b + 1, b)$ for some integers $a, b \geq 0$, or $(c_1, c_2) = (3a + 2, 0)$ for some integer $a \geq 1$. If we choose a coloring of G_1 and G_2 such that one active segment traverses the tubical part of G , then the residual graph H is a nanotube of type $(a + b, a)$ (in the first two cases) or $(a + 1, a)$ (in the third case). See Figure 1.43 for an illustration.

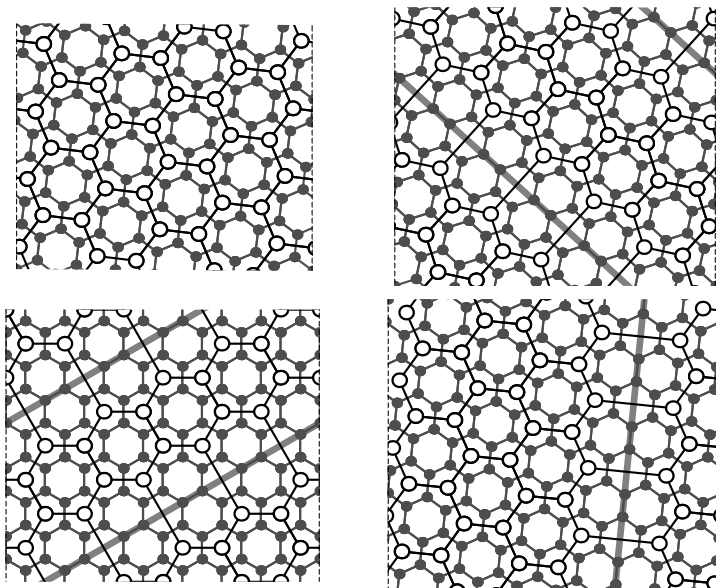


Figure 1.43: Clockwise, starting from top left: If G is a nanotube of type $(7, 1)$, $(7, 2)$, $(8, 1)$ or $(8, 0)$, respectively, then the residual graph is a nanotube of type $(3, 2)$.

This is the reason for excluding the aforementioned eight types of nanotubes. The following claim was verified by a computer.

Theorem 15 *Let $P \in \mathcal{P}$. Then P is parity-switching, unless P is one of the following exceptional patches:*

- the patch P_1 of curvature 1 having one pentagon,
- the patch P_2 of curvature 2 containing two adjacent pentagons,
- the patch P_3 with three pentagons sharing a common vertex,

- two patches P_4 and P_5 with four pentagons, depicted in Figure 1.44,
- four patches P_6 , P_7 , P_8 , P_9 with six pentagons, depicted in Figure 1.44.

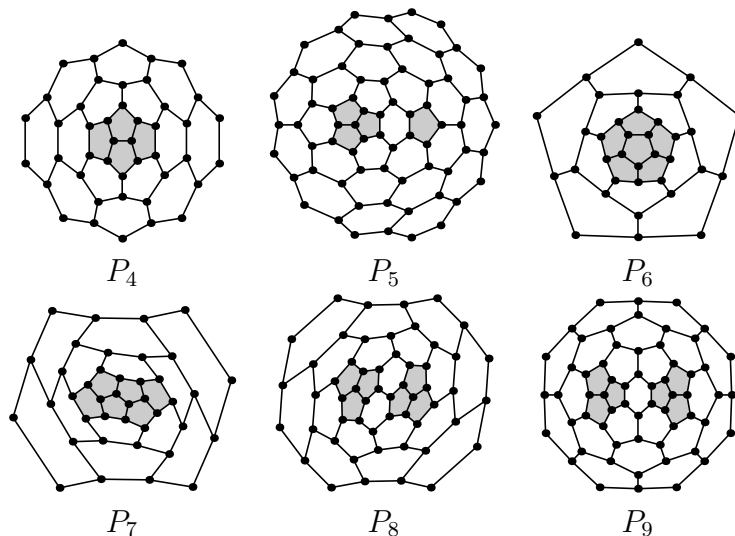


Figure 1.44: Patches with 4 and 6 pentagons for which it is not always possible to increase or decrease the number of black pentagons by 2.

Observe that P_4 and P_5 are both type-0 patches.

There is a combinatorial reason for the patches P_3 - P_9 not to be parity-switching: if three pentagons share a vertex, either one or two of them have to be black, so we do not have the freedom to change their colors independently.

As a consequence of Theorem 15, if a Barnette graph contains at least one parity-switching essential patch, we choose the coloring of G_1 and G_2 that allows to change the parity of the 2-factor inside the patch, and, by regularity, we are done.

It remains to consider Barnette graphs (in fact, fullerene graphs) only containing patches P_1 - P_9 and verify that we can use parity-switching operations using pentagons from different patches.

If a fullerene graph contains P_6 , it is a nanotube of type $(5, 0)$, and it is known to be Hamiltonian [53]. If a fullerene graph contains P_7 (P_8 , P_9 , respectively), then it is a nanotube of type $(4, 2)$ (of type $(6, 2)$, $(8, 0)$) – the patch itself already contains a corresponding ring. Out of all the possible patches (caps) to close the other end of the tube, P_7 (P_8 , P_9) is the only one that is not parity-switching, as it was verified by a computer. However, if both caps of a nanotube are P_7 (P_8 , P_9), then it has an

even number of hexagons and exactly 6 black and 6 white pentagons, so by Lemma 11 the number of cycles in the 2-factor is odd.

It remains to consider fullerene graphs only having patches P_1 , P_2 , P_3 , P_4 , and P_5 .

It was verified by computer (and also checked by hand) that for each of the five patches, for each active segment leaving the patch, the Γ -path can be transformed into a pair of Γ -paths (interconnected inside the patch or not) – it is nothing else than applying a half of one of the operations O_1 and O_2 (or its inverse) inside the patch and the other half inside another one. See Figure 1.45 for an illustration.

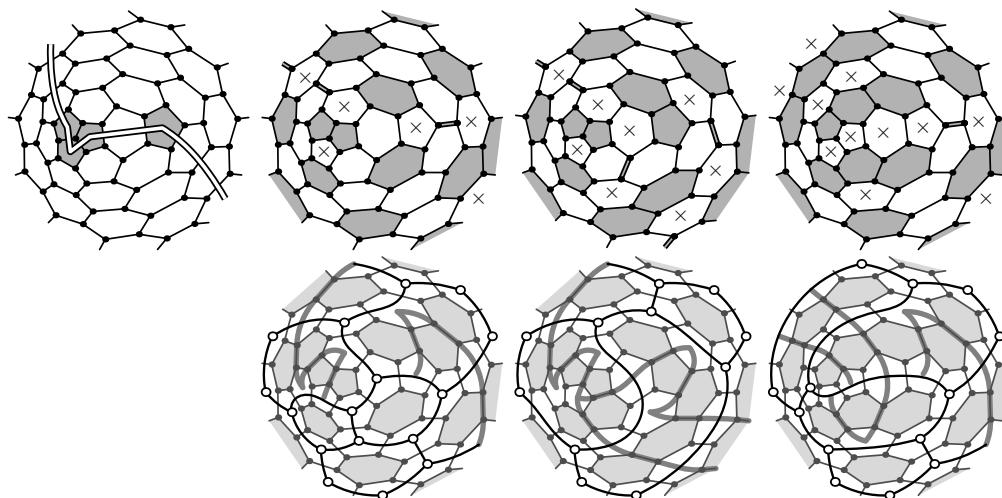


Figure 1.45: An example of a position of two segments leaving the patch P_5 (top left). For a fixed coloring of G_1 and G_2 (second column), for any active segment leaving the patch, a half of the operation O_1 or O_2 can be carried out inside the patch, changing (increasing or decreasing) the number of black pentagons by one (third and fourth column).

In each of the patches this modification corresponds to increasing or decreasing the number of black pentagons by one. In most of the cases both are possible. More precisely, for each segment leaving P_1 , P_2 , P_4 or P_5 , out of the nine possible colorings, for three colorings the segment is inactive, for at least two colorings it is possible to increase the number of black pentagons by one, and for at least four colorings it is possible to decrease the number of black pentagons by one. It means that if two of these patches are consecutive along C^* , then there exists a coloring such that we can decrease the number of black pentagons in each of them by one. See Figure 1.46 for an illustration.

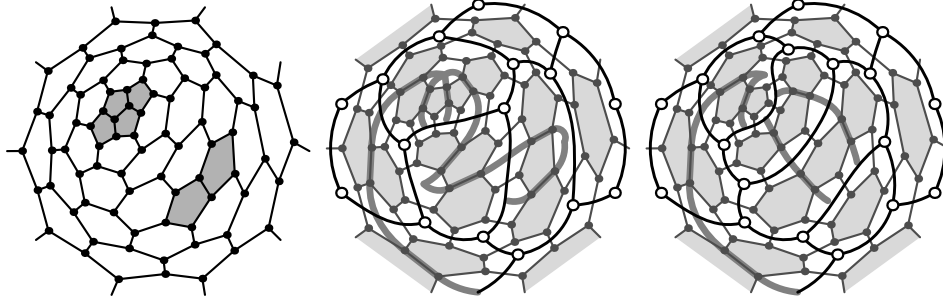


Figure 1.46: If two of the patches P_1, P_2, P_4, P_5 (here P_4 and P_2) are consecutive along C^* , then there exists a choice of coloring of G_1 and G_2 such that it is possible to decrease the number of black pentagons by one in each patch by applying the operation O_1 or O_2 involving one pentagon from each patch.

On the other hand, for P_3 , it is possible to increase the number of black pentagons by one for four colorings and decrease it for two of them. Again, if there are two such patches consecutive along C^* , then there exists a coloring such that we can increase the number of black pentagons in each of them by one.

It remains to consider fullerene graphs such that along C^* , the patches P_3 alternate with other types of patches among $\{P_1, P_2, P_4, P_5\}$. Since each P_3 contains three pentagons and there are twelve pentagons altogether, it is easy to see that the number of P_3 patches is either 2 or 3.

If there are two P_3 patches, the other two patches have six pentagons, and hence one of them is P_2 and the other one is either P_4 or P_5 . The patch with four pentagons has to be far from each of the P_3 patches, otherwise the graph would have at most 316 vertices (see Lemma 17). The patches P_4 and P_5 are both type 0. That is why we may omit the four-pentagon patch (P_4 or P_5) and search only for a cycle passing through the eight pentagons of the other three patches; we consider P_4 or P_5 as if no segment leaving it was active. As a consequence, we find two P_3 patches consecutive along C^* .

If there are three P_3 patches, the other three patches can only have one pentagon each. Moreover, the condition that for each segment joining a P_3 to a P_1 the two colorings allowing to decrease the number of black pentagons in P_3 correspond to the two colorings allowing to increase the number of black pentagons in the other patch implies that out of the nine colorings, there is one with no active segment joining a P_3 to a P_1 , there are four colorings with three active segments and three inactive segments alternating, and there are four colorings with all the six segments active. In all the cases there are three Γ -paths in G . (In the case of no active segments joining

different patches, there is still a Γ -path joining different pentagons inside each P_3 .)

If we replace a vertex incident to three pentagons inside each P_3 by a triangle temporarily, then the graph will contain three pentagons and three triangles (and all the other faces will be hexagons). Moreover, in the coloring of G_1 and G_2 all the six small faces have the same color.

By the structural theorem of Alexandrov (see e.g. [12], Theorem 23.3.1, or [59], Theorem 37.1), such a graph can be isometrically embedded onto a surface of a (possibly degenerate) convex polyhedron, say P . The polyhedron P has six vertices, and the cycle C^* is a Hamiltonian cycle in some triangulation of P .

The cycle C^* cuts the polyhedron P into two hexagons. In the two hexagons the angles at a fixed P_3 -vertex (center of a triangle) sum up to 180° , and hence they are both always convex (i.e. smaller than 180°). For the angles at the P_1 -vertices (centers of isolated pentagons), in at least one hexagon the angle is convex. Therefore, it is always possible to permute a P_1 patch with a P_3 patch to obtain a new cycle with two consecutive patches of the same type, which gives us a possibility to change the parity of the number of cycles. See Figure 1.47 for an illustration.

This concludes the proof of Theorem 10.

1.5.4 Concluding remarks

Similar technique could be used to prove Hamiltonicity of related graph classes: planar cubic graphs with only a few faces of size larger than six; projective-planar graphs with faces of size at most six (except, of course, for the Petersen graph), etc.

Let us recall that if G is a fullerene graph with an odd number of faces, then $L(G)$, the leapfrog of G , is a fullerene graph again. If Lemma 10 is applied to $L(G)$ with M being the complement of the 2-factor consisting of the disjoint union of the facial cycles corresponding to the faces of G , then the residual graph is nothing else but G . Clearly, G is cyclically 4-edge-connected (fullerene graphs are known to be cyclically 5-edge-connected), and so in this case, $L(G)$ is Hamiltonian.

Moreover, if we do not take Theorem 9 for granted, we can show that there are exponentially many decompositions of a given fullerene graph G with an odd number of faces into an induced tree and a stable set, providing an exponential lower bound on the number of Hamilton cycles in $L(G)$. More details can be found in [47].

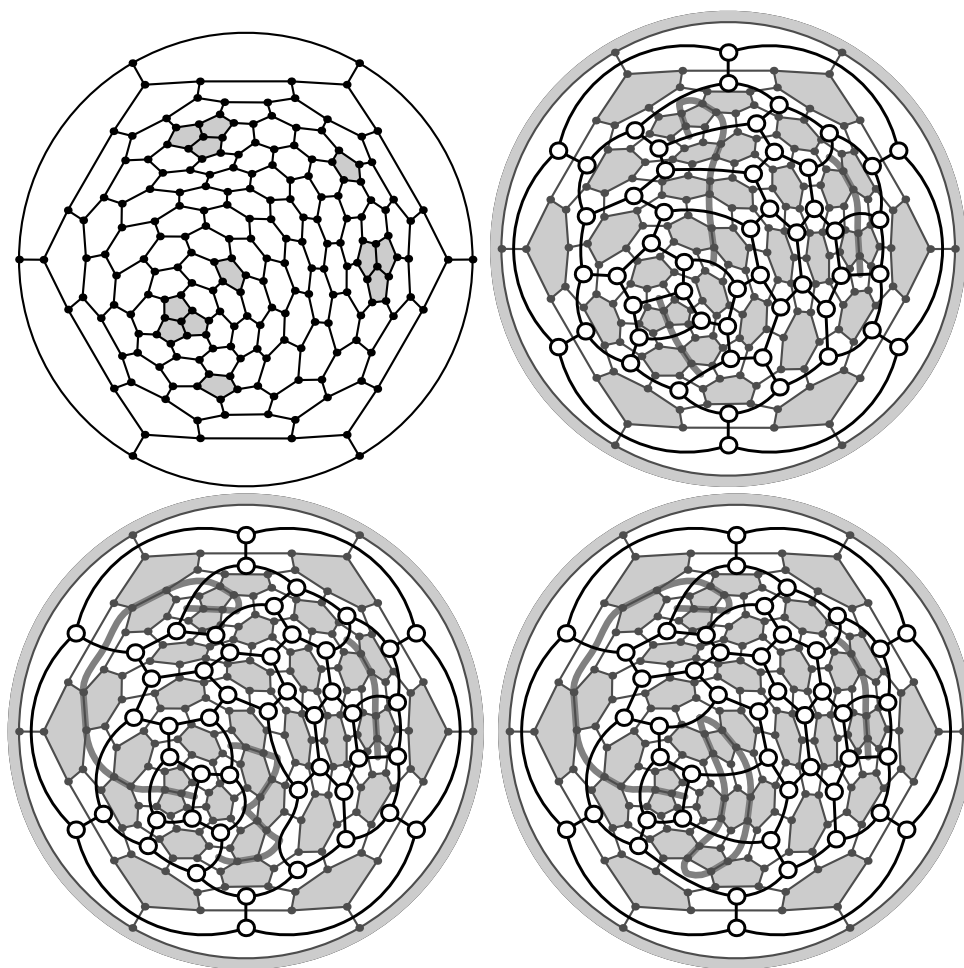


Figure 1.47: Top to bottom, left to right: An example of a fullerene graph on 198 vertices containing three patches P_3 and three patches P_1 . An even 2-factor with three Γ -paths without a possibility to apply O_1 or O_2 . Another even 2-factor obtained by switching the order of the patches. An odd 2-factor after applying O_1 .

Chapter 2

Perfect matchings in cubic graphs

One of the oldest results in graph theory is a theorem of Petersen, stating that every bridgeless cubic graph has a perfect matching. The original Petersen's proof is rather complicated; modern textbooks usually give a straightforward proof using Tutte theorem.

Why bridgeless? Clearly, by the handshaking lemma, if a cubic graph with a bridge admits a perfect matching, the bridge has to be included in the perfect matching. Therefore, it is very easy to find cubic graphs without a single perfect matching – it suffices to make two bridges adjacent, for instance.

On the other hand, being bridgeless seems to be a fundamental structural property of cubic graphs, since there are many open questions and conjectures about bridgeless cubic graphs which become false when the condition of bridgelessness is omitted. Several of them speak about interactions between perfect matchings. Let us recall a couple of the most famous ones.

Conjecture 2 (Berge-Fulkerson 1971 [31]) *Let G be a bridgeless cubic graph. Then there exists a collection of six perfect matchings, covering every edge exactly twice.*

The statement is trivially true for 3-edge-colorable graphs (it suffices to consider two copies of each color class). In particular, by the Four Color Theorem, all planar graphs satisfy Berge-Fulkerson Conjecture.

If Berge-Fulkerson conjecture is true, then it directly implies

Conjecture 3 (Fan-Raspaud 1994 [28]) *Let G be a bridgeless cubic graph. Then there exists a collection of three perfect matchings with empty intersection.*

These two conjectures on perfect matchings interactions were indirectly supported by yet another one, saying, informally speaking, that there is a big chance to find some perfect matchings with desired properties as long as there are plenty of them:

Conjecture 4 (Lovász-Plummer 1986 [61]) *Let G be a bridgeless cubic graph. Then G has exponentially many perfect matchings.*

While the first two conjectures remain open, the last one was settled by a group of five authors including myself in 2011.

As a special case, Voorhoeve [65] proved already in 1979 that Lovász-Plummer conjecture is true for bipartite bridgeless cubic graphs. In his proof, he uses an elegant elementary operation of *splitting a path* (to be defined below).

In this chapter, we will explain two main results. First, we will show that it is possible to use the path splitting (together with some careful analysis of short edge-cuts) beyond bipartite graphs (and so to prove the Lovász-Plummer conjecture for all bridgeless cubic graphs). Second, we will indicate how path splitting (again with some careful analysis of short edge-cuts) can be adapted to serve to approach the conjectures of Berge-Fulkerson and Fan-Raspaud.

All the graphs considered in this chapter are loopless. On the other hand, parallel edges are allowed, unless stated otherwise (or they are eliminated by stronger connectivity assumptions).

Let us introduce the main character of this chapter, namely the operation of splitting a path now.

Let $u_1v_1v_2u_2$ be a path in a cubic graph G . Let w_1 (w_2) be the third neighbor of v_1 (v_2 , respectively). Suppose that the vertices u_1 and w_1 are distinct from the vertices u_2 and w_2 . Let G' be the cubic graph obtained from G by deleting the vertices v_1 and v_2 together with all incident edges, and by adding the edges u_1u_2 and w_1w_2 . We say that G' is obtained by *splitting* the edge v_1v_2 . More precisely, if needed, we will also say that G' is obtained after a $(u_1u_2 : w_1w_2)_{v_1v_2}$ -reduction.

It is a known fact [26] that when applying this operation, the cyclic edge-connectivity of a cubic graph can drop by at most 2.

Note that if G is bipartite, then G' is bipartite too. Observe that two distinct graphs (usually non-isomorphic) can be obtained by splitting the same edge – in the definition, the vertices u_2 and w_2 might interchange their roles.

2.1 Counting perfect matching in bridgeless cubic graphs

Let us focus now on bounding the number of perfect matchings in bridgeless cubic graphs from above.

Throughout this section, we will use the following common notation.

Let $m(G)$ denote the number of perfect matchings of a graph G . Let $m_e(G)$ denote the number of perfect matchings of a graph G avoiding an edge e of G .

2.1.1 Bipartite cubic graphs and beyond

As a warmup, and also to appreciate how useful the edge splitting can be, let us start by reproving Voorhoeve's result.

Theorem 16 (Voorhoeve 1979 [65]) *Let G be a bipartite cubic graph on n vertices. Then*

$$m(G) \geq \frac{9}{4} \left(\frac{4}{3}\right)^{n/2}.$$

We will prove a stronger statement.

