Exercice 1. Simplified poker  Our first example is a finite duration game which is a simplified version of poker, inspired by Borel and von Neumann simplified poker [?]. This game is played with 4 cards \{♥, ♥, ♣, ♦\}.

— The goal of Eve and Adam is to win the content of a pot in which, initially, they both put 1 euro.
— Eve receives a private random card, unknown by Adam.
— Eve decides whether to "check" or "raise". If she "checks" then she wins the pot iff her card is ♥.
— If Eve raises then Adam has two options : "fold" or "call". If Adam folds then Eve receives the pot. If Adam raises then both player add two euros in the pot and Eve wins the pot iff her card is ♥.

Formalize this game and compute its value and optimal strategies for both players.

Solution using reduction to normal form. The simplified poker can be modelled as a stochastic game with signals. Actions of players are public signals sent to both players. Also their the payoff of Eve is publicly announced, when non-zero. Upon choosing whether to call or fold, Adam cannot distinguish between states ♥Raised and □Raised, in both cases he received the sequence of signals ◦,raise. A graphical representation is provided on Figure 1.

The game is played with 4 cards \{♥, ♥, ♣, ♦\}. We exploit the symmetry of payoffs with respect to \{♥, ♣, ♦\} and identify these three colours as a
single one, denoted ■, received initially by Eve with probability $\frac{3}{4}$. The set of vertices is an initial vertex Start, a terminal vertex End plus the four states

$$\{\heartsuit, \spadesuit\} \times \{\text{Play}, \text{Raised}\}.$$  

The set of colors are possible payoffs $C = \{0, -1, +1, -3, +3\}$. The set of actions $A$ is the union of actions of Eve $A_E = \{\cdot, \text{check}, \text{raise}\}$ and actions of Adam $A_A = \{\cdot, \text{call}, \text{fold}\}$. The set of signals is $\{\circ, \heartsuit, \spadesuit\}$ plus $\{\text{check}, \text{raise}, \text{call}, \text{fold}\} \times \{0, -1, +1, -3, +3\}$. The rules of the game, are defined by the set of legal transitions. Let $c \in \{\heartsuit, \spadesuit\}$. The following transitions are legal.

$$\Delta((\text{Start}, \cdot, \cdot)((c, \text{Play}), c, \circ, 0) = \begin{cases} \frac{1}{4} & \text{if } c = \heartsuit \\ \frac{3}{4} & \text{if } c = \spadesuit. \end{cases}$$

$$\Delta((c, \text{Play}), \text{check}, \cdot)((\text{End}, \text{check}_x, \text{check}_x, x) = 1 \text{ where } x = \begin{cases} +1 & \text{if } c = \heartsuit \\ -1 & \text{if } c = \spadesuit. \end{cases}$$

$$\Delta((c, \text{Play}), \text{raise}, \cdot)((c, \text{Raised}), \text{raise}_0, \text{raise}_0, 0) = 1$$

$$\Delta((c, \text{Raised}), \cdot, \text{call})((\text{End}, \text{call}_x, \text{call}_x, x) = 1 \text{ where } x = \begin{cases} +3 & \text{if } c = \heartsuit \\ -3 & \text{if } c = \spadesuit. \end{cases}$$

$$\Delta((c, \text{Raised}), \cdot, \text{fold})((\text{End}, \text{fold}_1, \text{fold}_1, +1) = 1$$

state End is absorbing with payoff 0.

To simplify the notations, we assumed in the general case that players share the same set of actions and signals. As a consequence, other transitions than the legal ones are possible. One can use a threat to guarantee that Eve plays check and raise after receiving her card, by setting a heavy loss of $-10$ if she plays another action instead. Same thing to enforce that Adam plays call or fold after receiving the signal raise. When targeting applications, legal moves should be explicitly specified, typically using an automaton to compute the set of legal actions depending on the sequence of signals.

