# When Can You Play Positionally?*

Hugo Gimbert and Wiesław Zielonka

LIAFA, Université Paris 7, case 7014
2, Place Jussieu
75251 Paris Cedex 05, France
{hugo,zielonka}@liafa.jussieu.fr

**Abstract.** We consider infinite antagonistic games over finite graphs. We present conditions that, whenever satisfied by the payoff mapping, assure for both players positional (memoryless) optimal strategies. To verify the robustness of our conditions we show that all popular payoff mappings, such as mean payoff, discounted, parity as well as several other payoffs satisfy them.

## 1 Introduction

We study antagonistic (zero sum) games played on finite oriented graphs $G$ by two players Max and Min. Each vertex of $G$ belongs to one of the players. If the current game position is a vertex $v$ then the owner of $v$ chooses an outgoing edge $e$ and the target of $e$ becomes a new game position. After an infinite number of moves we obtain an infinite path $p$ in $G$ that we call a play.

We suppose that the edges of $G$ are coloured by elements of some set $C$. Then each play yields an infinite sequence of colours of the edges traversed during the play. The payoff function indicates for each such infinite sequence of colours a real number: the amount that player Min pays to player Max at the end of the game.

Since the seminal paper of Shapley[12], this type of games is studied in game theory even in a much more general setting of stochastic games.

In general, optimal strategies of both players can depend on the whole past history. However, it turns out that for many games both players have optimal positional strategies, i.e. optimal strategies where the players' moves depend only on the current position. This type of strategies is particularly interesting in computer science since positional strategies allow us an easy and efficient implementation. Moreover, since the number of possible positional strategies is finite it is always possible to find optimal strategies by exhaustive search — note however that for most of the games we have much more efficient algorithms, see [11] for a recent survey of algorithmic problems.

Motivated by economic applications classical game theory studies mainly, but not exclusively, two payoff functions: mean-payoff and discounted [6]. The games

* This research was supported by European Research Training Network: Games and Automata for Synthesis and Validation

with lim sup payment are studied in the framework of gambling systems [8]. Since the paper of Shapley [12] it is known that even for stochastic discounted games both players have positional optimal strategies.

The existence of optimal positional strategies for mean-payoff deterministic games was established by Moulin [10] and Mycielski and Świerczkowski[4].

Let us note that recently discounted and even mean payoff games entered also in computer science, see [3, 2] for a nice exposition of the motivations behind discounting system properties.

Parity games have much more recent history. They appear in the work of Emerson and Jutla [5] in relation to the modal $\mu$-calculus and in Mostowski [9] in relation to the problem of determinizing finite automata over infinite trees (Rabin theorem). Again these games admit optimal positional strategies (even over infinite graphs).

As noted by Björklund et al. [1] it is possible to present highly similar proofs of the existence of optimal positional strategies for mean-payoff and parity games. Let us note however that [1] failed to extract explicitly the ingredients of both proofs that make them so similar. One can only guess that there are some common axioms hidden in the proof. Moreover the inductive method presented in [1] fails for discounted games.

All these results give rise to the following general question: what conditions should satisfy the payment mapping to assure the existence of optimal positional strategies for both players? The aim of our paper is to examine this problem in the simplest setting of deterministic perfect information games over finite graphs. Ideally, one would like to have conditions that are both sufficient and necessary.

In our paper we were able to provide a set of three conditions that are only sufficient. However, they seem to be quite robust in practice: virtually all payoff mappings that were discussed previously in the literature and that admit positional optimal strategies turn out to satisfy our conditions. It would be interesting to have more natural examples of "memoryless" payoff mappings to test the robustness of our conditions, we provide one such new mapping that can be of independent interest. We checked also our conditions against numerous trivial payoff mappings.

On the other hand, one may ask when our conditions fail, i.e., when a payoff mapping yielding optimal positional strategies does not satisfy our conditions. We have found one such example, however even in this case we can use our conditions indirectly to establish the existence of optimal positional strategies.

The conditions that we provide look also quite "natural", the examples of payoff mappings that are interesting in the context of the perfect information deterministic games over finite graphs but which do not satisfy our conditions are rare: the payoff mapping mentioned above is rather artificial, another prominent example are the games with Muller condition [7] but these games need memory.

Let us note finally that the inductive method developed in our paper was successfully adapted by the second author to perfect information stochastic games where it allowed us to show in a simpler way the existence of optimal positional pure strategies [13] for parity games.

An intriguing open question is whether our three conditions assure the existence of optimal positional pure strategies for perfect information *stochastic* games.

## 2 Preliminaries

For any (possibly infinite) set $C$, we write $C^+$ to denote the set of all finite non-empty words over $C$. An infinite word $c_0c_1c_2\ldots$ over $C$ is said to be *finitely generated* if there is a finite subset $A$ of $C$ such that for all $i$, $c_i \in A$. In our paper $C^\omega$ will stand always for a set of all *finitely generated infinite words* over $C$. Let us note that this is a departure from the standard notation where $C^\omega$ stands usually for the set of all infinite sequences (infinite words) over $C$. The difference appears of course only if $C$ is infinite. However, in our paper while it is useful to allow infinite alphabets, only finitely generated sequences are of interest.