Theorem 17 (Voorhoeve 1979 [65]) *Let G be a bipartite cubic graph on n vertices and let e be an edge of G . Then*

$$m_e(G) \geq \frac{3}{2} \left(\frac{4}{3}\right)^{n/2}.$$

Let us prove first that the latter implies the former. Let $\{e_1, e_2, e_3\}$ be three edges incident with the same vertex. Then every perfect matching of G is counted exactly twice in the numbers $m_{e_1}(G)$, $m_{e_2}(G)$, and $m_{e_3}(G)$. Therefore,

$$m(G) = \frac{1}{2} (m_{e_1}(G) + m_{e_2}(G) + m_{e_3}(G)) \geq \frac{3}{2} \cdot \frac{3}{2} \left(\frac{4}{3}\right)^{n/2} = \frac{9}{4} \left(\frac{4}{3}\right)^{n/2}.$$

Proof. (of Theorem 17) We do it by induction. There is only one bipartite cubic graph on 2 vertices, a graph that consists of three parallel edges. Clearly,

$$2 = m_e(G) \geq \frac{3}{2} \left(\frac{4}{3}\right)^{2/2} = 2$$

for every edge e , and so the claim holds.

Let G be a bipartite cubic graph on $n \geq 4$ vertices. If G is disconnected, let G_1 be the connected component of G containing e and let $G_2 = G \setminus G_1$. We may apply induction on G_1 (with e) and on G_2 (with an arbitrary edge e'). Every perfect matching of G_1 avoiding e combines with every perfect matching of G_2 avoiding e' into a perfect matching of G avoiding e . Therefore,

$$m_e(G) \geq m_e(G_1) \cdot m_{e'}(G_2) \geq \frac{3}{2} \left(\frac{4}{3}\right)^{n_1/2} \cdot \frac{3}{2} \left(\frac{4}{3}\right)^{n_2/2} > \frac{3}{2} \left(\frac{4}{3}\right)^{n/2}$$

where n_1 and n_2 are the orders of G_1 and G_2 , respectively.

We may assume that G is connected. In particular, there are no three parallel edges in G . Consider an edge e in a pair of parallel edges e, e' connecting two vertices u and v ; let e_u and e_v be the third edge incident with u and v , respectively. Let G' be the graph obtained by deleting the edge e and suppressing the vertices u and v of degree two. Clearly, G' is a bipartite cubic graph on $n - 2$ vertices. By induction, using the same argument as above, $m(G') \geq \frac{9}{4} \left(\frac{4}{3}\right)^{(n-2)/2}$. Moreover, every perfect matching of G avoiding e corresponds to a unique perfect matching of G' . Therefore,

$$m_e(G) = m(G') \geq \frac{9}{4} \left(\frac{4}{3}\right)^{(n-2)/2} = \frac{27}{16} \left(\frac{4}{3}\right)^{n/2} > \frac{3}{2} \left(\frac{4}{3}\right)^{(n-2)/2}.$$

Consider now an edge e adjacent to a pair of parallel edges f, f' . Let e' be the other edge adjacent to both f and f' . Let G' be the graph obtained by deleting f and suppressing two vertices of degree two. Clearly, G' is a bipartite cubic graph on $n - 2$ vertices. By induction, $m_e(G') \geq \frac{3}{2} \left(\frac{4}{3}\right)^{(n-2)/2}$. Moreover, every perfect matching of G' avoiding e extends to two distinct perfect matchings of G . Therefore,

$$m_e(G) = 2 \cdot m_e(G') \geq 2 \cdot \frac{3}{2} \left(\frac{4}{3}\right)^{(n-2)/2} = \frac{9}{4} \left(\frac{4}{3}\right)^{n/2} > \frac{3}{2} \left(\frac{4}{3}\right)^{(n-2)/2}.$$

It remains to consider the case when neither e nor any of its adjacent edges belong to a pair of parallel edges. Consider a bipartite graph G' on $n - 2$ edges obtained from G by splitting an edge adjacent to e . Let e denote the edge of G' corresponding to e (the one of the two added edges incident with an endvertex of e in G). It is easy to see that every perfect matching of G' avoiding e can be uniquely extended to a perfect matching of G .

Consider all four different graphs G'_1, G'_2, G'_3 , and G'_4 obtained by splitting one of the two edges adjacent to e incident with the same end-vertex. Then every perfect

matching of G avoiding e is obtained by extending a perfect matching avoiding e in precisely three out of the four smaller graphs, see Figure 2.1 for an illustration. Therefore,

$$m_e(G) = \frac{1}{3} \sum_{i=1}^4 m_e(G_i) \geq \frac{4}{3} \cdot \frac{3}{2} \left(\frac{4}{3}\right)^{(n-2)/2} = \frac{3}{2} \left(\frac{4}{3}\right)^{n/2}$$

as desired. □

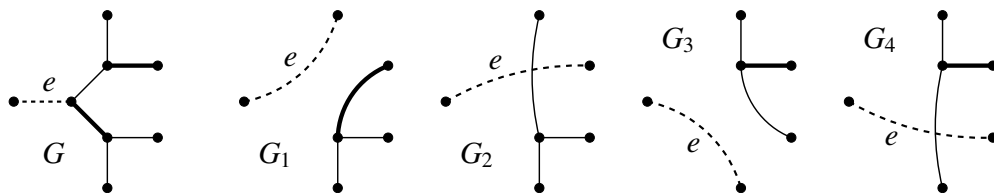


Figure 2.1: Splitting one of the two edges adjacent to e at the same end-vertex of G , in two different ways each, gives four different graphs G_1, \dots, G_4 .

Vorhoeve's trick with splitting a path cannot be applied directly in the general case, since splitting an edge in a bridgeless cubic graph might lead to a graph containing a bridge. Moreover, there are examples of bridgeless cubic graphs G containing an edge e with only a constant number of perfect matchings avoiding the edge e . (We will meet some of them later in the second part of this chapter when we will deal with Klee graphs.)

2.1.2 The general case: the context

Various different methods and tools had been used to find general lower bounds on the number of perfect matchings in bridgeless cubic graphs. Edmonds, Lovász and Pulleyblank [22] proved that any cubic bridgeless G contains at least $\frac{1}{4}|V(G)| + 2$ perfect matchings (see also [58]); this bound was later improved to $\frac{1}{2}|V(G)|$ [50] and then $\frac{3}{4}|V(G)| - 10$ [27]. The first bound, proved in 1982, is a direct consequence of a lower bound on the dimension of the perfect matching polytope, while the more recent bounds combine polyhedral arguments with analysis of brick and brace decompositions.

The fact that a progress on the constant in front of the linear term was considered as a significant progress and published in prestigious journal shows how difficult it seemed to grasp the conjecture at that time.

We started to work on this problem first with Louis Esperet and Dan Král' and we revisited the path-splitting method of Vorhoeve. The crucial observation that lead to a first super-linear lower bound on the number of perfect matching is the following: By splitting a path, the cyclic edge-connectivity cannot drop by more than 2; even if this is the case, it can only happen in at most one out of the four graphs obtained this way. Therefore, it suffices to find ad hoc arguments that yield a lower bound on the number of perfect matchings avoiding a given edge in cubic bridgeless graphs of low edge-connectivity (containing many 3- and 4-edge-cuts).

Finally, after Andrew King and Sergey Norine joined the team, we added a couple more ingredients: For graphs of low cyclic edge-connectivity the idea of counting perfect matchings avoiding a given edge was completely abandoned, and tools for finding linearly many disjoint alternating cycles are developped instead; for graphs with high cyclic edge-connectivity the perfect matchings *containing* a given edge are counted instead of those avoiding a given edge. To make the two approaches collaborate at the transition of the two states, probabilistic tools are introduced.

Let $m^*(G)$ denote the minimum, over all edges $e \in E(G)$, of the number of perfect matchings containing e . Our result is the following:

Theorem 18 *For every cubic bridgeless graph G we have $m(G) \geq 2^{|V(G)|/3656}$.*

We actually prove that at least one of two sufficient conditions applies:

Theorem 19 *For every cubic bridgeless graph G , at least one of the following holds:*

- [S1] $m^*(G) \geq 2^{|V(G)|/3656}$, or
- [S2] *there exist $M, M' \in \mathcal{M}(G)$ such that $M \triangle M'$ has at least $|V(G)|/3656$ components.*

To see that Theorem 19 implies Theorem 18, we can clearly assume that [S2] holds since $m^*(G) \leq m(G)$. Choose $M, M' \in \mathcal{M}(G)$ such that the set \mathcal{C} of components of $M \triangle M'$ has cardinality at least $|V(G)|/3656$, and note that each of these components is an even cycle alternating between M and M' . Thus for any subset $\mathcal{C}' \subseteq \mathcal{C}$, we can construct a perfect matching $M_{\mathcal{C}'}$ from M by flipping the edges on the cycles in \mathcal{C}' , i.e. $M_{\mathcal{C}'} = M \triangle \bigcup_{C \in \mathcal{C}'} C$. The $2^{|\mathcal{C}|}$ perfect matchings $M_{\mathcal{C}'}$ are distinct, implying Theorem 18.

We cannot discard either of the sufficient conditions [S1] or [S2] in the statement of Theorem 19. To see that [S2] cannot be omitted, consider the graph depicted in Figure 2.2 and observe that each of the four bold edges is contained in a unique perfect matching. To see that [S1] cannot be omitted, it is enough to note that there

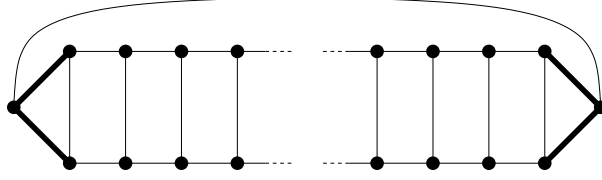


Figure 2.2: A graph cubic bridgeless graph G with $m^*(G) = 1$.

exist cubic graphs with girth logarithmic in their size (see [38] for a construction). Such graphs cannot have linearly many disjoint cycles, so condition [S2] does not hold.

2.1.2.1 Definitions and notation

For a graph G and a set $X \subseteq V(G)$, $G|X$ denotes the subgraph of G induced by X . For a set $X \subseteq V(G)$, let $\delta(X)$ denote the set of edges with exactly one endpoint in X , and let E_X denote the set of edges with at least one endpoint in X , i.e. $E_X = E(G|X) \cup \delta(X)$. The set $C = \delta(X) = E(X, Y)$ with $Y = V(G) \setminus X$ is an edge-cut, or a k -edge-cut, where $k = |C|$, and X and Y are the *sides* of C . A k -edge-cut is said to be *even* (resp. *odd*) if k is even (resp. odd). Observe that the parity of an edge-cut $\delta(X)$ in a cubic graph is precisely that of $|X|$. An edge-cut $\delta(X)$ is *cyclic* if both $G|X$ and $G|(V(G) \setminus X)$ contain a cycle. Observe that every 2-edge-cut in a cubic graph is cyclic. If G contains no edge-cut (resp. cyclic edge-cut) of size less than k , we say that G is k -edge-connected (resp. *cyclically k -edge-connected*).

Observe that the number of perfect matchings of a graph is the product of the number of perfect matchings of its connected components. Hence, in order to prove Theorem 18, we restrict ourselves to connected graphs for the remainder of this thesis (this means, for example, that we can consider the terms *2-edge-connected* and *bridgeless* to be interchangeable, and the sides of a cut are well-defined).

For a matching M and vertex set X , we say that M *covers* X or that X is *M -covered* if every vertex in X is an endpoint of an edge in M . Further, we use $M|X$ to denote the set $M \cap E(G|X)$.

2.1.2.2 Constants

Let $x := \log(\frac{4}{3})/\log(2)$. The following constants appear throughout the section:

$$\alpha := \frac{x}{314}, \quad \beta_1 := \frac{154x}{314}, \quad \beta_2 := \frac{74x}{314}, \quad \gamma := \frac{312x}{314}.$$

We avoid using the numerical values of these constants for the sake of clarity. Throughout the section we make use of the following inequalities, which can be routinely verified:

$$0 < \alpha \leq \beta_2 \leq \beta_1, \tag{2.1}$$

$$1/3656 \leq \frac{\alpha}{9\beta_1 + 3}, \tag{2.2}$$

$$\beta_2 + 6\alpha \leq \beta_1, \tag{2.3}$$

$$74\alpha \leq \beta_2, \tag{2.4}$$

$$146\alpha \leq \beta_1, \tag{2.5}$$

$$\beta_2 + 80\alpha \leq \beta_1, \tag{2.6}$$

$$6\alpha + \gamma \leq \log(6)/\log(2), \tag{2.7}$$

$$\gamma + 2\beta_1 + 7\alpha - \beta_2 \leq 1, \tag{2.8}$$

$$6\alpha + 2\beta_1 \leq \log(\frac{4}{3})/\log(2), \tag{2.9}$$

$$2\beta_1 + 4\alpha \leq \gamma. \tag{2.10}$$

The integer 3656 is chosen minimum so that the system of inequalities above has a solution. Inequalities (2.4), (2.6), (2.9), and (2.10) are tight.

2.1.3 Proof overview

In this section we sketch the proof of Theorem 19, postponing the proofs of two main lemmas until later sections. Our general approach to Theorem 19 is to reduce on cyclic 2-edge-cuts and cyclic 3-edge-cuts and prove inductively that either [S1] or [S2] holds. Dealing with [S1] is relatively straightforward – perfect matchings containing a given edge behave well with reductions on a cut, which is our main motivation for considering $m^*(G)$. To deal with [S2], we do not directly construct perfect matchings M and M' for which $M \Delta M'$ has many components. Instead, we prove the existence of a vector \mathbf{w} in the perfect matching polytope that in turn guarantees the existence of such perfect matchings M and M' . In order to do this, we define a special type of vertex set in which a given random perfect matching admits an alternating cycle with high probability (i.e. at least $\frac{1}{3}$). We call these sets *burls* and we call a set of disjoint burls a *foliage* – a large foliage will guarantee the existence of two perfect matchings with many components in their symmetric difference. In the end, the vector \mathbf{w} we seek in the perfect matching polytope will be uniformly valued $\frac{1}{3}$ except inside the burls.

2.1.3.1 Alternating sets and the perfect matching polytope

To define burls properly, we must first define three notions of a vertex set X being *alternating*. The first is simple. Given a matching M such that X is M -covered, we say that X is M -alternating if there is another matching M' such that X is M' -covered and $M \triangle M' \subseteq (G|X)$. The other two notions require consideration of random variables in $\mathcal{M}(G)$.

Let \mathbf{M} be a random perfect matching, i.e. a random variable \mathbf{M} in $\mathcal{M}(G)$, and let \mathbf{w} be a real edge weighting in $\mathbb{R}^{E(G)}$. We say that \mathbf{M} *corresponds to* \mathbf{w} (and vice-versa) if for every edge e , we have $\Pr[e \in \mathbf{M}] = \mathbf{w}(e)$. The *perfect matching polytope* $\mathcal{PMP}(G)$ is the set of edge weightings \mathbf{w} with at least one corresponding random variable $\mathbf{M}_{\mathbf{w}}$ on $\mathcal{M}(G)$. The second notion of an alternating set involves a weighting $\mathbf{w} \in \mathcal{PMP}(G)$. For such \mathbf{w} we say that X is \mathbf{w} -alternating if for every $\mathbf{M}_{\mathbf{w}}$ corresponding to \mathbf{w} , we have

$$\Pr[X \text{ is } \mathbf{M}_{\mathbf{w}}\text{-alternating}] \geq 1/3.$$

If $\{X_1, \dots, X_k\}$ is a collection of disjoint \mathbf{w} -alternating sets, then for a random variable $\mathbf{M}_{\mathbf{w}}$ in $\mathcal{M}(G)$ corresponding to \mathbf{w} , the probability that $\mathbf{M}_{\mathbf{w}}$ is X_i -alternating for at least $k/3$ values of i is non-zero. Thus [S2] is satisfied as long as we have a vector $\mathbf{w} \in \mathcal{PMP}(G)$ and a collection of at least $\frac{3}{3656} \cdot |V(G)|$ disjoint \mathbf{w} -alternating sets. Unfortunately the notion of \mathbf{w} -alternating sets has a troublesome shortcoming: When deciding whether or not X is a \mathbf{w} -alternating set, we want the freedom to ignore the weighting \mathbf{w} on edges not intersecting X .

Thus for the third notion of an alternating set, we look at partial edge weightings. Given a vertex set X , let \mathbf{w}_X be a weighting on the edges of E_X , i.e. those edges with at least one endpoint in X . Let $\mathcal{M}(G, X)$ denote the set of matchings contained in E_X and covering X . As with edge weightings in $\mathbb{R}^{E(G)}$, we say that a random variable $\mathbf{M}_{\mathbf{w}}^X \in \mathcal{M}(G, X)$ *corresponds to* \mathbf{w}_X (and vice-versa) if for every edge $e \in E_X$, we have $\Pr[e \in \mathbf{M}] = \mathbf{w}(e)$. We say that the set X is *strongly* \mathbf{w}_X -alternating if for every random variable $\mathbf{M}_{\mathbf{w}}^X$ on $\mathcal{M}(G, X)$ corresponding to \mathbf{w}_X , we have

$$\Pr[X \text{ is } \mathbf{M}_{\mathbf{w}}^X\text{-alternating}] \geq 1/3.$$

Given an edge weighting \mathbf{w} and an edge set E' such that \mathbf{w} gives each edge in E' a weight, let $\mathbf{w}|E'$ denote the restriction of \mathbf{w} to E' . Clearly, if we have a total edge weighting $\mathbf{w} \in \mathcal{PMP}(G)$ such that a vertex set X is strongly $(\mathbf{w}|E_X)$ -alternating, then X is \mathbf{w} -alternating.

We now extend this idea. We wish to take a collection of disjoint vertex sets $\{X_1, \dots, X_k\}$ and partial edge weightings \mathbf{w}_{X_i} such that each X_i is strongly \mathbf{w}_{X_i} -alternating, and construct from them a total edge weighting \mathbf{w} such that each X_i

is \mathbf{w} -alternating. To do this as simply as possible we want \mathbf{w} , which must be in $\mathcal{PMP}(G)$, to agree with each \mathbf{w}_{X_i} . Thus we certainly want the partial weightings to agree – this only concerns edges on the boundaries of the vertex sets – but we need more restrictions. To determine a sufficient set of restrictions for \mathbf{w}_{X_i} , we use Edmonds’ characterization of the perfect matching polytope:

Theorem 20 (Edmonds [20]) *Let G be a graph and let \mathbf{w} be a vector in $\mathbb{R}^{E(G)}$. Then \mathbf{w} is in $\mathcal{PMP}(G)$ precisely if the following hold:*

- (i) $0 \leq \mathbf{w}(e) \leq 1$ for each $e \in E(G)$,
- (ii) $\mathbf{w}(\delta(\{v\})) = 1$ for each vertex $v \in V$, and
- (iii) $\mathbf{w}(\delta(X)) \geq 1$ for each $X \subseteq V$ of odd cardinality.

This characterization immediately tells us that for any bridgeless cubic graph, the vector $\frac{1}{3}$, i.e. the vector valued $\frac{1}{3}$ on each edge, is in $\mathcal{PMP}(G)$. Given a vertex set X , let ∂X denote the set of vertices in X incident to edges in $\delta(X)$. We say that a partial edge weighting \mathbf{w}_X on E_X is *extendable from X* if it satisfies the following sufficient restrictions:

EXT1 $\mathbf{w}_X(e) \in \{0, \frac{1}{3}, \frac{2}{3}\}$ for each $e \in E_X$,

EXT2 $\mathbf{w}_X(\delta(\{v\})) = 1$ for each vertex $v \in X$,

EXT3 $\mathbf{w}_X(e) = \frac{1}{3}$ for each $e \in \delta(X)$,

EXT4 $\mathbf{w}_X(C) \geq \frac{1}{3}$ for every non-empty edge-cut C in $G|X$,

EXT5 if $\mathbf{w}_X(C) < 1$ for some edge-cut C in $G|X$ with $|C|$ odd then either $|C| = 1$ or one of the sides of C contains exactly one vertex in ∂X .

We are finally ready to formally define burls and foliages. A vertex set X is a *burl* if there exists a vector $\mathbf{w}_X \in \mathbb{R}^{E_X}$ such that (1) X is \mathbf{w}_X -alternating, and (2) \mathbf{w}_X is extendable from X . In this case we say that \mathbf{w}_X is a *certificate* for the burl X . Again, a collection of disjoint vertex sets $\{X_1, \dots, X_k\}$ is a *foliage* if each X_i is a burl.