Deterministic strategies of Eve are mappings $\sigma : \{\heartsuit, \spadesuit\} \rightarrow \{\text{check}, \text{raise}\}$. Adam has only two deterministic strategies: after the sequence $\circ$Raised, he should choose between actions call and fold.

The normal form is

<table>
<thead>
<tr>
<th>$\heartsuit$ $\rightarrow \text{check}$</th>
<th>$\spadesuit$ $\rightarrow \text{check}$</th>
<th>$\text{call}$</th>
<th>$\text{fold}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\heartsuit$ $\rightarrow \text{raise}$</td>
<td>$\spadesuit$ $\rightarrow \text{check}$</td>
<td>$0$</td>
<td>$-0.5$</td>
</tr>
<tr>
<td>$\heartsuit$ $\rightarrow \text{raise}$</td>
<td>$\spadesuit$ $\rightarrow \text{raise}$</td>
<td>$-1.5$</td>
<td>$+1$</td>
</tr>
<tr>
<td>$\heartsuit$ $\rightarrow \text{check}$</td>
<td>$\spadesuit$ $\rightarrow \text{raise}$</td>
<td>$-2$</td>
<td>$+1$</td>
</tr>
</tbody>
</table>

The first line corresponds to Eve never raising, thus her odds are +1 euro at 25% and -1 at 75% thus an expected payoff of $-0.5$. The third line cor-
responds to Eve always raising. If Adam calls then her odds are +3 at 25% and -3 at 75%, on average \(-1.5\). If Adam folds, she gets payoff +1.

Remark that the rows where Eve checks with ♦ are dominated by the corresponding row where Eve does not. Thus checking with ♦ (slow playing) has no strategic interest, and by elimination of weakly dominated strategies, the normal form game is equivalent to:

<table>
<thead>
<tr>
<th></th>
<th>call</th>
<th>fold</th>
</tr>
</thead>
<tbody>
<tr>
<td>♦ → raise, ■ → check</td>
<td>0</td>
<td>-0.5</td>
</tr>
<tr>
<td>♦ → raise, ■ → raise</td>
<td>-1.5</td>
<td>1</td>
</tr>
</tbody>
</table>

The value of this game is \(-\frac{1}{4}\). Eve has a unique optimal strategy which consists in playing the top row with probability \(\frac{5}{6}\). In other words, she should bluff with probability \(\frac{1}{6}\) when she receives ■. Adam has a unique optimal strategy which consists in calling or folding with equal probability \(\frac{1}{2}\).

**Solution using reduction to linear programming.** The second way to solve this example is to reduce to the linear program.

Maximise \(v\) subject to

\[
\forall r \in R_E, \ 0 \leq p_r \leq 1
\]

\[
p_{\Diamond, \text{check}} + p_{\Diamond, \text{raise}} = 1
\]

\[
p_{\spadesuit, \text{check}} + p_{\spadesuit, \text{raise}} = 1
\]

\[
v_e \leq v_0 \leq v_{\Diamond, \text{check}} + v_{\Diamond, \text{raise}}
\]

\[
v_{\Diamond, \text{check}} \leq \frac{1}{4} \cdot p_{\Diamond, \text{check}} \cdot (1) + \frac{3}{4} \cdot p_{\spadesuit, \text{check}} \cdot (-1)
\]

\[
v_{\Diamond, \text{raise}} \leq \frac{1}{4} \cdot p_{\Diamond, \text{raise}} \cdot (1) + \frac{3}{4} \cdot p_{\spadesuit, \text{raise}} \cdot (1)
\]

\[
v_{\spadesuit, \text{raise}} \leq \frac{1}{4} \cdot p_{\spadesuit, \text{raise}} \cdot (1) + \frac{3}{4} \cdot p_{\spadesuit, \text{raise}} \cdot (-3)
\]

Setting \(x = p_{\Diamond, \text{check}}\) and \(y = p_{\spadesuit, \text{check}}\), the solution is

\[
\frac{1}{4} \max_{(x,y) \in [0,1]^2} (x - 3y + \min((1-x) + 3(1-y), 3(1-x) - 9(1-y)))
\]

\[
= \frac{1}{4} \max_{(x,y) \in [0,1]^2} \min(4-6y,-6-2x+6y) = \frac{1}{4} \max_{y \in [0,1]} \min(4-6y,-6+6y)
\]

which is maximal when \(y = \frac{5}{6}\) and the solution, i.e. the value of the game, is \(-\frac{1}{4}\).