For $x \in C^+$, by $x^\omega = xxx\ldots$ we note the infinite concatenation of words $x$. We use also the standard mathematical notation: in particular for any sequence $a_n, n \geq 0$, of real numbers, $\limsup_n a_n = \lim_{n\to\infty} \sup_{i\geq n} a_i$.

$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ will denote the extended set of real numbers.

An *arena* is a tuple $G = (V_{\mathrm{Max}}, V_{\mathrm{Min}}, E, C, \varphi)$, where $(V_{\mathrm{Max}} \cup V_{\mathrm{Min}}, E)$ is a finite oriented graph with the set $V = V_{\mathrm{Max}} \cup V_{\mathrm{Min}}$ of vertices partitioned onto the set $V_{\mathrm{Max}}$ of vertices of player Max and the set $V_{\mathrm{Min}}$ of vertices belonging to player Min. $E \subseteq V \times V$ is the set of edges. We shall colour edges by means of a mapping $\varphi : E \to C$ which associates with each edge $e \in E$ a colour $\varphi(e) \in C$. Although the set of colours will be often infinite (for example $\mathbb{R}$ or $\mathbb{N}$) actually only finite subsets of $C$ will be used since we restrict our attention to finite arenas.

For any edge $e = (v, w) \in E$ we call $v$ the *source* and $w$ the *target* of $e$ and note $\mathrm{source}(e) = v$ and $\mathrm{target}(e) = w$. For a vertex $v \in V$, by $vE = \{e \in E \mid \mathrm{source}(e) = v\}$ we denote the set of outgoing edges.

We suppose that arenas have no dead-ends, i.e. each vertex has at least one outgoing edge.

A *path* in $G$ is a finite or infinite sequence of edges $p = e_0e_1e_2\ldots$ such that, for all $i \geq 0$, $\mathrm{target}(e_i) = \mathrm{source}(e_{i+1})$. The source of $p$ is the source of the first edge $e_0$. If $p$ is finite then $\mathrm{target}(p)$ is the target of the last edge in $p$.

Players Max and Min play on $G$ in the following way. If the current game position is a vertex $v \in V_{\mathrm{Max}}$ then player Max chooses an outgoing edge $e \in vE$ and vertex $\mathrm{target}(e)$ becomes a new game position. Otherwise, if the current game position $v$ belongs to player Min, $v \in V_{\mathrm{Min}}$, then Min chooses an outgoing edge $e \in vE$ and vertex $\mathrm{target}(e)$ becomes a new game position. If the initial position was $v$ then in this way the players construct an infinite path $\mathbf{e} = e_0e_1e_2\ldots$ of visited edges such that $\mathrm{source}(\mathbf{e}) = v$. Such an infinite path will be called sometimes a play in $G$. The set of all plays (infinite paths) in $G$ will be denoted $P_G^\omega$. The set of finite paths in $G$ will be noted $P_G^*$. Elements of $P_G^*$ will be sometimes called *histories* or finite plays, especially when they are used to encode the history of all movements of both players up to a current moment. It is convenient to assume that for each vertex $v$ there exists an empty path $\lambda_v \in P_G^*$ with no edges and such that $\mathrm{source}(\lambda_v) = \mathrm{target}(\lambda_v) = v$.

With any play $\mathbf{e} = e_0 e_1 e_2 \dots$ we associate the payoff sequence $\varphi(\mathbf{e}) = \varphi(e_0)\varphi(e_1)\varphi(e_2)\dots$ of visited colours. Note that we have extended in this way the colouring mapping to $\varphi : P_G^\omega \to C^\omega$. In a similar way, we set for a finite path $p = e_0 \dots e_k$, $\varphi(p) = \varphi(e_0)\dots\varphi(e_k)$.

A *payoff function* is a mapping

$$u : C^\omega \to \overline{\mathbb{R}}.$$

Intuitively, after an infinite play $p$ player Min pays to player Max the amount $u(\varphi(p))$ (with the natural interpretation that if $u(\varphi(p)) < 0$ then it is rather player Max that pays to player Min the amount $|u(\varphi(p))|$). Let us note that since our arenas are finite $\varphi(p)$ will be always finitely generated for any infinite path $p$. This is sometimes important since the definitions of some of the payoff mappings $u$ are meaningless for infinite colour words that are not finitely generated.

A *game* is a couple $\mathbf{G} = (G, u)$ made of an arena $G$ and a payoff function.

Let $G$ be an arena and $\mathfrak{p} \in \{\mathrm{Min}, \mathrm{Max}\}$ a player. A *strategy* for player $\mathfrak{p}$ in $G$ is a mapping $\sigma_\mathfrak{p}$ which indicates for each finite history $p$ such that $\mathrm{target}(p) \in V_\mathfrak{p}$ an edge outgoing from $\mathrm{target}(p)$ that player $\mathfrak{p}$ should choose. Thus formally

$$\sigma_\mathfrak{p} : \{p \in P_G^* \mid \mathrm{target}(p) \in V_\mathfrak{p}\} \to E\,,$$

where $\sigma_\mathfrak{p}(p) \in vE$ whenever $v = \mathrm{target}(p)$.