We already noted that $\frac{1}{3} \in \mathcal{PMP}(G)$. Thus we can easily verify that for any vertex set X , the partial weighting $\frac{1}{3}|E_X$ is extendable from X . Actually, much more is true. The following lemma clarifies our motivation for the definition of a foliage:

Lemma 20 *Let G be a cubic bridgeless graph, let $\mathcal{X} = \{X_1, \dots, X_k\}$ be a foliage, and for each i let \mathbf{w}_{X_i} be a certificate for X_i . Let \mathbf{w} be an edge weighting for G defined as*

$$\mathbf{w}(e) = \begin{cases} \mathbf{w}_{X_i}(e) & \text{if } e \in E(G|X_i) \\ 1/3 & \text{if } e \notin \bigcup_i E(G|X_i) \end{cases}$$

Then every set $X_i \in \mathcal{X}$ is \mathbf{w} -alternating.

Proof. Since every partial weighting \mathbf{w}_{X_i} is equal to $\frac{1}{3}$ on the boundary of X_i , we know that each X_i is strongly $(\mathbf{w}|_{E_{X_i}})$ -alternating. Therefore each X_i is \mathbf{w} -alternating. It remains to confirm that $\mathbf{w} \in \mathcal{PMP}(G)$. By Theorem 20 it suffices to check that \mathbf{w} satisfies conditions (i), (ii) and (iii). The first two conditions are satisfied by (EXT1), (EXT2) and (EXT3). To verify (iii), consider an odd $Y \subseteq V(G)$. We show that $\mathbf{w}(\delta(Y)) \geq 1$.

It follows from (EXT1) and (EXT2) that $3\mathbf{w}(\delta(Y))$ is an odd integer. Therefore, it is sufficient to verify that $\mathbf{w}(\delta(Y)) > 1/3$. Let $X_i \in \mathcal{X}$ be such that $C = \delta(Y) \cap E(G|X_i)$ is a non-empty edge-cut in $G|X_i$. (If no such X_i exists then $\mathbf{w}(\delta(Y)) \geq \frac{1}{3}|\delta(Y)| \geq 1$ by (EXT3).) It follows from (EXT1), (EXT2) and (EXT3) that $|C|$ and $3\mathbf{w}(C)$ have the same parity. Therefore, $\mathbf{w}(\delta(Y)) > 1/3$ by (EXT3) and (EXT4), unless $|C|$ is odd and $\delta(Y) = C$. In this last case, we have $|C| > 1$, as G is bridgeless and by (EXT5) one of the sides of C , without loss of generality $X_i \cap Y$, contains exactly one vertex in ∂X_i . Then $\delta(Y \setminus X_i)$ consists only of edges incident to this vertex, contradicting once again the fact that G is bridgeless. \square

In light of what we have already discussed, we get the following key fact as a consequence:

Corollary 4 *If a cubic bridgeless graph G contains a foliage \mathcal{X} , then there exist perfect matchings $M, M' \in \mathcal{M}(G)$ such that $M \triangle M'$ has at least $|\mathcal{X}|/3$ components.*

2.1.3.2 Burls, twigs, and foliage weight

We now introduce a special class of burls. Let G be a cubic bridgeless graph and let $X \subseteq V(G)$. We say that X is a *2-twig* if $|\delta(X)| = 2$, and X is a *3-twig* if $|\delta(X)| = 3$ and $|X| \geq 5$ (that is, X is not a triangle or a single vertex). A *twig* in G is a 2- or 3-twig. Before we prove that every twig is a burl, we need a simple lemma.

Lemma 21 *Let G be a cubic bridgeless graph. Then*

1. $m(G - e) \geq 2$ for every $e \in E(G)$, and

2. $m(G) \geq 4$ if $|V(G)| \geq 6$. In particular, for any $v \in V(G)$ there is an $e \in \delta(\{v\})$ contained in at least two perfect matchings.

Proof. The first item follows from the classical result mentioned in the introduction: every edge of a cubic bridgeless graph is contained in a perfect matching. The second is implied by the bound $m(G) \geq \frac{1}{4}|V(G)| + 2$ from [22]. \square

Lemma 22 *Every twig X in a cubic bridgeless graph G is a burl.*

Proof. We show that $\mathbf{w}_X = \frac{1}{3}|E_X|$ is a certificate for X . As we already noted, \mathbf{w}_X is extendable from X . Let \mathbf{M}_w^X be a random matching in $\mathcal{M}(G, X)$ corresponding to \mathbf{w}_X , as in the definition of a strongly alternating set.

If X is a 2-twig, let H be obtained from $G|X$ by adding an edge e joining the vertices in ∂X . Then H is cubic and bridgeless. By applying Lemma 21(1) to H , we see that the set X is M -alternating for every $M \in \mathcal{M}(G, X)$ such that $M \cap \delta(X) = \emptyset$. As $\Pr[\mathbf{M}_w^X \cap \delta(X) = \emptyset] \geq 1 - \mathbf{M}_w^X(\delta(X)) = 1/3$, we conclude that X is strongly \mathbf{w}_X -alternating.

Suppose now that X is a 3-twig. Let $\delta(X) = \{e_1, e_2, e_3\}$. Let H be obtained from G by identifying all the vertices in $V(G) - X$ (removing loops but preserving multiple edges). We apply Lemma 21(2) to H , which is again cubic and bridgeless. It follows that for some $1 \leq i \leq 3$, the edge e_i is in at least two perfect matchings of H . Therefore X is M -alternating for every $M \in \mathcal{M}(G, X)$ such that $M \cap \delta(X) = \{e_i\}$. Finally, $\Pr[\mathbf{M}_w^X \cap \delta(X) = \{e_i\}] = 1/3$ and thus X is strongly \mathbf{w}_X -alternating. \square

The *weight* of a foliage \mathcal{X} containing k twigs is defined as $fw(\mathcal{X}) := \beta_1 k + \beta_2(|\mathcal{X}| - k)$, that is each twig has weight β_1 and each non-twig burl has weight β_2 . Let $fw(G)$ denote the maximum weight of a foliage in a graph G .

2.1.3.3 Reducing on small edge-cuts

We now describe how we reduce on 2-edge-cuts and 3-edge-cuts, and consider how these operations affect $m^*(G)$ and foliages. Let C be a 3-edge-cut in a cubic bridgeless graph G . The two graphs G_1 and G_2 obtained from G by identifying all vertices on one of the sides of the edge-cut (removing loops but preserving multiple edges) are referred to as C -contractions of G and the vertices in G_1 and G_2 created by this identification are called *new*.

We need a similar definition for 2-edge-cuts. Let $C = \{e, e'\}$ be a 2-edge-cut in a cubic bridgeless graph G . The two C -contractions G_1 and G_2 are now obtained

from G by deleting all vertices on one of the sides of C and adding an edge joining the remaining ends of e and e' . The resulting edge is now called *new*.

In both cases we say that G_1 and G_2 are obtained from G by a cut-contraction. The next lemma provides some useful properties of cut-contractions.

Lemma 23 *Let G be a graph, let C be a 3- or a 2-edge-cut in G , and let G_1 and G_2 be the two C -contractions. Then*

1. G_1 and G_2 are cubic bridgeless graphs,
2. $m^*(G) \geq m^*(G_1) m^*(G_2)$, and
3. For $i = 1, 2$ let \mathcal{X}_i be a foliage in G_i such that for every $X \in \mathcal{X}_i$, if $|C| = 3$ then X does not contain the new vertex, and if $|C| = 2$ then $E(G_i|X)$ does not contain the new edge. Then $\mathcal{X}_1 \cup \mathcal{X}_2$ is a foliage in G . In particular, we have $fw(G) \geq fw(G_1) + fw(G_2) - 2\beta_1$.

Proof.

1. This can be confirmed routinely.
2. Consider first the case of the contraction of a 2-edge-cut $C = \delta(X)$ in G . Let e be an edge with both ends in $X = V(G_1)$. Every perfect matching of G_1 containing e combines either with $m^*(G_2)$ perfect matchings of G_2 containing the new edge of G_2 , or with $2m^*(G_2)$ perfect matchings of G_2 avoiding the new edge of G_2 . If e lies in C , note that perfect matchings of G_1 and G_2 containing the new edges can be combined into perfect matchings of G containing C . Hence, e is in at least $m^*(G_1) m^*(G_2)$ perfect matchings of G .
Now consider a 3-edge-cut $C = \delta(X)$. If e has both ends in $X \subset V(G_1)$, perfect matchings of G_1 containing e combine with perfect matchings of G_2 containing either of the 3 edges of C . If e is in C , perfect matchings containing e in G_1 and G_2 can also be combined into perfect matchings of G . In any case, e is in at least $m^*(G_1) m^*(G_2)$ perfect matchings of G .
3. By (EXT3), the coordinates of the new elements (if they are defined) in their respective certificates are precisely $1/3$, so assigning $1/3$ (if necessary) to the edges of C in G yields valid certificates for the elements of $\mathcal{X}_1 \cup \mathcal{X}_2$. Since $\beta_1 \geq \beta_2$, this implies $fw(G) \geq fw(G_1) + fw(G_2) - 2\beta_1$. \square

□

It is not generally advantageous to reduce on a 3-edge-cut arising from a triangle, unless this reduction leads to a chain of similar reductions. Thus we wish to get rid of certain triangles from the outset. We say that a triangle sharing precisely one edge with a cycle of length three or four in a graph G is *relevant*, and otherwise it is *irrelevant*. A graph G is *pruned* if it contains no irrelevant triangles. The following easy lemma shows that we can prune a bridgeless cubic graph by repeated cut-contraction without losing too many vertices.

Lemma 24 *Let G be a cubic bridgeless graph, and let k be the size of maximum collection of vertex-disjoint irrelevant triangles in G . Then one can obtain a pruned cubic bridgeless graph G' from G with $|V(G')| \geq |V(G)| - 2k$ by repeatedly contracting irrelevant triangles.*

Proof. We proceed by induction on k . Let a graph G'' be obtained from G by contracting an irrelevant triangle T . The graph G'' is cubic and bridgeless by Lemma 23(1). Since T is irrelevant in G , the unique vertex of G'' obtained by contracting T is not in a triangle in G'' . Therefore if \mathcal{T} is a collection of vertex disjoint irrelevant triangles in G'' then $\mathcal{T} \cup \{T\}$ is such a collection in G . (After the contraction of an irrelevant triangle, triangles that were previously irrelevant might become relevant, but the converse is not possible.) It follows that $|\mathcal{T}| \leq k - 1$. By applying the induction hypothesis to G'' , we see that the lemma holds for G . □

Corollary 5 *Let G be a cubic bridgeless graph. Then we can obtain a cubic bridgeless pruned graph G' from G with $|V(G')| \geq |V(G)|/3$ by repeatedly contracting irrelevant triangles.*

We wish to restrict our attention to pruned graphs, so we must make sure that the function $m^*(G)$ and the maximum size of a foliage does not increase when we contract a triangle.

Lemma 25 *Let G' be obtained from a graph G by contracting a triangle. Then $m^*(G') \leq m^*(G)$ and the maximum size of a foliage in G' is at most the maximum size of a foliage in G .*

Proof. Let xyz be the contracted triangle, and let e_x , e_y , and e_z be the edges incident with x , y , z and not contained in the triangle in G . Let t be the vertex of G' corresponding to the contraction of xyz . Every perfect matching M' of G' has a

canonical extension M in G : assume without loss of generality that e_x is the unique edge of M' incident to t . Then M consists of the union of M' and yz . Observe that perfect matchings in G containing yz necessarily contain e_x , so every edge of G is contained in at least $m^*(G')$ perfect matchings.

Now consider a burl X' in G' containing t , and let \mathbf{w}' be a certificate for X' . Let \mathbf{w} be the vector \mathbf{w}' with three new coordinates $\mathbf{w}(xy) = \mathbf{w}'(e_z)$, $\mathbf{w}(yz) = \mathbf{w}'(e_x)$ and $\mathbf{w}(xz) = \mathbf{w}'(e_y)$, then \mathbf{w} is a certificate showing that $X = X' \cup \{x, y, z\} \setminus t$ is a burl in G . Properties (EXT1), (EXT2), and (EXT3) are trivially satisfied. Now consider an edge-cut C in $G|X$. If $B = C \cap \{xy, yz, xz\}$ is empty, (EXT4) and (EXT5) follow directly from the fact that \mathbf{w}' is a certificate for X' . Otherwise B contains precisely two elements, say xy and yz . Then we have $\mathbf{w}(C) \geq \mathbf{w}(xy) + \mathbf{w}(yz) \geq \frac{1}{3}$ by (EXT1) and (EXT2), and therefore, (EXT4) follows. If $|C| \geq 3$ is odd and $\mathbf{w}(C) < 1$, then without loss of generality $\mathbf{w}(xy) = 0$. Using (EXT4) it can be checked that only one of the following two cases applies:

If $C \cap \{e_x, e_z\} = \emptyset$ then $C' = C \cup \{e_x, e_z\} \setminus \{xy, yz\}$ is an edge-cut of the same weight and cardinality as C in $G|X$, but also in $G'|X'$, and consequently, (EXT5) follows.

If $C \cap E_{\{x, y, z\}} = \{xy, yz, e_z\}$ then C has cardinality at least five and $C'' = C \cup \{e_x\} \setminus \{xy, yz, e_z\}$ is an odd edge-cut in $G|X$, but also in $G'|X'$ of cardinality at least 3 and weight $\mathbf{w}(C'') = \mathbf{w}(C)$. Since \mathbf{w}' satisfies (EXT5), \mathbf{w} also satisfies (EXT5) in this case.

Since a burl avoiding t in G' is also a burl in G , it follows from the analysis above that the maximum size of a foliage cannot increase when we contract a triangle. \square

2.1.3.4 Proving Theorem 19

We say that G has a core if we can obtain a cyclically 4-edge-connected graph G' with $|V(G')| \geq 6$ by applying a (possibly empty) sequence of cut-contractions to G (recall that this notion was defined in the previous subsection).

We will deduce Theorem 19 from the next two lemmas. This essentially splits the proof into two cases based on whether or not G has a core.

Lemma 26 *Let G be a pruned cubic bridgeless graph. Let $Z \subseteq V(G)$ be such that $|Z| \geq 2$ and $|\delta(Z)| = 2$, or $|Z| \geq 4$ and $|\delta(Z)| = 3$. Suppose that the $\delta(Z)$ -contraction G' of G with $Z \subseteq V(G')$ has no core. Then there exists a foliage \mathcal{X} in G with $\bigcup_{X \in \mathcal{X}} X \subseteq Z$ and*

$$fw(\mathcal{X}) \geq \alpha|Z| + \beta_2.$$

By applying Lemma 26 to a cubic graph G without a core and $Z = V(G) \setminus \{v\}$ for some $v \in V(G)$, we obtain the following.

Corollary 6 *Let G be a pruned cubic bridgeless graph without a core. Then*

$$fw(G) \geq \alpha(|V(G)| - 1) + \beta_2.$$

On the other hand, if G has a core, we will prove that either $fw(G)$ is linear in the size of G or every edge of G is contained in an exponential number of perfect matchings.

Lemma 27 *Let G be a pruned cubic bridgeless graph. If G has a core then*

$$m^*(G) \geq 2^{\alpha|V(G)| - fw(G) + \gamma}.$$

We finish this section by deriving Theorem 19 from Lemmas 26 and 27.

Proof. (of Theorem 19) Let $\epsilon := 1/3656$. By Corollary 5 there exists a pruned cubic bridgeless graph G' with $|V(G')| \geq |V(G)|/3$ obtained from G by repeatedly contracting irrelevant triangles. Suppose first that G' has a core. By Corollary 5 and Lemmas 25 and 27, condition [S1] holds as long as $\epsilon|V(G)| \leq \alpha|V(G)|/3 - fw(G')$. Therefore we assume $fw(G') \geq (\frac{\alpha}{3} - \epsilon)|V(G)|$. It follows from the definition of $fw(G')$ that G' has a foliage containing at least $(\frac{\alpha}{3} - \epsilon)|V(G)|/\beta_1$ burls. If G' has no core then by Corollary 6 and the fact that $\alpha \leq \beta_2$, $fw(G') \geq \alpha(|V(G')| - 1) + \beta_2 \geq \alpha|V(G')|$, so G' contains a foliage of size at least $\alpha|V(G')|/\beta_1 \geq \alpha|V(G)|/3\beta_1$. In both cases condition [S2] holds by Corollary 4 and Lemma 25, since Equation (2.2) tells us that $3\epsilon \leq (\frac{\alpha}{3} - \epsilon)/\beta_1$. \square

2.1.4 Cut decompositions

In this section we study cut decompositions of cubic bridgeless graphs. We mostly follow notation from [8], however we consider 2- and 3-edge-cuts simultaneously. Cut decompositions play a crucial role in the proof of Lemma 26 in the next section.

Let G be a graph. A *non-trivial cut-decomposition* of G is a pair (T, ϕ) such that:

- T is a tree with $E(T) \neq \emptyset$,
- $\phi : V(G) \rightarrow V(T)$ is a map, and
- $|\phi^{-1}(t)| + \deg_T(t) \geq 3$ for each $t \in V(T)$.

For an edge f of T , let T_1, T_2 be the two components of $T \setminus f$, and for $i = 1, 2$ let $X_i = \phi^{-1}(T_i)$. Thus (X_1, X_2) is a partition of $V(G)$ that induces an edge-cut denoted by $\phi^{-1}(f)$. If $|\phi^{-1}(f)| \in \{2, 3\}$ for each $f \in E(T)$ we call (T, ϕ) a *small-cut-decomposition* of G .

Let (T, ϕ) be a small-cut-decomposition of a 2-edge-connected cubic graph G , and let T_0 be a subtree of T such that $\phi^{-1}(V(T_0)) \neq \emptyset$. Let T_1, \dots, T_s be the components of $T \setminus V(T_0)$, and for $1 \leq i \leq s$ let f_i be the unique edge of T with an end in $V(T_0)$ and an end in $V(T_i)$. For $0 \leq i \leq s$, let $X_i = \phi^{-1}(V(T_i))$. Thus X_0, X_1, \dots, X_s form a partition of $V(G)$. Let G' be the graph obtained from G as follows. Set $G_0 = G$. For $i = 1, \dots, s$, take G_{i-1} and let G_i be the $(\phi^{-1}(f_i))$ -contraction containing X_0 . Now let G' denote G_s . Note that G' is cubic. We call G' the *hub of G at T_0* (with respect to (T, ϕ)). If $t_0 \in V(T)$ and $\phi^{-1}(t_0) \neq \emptyset$, by the *hub of G at t_0* we mean the hub of G at T_0 , where T_0 is the subtree of T with vertex set $\{t_0\}$.

Let \mathcal{Y} be a collection of disjoint subsets of $V(G)$. We say that a small-cut-decomposition (T, ϕ) of G *refines* \mathcal{Y} if for every $Y \in \mathcal{Y}$ there exists a leaf $v \in V(T)$ such that $Y = \phi^{-1}(v)$. Collections of subsets of $V(G)$ that can be refined by a small-cut decomposition are characterized in the following easy lemma.

Lemma 28 *Let G be a cubic bridgeless graph. Let \mathcal{Y} be a collection of disjoint subsets of $V(G)$. Then there exists a small-cut-decomposition refining \mathcal{Y} if $|\mathcal{Y}| \geq 2$ and $|\delta(Y)| \in \{2, 3\}$ for every $Y \in \mathcal{Y}$, and either*

1. $\mathcal{Y} = \emptyset$ and G is not cyclically 4-edge-connected, or
2. $\mathcal{Y} = \{Y\}$, and $|V(G) \setminus Y| > 1$, or
3. $|\mathcal{Y}| \geq 2$.

Proof. We only consider the case $|\mathcal{Y}| \geq 3$, as the other cases are routine. Take T to be a tree on $|\mathcal{Y}| + 1$ vertices with $|\mathcal{Y}|$ leaves $\{v_Y \mid Y \in \mathcal{Y}\}$ and a non-leaf vertex v_0 . The map ϕ is defined by $\phi(u) = v_Y$, if $u \in Y$ for some $Y \in \mathcal{Y}$, and $\phi(u) = v_0$, otherwise. Clearly, (T, ϕ) refines \mathcal{Y} and is a small-cut-decomposition of G . \square

We say that (T, ϕ) is \mathcal{Y} -*maximum* if it refines \mathcal{Y} and $|V(T)|$ is maximum among all small-cut decompositions of G refining \mathcal{Y} . The following lemma describes the structure of \mathcal{Y} -maximum decompositions. It is a variation of Lemma 4.1 and Claim 1 of Lemma 5.3 in [8].

