

Strong edge-colouring of graphs

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June 30th 2015

Université de Dijon, Le2i

université
de **BORDEAUX**



Definition (Fouquet and Jolivet, 1983)

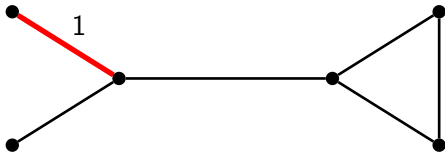
Strong edge-colouring (distance 2 edge-colouring) is an assignment of colours to edges of graph such that:

- *every two edges sharing a vertex have different colours (proper edge-colouring)*
- *every two edges joined by an edge are assigned distinct colours (distance 2 edge-colouring).*

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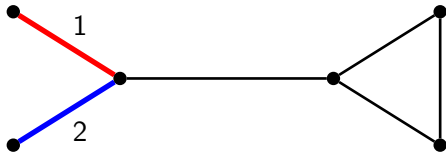
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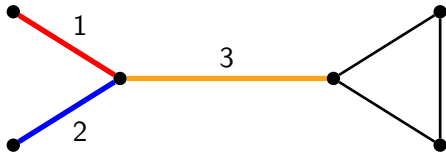
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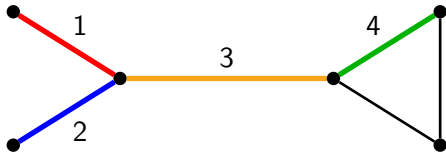
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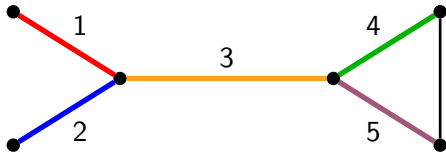
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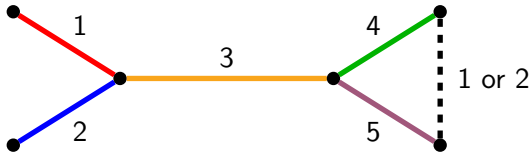
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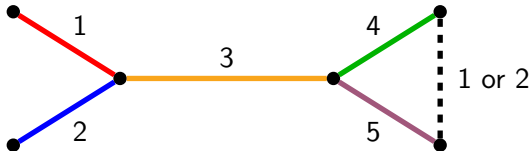
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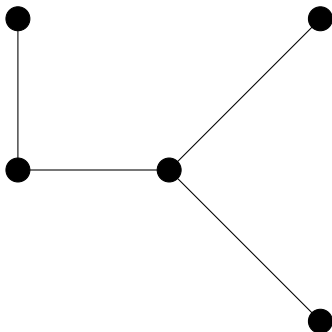
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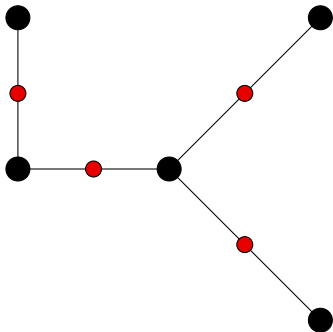
$\chi'_s(G)$ - minimum number of colours needed to obtain a strong edge-colouring of G

In other words...

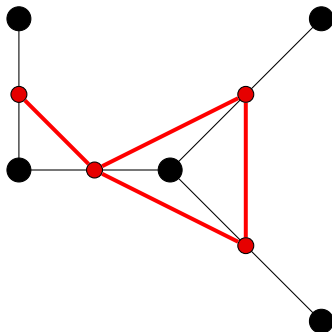
It can be seen as a proper vertex-colouring of the **square of the line graph**.



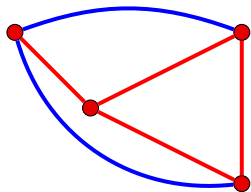
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Vertex coloring of the **line graph**.



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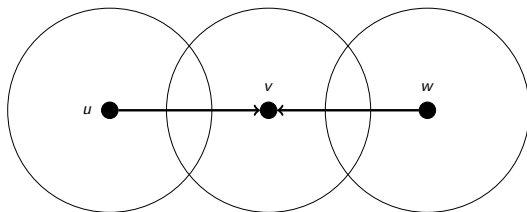


One of the motivations...