A finite or infinite path $\mathbf{e} = e_0 e_1 e_2 \dots$ is said to be *consistent* with the strategy $\sigma_\mathfrak{p}$ of player $\mathfrak{p}$ if, for all $i$, whenever $\mathrm{source}(e_i) \in V_\mathfrak{p}$ then $e_i = \sigma_\mathfrak{p}(e_0 ... e_{i-1})$.

$\Sigma_\mathfrak{p}$ denotes the set of strategies of a player $\mathfrak{p}$.

In this paper we are especially interested in the class of positional strategies.

A *positional* strategy for player $\mathfrak{p}$ is a mapping $\sigma_\mathfrak{p} : V_\mathfrak{p} \to E$ such that for all $v \in V_\mathfrak{p}$, $\sigma_\mathfrak{p}(v) \in vE$. Intuitively, when $\mathfrak{p}$ uses a positional strategy then his choice of the outgoing edge depends only on the current game position and not on the previous history.

Given a vertex $v$ and strategies $\sigma$ and $\tau$ for player Max and player Min respectively, there exists a unique play starting in $v$ and consistent with $\sigma$ and $\tau$. This play is denoted by $p^G(v, \sigma, \tau)$.

Strategies $\sigma^\sharp \in \Sigma_{\mathrm{Max}}$ and $\tau^\sharp \in \Sigma_{\mathrm{Min}}$ are said to be *optimal* if for all vertices $v \in V$ and all strategies $\sigma \in \Sigma_{\mathrm{Max}}$ and $\tau \in \Sigma_{\mathrm{Min}}$

$$u(\varphi(p^G(v, \sigma, \tau^\sharp))) \le u(\varphi(p^G(v, \sigma^\sharp, \tau^\sharp))) \le u(\varphi(p^G(v, \sigma^\sharp, \tau))). \tag{1}$$

If $\sigma^\sharp$ and $\tau^\sharp$ are optimal then the quantity $u(\varphi(p^G(v, \sigma^\sharp, \tau^\sharp)))$ is called the *value* of the game $\mathbf{G}$ at $v$ and is noted $\mathrm{val}(\mathbf{G}(v))$. Let us note that in our definition we require for positional strategies to be optimal independently of the starting vertex.

## 3 Fairly mixing Payoffs Yield Positional Strategies

The aim of this section is to present sufficient conditions for the payoff mapping $u$ assuring the existence of optimal positional strategies for both players.

**Definition 1.** *A payoff mapping* $u : C^\omega \to \overline{\mathbb{R}}$ *is said to be* fairly mixing *if the following conditions hold*

*(C1) for all* $x \in C^+, y_0, y_1 \in C^\omega$*, if* $u(y_0) \le u(y_1)$ *then* $u(xy_0) \le u(xy_1)$*,*
*(C2) for all* $x \in C^+, y \in C^\omega$*,* $\min\{u(x^\omega), u(y)\} \le u(xy) \le \max\{u(x^\omega), u(y)\}$*,*
*(C3) Suppose that* $(x_i)$*,* $i \in \mathbb{N}$*, is an infinite sequence of non-empty colour words* $x_i \in C^+$ *such that the infinite word* $x_0x_1x_2\ldots$ *is finitely generated. Let* $I \cup J = \mathbb{N}$ *be any partition of* $\mathbb{N}$ *onto two infinite sets* $I$ *and* $J$ *(*$I \cap J = \emptyset$*). Let* $U_I$ *be the set of all payoffs* $u(x_{i_0}x_{i_1}x_{i_2}\ldots)$*, where* $(i_k)_{k=0}^\infty$ *ranges over all sequences of elements of* $I$*. Similarly, let* $U_J$ *be the set of all payoffs* $u(x_{j_0}x_{j_1}x_{j_2}\ldots)$*, where* $(j_k)_{k=0}^\infty$ *ranges over all infinite sequences of elements of* $J$*. Then the following condition holds:*

$$\inf(U_I \cup U_J) \le u(x_0x_1x_2x_3\ldots) \le \sup(U_I \cup U_J). \tag{2}$$

Let us emphasize that in (C3) we suppose that (2) holds for any partition of $\mathbb{N}$ onto infinite sets $I$ and $J$. Moreover, in the definition of the set $U_I$ we use really *all* possible infinite sequences of elements of $I$, we do not limit ourselves to, for example, increasing sequences, neither we assume that the sequence $(i_k)_{k=0}^\infty$ enumerates all elements of $I$. The same remark is valid for the definition of $U_J$.

**Theorem 2.** *Let* $u : C^\omega \to \overline{\mathbb{R}}$ *be a fairly mixing payoff function. Then both players have optimal positional strategies in any game* $\mathbf{G} = (G, u)$ *over a finite arena* $G$ *coloured by* $C$*..*

For the sake of simplicity the following lemma is formulated only for player Max, however it should be clear that analogous characterisation holds also for player Min. This point will be discussed briefly in the sequel.