Lemma 29 *Let G be a cubic bridgeless graph. Let \mathcal{Y} be a collection of disjoint subsets of $V(G)$ and let (T, ϕ) be a \mathcal{Y} -maximum small-cut-decomposition of G . Then for every $t \in V(T)$ either $\phi^{-1}(t) = \emptyset$, or $\phi^{-1}(t) \in \mathcal{Y}$, or the hub of G at t is cyclically 4-edge-connected.*

Proof. Fix $t \in V(T)$ with $\phi^{-1}(t) \neq \emptyset$ and $\phi^{-1}(t) \notin \mathcal{Y}$. Let f_1, \dots, f_k be the edges of T incident with t , and let T_1, \dots, T_k be the components of $T \setminus \{t\}$, where f_i is incident with a vertex t_i of T_i for $1 \leq i \leq k$. Let $X_0 = \phi^{-1}(t)$, and for $1 \leq i \leq k$ let $X_i = \phi^{-1}(V(T_i))$. Let G' be the hub of G at t , and let G'' be the graph obtained from G' by subdividing precisely once every new edge e corresponding to the cut-contraction of a cut C with $|C| = 2$. The vertex on the subdivided edge e is called *the new vertex corresponding to the cut-contraction of C* , by analogy with the new vertex corresponding to the cut-contraction of a cyclic 3-edge-cut.

Note that G' is cyclically 4-edge-connected if and only if G'' is cyclically 4-edge-connected. Suppose for the sake of contradiction that $C = \delta(Z)$ is a cyclic edge-cut in G'' with $|C| \leq 3$. Then $|C| \in \{2, 3\}$ by Lemma 23(1), as G'' is a subdivision of G' and G' can be obtained from G by repeated cut-contractions. Let T' be obtained from T by splitting t into two vertices t' and t'' , so that t_i is incident to t' if and only if the new vertex of G'' corresponding to the cut-contraction of $\phi^{-1}(f_i)$ is in Z . Let $\phi'(t') = X_0 \cap Z$, $\phi'(t'') = X_0 \setminus Z$, and $\phi'(s) = \phi(s)$ for every $s \in V(T') \setminus \{t', t''\}$.

We claim that (T', ϕ') is a small-cut-decomposition of G contradicting the choice of T . It is only necessary to verify that $|\phi^{-1}(s)| + \deg_{T'}(s) \geq 3$ for $s \in \{t', t''\}$. We have $|\phi^{-1}(t')| + \deg_{T'}(t') - 1 = |Z \cap V(G'')| \geq 2$ as C is a cyclic edge-cut in G'' . It follows that $|\phi^{-1}(t')| + \deg_{T'}(t') \geq 3$ and the same holds for t'' by symmetry. \square

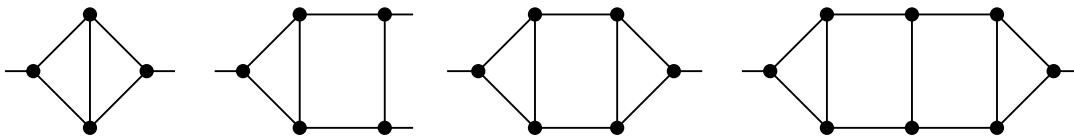


Figure 2.3: Isomorphism classes of subgraphs induced by elementary twigs.

We finish this section by describing a collection \mathcal{Y} to which we will be applying Lemma 29 in the sequel. In a cubic bridgeless graph G a union of the vertex set of a relevant triangle with the vertex set of a cycle of length at most four sharing an edge with it is called a *simple twig*. Note that simple twigs corresponding to distinct relevant triangles can intersect, but one can routinely verify that each simple twig intersects a simple twig corresponding to at most one other relevant triangle. An *elementary twig* is either a simple twig, that intersects no simple twig corresponding to a relevant triangle not contained in it, or the union of two intersecting simple twigs, corresponding to distinct relevant triangles. An elementary twig is, indeed, a twig, unless it constitutes the vertex set of the entire graph. Figure 2.3 shows all possible elementary twigs. The next corollary follows immediately from the observations above and Lemmas 28 and 29.

Corollary 7 *Let G be a cubic bridgeless graph that is not cyclically 4-edge-connected with $|V(G)| \geq 8$. Then there exists a collection \mathcal{Y} of pairwise disjoint elementary twigs in G such that every relevant triangle in G is contained in an element of \mathcal{Y} . Further, there exists a \mathcal{Y} -maximum small-cut-decomposition (T, ϕ) of G and for every $t \in V(T)$ either $\phi^{-1}(t) = \emptyset$, or $\phi^{-1}(t)$ is an elementary twig, or the hub of G at t is cyclically 4-edge-connected.*

2.1.5 Proof of Lemma 26

The proof of Lemma 26 is based on our ability to find burls locally in the graph. The following lemma is a typical example.

Lemma 30 *Let G be a cubic bridgeless graph and let $X \subseteq V(G)$ be such that $|\delta(X)| = 4$ and $m(G|X) \geq 2$. Then X contains a burl.*

Proof. Let $\mathbf{w} = \frac{1}{3}|E_X$. We already observed that \mathbf{w} is extendable from X . Note that if $M \in \mathcal{M}(G, X)$ contains no edges of $\delta(X)$ then X is M -alternating. As $M \cap \delta(X)$ is even for every $M \in \mathcal{M}(G, X)$ we have

$$\frac{4}{3} = \mathbb{E}[|\mathbf{M}_{\mathbf{w}} \cap \delta(X)|] \geq 2 \Pr[\mathbf{M}_{\mathbf{w}} \cap \delta(X) \neq \emptyset].$$

Therefore $\Pr[\mathbf{M}_{\mathbf{w}} \cap \delta(X) = \emptyset] \geq 1/3$, and so X is strongly \mathbf{w} -alternating. \square

The proof of Lemma 26 relies on a precise study of the structure of small-cut trees for graphs with no core. The following two lemmas indicate that long paths in such trees necessarily contain some burls.

Lemma 31 *Let (T, ϕ) be a small-cut-decomposition of a cubic bridgeless graph G , and let P be a path in T with $|V(P)| = 10$. If we have*

- $\deg_T(t) = 2$ for every $t \in V(P)$,
- the hub of G at t is isomorphic to K_4 for every $t \in V(P)$, and
- $|\phi^{-1}(f)| = 3$ for every edge $f \in E(T)$ incident to a vertex in $V(P)$,

then $\phi^{-1}(P)$ contains a burl.

Proof. Let $P' = v_{-1}v_0 \dots v_9v_{10}$ be a path in T such that $P = v_0 \dots v_{10}$. Let $f_i = v_{i-1}v_i$ and let $C_i = \phi^{-1}(f_i)$, $0 \leq i \leq 10$. Let $X := \phi^{-1}(V(P))$. We assume without loss of generality that $G|X$ contains no cycles of length 4, as otherwise

the lemma holds by Lemma 30. Let A be the set of ends of edges in C_0 outside of X , and let B be the set of ends of edges in C_{10} outside of X . Observe that E_X consists of 3 internally vertex-disjoint paths from A to B , as well as one edge in $G[\phi^{-1}(\{v_i\})]$ for $0 \leq i \leq 9$. Let R_1, R_2 and R_3 be these three paths from A to B , and let u_j be the end of R_j in A for $j = 1, 2, 3$. For $0 \leq i \leq 9$, we have $\phi^{-1}(v_i) = \{x_i, y_i\}$ so that $x_i \in V(R_j), y_i \in V(R_{j'})$ for some $\{j, j'\} \subseteq \{1, 2, 3\}$ with $j \neq j'$, and $e_i := x_i y_i \in E(G)$. Let the *index* of i be defined as $(\{j, j'\}, \text{sgn}(i))$, where $\text{sgn}(i) = 0$ if the number of vertices in R_j between u_j and x_i and the number of vertices in $R_{j'}$ between $u_{j'}$ and y_i have the same parity, and $\text{sgn}(i) = 1$ otherwise. There are 6 possible indices, so there exist $1 \leq i < i' \leq 7$ with the same indices. Without loss of generality we assume that those indices are $(\{1, 2\}, 0)$ or $(\{1, 2\}, 1)$.

To show that X is a burl, we construct a certificate \mathbf{w} on E_X . We first set $\mathbf{w}(e) = \frac{1}{3}$ for every $e \in \delta(X)$. We then set $\mathbf{w}(e_{i''}) = 0$ for $i < i'' < i'$ and $\mathbf{w}(e_{i''}) = \frac{1}{3}$ for $0 \leq i'' \leq i$ and $i' < i'' \leq 9$. On the edges of R_1, R_2, R_3 , and $e_{i'}$, we let \mathbf{w} be the unique assignment of weights that satisfies conditions (EXT2) and (EXT3), which gives each such edge weight $\frac{1}{3}$ or $\frac{2}{3}$ on the paths and gives $\mathbf{w}(e_{i'})$ weight either 0 or $\frac{1}{3}$, depending on the parity of $i' - i$. Two examples are shown in Figure 2.4.

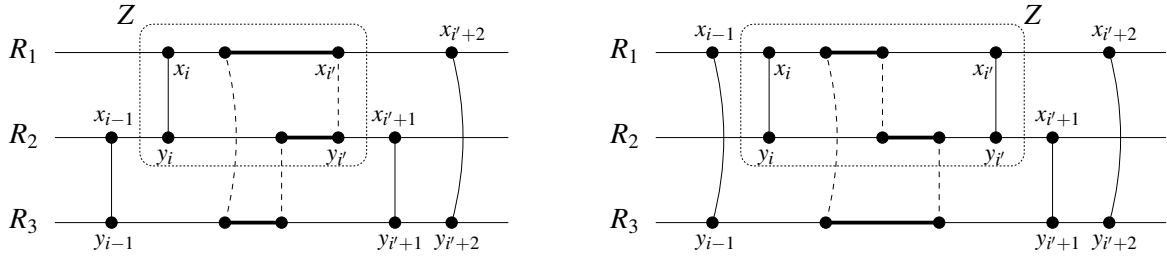


Figure 2.4: Certificates for the burl X when $i' - i$ is odd (left) and when $i' - i$ is even (right). Horizontal paths are R_1, R_2 and R_3 , solid edges correspond to the value $1/3$ of \mathbf{w} , bold edges to value $2/3$ and dashed edges to value 0.

We claim that \mathbf{w} is a certificate for X . Let Z consist of $x_i, y_i, x_{i'}, y_{i'}$ and vertex sets of segments of R_1 and R_2 between these vertices. The only edges in support of \mathbf{w} in $\delta(Z)$ belong to either R_1 or R_2 . As $|Z|$ is even, repeating the argument in the proof of Lemma 30, we deduce that $\Pr[\mathbf{M}_{\mathbf{w}} \cap \delta(Z) = \emptyset] \geq 1/3$. As $G[Z]$ contains a spanning even cycle, and therefore at least two perfect matchings, we conclude that Z , and consequently X , are strongly \mathbf{w} -alternating. It is easy to see that \mathbf{w} satisfies (EXT4). By our assumption that $\phi^{-1}(P)$ contains no cycles of length 4, the edges $e_{i-1}, e_{i'+1}$ have ends on R_3 and both R_1 and R_2 contain an end of one of the edges e_{i+1} and $e_{i'+2}$ (we insist that P contains 10 rather than 7 vertices to ensure this

property). Using this fact, one can routinely verify that \mathbf{w} satisfies (EXT5) and is therefore extendable from X . \square

Lemma 32 *Let (T, ϕ) be a small-cut-decomposition of a cubic bridgeless graph G . Let $t_1, t_2 \in V(T)$ be a pair of adjacent vertices of degree 2. Suppose that $|\phi^{-1}(f)| = 2$ for every edge $f \in E(T)$ incident to t_1 or t_2 . Then $\phi^{-1}(\{t_1, t_2\})$ contains a burl.*

Proof. Let $t_0 t_1 t_2 t_3$ be a subpath of T and let $C_i = \phi^{-1}(t_{i-1} t_i)$ for $i = 1, 2, 3$ be an edge-cut of size 2. Assume that both $G|_{\phi^{-1}(t_1)}$ and $G|_{\phi^{-1}(t_2)}$ have at most one perfect matching. By Lemma 30 it suffices to show that $G|_{\phi^{-1}(\{t_1, t_2\})}$ has at least two perfect matchings. As the hub G_1 over t_1 is cubic and bridgeless it contains at least 2 perfect matching avoiding any edge. Let $e_1, e_2 \in E(G_1)$ be the edges in $E(G_1) - E(G)$ corresponding to C_1 - and C_2 -contraction, respectively. By assumption, at most one perfect matching of G_1 avoids both e_1 and e_2 . It follows that either two perfect matchings of G_1 avoid e_1 and contain e_2 , or one avoids e_1 and e_2 and one avoids e_1 and contains e_2 . Let G_2 be the hub over t_2 . The symmetric statement holds for G_2 . In any case, the perfect matchings in G_1 and G_2 can be combined to obtain at least two perfect matchings of $G|_{\phi^{-1}(\{t_1, t_2\})}$. \square

From the definition of a small-cut-decomposition, we immediately get the following corollary:

Corollary 8 *Let (T, ϕ) be a small-cut-decomposition of a cubic bridgeless graph G , and let P be a path in T in which every vertex has degree 2. Suppose there exist three edges f_1, f_2, f_3 of T incident to vertices of P such that $|\phi^{-1}(f_1)| = |\phi^{-1}(f_2)| = |\phi^{-1}(f_3)| = 2$. Then $\phi^{-1}(P)$ contains a burl.*

Let B_3 denote the cubic graph consisting of two vertices joined by three parallel edges. Lemmas 31 and 32 imply the following.

Corollary 9 *Let (T, ϕ) be a small-cut-decomposition of a cubic bridgeless graph G and let P be a path in T with $|V(P)| = 32$. If for every $t \in V(P)$, $\deg_T(t) = 2$ and the hub of G at t is isomorphic to K_4 or B_3 , then $\phi^{-1}(P)$ contains a burl.*

Proof. If at least three edges incident to vertices in $V(P)$ correspond to edge-cuts of size 2 in G then the corollary holds by Corollary 8. Otherwise, since there are 33 edges of T incident to vertices of P , there must be 11 consecutive edges incident to vertices in P corresponding to edge-cuts of size 3. In this case, the result follows from Lemma 31. \square

Proof. (of Lemma 26) We proceed by induction on $|Z|$. If $|Z| \leq 6$ then Z is a twig. In this case the lemma holds since $\beta_1 \geq \beta_2 + 6\alpha$ by **(2.3)**. We assume for the remainder of the proof that $|Z| \geq 7$. It follows that G' is not cyclically 4-edge-connected, as G' has no core. Therefore Corollary 7 is applicable to G' . Let \mathcal{Y} be a collection of disjoint elementary twigs in G' such that every relevant triangle in G' is contained in an element of \mathcal{Y} , and let (T, ϕ) be a \mathcal{Y} -maximum small-cut decomposition of G' . By Corollary 7, the hub at every $t \in V(T)$ with $|\phi^{-1}(t)| \neq \emptyset$ is either an elementary twig, in which case t is a leaf of T , or is cyclically 4-edge-connected, in which case it is isomorphic to either K_4 or B_3 .

In calculations below we will make use of the following claim: If $\deg_T(t) = 2$ for some $t \in V(T)$, then $|\phi^{-1}(t)| \leq 2$. If this is not the case, the hub at t is isomorphic to K_4 , and at least three of its vertices must be vertices of G . It follows that there is an edge $f \in E(T)$ incident to t for which $|\phi^{-1}(f)| = 2$. Let $v \in \phi^{-1}(t)$ be a vertex incident to an edge in $\phi^{-1}(f)$. Then $C = \phi^{-1}(f) \triangle \delta(v)$ is a 3-edge-cut in G . As in the proof of Lemma 29 we can split t into two vertices t', t'' with $\phi^{-1}(t') = \{v\}$ and $\phi^{-1}(t'') = \phi^{-1}(t) \setminus v$. We now have $\phi^{-1}(t't'') = C$ and the new small-cut-decomposition contradicts the maximality of (T, ϕ) . This completes the proof of the claim.

Let $t_0 \in V(T)$ be such that $\phi^{-1}(t_0)$ contains the new vertex or one of the ends of the new edge in G' . Since G is pruned, G' contains at most one irrelevant triangle, and if such a triangle exists, at least one of its vertices lies in $\phi^{-1}(t_0)$. As a consequence, for any leaf $t \neq t_0$ of T , $\phi^{-1}(t)$ is a twig. Let $t^* \in V(T) \setminus \{t_0\}$ be such that $\deg_T(t^*) \geq 3$ and, subject to this condition, the component of $T \setminus \{t^*\}$ containing t_0 is maximal. If $\deg_T(t) \leq 2$ for every $t \in V(T) \setminus \{t_0\}$, we take $t^* = t_0$ instead.

Let T_1, \dots, T_k be all the components of $T \setminus \{t^*\}$ not containing t_0 . By the choice of t^* , each T_i is a path. If $|V(T_i)| \geq 33$ for some $1 \leq i \leq k$ then let T' be the subtree of T_i containing a leaf of T and exactly 32 other vertices. Let f be the unique edge in $\delta(T')$. Let H (resp. H') be the $\phi^{-1}(f)$ -contraction of G (resp. G') containing $V(G') \setminus \phi^{-1}(T')$, and let Z' consist of $V(H') \cap Z$ together with the new vertex created by $\phi^{-1}(f)$ -contraction (if it exists). If H is not pruned then it contains a unique irrelevant triangle and we contract it, obtaining a pruned graph. By the induction hypothesis, either $|Z'| \leq 6$ or we can find a foliage \mathcal{X}' in Z' with $fw(\mathcal{X}') \geq \alpha(|Z'| - 2) + \beta_2$. If $|Z'| \leq 6$ let $\mathcal{X}' := \emptyset$.

Let t' be a vertex of T' which is not a leaf in T . Since $\deg_T(t') = 2$, $|\phi^{-1}(t')| \neq \emptyset$. Therefore $\phi^{-1}(t')$ is isomorphic to B_3 or K_4 and we can apply Corollary 9. This implies that $\phi^{-1}(T')$ contains an elementary twig and a burl that are vertex-disjoint, where the elementary twig is the preimage of the leaf. Further, we have $|\phi^{-1}(T')| \leq 8 + 2 \cdot 32 = 72$, since an elementary twig has size at most 8 and the preimage of

every non-leaf vertex of T' has size at most 2 by the claim above. By Lemma 23(3), we can obtain a foliage \mathcal{X} in Z by adding the twig and the burl to \mathcal{X}' and possibly removing a burl (which can be a twig) containing the new element of H' created by $\phi^{-1}(f)$ -contraction. It follows that if $|Z'| \geq 7$ then

$$fw(\mathcal{X}) \geq \alpha(|Z'| - 2) + 2\beta_2 \geq (\alpha|Z| + \beta_2) - 74\alpha + \beta_2 \geq \alpha|Z| + \beta_2,$$

by (2.4), as desired. If $|Z'| \leq 6$ then $|Z| \leq 78$ and

$$fw(\mathcal{X}) \geq \beta_1 + \beta_2 \geq 78\alpha + \beta_2 \geq \alpha|Z| + \beta_2,$$

by (2.5).

It remains to consider the case when $|V(T_i)| \leq 32$ for every $1 \leq i \leq k$. Suppose first that $t^* \neq t_0$ and that $|\phi^{-1}(T_0)| \geq 7$, where T_0 denotes the component of $T \setminus t^*$ containing t_0 . Let f_0 be the edge incident to t^* and a vertex of T_0 . We form the graphs H , H' and a set Z' by a $\phi^{-1}(f_0)$ -contraction as in the previous case, and possibly contract a single irrelevant triangle. As before, we find a foliage \mathcal{X}' in Z' with $fw(\mathcal{X}') \geq \alpha(|Z'| - 2) + \beta_2$. Note that $\phi^{-1}(T_i)$ contains a twig for every $1 \leq i \leq k$. By Lemma 23(3), we now obtain a foliage \mathcal{X} in Z from \mathcal{X}' , adding $k \geq 2$ twigs and possibly removing one burl (which can be a twig) from \mathcal{X}' . We have $|\phi^{-1}(T_i)| \leq 8 + 31 \cdot 2 = 70$ for every $1 \leq i \leq k$, and $|\phi^{-1}(t^*)| \leq 4$. Therefore $|Z| \leq |Z'| + 70k + 4$. It follows from (2.5) that

$$fw(\mathcal{X}) \geq \alpha(|Z'| - 2) + \beta_2 + (k-1)\beta_1 \geq \alpha|Z| + \beta_2 - 76\alpha + (k-1)(\beta_1 - 70\alpha) \geq \alpha|Z| + \beta_2.$$

Now we can assume $t^* = t_0$ or $|\phi^{-1}(T_0)| \leq 6$. First suppose $t^* \neq t_0$ but $|\phi^{-1}(T_0)| \leq 6$. Then again $|\phi^{-1}(t^*)| \leq 4$, so we have $|Z| \leq 70k + 10$. Let \mathcal{X} be the foliage consisting of twigs in T_1, \dots, T_k . Thus by (2.6), we have

$$fw(\mathcal{X}) = k\beta_1 \geq (\alpha|Z| + \beta_2) + k(\beta_1 - 70\alpha) - 10\alpha - \beta_2 \geq \alpha|Z| + \beta_2.$$

Finally we can assume $t^* = t_0$. Then $|\phi^{-1}(t^*)| \leq 4$, unless $k = 1$ and $\phi^{-1}(t^*)$ is an elementary twig. In either case, $|Z| \leq 70k + 8$ and the equation above applies. \square

2.1.6 Proof of Lemma 27

The following lemma is a direct consequence of a theorem of Kotzig, stating that any graph with a unique perfect matching contains a bridge (see [27]).