**Exercice 2. Poker avec un signal privé pour chaque joueur.** Assume now that Eve does not have the perfect poker face: whenever she has ♦ she scratches her nose with probability \(\frac{1}{2}\) whereas in general it happens only with probability \(\frac{1}{6}\). Only Adam is aware of this sign, which he receives as a private signal \(s\) (scratch) or \(n\) (no scratch).

Compute the value of the game and optimal strategies.
**Solution using reduction to normal form.** Compared to the perfect poker face situation, the situation is much better for Adam : the value drops from $-\frac{1}{2}$ to $-(\frac{1}{2} - \frac{1}{6})$. The optimal bluffing frequency of Eve decreases from $\frac{1}{6}$ to $\frac{1}{5}$.

Deterministic strategies of Eve can be represented as mappings $\sigma : \{\heartsuit, \spadesuit\} \to \{\text{check}(c), \text{raise}(r)\}$. Deterministic strategies of Adam can be represented as mappings $\tau : \{s, n\} \to \{\text{call}(c), \text{fold}(f)\}$

The normal form is

<table>
<thead>
<tr>
<th></th>
<th>(s, c)(n, c)</th>
<th>(s, f)(n, f)</th>
<th>(s, c)(n, f)</th>
<th>(s, f)(n, c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\heartsuit c, \clubsuit c$</td>
<td>-0.5</td>
<td>-0.5</td>
<td>-0.5</td>
<td>-0.5</td>
</tr>
<tr>
<td>$\heartsuit c, \spadesuit c$</td>
<td>0</td>
<td>-0.5</td>
<td>$\frac{1}{4} \left( \frac{1}{2} (+3) + \frac{1}{2} (+1) \right)$</td>
<td>$\frac{1}{4} \left( \frac{1}{2} (+1) + \frac{1}{2} (+3) \right)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>+$\frac{3}{4} (-1)$</td>
<td></td>
<td>+$\frac{3}{4} (-1)$</td>
</tr>
<tr>
<td>$\heartsuit c, \spadesuit r$</td>
<td>-1.5</td>
<td>+1</td>
<td>$\frac{1}{4} \left( \frac{1}{2} (+3) + \frac{1}{2} (+1) \right)$</td>
<td>$\frac{1}{4} \left( \frac{1}{2} (+1) + \frac{1}{2} (+3) \right)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>+$\frac{3}{4} \left( \frac{1}{6} (-3) + \frac{5}{6} (+1) \right)$</td>
<td></td>
<td>+$\frac{3}{4} \left( \frac{1}{6} (+1) + \frac{5}{6} (-3) \right)$</td>
</tr>
<tr>
<td>$\heartsuit c, \spadesuit r$</td>
<td>-2</td>
<td>+1</td>
<td>$\frac{1}{4} (+1)$</td>
<td>$\frac{1}{4} (+1)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>+$\frac{3}{4} \left( \frac{1}{6} (-3) + \frac{5}{6} (+1) \right)$</td>
<td></td>
<td>+$\frac{3}{4} \left( \frac{1}{6} (+1) + \frac{5}{6} (-3) \right)$</td>
</tr>
</tbody>
</table>