**Lemma 3.** *Let* $\mathbf{G} = (G, u)$ *be a game with fairly mixing payoff function* $u$*. Suppose that*

*(1) there exists a vertex* $v \in V_{\mathrm{Max}}$ *such that* $|vE| > 1$*, where* $vE = \{e \in E \mid \mathrm{source}(e) = v\}$ *is the set of all edges with the source* $v$*,*
*(2)* $vE = A_1 \cup A_2$ *is a partition of the set* $vE$ *onto two non-empty sets* $A_1$ *and* $A_2$*,*
*(3)* $E' = \{e \in E \mid \mathrm{source}(e) \ne v\}$*,* $E_i = E' \cup A_i$ *and* $G_i = (V_{\mathrm{Max}}, V_{\mathrm{Min}}, E_i, C, \varphi)$*,* $i = 1, 2$*, are arenas obtained from* $G$ *by keeping only the edges of* $E_i$ *(i.e., by removing the edges of* $A_{3-i}$*) and keeping all vertices* $V$*,*
*(4) players* Max *and* Min *have optimal positional strategies in the games* $\mathbf{G}_1 = (G_1, u)$ *and* $\mathbf{G}_2 = (G_2, u)$*.*

*Then* Max *and* Min *have optimal strategies* $\sigma^\sharp$ *and* $\tau^\sharp$ *in the game* $\mathbf{G} = (G, u)$*. More exactly, we can assume without loss of generality that*

$$\mathrm{val}(\mathbf{G}_2(v)) \leq \mathrm{val}(\mathbf{G}_1(v)) \tag{3}$$

*and then*

*(i) the optimal positional strategy of player* Max *in* $\mathbf{G}_1$ *is also optimal for the same player in the game* $\mathbf{G}$*,*
*(ii) for all* $w \in V$*,* $\mathrm{val}(\mathbf{G}(w)) = \mathrm{val}(\mathbf{G}_1(w))$*.*

*Proof.* Let $\sigma_i^\sharp$, $\tau_i^\sharp$ be optimal positional strategies of players Max and Min respectively in the games $\mathbf{G}_i = (G_i, u)$, $i = 1, 2$. We then set

$$\sigma^\sharp = \sigma_1^\sharp \,. \tag{4}$$

It is clear that $\sigma^\sharp$ is a positional strategy for Max not only in the game $\mathbf{G}_1$ but also in the game $\mathbf{G}$. We claim that $\sigma^\sharp$ is optimal for Max in the game $\mathbf{G} = (G, u)$ and that for all vertices $w \in V$, $\mathrm{val}(\mathbf{G}(w)) = \mathrm{val}(\mathbf{G}_1(w))$.

The following remark holds under the hypotheses of Lemma 3 and conditions (3) and (4):

*Remark 4.* Let $\tau$ be any strategy of Min in the game $\mathbf{G}$ and $w$ an initial vertex. Then $u(\varphi(p^G(w, \sigma^\sharp, \tau))) \geq \mathrm{val}(\mathbf{G}_1(w))$, i.e., for the games starting at $w$ the strategy $\sigma^\sharp$ can assure to player Max the payoff of at least $\mathrm{val}(\mathbf{G}_1(w))$.

Indeed, if we restrict the strategy $\tau$ to finite paths in $G_1$ then we obtain a strategy $\tau_1$ of Min in the game $\mathbf{G}_1$. Now note that if player Max plays according to $\sigma^\sharp = \sigma_1^\sharp$ then he never uses the edges of $A_2$ and the resulting game looks for his adversary like the game on $G_1$. In particular, the play in $\mathbf{G}$ that is consistent with $\sigma^\sharp$ and $\tau$ is the same as the play in $\mathbf{G}_1$ consistent with $\sigma_1^\sharp = \sigma^\sharp$ and $\tau_1$, $p^G(w, \sigma^\sharp, \tau) = p^{G_1}(w, \sigma_1^\sharp, \tau_1) \geq \mathrm{val}(\mathbf{G}_1(w))$, which proves Remark 4.

To finish the proof of Lemma 3 we should construct a strategy $\tau^\sharp$ for player Min assuring that for any strategy $\sigma$ of player Max in $\mathbf{G}$ and any initial vertex $w$:

$$u(\varphi(p^G(w, \sigma, \tau^\sharp))) \leq \mathrm{val}(\mathbf{G}_1(w)), \tag{5}$$

i.e., player Max cannot win more that $\mathrm{val}(\mathbf{G}_1(w))$ in the game $\mathbf{G}$ against the strategy $\tau^\sharp$.

We define first a mapping $b : P_G^* \to \{1, 2\}$. For a finite path $p \in P_G^*$ in $G$ we set

$$b(p) = \begin{cases} 1 & \text{if either } p \text{ does not contain any edge with the source } v \text{ or} \\ & \text{the last edge of } p \text{ with the source } v \text{ belongs to } G_1, \\ 2 & \text{if the last edge of } p \text{ with the source } v \text{ belongs to } G_2. \end{cases} \tag{6}$$