Lemma 33 *Every edge of a cyclically 4-edge-connected cubic graph with at least six vertices is contained in at least two perfect matchings.*

Let G be a cubic graph. For a path $v_1v_2v_3v_4$, the graph obtained from G by *splitting along the path* $v_1v_2v_3v_4$ is the cubic graph G' obtained as follows: remove the vertices v_2 and v_3 and add the edges v_1v_4 and $v'_1v'_4$ where v'_1 is the neighbor of v_2 different from v_1 and v_3 and v'_4 is the neighbor of v_3 different from v_2 and v_4 . The idea of this construction (and its application to the problem of counting perfect matchings) originally appeared in [65]. We say that a perfect matching M of G is a *canonical extension* of a perfect matching M' of G' if $M \triangle M' \subseteq E(G) \triangle E(G')$, i.e. M and M' agree on the edges shared by G and G' .

Lemma 34 *Let G be a cyclically 4-edge-connected cubic graph with $|V(G)| \geq 6$. If G' is the graph obtained from G by splitting along some path $v_1v_2v_3v_4$, then*

1. G' is cubic and bridgeless;
2. G' contains at most 2 irrelevant triangles;
3. $fw(G) \geq fw(G') - 2\beta_1$;
4. Every perfect matching M' of G' avoiding the edge v_1v_4 has a canonical extension in G .

Proof.

1. The statement is a consequence of an easy lemma in [26], stating that the cyclic edge-connectivity can drop by at most two after a splitting.
2. Since G is cyclically 4-edge-connected and has at least six vertices, it does not contain any triangle. The only way an irrelevant triangle can appear in G' is that v_1 and v_4 (or v'_1 and v'_4) have precisely one common neighbor (if they have two common neighbors, the two arising triangles share the new edge v_1v_4 or $v'_1v'_4$ and hence, are relevant).
3. At most two burls from a foliage of G' intersect the edge v_1v_4 or the edge $v'_1v'_4$. Therefore, a foliage of G can be obtained from any foliage of G' by removing at most two burls.
4. The canonical extension is obtained (uniquely) from $M' \cap E(G)$ by adding either v_2v_3 if $v'_1v'_4 \notin M'$ or $\{v'_1v_2, v_3v'_4\}$ if $v'_1v'_4 \in M'$. \square

□

Proof. (of Lemma 27) We proceed by induction on $|V(G)|$. The base case $|V(G)| = 6$ holds by Lemma 33 and (2.7).

For the induction step, consider first the case that G is cyclically 4-edge-connected. Fix an edge $e = uv \in E(G)$. Our goal is to show that e is contained in at least $2^{\alpha|V(G)|-fw(G)+\gamma}$ perfect matchings.

Let $w \neq u$ be a neighbor of v and let w_1 and w_2 be the two other neighbors of w . Let x_i, y_i be the neighbors of w_i distinct from w for $i = 1, 2$. Let G_1, \dots, G_4 be the graphs obtained from G by splitting along the paths $vw w_1 x_1$, $vw w_1 y_1$, $vw w_2 x_2$ and $vw w_2 y_2$. Let G'_i be obtained from G_i by contracting irrelevant triangles for $i = 1, \dots, 4$. By Lemma 34(2) we have $|V(G'_i)| \geq |V(G)| - 6$.

Suppose first that one of the resulting graphs, without loss of generality G'_1 , does not have a core. By Corollary 6, Lemma 25 and Lemma 34, we have

$$\alpha|V(G)| \leq \alpha(|V(G'_1)|+6) \leq fw(G'_1)+7\alpha-\beta_2 \leq fw(G_1)+7\alpha-\beta_2 \leq fw(G)+2\beta_1+7\alpha-\beta_2.$$

Therefore

$$\alpha|V(G)| - fw(G) + \gamma \leq \gamma + 2\beta_1 + 7\alpha - \beta_2 \leq 1$$

by (2.8) and the lemma follows from Lemma 33.

We now assume that all four graphs G'_1, \dots, G'_4 have a core. By Lemma 34(4), every perfect matching containing e in G_i canonically extends to a perfect matching containing e in G . Let S be the sum of the number of perfect matchings of G_i containing e , for $i \in \{1, 2, 3, 4\}$. By induction hypothesis and Lemmas 25 and 34, $S \geq 4 \cdot 2^{\alpha(|V(G)|-6)-fw(G)-2\beta_1+\gamma}$. On the other hand, a perfect matching M of G containing e is the canonical extension of a perfect matching containing e in precisely three of the graphs G_i , $i \in \{1, 2, 3, 4\}$. For instance if $w_1 y_1, w w_2 \in M$, then G_2 is the only graph (among the four) that does not have a perfect matching M' that canonically extends to M (see Figure 2.5). As a consequence, there are precisely $S/3$ perfect matchings containing e in G . Therefore,

$$m^*(G) \geq \frac{4}{3} \cdot 2^{\alpha(|V(G)|-6)-fw(G)-2\beta_1+\gamma} \geq 2^{\alpha|V(G)|-fw(G)+\gamma},$$

by (2.9), as desired.

It remains to consider the case when G contains a cyclic edge-cut C of size at most 3. Suppose first that for such edge-cut C , both C -contractions H_1 and H_2 have a core. Then, by Lemma 23(3), $fw(G) \geq fw(H_1) + fw(H_2) - 2\beta_1$ and, by induction

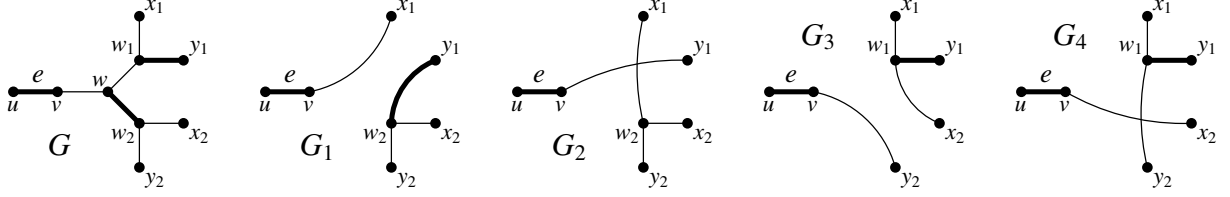


Figure 2.5: Perfect matchings in only three of the G_i 's canonically extend to a given perfect matching of G containing e .

hypothesis, applied to H_1 and H_2 (after possibly contracting one irrelevant triangle in each) and Lemma 23,

$$m^*(G) \geq m^*(H_1)m^*(H_2) \geq 2^{\alpha|V(G)|-4\alpha-fw(G)-2\beta_1+2\gamma} \geq 2^{\alpha|V(G)|-fw(G)+\gamma},$$

by (2.10), as desired. Finally, if for every cyclic edge-cut C of size at most 3 only one C -contraction has a core, we apply Corollary 7 to G . Let (T, ϕ) be the resulting small-cut-decomposition of G . There exists a unique vertex $t \in V(T)$ such that the hub H of G at t is cyclically 4-edge-connected with $|V(H)| \geq 6$. Let T_1, \dots, T_k be the components of $T - t$ and let $Z_i = \phi^{-1}(V(T_i))$. We apply Lemma 26 to Z_1, \dots, Z_k . Note that Lemma 26 is indeed applicable, as G is pruned, and therefore every triangle in G belongs to an elementary twig. Consequently, no edge-cut corresponding to an edge of (T, ϕ) separates exactly 3 vertices of G .

Let $\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_k$ be the foliages satisfying the lemma. Let \mathcal{X}_0 be the maximal foliage in H avoiding new vertices and edges created by contraction of the edge-cuts $\delta(Z_1), \dots, \delta(Z_k)$. Then $fw(\mathcal{X}_0) \geq fw(H) - k\beta_2$, as H contains no twigs (it is cyclically 4-edge-connected). Since $\mathcal{X}_0 \cup \mathcal{X}_1 \cup \dots \cup \mathcal{X}_k$ is a foliage in G we have

$$fw(G) \geq fw(H) - k\beta_2 + \sum_{i=1}^k fw(\mathcal{X}_i) \geq fw(H) + \alpha \sum_{i=1}^k |Z_i|,$$

by the choice of $\mathcal{X}_1, \dots, \mathcal{X}_k$. It remains to observe that

$$m^*(G) \geq m^*(H) \geq 2^{\alpha|V(H)|-fw(H)+\gamma} \geq 2^{\alpha(|V(G)|-\sum_{i=1}^k |Z_i|)-fw(H)+\gamma} \geq 2^{\alpha|V(G)|-fw(G)+\gamma},$$

by the above. \square

2.1.7 Concluding remarks

2.1.7.1 Improving the bound.

We expect that the bound in Theorem 18 can be improved at the expense of more careful case analysis. In particular, it is possible to improve the bound on the length of the path in Corollary 9. We have chosen not to do so in an attempt to keep the argument as short and linear as possible.

In [11] it is shown that for some constant $c > 0$ and every integer n there exists a cubic bridgeless graph on at least n vertices with at most $c2^{n/17.285}$ perfect matchings.

2.1.7.2 Number of perfect matchings in k -regular graphs.

In [61, Conjecture 8.1.8] the following generalization of the conjecture considered in this work is stated. A graph is said to be *matching-covered* if every edge of it belongs to a perfect matching.

Conjecture 5 *For $k \geq 3$ there exist constants $c_1(k), c_2(k) > 0$ such that every k -regular matching covered graph contains at least $c_2(k)c_1(k)^{|V(G)|}$ perfect matchings. Furthermore, $c_1(k) \rightarrow \infty$ as $k \rightarrow \infty$.*

While our proof does not seem to extend to the proof of this conjecture, the following weaker statement can be deduced from Theorem 18. We are grateful to Paul Seymour for suggesting the idea of the following proof.

Theorem 21 *Let G be a k -regular $(k-1)$ -edge-connected graph on n vertices for some $k \geq 4$. Then*

$$\log_2 m(G) \geq (1 - \frac{1}{k})(1 - \frac{2}{k})\frac{n}{3656}.$$

Proof. Let \mathbf{w} be an edge-weighting of G assigning weight $1/k$ to every edge. It is easy to deduce from Theorem 20 that $\mathbf{w} \in \mathcal{PMP}(G)$. Let $\mathbf{M}_{\mathbf{w}}$ be a random variable in $\mathcal{M}(G)$ corresponding to \mathbf{w} . We choose a triple of perfect matchings of G as follows. Let $M_1 \in \mathcal{M}(G)$ be arbitrary. We have

$$\mathbb{E}[|\mathbf{M}_{\mathbf{w}} \cap M_1|] = \frac{n}{2k}.$$

Therefore we can choose $M_2 \in \mathcal{M}(G)$ so that $|M_2 \cap M_1| \leq \frac{n}{2k}$. Let $Z \subseteq V(G)$ be the set of vertices not incident with an edge of $M_1 \cap M_2$. Then $|Z| \geq (1 - \frac{1}{k})n$. For each $v \in Z$ we have

$$\Pr[\mathbf{M}_{\mathbf{w}} \cap \delta(\{v\}) \cap (M_1 \cup M_2) = \emptyset] = 1 - \frac{2}{k}.$$

Therefore the expected number of vertices whose three incident edges are in \mathbf{M}_w , M_1 and M_2 respectively, is at least $(1 - \frac{1}{k})(1 - \frac{2}{k})n$. It follows that we can choose $M_3 \in \mathcal{M}(G)$ so that the subgraph G' of G with $E(G') = M_1 \cup M_2 \cup M_3$ has at least $(1 - \frac{1}{k})(1 - \frac{2}{k})n$ vertices of degree three. Note that G' is by definition matching-covered. It follows that the only bridges in G' are edges joining pairs of vertices of degree one. Let G'' be obtained from G' by deleting vertices of degree one and replacing by an edge every maximal path in which all the internal vertices have degree two. The graph G'' is cubic and bridgeless and therefore by Theorem 18 we have

$$\log_2 m(G) > \log_2 m(G') \geq \log_2 m(G'') \geq \frac{1}{3656} |V(G'')| \geq (1 - \frac{1}{k})(1 - \frac{2}{k}) \frac{n}{3656},$$

as desired. \square

2.2 Perfect matchings interactions

In this section we will show another context where the operation of path splitting, together with careful analysis of short edge-cuts, can be applied in order to obtain interesting results.

In 2013, Giuseppe Mazzuoccolo [56] proposed three conjectures about the interactions between perfect matchings in bridgeless cubic graphs. His first conjecture, implied by the Berge-Fulkerson Conjecture [31] and Fan-Raspaud Conjecture [28], is the following.

Conjecture 6 (Mazzuoccolo, 2013 [56]) *Let G be a bridgeless cubic graph. Then, there exist two perfect matchings of G such that the complement of their union is a bipartite graph.*

Suppose there are three perfect matchings such that every edge is in at most two of them, as by Fan and Raspaud (or take any three out of the six by Berge and Fulkerson). Then any two of them have the desired property: Indeed, if the complement of their union contained an odd cycle, then the third perfect matching could only cover the vertices of this cycle by the edges of the cycle, which is impossible.

This last conjecture was presented by Jean Paul Zerafa at a workshop on exceptional structures in discrete mathematics in Modra (Slovakia), and together with Edita Máčajová, the three of us started to work on this problem there. We managed to prove the conjecture, more precisely, we proved the following stronger statement.

Theorem 22 (Kardoš, Máčajová, and Zerafa, 2023 [46]) *Let G be a bridgeless cubic graph. Let F be a 1^+ -factor of G and let $e \in E(G)$. Then, there exists a perfect matching M of G such that $e \in M$, and $G \setminus (F \cup M)$ is bipartite.*

Note that a 1^+ -factor of G is the edge set of a spanning subgraph of G such that its vertices have degree 1, 2 or 3. Theorem 22 not only shows the existence of two perfect matchings of G whose deletion leaves a bipartite subgraph of G , but that for every perfect matching of G there exists a second one such that the deletion of the two leaves a bipartite subgraph of G . In particular, Theorem 22 also implies that for every collection of disjoint odd circuits of G , there exists a perfect matching which intersects at least one edge from each odd circuit (this was posed as an open problem by Mazzuoccolo and Zerafa in [57], see also [69]).

Mazzuoccolo moved on to propose two stronger conjectures, with Conjecture 8 being the strongest of all three.

Conjecture 7 (Mazzuoccolo, 2013 [56]) *Let G be a bridgeless cubic graph. Then, there exist two perfect matchings of G such that the complement of their union is an acyclic graph.*

Conjecture 8 (Mazzuoccolo, 2013 [56]) *Let G be a bridgeless cubic graph. Then, there exist two perfect matchings of G such that the complement of their union is an acyclic graph, whose components are of order 2 or 3.*

Clearly, these last two conjectures are true for 3-edge-colourable cubic graphs, and Janos Hägglund verified the strongest of these conjectures (Conjecture 8) by computer for all non-trivial snarks (non 3-edge-colourable cubic graphs) of order at most 34 [56]. However, 5 years later, Jin, Steffen, and Mazzuoccolo [40] gave a counterexample to Conjecture 7. Their counterexample contains a lot of 2-edge-cuts and the authors state that the conjecture "could hold true for 3-connected or cyclically 4-edge-connected cubic graphs". In fact, we reviewed our proof of Theorem 22, and we realized that most of the arguments can be adapted to show the following stronger statement.

Theorem 23 *Let G be a cyclically 3-edge-connected cubic graph, which is not a Klee-graph. Then, for any $e \in E(G)$ and any 1^+ -factor F of G , there exists a perfect matching M of G containing e such that $G \setminus (F \cup M)$ is acyclic.*

We remark that Klee-graphs (see Definition 1 below), which are to be discussed further later (see section 2.2.1), are 3-edge-colourable cubic graphs and so are not a

counterexample to Conjecture 7. However, the stronger statement given in Theorem 23 does not hold for this class of graphs, and this is the reason why we exclude them.

Although Theorem 23 is not a direct consequence of the Berge–Fulkerson Conjecture, we believe that the results presented here and in [46] are valuable steps towards trying to decipher long-standing conjectures about interactions between perfect matchings in bridgeless cubic graphs.

In fact, we will prove the following statement, which is equivalent to Theorem 23.

Theorem 24 *Let G be a cyclically 3-edge-connected cubic graph, which is not a Klee-graph. Then, for any $e \in E(G)$ and any collection of disjoint circuits \mathcal{C} , there exists a perfect matching M of G containing e such that every circuit in \mathcal{C} contains an edge from M .*

Indeed, given a collection of disjoint circuits \mathcal{C} , its complement is a 1^+ -factor, say $F_{\mathcal{C}}$. A perfect matching M containing e such that $G \setminus (F_{\mathcal{C}} \cup M)$ is acyclic must contain an edge from every circuit in \mathcal{C} . On the other hand, given a 1^+ -factor F , its complement is a collection of disjoint paths and circuits, and so it suffices to consider the collection \mathcal{C}_F of circuits disjoint from F . A perfect matching M containing e such that every circuit in \mathcal{C}_F contains an edge from M , clearly makes $G \setminus (F \cup M)$ acyclic.

2.2.1 Klee-graphs

Definition 1 (as in [11]) *A graph G is a Klee-graph if G is the complete graph on 4 vertices K_4 or there exists a Klee-graph G_0 such that G can be obtained from G_0 by replacing a vertex by a triangle (see Figure 2.6).*

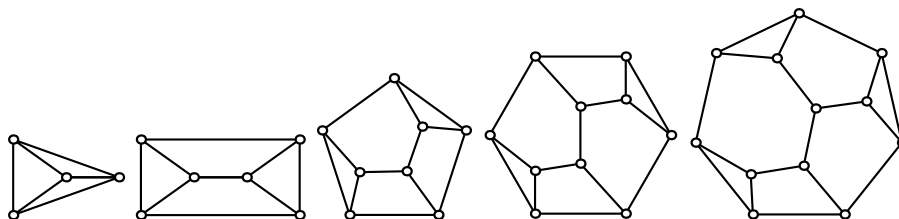


Figure 2.6: Examples of Klee-graphs on 4 upto 12 vertices, left to right.

For simplicity, if a graph G is a Klee-graph, we shall sometimes say that G is Klee. We note that there is a unique Klee-graph on 6 vertices (the graph of a 3-sided prism), and a unique Klee-graph on 8 vertices. These two graphs are examples of so-called Klee ladders, and shall be respectively denoted as KL_6 and KL_8 .

Lemma 35 (Cygan, Pilipczuk, and Škrekovski 2013 [11]) *The edge set of any Klee-graph can be uniquely partitioned into three pairwise disjoint perfect matchings. In other words, any Klee-graph is 3-edge-colourable, and the colouring is unique up to a permutation of the colours.*

Since Klee-graphs are 3-edge-colourable, they easily satisfy the statement of Conjecture 7.

Proposition 2 *Let G be a Klee-graph. Then, G admits two perfect matchings M_1 and M_2 such that $G \setminus (M_1 \cup M_2)$ is acyclic.*

The new graph obtained after expanding a vertex of a Hamiltonian graph (not necessarily Klee) into a triangle is still Hamiltonian, and so, since K_4 is Hamiltonian, all Klee-graphs are Hamiltonian. Hamiltonian cubic graphs have the following distinctive property.