after simplification

<table>
<thead>
<tr>
<th></th>
<th>(s, c)(n, c)</th>
<th>(s, f)(n, f)</th>
<th>(s, c)(n, f)</th>
<th>(s, f)(n, c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\heartsuit c, \clubsuit c$</td>
<td>-0.5</td>
<td>-0.5</td>
<td>-0.5</td>
<td>-0.5</td>
</tr>
<tr>
<td>$\heartsuit c, \spadesuit c$</td>
<td>0</td>
<td>-0.5</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-$\frac{1}{4}$</td>
<td></td>
<td>-$\frac{1}{4}$</td>
</tr>
<tr>
<td>$\heartsuit c, \spadesuit r$</td>
<td>-1.5</td>
<td>+1</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>+$\frac{1}{4}$</td>
<td></td>
<td>+$\frac{1}{4}$</td>
</tr>
<tr>
<td>$\heartsuit c, \spadesuit r$</td>
<td>-2</td>
<td>+1</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>+$\frac{1}{4}$</td>
<td></td>
<td>+$\frac{1}{4}$</td>
</tr>
</tbody>
</table>

after simplification and multiplication by 4

<table>
<thead>
<tr>
<th></th>
<th>(s, c)(n, c)</th>
<th>(s, f)(n, f)</th>
<th>(s, c)(n, f)</th>
<th>(s, f)(n, c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\heartsuit c, \clubsuit c$</td>
<td>-2</td>
<td>-2</td>
<td>-2</td>
<td>-2</td>
</tr>
<tr>
<td>$\heartsuit c, \spadesuit c$</td>
<td>0</td>
<td>-2</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\heartsuit c, \spadesuit r$</td>
<td>-6</td>
<td>+4</td>
<td>3</td>
<td>-5</td>
</tr>
<tr>
<td>$\heartsuit c, \spadesuit r$</td>
<td>-8</td>
<td>+4</td>
<td>2</td>
<td>-6</td>
</tr>
</tbody>
</table>

Again, calling with heart is dominated by raising with heart : the pure strategy $\heartsuit c, \spadesuit r$ is weakly dominated by the pure strategy $\heartsuit r, \spadesuit r$ and $\heartsuit c, \spadesuit c$ is weakly dominated by $\heartsuit c, \spadesuit c$ thus the game is equivalent to (a $\frac{1}{4}$ of)

<table>
<thead>
<tr>
<th></th>
<th>(s, c)(n, c)</th>
<th>(s, f)(n, f)</th>
<th>(s, c)(n, f)</th>
<th>(s, f)(n, c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\heartsuit r, \spadesuit c$</td>
<td>0</td>
<td>-2</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\heartsuit r, \spadesuit r$</td>
<td>-6</td>
<td>+4</td>
<td>3</td>
<td>-5</td>
</tr>
</tbody>
</table>

The pure strategy (s, c)(n, f) is weakly dominated by the pure strategy (s, f)(n, c) thus the game is equivalent to
The strategy of Eve is parametrized by the probability $0 \leq \alpha \leq 1$ of checking with $\blacksquare$. Then Adam minimize over the three columns. In other word we compute

$$\max_{\alpha \in [0,1]} \min \{-6(1-\alpha), 4-6\alpha, -5+4\alpha\}$$

We plot the three functions using chalk.

We find the optimum when $\alpha \rightarrow 4 - 6\alpha$ intersects $\alpha \rightarrow -5 + 4\alpha$ i.e., when $\alpha = \frac{9}{10}$. Thus the value is $\frac{1}{4}(4 - 6 \cdot \frac{9}{10}) = -\frac{7}{20}$.