Then the strategy $\tau^\sharp$ of Min in $\mathbf{G}$ is defined as:

$$\tau^\sharp(p) = \begin{cases} \tau_1^\sharp(\mathrm{target}(p)) & \text{if } b(p) = 1 \\ \tau_2^\sharp(\mathrm{target}(p)) & \text{if } b(p) = 2 \end{cases} \tag{7}$$

for any finite path $p$ with $\mathrm{target}(p) \in V_{\mathrm{Min}}$. In other words, playing in $\mathbf{G}$ player Min applies either his optimal strategy $\tau_1^\sharp$ from the game $\mathbf{G}_1$ or his optimal strategy $\tau_2^\sharp$ from the game $\mathbf{G}_2$. Initially, up to the first visit to $v$, player Min uses the strategy $\tau_1^\sharp$. After the first visit at $v$ the choice between $\tau_1^\sharp$ and $\tau_2^\sharp$ depends on whether the last time when visiting $v$ his adversary Max chose an outgoing edge in $E_1$ or an edge in $E_2$. Intuitively, if the last time at $v$ player Max chose an outgoing edge from $E_1$ then it means that the play from this moment is like a play in $\mathbf{G}_1$ thus player Min tries to respond with his optimal strategy from $\mathbf{G}_1$. Symmetrically, if during the last visit at $v$ player Max chose an outgoing edge from $E_2$ then from this moment onward the play is like a play in $\mathbf{G}_2$ and player Min tries to counter with his optimal strategy from $\mathbf{G}_2$.

It should be clear that the strategy $\tau^\sharp$ needs in fact just two valued memory $\{1, 2\}$ for player Min to remember if during the last visit to $v$ an edge of $E_1$ or an edge of $E_2$ was chosen by his adversary. This memory is initialised to 1 and updated only when the vertex $v$ is visited.

In the sequel we shall say that a finite or infinite path $p$ in $G$ is *homogeneous* if one of the following three conditions folds: (1) $p$ never visits $v$ or (2) each edge $e$ of $p$ with source $v$ belongs to $E_1$ or (3) each edge $e$ of $p$ with source $v$ belongs to $E_2$.

The proof of (5) is divided on four cases.

**Case 1: $w = v$ and the memory state of player** Min **is ultimately constant during the play $p = p^G(v, \sigma, \tau^\sharp)$.**
This means that $p$ can be factorised as $p = p_0 p_1 ... p_n q$, where $p_i$ are finite non-empty homogeneous paths such that $\mathrm{source}(p_i) = \mathrm{target}(p_i) = v$ and $q$ is an infinite homogeneous path with source $v$.

Since $p$ is consistent with $\tau^\sharp$ and $p_i$ are homogeneous, each infinite play $p_i^\omega = p_i p_i p_i \dots$ is consistent either with $\tau_1^\sharp$ (if $p_i$ contains only the edges of $G_1$) or with $\tau_2^\sharp$ (if $p_i$ contains only edges of $G_2$).

By optimality of strategies $\tau_1^\sharp$ and $\tau_2^\sharp$, we get that either $u(\varphi(p_i^\omega)) \le \mathrm{val}(\mathbf{G}_1(v))$ or $u(\varphi(p_i^\omega)) \le \mathrm{val}(\mathbf{G}_2(v))$. Thus, by (3),

$$\text{for all } i,\ 0 \le i \le n, \quad u(\varphi(p_i^\omega)) \le \mathrm{val}(\mathbf{G}_1(v)). \tag{8}$$

Similarly, the infinite path $q$ is consistent either with $\tau_1^\sharp$ or with $\tau_2^\sharp$ implying that $u(\varphi(q))$ cannot be greater than either $\mathrm{val}(\mathbf{G}_1(v))$ or $\mathrm{val}(\mathbf{G}_2(v))$. Thus, again by (3),

$$u(\varphi(q)) \le \mathrm{val}(\mathbf{G}_1(v)) \tag{9}$$

From (C2) of Definition 1, by a trivial induction on $n$, we can deduce that $u(x_0 \dots x_n y) \le \max\{u(x_0^\omega), \dots, u(x_n^\omega), u(y))\}$ for any $x_i \in C^+$ and $y \in C^\omega$.

This inequality, and (8) and (9) imply

$$u(\varphi(p)) = u(\varphi(p_0)\dots\varphi(p_n)\varphi(q)) \leq \\ \max\{u(\varphi(p_0)^\omega),\dots,u(\varphi(p_n)^\omega),u(\varphi(q))\} \leq \mathrm{val}(\mathbf{G}_1(v)).$$

**Case 2:** $w = v$ **and the memory state of player** Min **changes infinitely often during the play** $p = p^G(v,\sigma,\tau^\sharp)$**.**

Thus, in particular, $p$ visits infinitely often the vertex $v$. Let $p = p_0p_1p_2\dots$ the the unique factorization of $p$ such that for all $i \in \mathbb{N}$: (1) each finite path $p_i$ is non-empty and (2) $\mathrm{source}(p_i) = \mathrm{target}(p_i) = v$ and the first edge of $p_i$ is the only edge of $p_i$ with the source $v$.

We set $I = \{i \in \mathbb{N} \mid p_i \text{ begins with an edge belonging to } A_1\}$ and $J = \{j \in \mathbb{N} \mid p_j \text{ begins with an edge belonging to } A_2\}$. Thus $I \cup J = \mathbb{N}$ forms a partition of $\mathbb{N}$.