Proposition 3 *Let G be a Hamiltonian cubic graph. Then, for any collection of disjoint circuits \mathcal{C} of G there exists a perfect matching M of G which intersects at least one edge of every cycle in \mathcal{C} .*

Proof. Since G is Hamiltonian, it admits three disjoint perfect matchings M_1, M_2, M_3 covering $E(G)$ such that at least two of them induce a Hamiltonian cycle. Without loss of generality, assume that $M_2 \cup M_3$ induce a Hamiltonian cycle. Let \mathcal{C} be a collection of disjoint cycles of G for which the statement of the proposition does not hold. In particular, this implies that M_1 does not intersect all the cycles in \mathcal{C} — since the complement of M_1 is a Hamiltonian cycle, \mathcal{C} consists of exactly one cycle. However, this means that M_2 (or M_3) intersects the only cycle in \mathcal{C} , contradicting our initial assumption. \square

Corollary 10 *For any collection of disjoint cycles \mathcal{C} of a Klee-graph G there exists a perfect matching M of G which intersects at least one edge of every cycle in \mathcal{C} .*

On the other hand, we have to exclude Klee-graphs from Theorem 23 (and Theorem 24) since for some Klee-graphs there are edges contained in a unique perfect matching, as we will see below.

Lemma 36 (Cygan, Pilipczuk, and Škrekovski 2013 [11]) *Let G be a Klee-graph on at least 6 vertices. Then, G has at least two triangles and all its triangles are vertex-disjoint.*

Indeed, expanding a vertex into a triangle can only destroy triangles containing the vertex to be expanded.

We will now define a series of particular Klee-graphs, which we will call *Klee ladders*. Let KL_4 be the complete graph on 4 vertices, and let u_4v_4 be an edge of KL_4 . For any even $n \geq 4$, let KL_{n+2} be the Klee-graph obtained from KL_n by expanding the vertex u_n into a triangle. In the resulting graph KL_{n+2} , we denote the vertex corresponding to v_n by v_{n+2} , and denote the vertex of the new triangle adjacent to v_{n+2} by u_{n+2} .

In other words, the graph KL_{2k+2} consists of the Cartesian product $P_2 \square P_k$ (where P_t denotes a path on t vertices) with two additional vertices u_{2k+2} and v_{2k+2} adjacent to each other, such that u_{2k+2} (v_{2k+2}) is adjacent to the two vertices in the first (last, respectively) copy of P_2 in $P_2 \square P_k$ (see Figure 2.7).

Klee ladders can be used to illustrate why we have to exclude Klee-graphs from our main result. For a given Klee ladder G there exists an edge e such that e is contained in a unique perfect matching of G , and therefore there is no hope for a statement like Theorem 24 to be true.

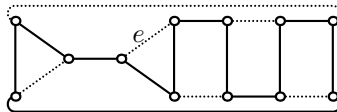


Figure 2.7: An example of a Klee ladder, KL_{12} . There is a unique perfect matching (here depicted using dotted lines) containing the edge e . The complement of this perfect matching is a Hamiltonian cycle.

We will frequently use the following structural property of certain Klee-graphs.

Lemma 37 *Let G be a Klee-graph on at least 8 vertices having exactly two (disjoint) triangles. Then,*

- (i) *exactly one edge of each triangle lies on a 4-cycle; and*
- (ii) *if G admits an edge joining the two triangles, then G is a Klee ladder.*

Proof. We prove this by induction. Claim (i) is obvious for KL_8 , the only Klee-graph on 8 vertices, so let G be a Klee-graph on $n \geq 10$ vertices. By definition, it can be obtained from a smaller one, say G_0 , by expanding a vertex into a triangle. Since G only has two triangles, this operation must have destroyed a (single) triangle

of G_0 , which in turn gives rise to a 4-cycle containing exactly one of the edges of the new triangle.

Moreover, if G admits an edge e joining the two triangles, then the corresponding edge e_0 in G_0 joins a triangle to a vertex contained in a (distinct) triangle, so it joins the two triangles of G_0 . By induction, G_0 is a Klee ladder, say KL_{n-2} , for some $n \geq 8$, and the edge e_0 is the edge $u_{n-2}v_{n-2}$ (see the definition of Klee ladders above). Claim (ii) follows immediately. \square

2.2.2 The general case

Proof. (of Theorem 24) Let G be a minimum counterexample to the statement of Theorem 24. Since K_4 is Klee, G has at least six vertices. There are only two 3-connected cubic graphs on six vertices, namely KL_6 and $K_{3,3}$. The former is Klee. For the latter, $K_{3,3}$, a collection of disjoint cycles can only contain one cycle on either four or six vertices and in both cases it is easy to check that every edge is contained in a perfect matching intersecting the prescribed cycle. Therefore, G has at least eight vertices.

Let $e \in E(G)$ be an edge of G such that there exists a collection of disjoint cycles such that for every perfect matching M containing e there is a cycle in the collection containing no edge from M . Amongst all such collections, we can choose an inclusion-wise minimal one, denoted by \mathcal{C} . By the choice of \mathcal{C} , we may assume that $e \notin C$ for any $C \in \mathcal{C}$.

In the sequel, we will prove progressively a series of structural properties of G . Before that, we need to define three additional graph families. Let KL_{2k-2} be the Klee ladder on $2k-2$ vertices with $k \geq 4$; let u_{2k-2} and v_{2k-2} be the two vertices contained in the two triangles, say $u_{2k-2}u_1u_2$ and $v_{2k-2}v_1v_2$, which are adjacent to each other. Moreover, we may assume that $KL_{2k-2} \setminus \{u_{2k-2}, v_{2k-2}\}$ contains two disjoint paths of length $k-3$, one from u_1 to v_1 and the other from u_2 to v_2 .

We remove the vertices u_{2k-2} and v_{2k-2} and replace them by four vertices, say u'_1, u'_2, v'_1 , and v'_2 , adjacent to u_1, u_2, v_1 , and v_2 , respectively, and we add a 4-cycle passing through the four new vertices. In fact, we can see the last operation as adding a complete graph on 4 vertices and removing a perfect matching. Up to symmetry, only three outcomes are possible.

- A *ladder* L_{2k} is obtained if the edges $u'_1v'_2$ and $u'_2v'_1$ are missing.
- A *Möbius ladder* ML_{2k} is obtained if the edges $u'_1v'_1$ and $u'_2v'_2$ are missing.
- A *quasi-ladder* QL_{2k} is obtained if the edges $u'_1u'_2$ and $v'_1v'_2$ are missing.

Observe that the ladder L_{2k} is the graph of a k -sided prism. Observe that ladders and Möbius ladders are vertex-transitive.

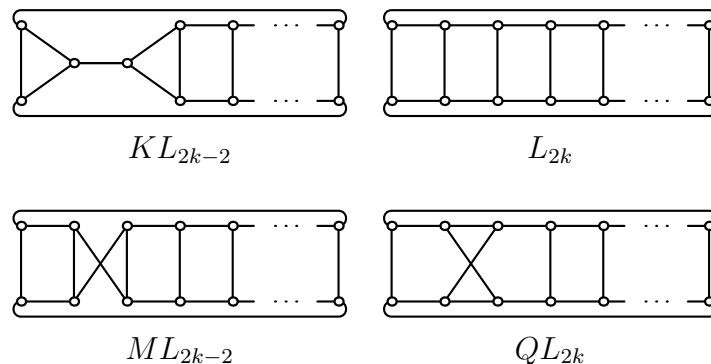


Figure 2.8: An illustration of a Klee ladder, a ladder, a Möbius ladder, and a quasi-ladder.

Claim 1. The graph G is not a ladder, a Möbius ladder, nor a quasi-ladder.

Proof of Claim 1. We proceed by contradiction. Suppose that $G \in \{L_{2n}, ML_{2n}, QL_{2n} : n \geq 4\}$, and let \mathcal{C} be a collection of disjoint cycles in G . We prove that for every edge e there exists a perfect matching M_e containing e such that its complement is a Hamiltonian cycle, say C_e ; moreover, there exists yet another perfect matching M'_e containing e . The first perfect matching can be used to prove Theorem 24 unless $\mathcal{C} = \{C_e\}$. If this is the case, then we can use M'_e .

In most of the cases, the second perfect matching M'_e can be obtained from M_e by the following operation: we find a 4-cycle consisting of the edges e_1, e_2, e_3, e_4 (in this cyclic order) avoiding e and containing exactly two edges from M_e , say e_1 and e_3 . We then set $M'_e = (M_e \setminus \{e_1, e_3\}) \cup \{e_2, e_4\}$. In other words, M'_e is obtained as the symmetric difference of M_e and a suitable 4-cycle.

If G is a ladder or a Möbius ladder, then G is vertex-transitive, and there are only two edge orbits. It suffices to distinguish between e being an edge contained in two 4-cycles (vertical according to Figure 2.8) or in a single one (horizontal or diagonal). An example of a pair of perfect matchings M_e and M'_e having the desired properties is depicted in Figure 2.9.

Let $G = QL_{2k}$ for some $k \geq 4$. If e is an edge of the subgraph $P_2 \square P_{k-2}$ or an edge of the 4-cycle $u'_1 v'_1 u'_2 v'_2$ (see the definition of a quasi-ladder for the notation), then a pair of perfect matchings M_e and M'_e having the desired properties can be found in a same way as in the previous case, see Figure 2.10 for an illustration.

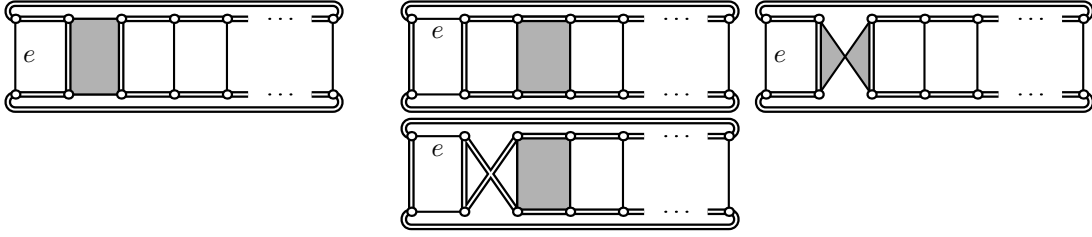


Figure 2.9: An example of a Hamiltonian cycle C_e (drawn using double lines) avoiding a given edge e whose complement is a perfect matching M_e containing e , for both possible positions of the prescribed edge e in a ladder (top line) or a Möbius ladder (bottom line). A second perfect matching M'_e can be obtained by the symmetric difference with the grey 4-cycle.

Otherwise, let $e = u_1u'_1$ (for the remaining three edges the situation is symmetric). There is a unique Hamiltonian cycle C_e avoiding e and containing $u_2u'_2$, see Figure 2.11 for an illustration. In this case, there is another perfect matching M'_e containing $\{u_1u'_1, u_2u'_2, v_1v'_1, v_2v'_2\}$ and all the vertical edges of the subgraph $P_2 \square P_{k-2}$ except for the first and the last one. ■

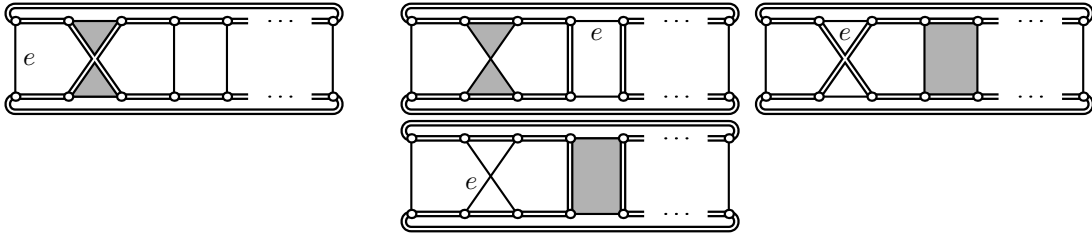


Figure 2.10: An example of a Hamiltonian cycle C_e avoiding a given edge e (drawn using double lines) whose complement is a perfect matching M_e containing e , for an edge e contained in the grid $P_2 \square P_{k-2}$ (top line) and in the 4-cycle outside the grid (bottom line) of a quasi-ladder. A second perfect matching M'_e can be obtained by the symmetric difference with the grey 4-cycle.

Claim 2. The graph G does not have any cyclic 3-edge-cuts.

Proof of Claim 2. Suppose that G admits a cyclic 3-edge-cut $E(V', V'')$ with $E(V', V'') = \{f_1, f_2, f_3\} =: X$, where each $f_i = v'_i v''_i$, for some $v'_1, v'_2, v'_3 \in V'$ and $v''_1, v''_2, v''_3 \in V''$. Since G has no 2-edge-cuts, the vertices $v'_1, v'_2, v'_3, v''_1, v''_2, v''_3$ are all distinct.

Either there is no cycle in \mathcal{C} intersecting X , or the cut X is intersected by a

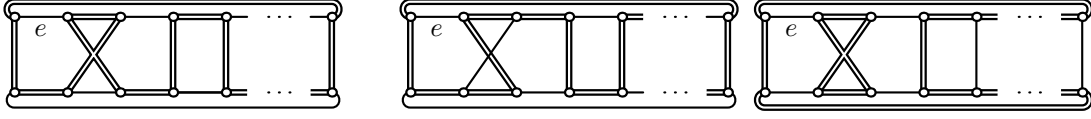


Figure 2.11: An example of a Hamiltonian cycle C_e avoiding a given edge e (drawn using double lines) whose complement is a perfect matching M_e containing e , for an edge e joining a vertex in the grid $P_2 \square P_{k-2}$ to a vertex of the 4-cycle outside the grid in a quasi-ladder (top line, two cases depending on the parity of the length of the grid). A second perfect matching M'_e (bottom).

unique cycle C_X in \mathcal{C} . Without loss of generality, we shall assume that when C_X exists, $X \cap C_X = \{f_2, f_3\}$.

Let G' and G'' be the two graphs obtained from G after deleting X and joining the vertices v'_i to a new vertex v' , and the vertices v''_i to a new vertex v'' . For each $i \in [3]$, let $e'_i = v'_i v'$ and $e''_i = v''_i v''$.

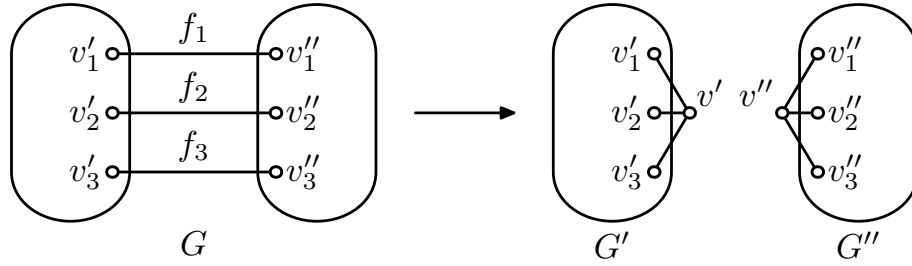


Figure 2.12: The graphs G' and G'' when G admits a cyclic 3-edge-cut $\{f_1, f_2, f_3\}$.

Let

$$\mathcal{C}' = \begin{cases} \{C \in \mathcal{C} \setminus \{C_X\} : C \cap E(G') \neq \emptyset\} \cup \{(C_X \cap E(G')) \cup \{e'_2, e'_3\}\} & \text{if } C_X \text{ exists,} \\ \{C \in \mathcal{C} : C \cap E(G') \neq \emptyset\} & \text{otherwise.} \end{cases}$$

Similarly, let

$$\mathcal{C}'' = \begin{cases} \{C \in \mathcal{C} \setminus \{C_X\} : C \cap E(G'') \neq \emptyset\} \cup \{(C_X \cap E(G'')) \cup \{e''_2, e''_3\}\} & \text{if } C_X \text{ exists,} \\ \{C \in \mathcal{C} : C \cap E(G'') \neq \emptyset\} & \text{otherwise.} \end{cases}$$

It is not hard to see that \mathcal{C}' (\mathcal{C}'') is a collection of disjoint cycles in G' (in G'' , respectively). Every cycle $C \neq C_X$ in \mathcal{C} corresponds to a cycle either in \mathcal{C}' or in

\mathcal{C}'' . The cycle C_X (if it exists) corresponds to two cycles C'_X and C''_X in \mathcal{C}' and \mathcal{C}'' , respectively.

Case A. We first consider the case when G does not admit any triangles, and claim that G' (similarly G'') is not Klee. For, suppose that G' is Klee. Since G has no triangles, $|V(G')| \geq 6$, and so, by Lemma 36, G' must admit two disjoint triangles. This is impossible since any triangle in G' must contain the vertex v' . Hence, when G does not admit any triangles, G' and G'' are both not Klee.

Without loss of generality, we can also assume that at least one of the endvertices of e corresponds to a vertex in V' . We consider two cases, depending on the existence of C_X .

Case A1. First, consider the case when C_X does not exist. When $e \in X$, say $e = f_1$, then, by minimality of G , there exists a perfect matching M' of G' (M'' of G'') containing e'_1 (e''_1), intersecting every cycle in \mathcal{C}' (in \mathcal{C}'' , respectively). Consequently, $M = M' \cup M'' \cup \{f_1\} \setminus \{e'_1, e''_1\}$ is a perfect matching of G containing $e = f_1$, intersecting every cycle in \mathcal{C} .

It remains to consider the case when $e \notin X$, and so the endvertices of e both correspond to vertices in G' . Once again, for simplicity, we shall refer to this edge as e . Let M' be a perfect matching of G' containing e intersecting every cycle in \mathcal{C}' . Without loss of generality, assume that $e'_1 \in M'$. Let M'' be a perfect matching of G'' containing e''_1 intersecting every cycle in \mathcal{C}'' . Let $M = M' \cup M'' \cup \{f_1\} \setminus \{e'_1, e''_1\}$. This is a perfect matching of G containing e , intersecting every cycle in \mathcal{C} , a contradiction.

Case A2. Suppose that C_X exists. When $e \in X$, we have that $e = f_1$ by the choice of C_X , and so, by the minimality of G , there exists a perfect matching M' of G' (M'' of G'') containing e'_1 (e''_1), intersecting every cycle in \mathcal{C}' (in \mathcal{C}'' , respectively). Consequently, $M = M' \cup M'' \cup \{f_1\} \setminus \{e'_1, e''_1\}$ is a perfect matching of G containing $e = f_1$. Clearly, every cycle in $\mathcal{C} \setminus \{C_X\}$ is intersected by M . The cycle C_X must be intersected by M since C'_X (C''_X) contains an edge of M' (M''), not incident to v' (v'' , respectively).

When $e \notin X$, the endvertices of e both correspond to vertices in G' . Once again, for simplicity, we shall refer to this edge as e . Let M' be a perfect matching of G' containing e intersecting every cycle in \mathcal{C}' . We have $e'_i \in M'$ for some $i \in [3]$. Let M'' be a perfect matching of G'' containing e''_i intersecting every cycle in \mathcal{C}'' . Let $M = M' \cup M'' \cup \{f_i\} \setminus \{e'_i, e''_i\}$. This is a perfect matching of G containing e . As before, M intersects every cycle in \mathcal{C} unless $i = 1$ and no edge of G' or G'' corresponding to an edge of C_X is in M' or M'' , which is impossible since C'_X (C''_X) is a cycle in \mathcal{C}' (\mathcal{C}''), so it contains an edge of M' (M''), not incident to v' (v'' , respectively).

Case B. What remains to be considered is the case when G admits a triangle. Consequently, without loss of generality, we can assume that G'' is K_4 . We note that

in this case, G' cannot be Klee because otherwise G itself would be Klee. Thus, the inductive hypothesis can only be applied to G' but not to G'' . As in Case A, we can assume that at least one of the endvertices of e corresponds to a vertex in V' , since if the endvertices of e both belong to V'' , say $e = v''_i v''_j$, a perfect matching of G contains e if and only if it contains f_k , where $\{i, j, k\} = [3]$. We proceed as in Case A and note that the perfect matching M' containing (the edge corresponding to) e intersecting every cycle in \mathcal{C}' obtained after applying the inductive hypothesis to G' can be easily extended to a perfect matching M of G containing e . What remains to show is that M intersects every cycle in \mathcal{C} . The only cycle possibly not intersected by M is C_X , if it exists. However, this can only happen if $i = 1$, and, if this is the case, then, in particular, C'_X is a cycle in G' and so contains an edge of M' not incident to v' . This implies that C_X contains the corresponding edge of M in G , a contradiction. ■

Claim 3. The graph G does not have any cyclic 4-edge-cuts.

Proof of Claim 3. Suppose first that, in particular, G has a 4-cycle $C = (v''_1, v''_2, v''_3, v''_4)$. Let v'_1, v'_2, v'_3, v'_4 be the vertices in $G - C$ respectively adjacent to $v''_1, v''_2, v''_3, v''_4$, let $f_i = v'_i v''_i$ for $i \in \{1, 2, 3, 4\}$ and let $X = \{f_1, f_2, f_3, f_4\}$. The vertices v'_i are pairwise distinct since G does not have any cyclic 3-cuts.

Let $\{i, j, k\} = \{2, 3, 4\}$. We denote by G_{1i} the graph obtained after adding two new vertices x and y to $G - C$, such that:

- x and y are adjacent;
- v'_1 and v'_i are adjacent to x ; and
- v'_j and v'_k are adjacent to y .

