

Exercise 1

(borrowed from Stefan Felsner, TU Berlin)

Let L be a list assignment of colors to vertices of a graph G such that

- there are at least ℓ colors available for each vertex, and
- for every vertex v and every color $c \in L(v)$, the color c appears in the lists of at most $\ell/8$ neighbors of v .

We want to prove that G is then L -colorable.

For simplicity, we may assume that $|L(v)| = \ell$ for each vertex v (we may remove unnecessary surplus colors).

Let us introduce a randomized procedure as follows : For every vertex, choose a color from its list uniformly at random and independently from other vertices. We want to use LLL to prove that with non-zero probability a proper coloring occurs. In order to do that, there are several ways to define the bad events :

1. For every vertex v , let A_v be the event that v receives the same color as (at least) one of its neighbors.
2. For every edge e , let B_e be the event that the endpoints of e receive the same color.
3. For every edge $e = uv$ and for every color $i \in L(u) \cup L(v)$, let $C_{e,i}$ be the event that both u and v are colored with i

Determine which of the three settings is the right one, allowing us to use the Lemma.

Exercise 2

A k -CNS formula is a conjunction of clauses, where each clause is a disjunction of k literals, with a literal either a boolean variable or a negation of one ; we require that each variable appears at most once (including its negation) in each clause.

A k -CNS formula is *satisfiable* if there exists an assignement of boolean values to variables such that every clause has at least one true literal.

1. Use linearity of expectation to prove that an unsatisfiable k -CNS formula has at least 2^k clauses. Find an unsatisfiable formula with exactly 2^k clauses.
2. Find a function $f(k)$ such that if for every clause C , there are at most $f(k)$ other clauses sharing at least one variable with C , then the whole formula admits a satisfying assignment.
3. A k -CNS formula is *linear* if any two clauses share at most one variable. Find a function $g(k)$ (faster than 2^k) such that every linear k -CNS formula with at most $g(k)$ clauses is satisfiable.

More reading : H. Gebauer, R.A. Moser, D. Scheder, E. Welzl, The Lovász Local Lemma and Satisfiability, in S. Albers, H. Alt, S. Näher (eds.), Efficient Algorithms, Lecture Notes in Computer Science 5760, Springer 2009.

Exercise 3

Let G be a graph with maximum degree Δ and let $V(G) = V_1 \cup V_2 \cup \dots \cup V_r$ be a partition of its vertex set into r pairwise disjoint subsets called *parts* of G . An *independent transversal* of G with respect to the partition $\{V_i\}_{i \in [r]}$ is an independent set of vertices which contains exactly one vertex from each part V_i .

The best known ratio so far between the part size and maximum degree is due to Haxell :

Theorem 1 (Haxell 2001) *Let G be a graph with maximum degree at most Δ , whose vertex set is partitioned into parts of size at least 2Δ . Then G admits an independent transversal.*

1. Prove a similar result for parts of size at least $c \cdot \Delta$ (for a suitable constant c) using LLL.
2. Prove a similar result for parts of size at least $c \cdot \Delta$ (for a suitable constant c) using Entropy Compression.