For any sequence $(i_k)_{k=0}^\infty$ of elements of $I$ the infinite concatenation $p_{i_0}p_{i_1}p_{i_2}\dots$ of finite paths is in fact an infinite path in the arena $G_1$. Moreover, directly from the definition of $\tau^\sharp$ it follows that during the passage through $p_{i_k}$ in $p$ player Min played using the strategy $\tau_1^\sharp$ and therefore $p_{i_0}p_{i_1}p_{i_2}\dots$ is consistent with $\tau_1^\sharp$ implying that $u(\varphi(p_{i_0}p_{i_1}p_{i_2}\dots)) \leq \mathrm{val}(\mathbf{G}_1(v))$ by optimality of $\tau_1^\sharp$ in $\mathbf{G}_1$.

A similar reasoning shows that for any sequence $(j_k)_{k=0}^\infty$ of elements of $J$ we have $u(\varphi(p_{j_0}p_{j_1}p_{j_2}\dots)) \leq \mathrm{val}(\mathbf{G}_2(v))$.

These two bounds imply, by (C3) and (3), that
$u(\varphi(p)) \leq \max\{\mathrm{val}(\mathbf{G}_1(v)), \mathrm{val}(\mathbf{G}_2(v))\} = \mathrm{val}(\mathbf{G}_1(v))$.

**Case 3:** $w \neq v$ **and the play** $p = p^G(w,\sigma,\tau^\sharp)$ **never visits the vertex** $v$**.**

If we define $\sigma_1$ to be the restriction of the strategy $\sigma$ to the paths in $G_1$ then $\sigma_1$ is a strategy of Max in $G_1$. Moreover, since $v$ is never visited, player Min using $\tau^\sharp$ in fact applies always the strategy $\tau_1^\sharp$, which is optimal for him in $\mathbf{G}_1$. Thus $p$ can be seen as a play in $G_1$ consistent with $\sigma_1$ and with $\tau_1^\sharp$ and by optimality of $\tau_1^\sharp$ we get $u(\varphi(p)) \leq \mathrm{val}(\mathbf{G}_1(w))$.

**Case 4:** $w \neq v$ **and the play** $p = p^G(w,\sigma,\tau^\sharp)$ **visits at least once the vertex** $v$**.**

Let us factorise $p$: $p = rq$, where $r$ is the finite prefix of $p$ until the first visit to $v$, i.e. $r$ is the shortest prefix of $p$ with $\mathrm{target}(r) = v$.

Let $q^\sharp = p^{G_1}(v,\sigma_1^\sharp,\tau_1^\sharp)$, where $\sigma_1^\sharp$ is the optimal positional strategy of Max in $\mathbf{G}_1$. Thus, since both $\sigma_1^\sharp$ and $\tau_1^\sharp$ are optimal in $\mathbf{G}_1$, we have $u(\varphi(q^\sharp)) = \mathrm{val}(\mathbf{G}_1(v))$.

Now note that $q$ is in fact a play in $G$ starting at $v$ and consistent with $\tau^\sharp$. This situation was already examined above (Cases 1 and 2) and we have learned there that $u(\varphi(q)) \leq \mathrm{val}(\mathbf{G}_1(v))$.

In this way we get $u(\varphi(q)) \leq \mathrm{val}(\mathbf{G}_1(v)) = u(\varphi(q^\sharp))$ which, by condition (C1) of fairly mixing payoffs, yields

$$u(\varphi(r)\varphi(q)) \leq u(\varphi(r)\varphi(q^\sharp)) = u(\varphi(rq^\sharp)). \tag{10}$$

However, $rq^\sharp$ is an infinite play in $G_1$ starting at $w$ and consistent with $\tau_1^\sharp$ ($r$ is consistent with $\tau_1^\sharp$ since until the first visit to $v$ in $p = rq$ player Min plays according to $\tau_1^\sharp$ while $q^\sharp$ is consistent with $\tau_1^\sharp$ just by definition). Thus, by optimality of $\tau_1^\sharp$ in $\mathbf{G}_1$, we get $u(\varphi(rq^\sharp)) \leq \mathrm{val}(\mathbf{G}_1(w))$. This and (10) imply $u(\varphi(p)) = u(\varphi(rq)) \leq \mathrm{val}(\mathbf{G}_1(w))$.

This concludes the proof of Lemma 3.

□

Before applying Lemma 3 let us note what happens if all hypotheses of Lemma 3 are satisfied except that the vertex $v$ belongs rather to player Min. Suppose also that, as in Lemma 3, $\mathrm{val}(\mathbf{G}_2(v)) \leq \mathrm{val}(\mathbf{G}_1(v))$. Then it is the optimal strategy $\tau_2^\sharp$ of Min from $\mathbf{G}_2$ that is optimal in $\mathbf{G}$. This can be deduced immediately from Lemma 3 since player Min becomes the maximizing player for the payoff $-u$ (and $-u$ is fairly mixing iff $u$ is).

**Proof of Theorem 2**

Let $G = (V_{\mathrm{Max}}, V_{\mathrm{Min}}, E, C, \varphi)$ be a finite arena and $\mathbf{G} = (G, u)$ with fairly mixing payoff $u$.

We prove the theorem by induction on $n_G = |E| - |V|$.

If $n_G = 0$ then, since $G$ has no dead ends, each vertex of $V$ has only one outgoing edge. Thus the players have no choice and there is only one possible strategy for each of them and such a strategy is positional and, obviously, also optimal.