It is known that the graph G_{1i} is 3-connected whenever G is cyclically 4-edge-connected [26]. We claim that G_{1i} is not Klee, for any $i \in \{2, 3, 4\}$. For, suppose not. Since G does not admit any cyclic 3-cuts, by Lemma 36, the only two possible triangles in G_{1i} are (v'_1, v'_i, x) and (v'_j, v'_k, y) . Moreover, since x is adjacent to y , by Lemma 37, G_{1i} is a Klee ladder. For every $i \in \{2, 3, 4\}$, this implies that G is a graph isomorphic to a ladder, a Möbius ladder, or a quasi-ladder — this is a contradiction.

We proceed by considering whether e belongs to $E(C)$, X , or $E(G - C)$.

Case A. When $e \in E(C)$, then for every $i \in \{2, 3, 4\}$, every perfect matching of G_{1i} containing $e' = xy$ extends to a perfect matching of G containing e . The cut X contains an even number of edges belonging to some cycle in \mathcal{C} . In particular, $E(C)$ can contain at most three cycle edges belonging to some cycle in \mathcal{C} , and so $C \notin \mathcal{C}$.

If X contains no cycle edges, then we can set $\mathcal{C}' = \mathcal{C}$ and apply induction on any $G' = G_{1i}$ to find a perfect matching M' containing e' intersecting every cycle in \mathcal{C}' ,

which readily extends to a perfect matching M containing e intersecting every cycle in \mathcal{C} . Note that the cycle C , in particular, is always intersected by at least one edge of M .

If there is a single cycle intersecting X exactly twice, say C_X passing through the edges f_j and f_k , then we apply induction on the graph $G' = G_{1i}$ where $\{1, i\} = \{j, k\}$ (if $1 \in \{j, k\}$), or $|\{1, i, j, k\}| = 4$ (otherwise). The cycle C'_X in G' corresponding to C_X contains two edges both incident to either x or y . Hence, if a perfect matching M' containing $e' = xy$ intersects every cycle in $\mathcal{C}' = (\mathcal{C} \setminus \{C_X\}) \cup \{C'_X\}$, then it extends to a perfect matching containing e intersecting every cycle in \mathcal{C} , since C'_X contains an edge in M' not incident to x (nor y).

If there are two distinct cycles each intersecting X twice, say C_X passing through the edges f_1 and f_2 , and D_X passing through the edges f_3 and f_4 , or if there is a single cycle intersecting X four times, say C_X passing through the vertices v'_1, v''_1, v''_2, v'_2 , and also v'_3, v''_3, v''_4, v'_4 , then we can apply induction on the graph G_{12} with $e' = xy$ just like in the previous case.

Case B. When $e \in X$, say $e = f_1$, then every perfect matching of $G' = G_{13}$ containing $e' = xu_1$ extends to a perfect matching of G containing e in a unique way. If there is no cycle in \mathcal{C} intersecting X , then we can set $\mathcal{C}' = \mathcal{C} \setminus \{C\}$ and apply induction directly. If there is a cycle in \mathcal{C} intersecting X , say C_X , then $|C_X \cap X| = 2$. The corresponding cycle C'_X in G' is well-defined: it always contains y and eventually also x (when $C_X \cap X \neq \{f_2, f_4\}$). A perfect matching M' in G' containing e' intersecting every cycle in $\mathcal{C}' = (\mathcal{C} \setminus \{C_X\}) \cup \{C'_X\}$ intersects C'_X at a cut-edge incident to y or an edge of $G - C$. In both cases, the corresponding perfect matching M in G containing e intersects every cycle in \mathcal{C} , since M intersects C_X at an edge in X or an edge of $G - C$.

Case C. It remains to consider the case when $e \in E(G - C)$. Let $G' = G_{13}$ and let e' be the edge of G' corresponding to e in G . Every perfect matching M' of G' containing e' and not containing xy extends to a perfect matching M of G containing e in a unique way; every perfect matching M' of G' containing e' and xy extends to a perfect matching M of G in two distinct ways, whose symmetric difference is the 4-cycle C . In all the cases, we obtain a perfect matching M of G containing at least one edge of C .

If X contains no edges belonging to any cycle in \mathcal{C} , then we can set $\mathcal{C}' = \mathcal{C} \setminus \{C\}$ and apply induction directly. The cycle C in particular (if it is in \mathcal{C}) is always intersected by at least one edge of M .

If there is a single cycle intersecting X exactly twice, say C_X , passing through the edge f_1 and f_i for some $i \in \{2, 3, 4\}$, then the corresponding cycle C'_X in G' is well-defined: it always contains x and eventually also y (when $C_X \cap X \neq \{f_1, f_3\}$).

We can set $\mathcal{C}' = (\mathcal{C} \setminus \{C_X\}) \cup \{C'_X\}$ and apply induction. If M' contains an edge of C'_X not incident to x nor y , then M contains an edge of C_X not incident to any vertex of C . If M' contains the edge xy , then amongst the two possible extensions of M' into M we can always choose one that contains at least one edge of C_X . If M' contains an edge incident to x or to y distinct from xy , then M contains the corresponding edge in X . In all the cases, it is possible to extend a perfect matching M' of G' containing e' and intersecting every cycle in \mathcal{C}' into a perfect matching M of G containing e and intersecting every cycle in \mathcal{C} .

If there are two distinct cycles each intersecting X twice, say C_X passing through the edges f_1 and f_2 and D_X passing through the edges f_3 and f_4 , then we apply induction on G' with $\mathcal{C}' = \mathcal{C} \setminus \{C_X, D_X\}$. If the perfect matching M' containing e' and intersecting every cycle in \mathcal{C}' obtained by induction also contains xy , then we can choose M to contain both $v''_1 v''_2$ and $v''_3 v''_4$, and so it intersects both C_X and D_X as well. If M' does not contain xy , then $|M \cap \{f_1, f_2, f_3, f_4\}| = 2$. If M contains exactly one of f_1 and f_2 then it also contains one of f_3 and f_4 , and so M intersects both C_X and D_X . If $\{f_1, f_2\} \subset M$, then $v''_3 v''_4 \in M$; similarly, if $\{f_3, f_4\} \subset M$, then $v''_1 v''_2 \in M$. In all the cases M intersects both C_X and D_X , as desired.

If there is a single cycle intersecting X four times, say C_X passing through $v''_1 v''_2 v''_2 v''_1$ and also $v''_3 v''_3 v''_4 v''_4$, then we can apply induction on the graph G' with $\mathcal{C}' = \mathcal{C} \setminus \{C_X\}$ just like in the previous case.

From this point on we may assume that G does not contain any 4-cycles. In particular, for every cyclic 4-edge-cut $E(V', V'')$ both sides have at least six vertices. Suppose that G admits a cyclic 4-edge-cut $E(V', V'')$ with $E(V', V'') = \{f_1, f_2, f_3, f_4\} =: X$, where each $f_i = v'_i v''_i$, for some $v'_1, v'_2, v'_3, v'_4 \in V'$ and $v''_1, v''_2, v''_3, v''_4 \in V''$. Since G has no 3-edge-cuts, the vertices $v'_1, v'_2, v'_3, v'_4, v''_1, v''_2, v''_3, v''_4$ are all distinct.

We define graphs G'_{1i} and G''_{1i} for $i \in \{2, 3, 4\}$ analogously as in the previous part. We denote by x' and y' (x'' and y'') the two new vertices in G'_{1i} (in G''_{1i}), and by e'_1, e'_2, e'_3, e'_4 ($e''_1, e''_2, e''_3, e''_4$) the edges of G'_{1i} (of G''_{1i} , respectively) corresponding to f_1, f_2, f_3, f_4 , respectively, for $i \in \{2, 3, 4\}$. These graphs are all 3-connected [26]. None of these graphs can be a Klee-graph: if this was the case, it would have to be a Klee ladder on at least eight vertices, but there are no 4-cycles at all in G , so this is impossible.

Consider first the case when $e \in X$, say $e = f_1$. If there is a cycle C_X in \mathcal{C} intersecting X , then $e \notin C_X$ and $|C_X \cap X| = 2$. We may assume that $C_X \cap X = \{f_2, f_3\}$. We consider all the three graphs G'_{12}, G'_{13} , and G'_{14} (and all the three graphs G''_{12}, G''_{13} , and G''_{14}) at the same time. The cycle C_X (if it exists) corresponds to a cycle C'_X (C''_X) in each of them in a natural way, covering either one or two vertices amongst x' and y' (x'' and y'' , respectively). If C_X does not exist, we shall proceed in the same

manner, but letting C_X , C'_X , and C''_X be equal to \emptyset . We apply induction with $e' = e'_1$ ($e'' = e''_1$) and $\mathcal{C}' = ((\mathcal{C} \setminus \{C_X\}) \cap E(G'_{1i})) \cup \{C'_X\}$ ($\mathcal{C}'' = ((\mathcal{C} \setminus \{C_X\}) \cap E(G''_{1i})) \cup \{C''_X\}$, respectively). Let M'_i (M''_i) be a perfect matching in G'_{1i} (G''_{1i}) containing e' (e'') intersecting every cycle in \mathcal{C}' (in \mathcal{C}'' , respectively). Every perfect matching amongst M'_2 , M'_3 , and M'_4 contains exactly one edge e'_k corresponding to a cut edge f_k for some $k \in \{2, 3, 4\}$ (besides the edge e' corresponding to f_1) and the three values of k cannot all be the same for the three perfect matchings. The same thing holds for the other three perfect matchings M''_2 , M''_3 , and M''_4 . Therefore, for some $k \in \{2, 3, 4\}$ there exist two perfect matchings M'_i and M''_j containing the edge e'_k and e''_k , respectively. We can combine them together into a perfect matching M containing e and f_k , intersecting every cycle in \mathcal{C} . In particular, if C_X exists, then it can only be avoided by M if $k = 4$, but then M'_i (M''_j) cannot contain any edge of C'_X (C''_X) incident to x' or to y' (to x'' or to y''), so it intersects C'_X inside $G[V']$ (C''_X inside $G[V'']$, respectively). Consequently, M intersects C_X inside $G[V']$ and $G[V'']$.

Consider next the case where $e \notin X$. We may assume that $e \in G[V']$. Let \mathcal{C}_X be the set of cycles in \mathcal{C} intersecting X . We have $|\mathcal{C}_X| \leq 2$, and even if there is a single cycle in \mathcal{C}_X , it may contain all four edges of X . Let \mathcal{C}'_0 (\mathcal{C}''_0) be the set of cycles from \mathcal{C} within $G[V']$ ($G[V'']$, respectively). Given $G' = G'_{1i}$ ($G'' = G''_{1i}$) for some arbitrary $i \in \{2, 3, 4\}$, let \mathcal{C}'_X (\mathcal{C}''_X) be the set of cycles obtained from the subpaths of cycles in \mathcal{C}_X contained in $G[V']$ (in $G[V'']$) by adding the necessary edges from $\{e'_1, e'_2, e'_3, e'_4\}$ (from $\{e''_1, e''_2, e''_3, e''_4\}$) and eventually also the edge $x'y'$ ($x''y''$, respectively), if needed. Observe that $|\mathcal{C}'_X| = 2$ ($|\mathcal{C}''_X| = 2$) is possible when $|\mathcal{C}_X| = 1$, and vice-versa. Finally, let $\mathcal{C}' = \mathcal{C}'_0 \cup \mathcal{C}'_X$ and $\mathcal{C}'' = \mathcal{C}''_0 \cup \mathcal{C}''_X$.

Let e' be the edge in G' corresponding to e in G . By induction, we obtain a perfect matching M' containing e' intersecting every cycle in \mathcal{C}' .

Consider first the case when $x'y' \in M'$. We apply induction to obtain a perfect matching M'' of $G'' = G'_{1i}$, for any $i \in \{2, 3, 4\}$, containing $x''y''$ intersecting every cycle in \mathcal{C}'' . Then, $M = (M' \setminus \{x'y'\}) \cup (M'' \setminus \{x''y''\})$ is a perfect matching of G containing e . It is easy to check that M intersects every cycle in \mathcal{C}'_0 and in \mathcal{C}''_0 ; it remains to certify that M intersects all the cycles in \mathcal{C}_X . If $|\mathcal{C}_X| \leq 1$, then we choose G'_{1i} in such a way that $x''y''$ does not belong to any cycle in \mathcal{C}''_X , and so M'' contains at least one edge (not incident to x'' or y'') of every cycle in \mathcal{C}''_X , and so the cycle in \mathcal{C}_X will contain at least one edge from M . If $|\mathcal{C}_X| = 2$, then it suffices to choose G'_{1i} in such a way that \mathcal{C}''_X contains two distinct cycles (avoiding $x''y''$), and then each of them will contain at least one edge (not incident to x'' or y'') from M'' , and thus each cycle in \mathcal{C}_X will contain at least one edge from M , as desired.

It remains to consider the case when for every choice of $G' = G'_{1i}$, a perfect matching M'_i of G' containing e' and intersecting every cycle in \mathcal{C}' never contains

the edge $x'y'$. Without loss of generality, we may assume that for G'_{12} the perfect matching M'_2 contains the edges e'_1 and e'_3 . We then consider G'_{13} . Again, without loss of generality, the perfect matching M'_3 contains the edges e'_1 and e'_2 . Finally, we apply induction on $G'' = G'_{14}$ with $e'' = e'_1$. Every perfect matching M'' of G'' containing e'' contains either e''_2 or e''_3 , so it can be combined with either M'_2 or M'_3 to give a perfect matching M of G containing e . We may assume that $e''_2 \in M''$. It is easy to check that such a perfect matching M intersects all the cycles in \mathcal{C}'_0 and in \mathcal{C}''_0 ; it remains to certify that M intersects all the cycles in \mathcal{C}_X . The only cycle from \mathcal{C}_X potentially not intersected by M is the one containing the edges f_3 and f_4 , say C_X . However, the corresponding cycles C'_X and C''_X in \mathcal{C}'_X and \mathcal{C}''_X (there is exactly one on each side) respectively contain an edge of M'_3 (not incident to x' or y') and an edge of M'' (not incident to x'' or y''). Therefore, M intersects C_X at least twice, which is more than what is desired. ■

From this point on we may assume that G is cyclically 5-edge-connected. We now consider the edges at distance 2 from e (distance measured as the length of a shortest path joining corresponding vertices in the line graph of G).

Claim 4. Let f be an edge at distance 2 from e . Then, $f \notin C$ for any $C \in \mathcal{C}$.

Proof of Claim 4. Now it is time to do some path splitting. Let $f = uv$, let the neighbours of u distinct from v be α and γ , and let the neighbours of v distinct from u be β and δ . In particular, since G is cyclically 5-edge-connected, these four vertices are all distinct and non-adjacent. Without loss of generality, we may assume that α is an endvertex of e .

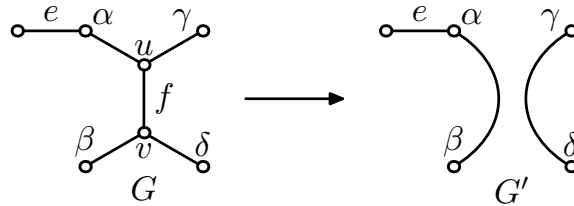


Figure 2.13: The vertices $\alpha, \beta, \gamma, \delta$ and an $(\alpha\beta : \gamma\delta)_{uv}$ -reduction.

As shown in Figure 2.13, we obtain a smaller graph by deleting the endvertices of f (together with all edges incident to them) and adding the edges $\alpha\beta$ and $\gamma\delta$ – by splitting the path $\alpha uv\beta$. Let this resulting graph, obtained after an $(\alpha\beta : \gamma\delta)_{uv}$ -reduction, be G' . Since G is cyclically 5-edge-connected, G' is cyclically 3-edge-connected.

Let the edge in G' corresponding to e , and the vertices in G' corresponding to $\alpha, \beta, \gamma, \delta$ be denoted by the same name. We recall that any perfect matching of G' which contains e can be extended to a perfect matching of G containing the edge e (see also Figure 2.14). In fact, let M' be a perfect matching of G' containing e . This is extended to a perfect matching M of G containing e as follows:

$$M = \begin{cases} M' \cup \{u\gamma, v\delta\} \setminus \{\gamma\delta\} & \text{if } \gamma\delta \in M', \\ M' \cup \{f\} & \text{otherwise.} \end{cases}$$

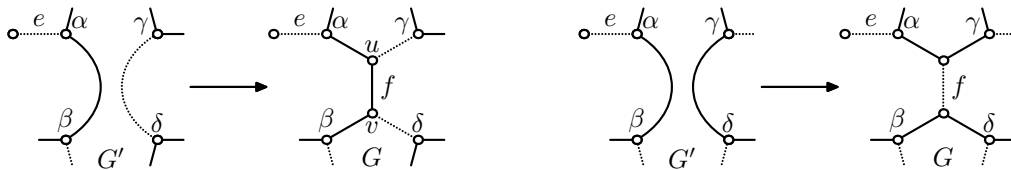


Figure 2.14: Extending a perfect matching of G' containing e to a perfect matching of G containing e . Dotted lines represent edges in M or M' .

Suppose that for some edge f at distance 2 from e , f is in some cycle C_f in \mathcal{C} . This means that exactly one of $u\alpha$ and $u\gamma$, and exactly one of $v\beta$ and $v\delta$ belong to C_f . Without loss of generality, we may assume that $u\alpha \in C_f$ if and only if $v\beta \in C_f$ (otherwise, we rename β and δ). Let G' be the graph obtained from G after an $(\alpha\beta : \gamma\delta)_{uv}$ -reduction. Let C'_f be the cycle in G' corresponding to C_f in G obtained by replacing the 3-edge path passing through u and v by a single edge. Since G is of girth 5, C_f is a cycle of length at least 3. Let $\mathcal{C}' = (\mathcal{C} \setminus \{C_f\}) \cup \{C'_f\}$ be the collection of disjoint cycles of G' obtained by this reduction. This is portrayed in Figure 2.15.



Figure 2.15: If $f \in C_f$, then we apply induction on G' — the graph obtained from G after an $(\alpha\beta : \gamma\delta)_{uv}$ -reduction. Dashed lines represent edges outside \mathcal{C} or \mathcal{C}' , respectively.

Let us first assume that G' is not a Klee-graph. Since G' is cyclically 3-edge-connected and its order is strictly less than G , it is not a counterexample. Let M' be a perfect matching of G' containing e intersecting all the cycles in \mathcal{C}' . We extend this

perfect matching to a perfect matching M of G containing e as described above (see Figure 2.14), and claim that it intersects all the cycles in \mathcal{C} . Every cycle $C' \neq C'_f$ in \mathcal{C}' is hit by an edge of M' in G' , and so the corresponding cycle C is hit by the corresponding edge of M in G . The cycle C'_f is hit by an edge M' in G' , and so the corresponding cycle C_f is hit by the corresponding edge in G , unless $\gamma\delta \in E(C_f)$ and the hitting edge is $\gamma\delta$, but then C_f is hit by both edges γu and $v\delta$. Observe that M' cannot contain $\alpha\beta$ because $e \in M'$.

Therefore, G' must be Klee. Since G' is obtained after an $(\alpha\beta : \gamma\delta)_{uv}$ -reduction, and G is cyclically 5-edge-connected, by Lemma 36, the graph G' admits exactly two (disjoint) triangles T_ℓ and T_r such that $V(T_\ell) = \{v_\ell, \alpha, \beta\}$ and $V(T_r) = \{v_r, \gamma, \delta\}$, for some v_ℓ and v_r in G' . Let a, b, c, d be the vertices in $G' - \{v_\ell, v_r, \alpha, \beta, \gamma, \delta\}$ which are adjacent to $\alpha, \beta, \gamma, \delta$, respectively (see Figure 2.16). Furthermore, since G is cyclically 5-edge-connected, by Lemma 37 the edge $\alpha\beta$ ($\gamma\delta$) is the only edge in T_ℓ (in T_r) which lies on a 4-cycle. Therefore, (α, β, b, a) and (γ, δ, d, c) are 4-cycles in G' . Next we show that a, b, c, d are pairwise distinct. Clearly, $a \neq b$, and $c \neq d$. Moreover, $a \neq c$, and $b \neq d$, otherwise G would admit a 4-cycle. What remains to show is that $a \neq d$, and $b \neq c$. We first note that since G is cubic, and $a = d$ if and only if $b = c$. Indeed, if $a = d$ and $b \neq c$, then, a is adjacent to α, β, c, δ , a contradiction. Moreover, since G is cyclically 4-edge-connected, if $a = d$, G would be the Petersen graph. However, it is an easy exercise to check that the Petersen graph is not a counterexample.