**Solution using linear programming.** The linear program is.
Maximise \( v_\epsilon \) subject to
\[
\forall u \in R_E, \ 0 \leq p_u \leq 1
\]
\[
p_{\diamondsuit,c} + p_{\diamondsuit,r} = 1 \quad p_{\spadesuit,c} + p_{\spadesuit,r} = 1
\]
\[
v_\epsilon \leq v_s + v_n \quad v_s \leq v_{sc} + v_{sr} \quad v_n \leq v_{nc} + v_{nr}
\]
\[
v_{sc} \leq \frac{1}{4} \cdot \frac{1}{2} \cdot p_{\diamondsuit,c} \cdot (+1) + \frac{3}{4} \cdot \frac{1}{6} \cdot p_{\spadesuit,c} \cdot (-1)
\]
\[
v_{nc} \leq \frac{1}{4} \cdot \frac{1}{2} \cdot p_{\diamondsuit,c} \cdot (+1) + \frac{3}{4} \cdot \frac{5}{6} \cdot p_{\spadesuit,c} \cdot (-1)
\]
\[
v_{sr} \leq \frac{1}{4} \cdot \frac{1}{2} \cdot p_{\diamondsuit,r} \cdot (+1) + \frac{3}{4} \cdot \frac{1}{6} \cdot p_{\spadesuit,r} \cdot (+1)
\]
\[
v_{nr} \leq \frac{1}{4} \cdot \frac{1}{2} \cdot p_{\diamondsuit,r} \cdot (+3) + \frac{3}{4} \cdot \frac{5}{6} \cdot p_{\spadesuit,r} \cdot (-3)
\]
We set \( x = p_{\diamondsuit,c} \) and \( y = p_{\spadesuit,c} \) and get

Maximise \( \frac{1}{4} v_\epsilon \) subject to
\[
0 \leq x \leq 1
\]
\[
0 \leq y \leq 1
\]
\[
v_\epsilon \leq \frac{1}{2} \cdot x - \frac{1}{2} \cdot y + \frac{1}{2} x - \frac{5}{2} y
\]
\[
+ \min \left( \frac{1}{2} (1 - x) + \frac{1}{2} (1 - y), \frac{3}{2} (1 - x) - \frac{3}{2} (1 - y) \right)
\]
\[
+ \min \left( \frac{1}{2} (1 - x) + \frac{15}{6} (1 - y), \frac{3}{2} (1 - x) - \frac{9}{6} (1 - y) \right)
\]
Further simplification leads to

Maximise \( \frac{1}{8} v_\epsilon \) subject to
\[
0 \leq x \leq 1
\]
\[
0 \leq y \leq 1
\]
\[
v_\epsilon \leq 2x - 6y
\]
\[
+ \min (2 - x - y, -3x + 3y)
\]
\[
+ \min (6 - x - 5y, -12 - 3x + 15y)
\]
Further simplification (inject $x - 3y$ in both sides of both $\min$) leads to:

\[
\text{Maximise } \frac{1}{8} v_e \text{ subject to } \\
0 \leq x \leq 1 \\
0 \leq y \leq 1 \\
v_e \leq \min (2 - 4y, -2x) + \min (6 - 8y, -12 - 2x + 12y)
\]

Since the coefficient of $x$ in all sides of the $\min$ is zero or negative, we shall maximize with $x = 0$, i.e. there is no strategic advantage for Eve to check with $\heartsuit$, hence:

\[
\text{Maximise } \frac{1}{8} v_e \text{ subject to } \\
0 \leq y \leq 1 \\
v_e \leq \min (2 - 4y, 0) + \min (6 - 8y, -12 + 12y)
\]

Since $\min(a, b) + \min(c, d) = \min(a + c, a + d, b + c, b + d)$, we get:

\[
\text{Maximise } \frac{1}{8} v_e \text{ subject to } \\
0 \leq y \leq 1 \\
v_e \leq \min (8 - 12y, -10 + 8y, 6 - 8y, -12 + 12y)
\]

Using wolfram alpha we get:

We can see that the optimum is when $y \rightarrow 8y - 10$ intersects $y \rightarrow 8 - 12y$ i.e. when $y \rightarrow \alpha = \frac{9}{10}$. The optimum is:

\[
\frac{1}{8} \left( 8 \left( \frac{9}{10} \right) - 10 \right) = -\frac{7}{20}.
\]