Let $n_G > 0$ and suppose that the thesis holds for each game $\mathbf{G}'$ over an arena such that $n_{G'} < n_G$.

If all vertices $v \in vM$ of player Max have only one outgoing edge then Max has only one possible strategy and this strategy is positional. Obviously, this unique strategy is also optimal for Max.

Now suppose that there exists $v \in vM$ having at least two outgoing edges. We decompose $G$ onto two subarenas $G_1$ and $G_2$ exactly as in Lemma 3. Since $G_1$ and $G_2$ have the same number of vertices as $G$ but their number of edges is strictly less than $|E|$ we can apply to $\mathbf{G}_i = (G_i, u)$, $i = 1, 2$, the induction hypothesis to deduce the existence of optimal positional strategies $\sigma_i^\sharp$ for Max in $\mathbf{G}_i$, $i = 1, 2$. Again by Lemma 3, either $\sigma_1^\sharp$ or $\sigma_2^\sharp$ is an optimal positional strategy of Max in $\mathbf{G}$, depending on whether $\mathrm{val}(\mathbf{G}_2(v)) \leq \mathrm{val}(\mathbf{G}_1(v))$ or the inverse inequality holds.

The existence of an optimal positional strategy for player Min follows by a symmetric argument.

□

## 4 Applications

In this section we show that virtually all popular (as well as some less popular) payoff mappings satisfy conditions (C1)-(C3). This implies immediately the existence of positional optimal strategies due to Theorem 2. Due to space restrictions the proofs, in most cases rather straightforward, are omitted.

If not stated explicitly otherwise, in the examples examined below we suppose that $C = \mathbb{R}$, i.e. the edges are labelled by real numbers. In particular, $\mathbb{R}^+$ will stand always for the set of non-empty finite sequences of real numbers and $\mathbb{R}^\omega$ is the set of finitely generated infinite sequences of reals.

*Sup Game.* Max wins the highest value seen during the play, i.e. the payoff is

$$u_s(c_0c_1\ldots) = \sup\{c_0, c_1, \ldots\}, \quad \text{where } c_i \in C.$$

*Limsup Game.* Now we suppose that Max wins the highest value seen infinitely often during the play, i.e. the payoff is given by

$$u_l(c_0c_1\ldots) = \limsup_i c_i.$$

This payoff is used for example in gambling systems [8].

*Total Reward Game.* In the total reward game player Max accumulates the one day payoffs:

$$u_t(c_0c_1\ldots) = \limsup_n \sum_{i=0}^{n} c_i.$$

Note that in this case the payoff can take infinite values $\pm\infty$. This type of payoffs is classical in game theory [6].

*Parity Game.* $C = \mathbb{N}$ is the set of non-negative integers. The payoff is defined as

$$u_p(c_0c_1\ldots) = (\limsup_{i\in\mathbb{N}} c_i) \mod 2$$

In other words, player Max wins 1 if the highest colour visited infinitely often is odd, otherwise his payoff is 0. This is the most relevant type of payoff for computer science, [5].

*Weighted Reward Game.* $C = \mathbb{R}$ is again the set of real numbers. The payoff is given by

$$u_\lambda(c_0c_1...) = \lambda \cdot \liminf_{i\in\mathbb{N}} c_i + (1-\lambda) \cdot \limsup_{i\in\mathbb{N}} c_i,$$

where $\lambda \in [0, 1]$ is any fixed constant from the closed interval $[0, 1]$.

The interpretation for $u_\lambda$ is the following. If $c_i$ is the capital of player Max on the day $i$ then using the coefficient $\lambda$ he can weight relatively his good and bad fortune. For example, for payoffs with $\lambda$ close to 1 the bad fortune counts more than the good one. This type of payoff is a natural extension of Limsup payoff but it seems not be considered before.

Note also that the payoff $u = 2 \cdot u_{\frac{1}{2}}$ can be seen as a generalisation of the parity payoff. To see this let us take a parity game $\mathbf{G}$ with an underlying arena $G$. Let us replace in $G$ each odd label $c$ by $-c$ and now consider over this modified arena the game $\mathbf{G}' = (G', u)$ with the payoff $u$ defined above. Now it suffices to note that if the game value of $\mathbf{G}'(s)$ is non-negative then in the original parity game $\mathbf{G}(s)$ it is the player 0 that wins. On the other hand, if the game value of $\mathbf{G}'(s)$ is negative then in the original parity game $\mathbf{G}(s)$ it is the player 1 that wins. The game $\mathbf{G}'$ can be seen as quantitative version of the parity game. In the parity game we examine only if the maximal infinitely often visited colour is even or odd, in the game $\mathbf{G}'$ we measure more precisely the "distance" between the greatest even and odd colours visited infinitely often.

*Mean Payoff Game.* Again $C = \mathbb{R}$. With any finite sequence $x \in \mathbb{R}^+$ of elements of $\mathbb{R}$ we associate their mean value

$$\text{mean}(x) = \frac{1}{|x|}\mathfrak{S}(x),$$

where $\mathfrak{S}(x)$ denotes the sum of all elements of $x$ while $|x|$ stands for the length of $x$. The mean payoff mapping is defined by

$$u_m(c_0c_1\ldots) = \limsup_{n\in\mathbb{N}}(\text{mean}(c_0\ldots c_{n-1}))$$

Let us note that the apparently similar game with the payoff $g(c_0c_1\ldots) = \sup_{n\in\mathbb{N}}(\text{mean}(c_0\ldots c_{n-1}))$ does not satisfy (C1) and in fact there are finite graphs for which both players need a memory to play optimally for the payoff $g$.