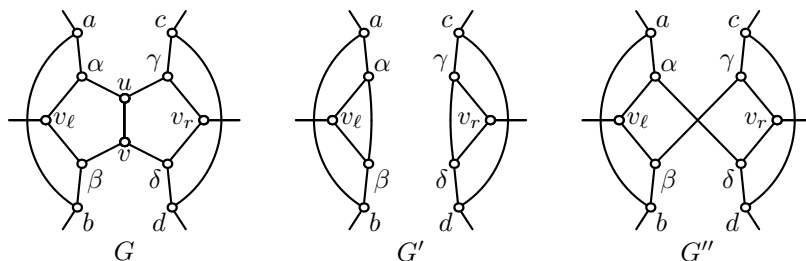


Figure 2.16: If for some graph G , the graph G' obtained after an $(\alpha\beta : \gamma\delta)_{uv}$ -reduction is a Klee graph, then the graph G'' obtained after an $(\alpha\delta : \beta\gamma)_{uv}$ -reduction is not.

Hence, a, b, c, d are four distinct vertices. Consequently, if we apply an $(\alpha\delta : \beta\gamma)_{uv}$ -reduction to G we can be sure that $\alpha\delta$ and $\beta\gamma$ do not lie on a triangle in the resulting graph G'' . In particular, G'' is not Klee. Let $\mathcal{C}'' = \mathcal{C} \setminus \{C_f\}$. By the inductive hypothesis, G'' admits a perfect matching M'' containing e intersecting every cycle in \mathcal{C}'' . By extending the perfect matching M'' to a perfect matching M of G containing e as described above (see Figure 2.14), we can deduce that M intersects every cycle

in \mathcal{C} , because, in particular, M contains exactly one edge from $E(C_f) \cap \{u\gamma, uv, v\beta\}$. ■

From this point on we may assume that no edge f at distance 2 from e is contained in a cycle in \mathcal{C} . As a consequence, we have that no edge at distance at most 2 from e is contained in a cycle in \mathcal{C} .

Claim 5. Every vertex at distance 2 from e is traversed by a cycle in \mathcal{C} .

Proof of Claim 5. Once again, let us consider an edge $f = uv$ at distance 2 from e , with vertices denoted $\alpha, \beta, \gamma, \delta$ as above. In particular, there can be a cycle in \mathcal{C} passing through an endvertex of f only if it passes through the edges βv and $v\delta$.

Suppose that there is no such cycle. As we have seen in the above claim, at least one of the the resulting graphs obtained by an $(\alpha\beta : \gamma\delta)_{uv}$ -reduction or an $(\alpha\delta : \gamma\beta)_{uv}$ -reduction is not Klee, and so, without loss of generality, we can assume that the graph G' obtained after an $(\alpha\beta : \gamma\delta)_{uv}$ -reduction is not Klee. In G' , there is a perfect matching M' containing e and intersecting every cycle in $\mathcal{C}' = \mathcal{C}$. It is easy to see that the perfect matching M (of G) containing e obtained as an extension of M' still intersects all the cycles in \mathcal{C} , a contradiction. ■

As a consequence of Claim 5, we have the following.

Claim 6. The edge e does not belong to a 5-cycle.

Proof of Claim 6. Suppose that e belongs to a 5-cycle $C = (t_1, t_2, t_3, t_4, t_5)$. Let the vertices in $G - V(C)$ which are adjacent to some vertex in C be v_1, v_2, v_3, v_4, v_5 , such that v_i is adjacent to t_i and $e = t_1 t_2$. Since G is cyclically 5-edge-connected, the v_i s are pairwise distinct. Moreover, by Claim 4, no edge in C can be contained in a cycle of \mathcal{C} , but by Claim 5, the vertex t_4 must be traversed by a cycle of \mathcal{C} , which is clearly impossible. ■

We also show that e cannot be at distance 2 from a 5-cycle.

Claim 7. Edges at distance 2 from e do not belong to a 5-cycle.

Proof of Claim 7. Suppose the above assertion is false and let $C = (t_1, t_2, t_3, t_4, t_5)$ be such a 5-cycle, with t_1 being an endvertex of an edge adjacent to e . We obtain a smaller graph G' by deleting the edge $t_3 t_4$, and smooth the vertices t_3 and t_4 . Let the resulting graph be denoted by G' . It can be easily seen that G' is cyclically 3-edge-connected and that it does not admit any triangles (and therefore not Klee), because otherwise, G would contain 4-cycles. For each $i \in [5]$, let the vertex in $V(G) - V(C)$ adjacent to t_i be denoted by t'_i . We proceed by first showing that a

perfect matching M' of G' containing e can be extended to a perfect matching M of G containing e . Without loss of generality, assume that $t_1t_2 \in M'$. We extend this to a perfect matching of G as follows:

$$M = \begin{cases} M' \cup \{t_3t_4\} & \text{if } t_5t'_5 \in M', \\ M' \cup \{t_1t_5, t_2t_3, t_4t'_4\} \setminus \{t_1t_2, t'_4t_5\} & \text{otherwise.} \end{cases}$$

Next, since in G , no edge at distance at most 2 from e belongs to a cycle in \mathcal{C} , and every vertex at distance 2 from an endvertex of e is traversed by one, we have that t_2t_3 and t_4t_5 belong to some cycle in \mathcal{C} (possibly the same). We consider two cases depending on whether the edge t_3t_4 is in a cycle edge or not.

- (i) When t_3t_4 is a cycle edge, then the vertices t_2, t_3, t_4, t_5 are consecutive vertices on some cycle C_X in \mathcal{C} . In this case, we let $E(C'_X) = E(C_X) \cup \{t_1t_2, t_1t_5\} \setminus \{t_2t_3, t_3t_4, t_4t_5\}$ and $\mathcal{C}' = (\mathcal{C} \setminus \{C_X\}) \cup \{C'_X\}$ to be a collection of disjoint cycles in G' . By the inductive hypothesis there exists a perfect matching M' containing e intersecting every cycle in \mathcal{C}' . The perfect matching M of G containing e obtained from M' as explained above clearly intersects every cycle in \mathcal{C} (it contains either t_2t_3 or t_3t_4). This contradicts our initial assumption and so we must have the following case.
- (ii) When t_3t_4 is not a cycle edge, we let C_X and C_Y (with X not necessarily distinct from Y) to be the cycles containing the edges t_2t_3 and t_4t_5 , respectively. The corresponding cycles C'_X and C'_Y in G' are obtained by smoothing out the vertices t_3 and t_4 . We then set $\mathcal{C}' = (\mathcal{C} \setminus \{C_X, C_Y\}) \cup \{C'_X, C'_Y\}$. Note that the edges $t_2t'_2$ and $t_5t'_5$ belong to distinct cycles in \mathcal{C}' if and only if they belong to distinct cycles in \mathcal{C} . By the inductive hypothesis, there exists a perfect matching M' of G' containing e intersecting every cycle in \mathcal{C}' . Without loss of generality, assume that $t_1t_2 \in M'$. The perfect matching M' contains either $t_5t'_5$ or $t_5t'_4$, and consequently, so does the perfect matching M obtained from M' as explained above. This implies that C_Y is intersected by M . If $C'_X \neq C'_Y$, then M' contains an edge of C'_X not incident to t_2 in G' , and so M contains the corresponding edge of C_X in G . Altogether, the perfect matching M obtained from M' as shown above intersects every cycle in \mathcal{C} . This is again a contradiction to our initial assumption that G is a counterexample — thus proving our claim. ■

Let's get back to analysing an edge $f = uv$ at distance 2 from e . We cannot use the reduction portrayed in Figure 2.13 as we do not have a guarantee that we

can obtain a perfect matching M intersecting the cycle in \mathcal{C} containing the edges $v\beta$ and $v\delta$, which we shall denote by C_v . Since G is cyclically 5-edge-connected, this latter cycle is of length at least 5. Let δ, v, β, y, z be consecutive and distinct vertices on this cycle (see Figure 2.17). Moreover, let w and x be the vertices in G respectively adjacent to β and y , such that $w\beta, xy \notin E(C_v)$. We proceed by applying an $(\alpha\beta : \gamma\delta)_{uv}$ -reduction followed by an $(\alpha x : wz)_{\beta y}$ -reduction as portrayed in Figure 2.17.

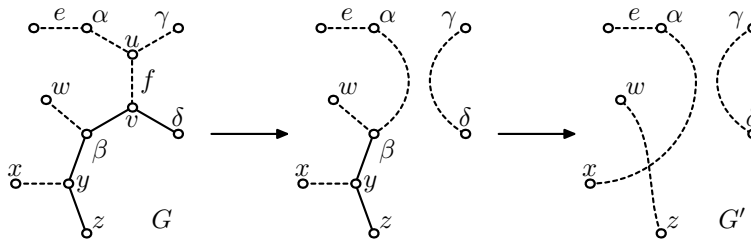


Figure 2.17: An $(\alpha\beta : \gamma\delta)_{uv}$ -reduction followed by an $(\alpha x : wz)_{\beta y}$ -reduction. Dashed edges represent edges outside \mathcal{C} or \mathcal{C}' .

Let the resulting graph after these two reductions be denoted by G' , and let $\mathcal{C}' = \mathcal{C} \setminus \{C_v\}$. Since G' is obtained by applying twice the reduction at an edge at distance 2 from e , a perfect matching of G' containing e can always be extended to a perfect matching of G containing e (recall that βy cannot be adjacent to e , since $\beta y \in C_v$). Moreover, any such matching contains either the edge βy or the edge yz , and so it also contains at least one edge of the cycle C_v . Therefore, as long as G' is cyclically 3-edge-connected and not Klee, by minimality of G , there exists a perfect matching M' of G' containing e intersecting every cycle in \mathcal{C}' , which extends to a perfect matching M of G containing e intersecting every cycle in \mathcal{C} . This contradicts our initial assumption that G is a counterexample.

Therefore, G' is either Klee or admits a (cyclic) edge-cut of size at most 2 (more details after proof of Claim 8).

Claim 8. The graph G' is not Klee.

Proof of Claim 8. Suppose that G' is Klee. The edge $\gamma\delta$ cannot be on a triangle otherwise the edges uv and $u\gamma$ at distance 2 from e belong to a 5-cycle, contradicting Claim 7. Since G is cyclically 5-edge-connected, if G' is Klee then we must have that αx and wz each lie on a triangle (see Lemma 36). Therefore, in particular, if G' is Klee, α and x must have a common neighbour (in both G' and G , so it is not u). This common neighbour cannot be u' , the neighbour of α not incident to e and distinct from u , either, since x would then be a vertex at distance 2 from the

endvertex α of e (via u') and so, by Claim 5, it would be traversed by a cycle in \mathcal{C} . However, the edges xu' and xy are not in any cycle in \mathcal{C} , a contradiction. Therefore, the common neighbour of x and α is α' , the other endvertex of e . By Lemma 37, one edge of the triangle (x, α, α') lies on a 4-cycle in G' , which is not present in G . First, consider the case when exactly one of $\alpha'x$ and $\alpha'\alpha$ lie on a 4-cycle, say (α', s, t, x) or (α', s, t, α) accordingly. Since the edges $\alpha's, xt, \alpha'x, \alpha'\alpha, \alpha t$ all belong to G , s and t cannot be adjacent in G , and so $\{s, t\}$ is equal to $\{w, z\}$ or $\{\gamma, \delta\}$. If $\{s, t\} = \{\gamma, \delta\}$, then we either have that $\alpha'\gamma \in E(G)$, implying that $(\alpha', \gamma, u, \alpha)$ is a 4-cycle in G , or that $\alpha'\delta \in E(G)$, implying that $(\alpha', \alpha, u, v, \delta)$ is a 5-cycle in G containing e , both a contradiction. Hence, $\{s, t\} = \{w, z\}$. Since, G is cyclically 5-edge-connected, x cannot be adjacent to z nor w , and so, we must have that the edge lying on the 4-cycle with w and z is $\alpha'\alpha$. However, this is impossible since z cannot be adjacent to an endvertex of e . Consequently, we must have that the edge of the triangle (x, α, α') lying on a 4-cycle in G' is αx . In this case, st cannot be an edge in G , otherwise $(\alpha', \alpha, s, t, x)$ would be a 5-cycle in G containing e , contradicting Claim 6. Thus, $\{s, t\}$ is equal to $\{w, z\}$ or $\{\gamma, \delta\}$ once again. As before, x cannot be adjacent to z or w , implying that α being adjacent to γ or δ , respectively giving rise to (α, u, γ) or (α, u, v, δ) in G , a contradiction. Therefore, G' is not Klee. \blacksquare

Consequently, G' must admit some (cyclic) edge-cut of size at most 2. Whenever G is cyclically 4-edge-connected, by the analysis done at the end of the main result in [46], we know the graph G' is bridgeless, so if G' is not (cyclically) 3-edge-connected, it admits a 2-edge-cut. We next show that this cannot be the case, that is, if G' admits a 2-edge-cut, then G is not a counterexample to our statement.

Claim 9. G' is cyclically 3-edge-connected, unless α is adjacent to x in G .

Proof of Claim 9. Suppose that G' admits a 2-edge-cut $X_2 = \{g_1, g_2\}$. Let $\Omega_1 = \{\alpha, \gamma, \delta, w, x, z\}$ and let $\Omega_2 = \{u, v, \beta, y\}$.

We label the vertices of $G' \setminus X_2$ with labels A and B depending in which connected component of $G' \setminus X_2$ they belong to. Consequently, G' has exactly two edges which are not monochromatic: g_1 and g_2 . We consider different cases depending on the number of vertices in Ω_1 labelled A in G' , and show that, in each case, a 2-edge-cut in G' would imply that G is not cyclically 5-edge-connected or not a counterexample to our statement. Without loss of generality, we shall assume that the number of vertices in Ω_1 labelled A is at least the number of vertices in Ω_1 labelled B . We consider four cases.

(B0) All the vertices in Ω_1 are labelled A in G' .

First, we extend this labelling of $V(G')$ to a partial labelling of $V(G)$ by giving to the vertices in $V(G) - \Omega_2$ the same label they had in G' . We then give label A to all the vertices in Ω_2 . However, this means that G has exactly two edges, corresponding to the edges in X_2 , which are not monochromatic, a contradiction, since G does not admit any 2-edge-cuts.

(B1) Exactly 5 vertices in Ω_1 are labelled A in G' .

This means that exactly one of the edges in $\{\alpha x, wz, \gamma \delta\}$ belongs to X_2 , say g_1 , without loss of generality. Once again, we extend this labelling to a partial labelling of G , and then give label A to all the vertices in Ω_2 . However, this means that G has an edge which has exactly one endvertex in Ω_1 labelled B and exactly one endvertex in Ω_2 labelled A , which together with the edge g_2 gives a 2-edge-cut in G , a contradiction once again.

(B2) Exactly 4 vertices in Ω_1 are labelled A in G' .

We consider two cases depending on whether there is one or three monochromatic edges in $\{\alpha x, wz, \gamma \delta\}$. First, consider the case when $\{\alpha x, wz, \gamma \delta\}$ has exactly one monochromatic edge, meaning that $X_2 \subset \{\alpha x, wz, \gamma \delta\}$. As in the previous cases, we extend this labelling to a partial labelling of $V(G)$, and then give label A to all the vertices in Ω_2 . However, this means that G has exactly two edges each having exactly one endvertex in Ω_1 labelled B and exactly one endvertex in Ω_2 labelled A , meaning that G admits a 2-edge-cut, a contradiction.

Therefore the edges $\{\alpha x, wz, \gamma \delta\}$ are all monochromatic: two edges with all their endvertices coloured A , and one edge with its endvertices coloured B . We extend this labelling to a partial labelling of $V(G)$, and then give label A to all the vertices in Ω_2 . This gives rise to exactly two edges each having exactly one endvertex in Ω_1 labelled B and exactly one endvertex in Ω_2 labelled A . These two edges together with the two edges in X_2 form a 4-edge-cut X_4 of G . Since the latter is cyclically 5-edge-connected this 4-edge-cut is not cyclic — it separates two adjacent vertices from the rest of the graph. Since $w \neq z$ and $\gamma \neq \delta$ (otherwise there would be a 3-cycle in G) and also $\alpha \neq x$ (otherwise C_v would contain edges at distance 1 from e), these two adjacent vertices in G are endvertices of exactly one of $\alpha x, wz, \gamma \delta$ in G' , and the 2-edge-cut in G' separates a 2-cycle from the rest of the graph. However, G can contain neither the edge wz nor $\gamma \delta$, since G has no 4-cycles. Thus, we must have that α is adjacent to x in G .

(B3) Exactly 3 vertices in Ω_1 are labelled A in G' .

Since G' has exactly two edges which are not monochromatic, there is exactly one edge in $\{\alpha x, wz, \gamma \delta\}$ which is not monochromatic. The latter corresponds to one of the edges in X_2 , say g_1 , without loss of generality. As before, we extend this labelling to a partial labelling of $V(G)$, and then give label A to all the vertices in Ω_2 . This gives rise to exactly three edges each having exactly one endvertex in Ω_1 labelled B and exactly one endvertex in Ω_2 labelled A , which together with the edge g_2 from X_2 form a 4-edge-cut X_4 of G . As in the previous case, X_4 separates two adjacent vertices from the rest of the graph — G has exactly two vertices labelled B . As in the previous case, the endvertices of the monochromatic edge belonging to $\{\alpha x, wz, \gamma \delta\}$ (in G') which are labelled B must be either equal or adjacent in G , which is only possible if α is adjacent to x in G . ■

Therefore, given any edge $f = uv$ at distance 2 from e , applying an $(\alpha\beta : \gamma\delta)_{uv}$ -reduction followed by an $(\alpha x : wz)_{\beta y}$ -reduction would lead to α being adjacent to x in G . Let y' and z' be the two consecutive vertices on C_v such that y' is adjacent to δ (note that y' or z' are possibly equal to z). Let w' and x' respectively be the vertices in $G - C_v$ adjacent to δ and y' , and let α' be the other endvertex of e . Applying an $(\alpha\delta : \gamma\beta)_{uv}$ -reduction followed by an $(\alpha x' : w'z')_{\delta y'}$ -reduction leads to α being adjacent to x' . Therefore, x' can be equal to α', u or x . If $x' = \alpha'$, then the edge $\delta y'$ would be an edge belonging to C_v at distance 2 from the edge e , and if $x' = u$, then (u, v, δ, y') would be a 4-cycle in G , with both cases leading to a contradiction. Therefore, $x = x'$.

Let G' be the graph obtained after an $(\alpha\beta : \gamma\delta)_{uv}$ -reduction and let $\mathcal{C}' = \mathcal{C} \setminus \{C_v\}$. By induction, there exists a perfect matching M' containing e intersecting all the cycles in \mathcal{C}' , and it can be extended into a perfect matching M of G containing e . Suppose that M does not intersect C_v (it is the only cycle in \mathcal{C} that M could possibly avoid). To cover vertices y and y' , we must have $\{xy, xy'\} \subset M$ — which is impossible. □

Here are some consequences of Theorem 24. Corollary 11 follows by the above result and Corollary 10.

Corollary 11 *Let G be a cyclically 3-edge-connected cubic graph and let \mathcal{C} be a collection of disjoint cycles of G . Then, there exists a perfect matching M such that $M \cap E(C) \neq \emptyset$, for every $C \in \mathcal{C}$.*

Corollary 12 *Let G be a cyclically 3-edge-connected cubic graph. For every perfect matching M_1 of G , there exists a perfect matching M_2 of G such that $G \setminus (M_1 \cup M_2)$ is acyclic.*

Conclusion

In this thesis, I have aimed to showcase a selection of my scientific results, including the two I am most proud of: the proofs of Barnette-Goodey conjecture and Lovász-Plummer conjecture. I believe that this selection also highlights the breadth and variety of techniques I have employed in my work, ranging from probabilistic tools and discharging methods to the use of computers for exhaustive enumeration and property verification.

Regarding my future work, if I were to name a single open problem that particularly draws my attention within the areas covered by this manuscript, it would be one situated at the intersection of the two main chapters, as it concerns both perfect matchings and fullerene graphs. The *Fries number* of a fullerene graph G is defined as the maximum number of resonant hexagons with respect to a fixed perfect matching of G . Similarly, the *Clar number* is the maximum number of *disjoint* resonant hexagons with respect to a fixed perfect matching. Asymptotically tight lower and upper bounds for both parameters are known. However, despite the effort of several research groups, the exact computational complexity of determining either value remains open. Thus, the central question is: *Is there a polynomial-time algorithm to find the value of the Fries or Clar number of a given fullerene graph?*

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