*Discounted Game.* The set of colours is $C = [0,1) \times \mathbb{R}$. For any finitely generated infinite word $(\lambda_0, a_0)(\lambda_1, a_1)\ldots \in C^\omega$ we set

$$u_d((\lambda_0, a_0)(\lambda_1, a_1)\ldots) = \lambda_0 a_0 + \lambda_0\lambda_1 a_1 + \lambda_0\lambda_1\lambda_2 a_2 + \cdots$$

Usually in discounted games there is one discount factor $\lambda$ and then $u_d(a_0a_1\ldots) = \sum_{i\geq 0}\lambda^i a_i$. Allowing different discount factors is in the spirit of the original paper of Shapley [12].

**Theorem 5.** *All payoff mappings listed above satisfy conditions (C1)-(C3) and therefore both players have optimal positional strategies.*

Let us note finally that the conditions of Definition 1 are not necessary. Let $(c_i)_{i=0}^\infty$ be a finitely generated sequence of real numbers. Consider the payoff $f(c_0c_1c_2\ldots) = \text{sgn}(\inf_{n\geq 0}\sum_{i=0}^n c_i)$, where $\text{sgn}(x)$ is equal either $-1$, or 0, or 1 depending on whether $x$ is negative, zero or positive. This payoff mapping admits optimal positional strategies but does not satisfy (C1). Nevertheless, even in this case we can use indirectly Theorem 2. Indeed, the payoff $g(c_0c_1\ldots) = \inf_{n\geq 0}\sum_{i=0}^n c_i$ is fairly mixing. Now note the following trivial observation:

*Remark 6.* If a payoff $u$ admits optimal positional strategies and and $h : \mathbb{R} \to \mathbb{R}$ is a non-decreasing mapping then the composition $h \circ u$ admits optimal positional strategies, in fact the strategies optimal for $u$ remain optimal for $h \circ u$.

Thus from the fact that $g$ is fairly mixing and that sgn is non-decreasing we can conclude from Remark 6 that $f = \mathrm{sgn} \circ g$ admits optimal positional strategies.

This example leads immediately to the following question:

Let $u$ be a payoff mappings admitting optimal positional strategies for both players for all finite arenas. Is it true that there exist always a fairly mixing mapping $g$ and a non-decreasing mapping $h : \mathbb{R} \to \mathbb{R}$ such that $u = h \circ g$?

We ignore the answer to this problem.

## References


1. Henrik Björklund, Sven Sandberg, and Sergei Vorobyov. Memoryless determinacy of parity and mean payoff games: a simple proof. *Theor. Computer Science*, 310:365–378, 2004.
2. Luca de Alfaro, Marco Faella, Thomas A. Henzinger, Rupak Majumdar, and Mariëlle Stoelinga. Model checking discounted temporal properties. In *TACAS 2004*, volume 2988 of *LNCS*, pages 77–92. Springer, 2004.
3. Luca de Alfaro, Thomas A. Henzinger, and Rupak Majumdar. Discounting the future in systems theory. In *ICALP 2003*, volume 2719 of *LNCS*, pages 1022–1037. Springer, 2003.
4. Andrzej Ehrenfeucht and Jerzy Mycielski. Positional strategies for mean payoff games. *Intern. J. of Game Theory*, 8:109–113, 1979.
5. E.A. Emerson and C. Jutla. Tree automata, $\mu$-calculus and determinacy. In *FOCS'91*, pages 368–377. IEEE Computer Society Press, 1991.
6. Jerzy Filar and Koos Vrieze. *Competitive Markov Decision Processes*. Springer, 1997.
7. Y. Gurevich and L. Harrington. Trees, automata and games. In *Proc. 17th Symp. on the Theory of Comp.*, pages 60–65. IEEE Computer Society Press, 1982.
8. Ashok P. Maitra and William D. Sudderth. *Discrete Gambling and Stochastic Games*. Springer.
9. A.W. Mostowski. Games with forbidden positions. Technical Report 78, Uniwersytet Gdański, Instytut Matematyki, 1991.
10. Hervé Moulin. Prolongements des jeux á deux joueurs de somme nulle. Une théorie abstraite des duels. *Bull. Soc. math. France*, pages 1–111, 1976. Mémoire 45.
11. T.E.S Raghavan. Finite-step algorithms for single-controller and perfect information stochastic games. In A. Neyman and S. Sorin, editors, *Stohastic Games and Applications*, volume 570 of *NATO Science Series C, Mathematical and Physical Sciences*, pages 227–251. Kluwer Academic Publishers, 2003.
12. L. S. Shapley. Stochastic games. *Proceedings Nat. Acad. of Science USA*, 39:1095–1100, 1953.
13. Wiesław Zielonka. Perfect-information stochastic parity games. In *FOSSACS 2004*, volume 2987 of *LNCS*, pages 499–513. Springer, 2004.