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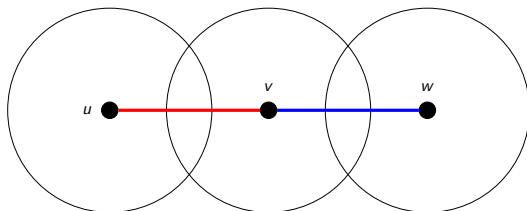
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Conflict at node v

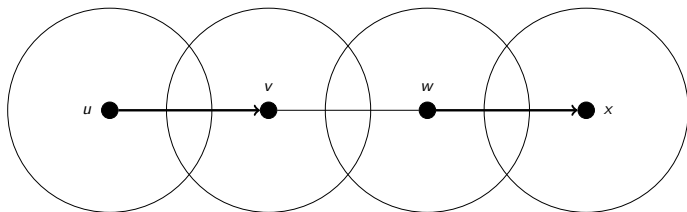
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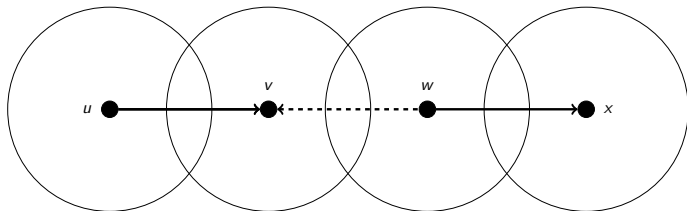
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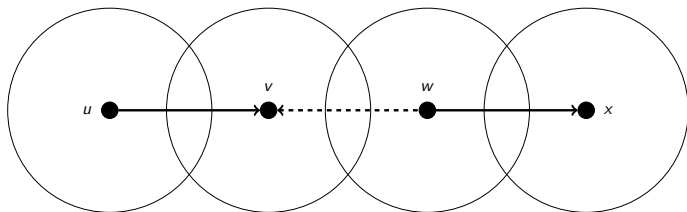
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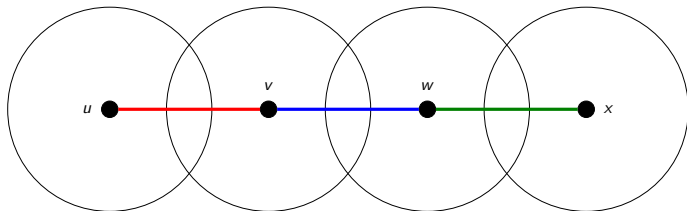
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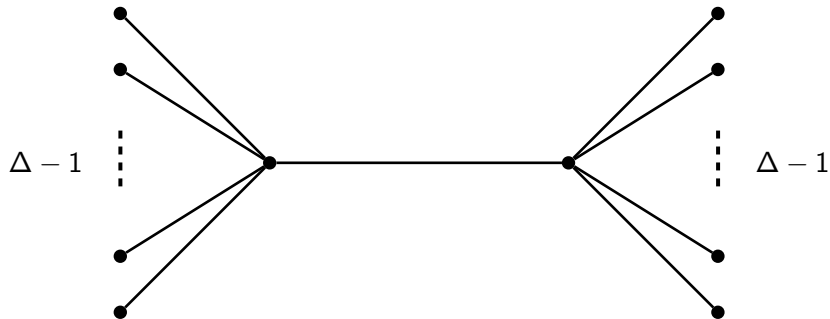
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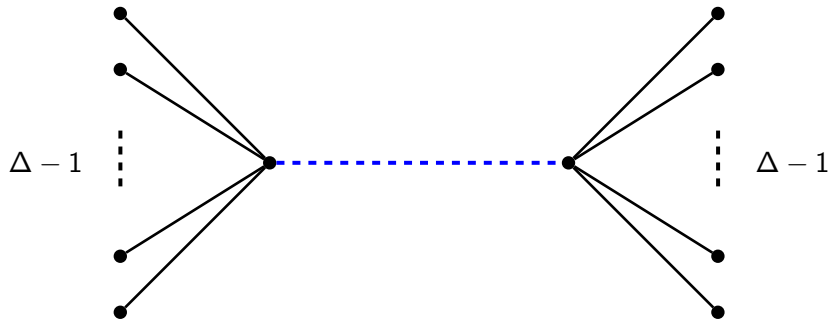
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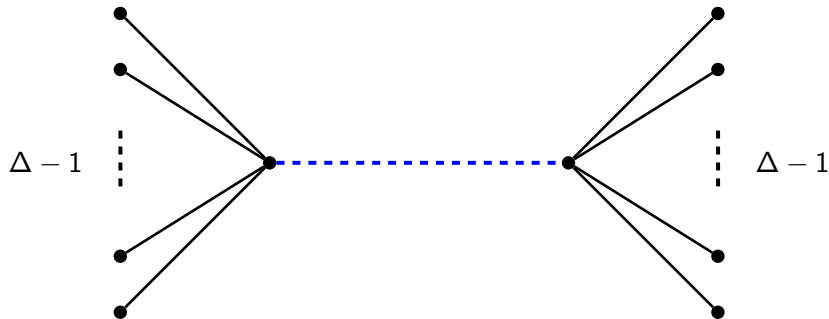
Starting point



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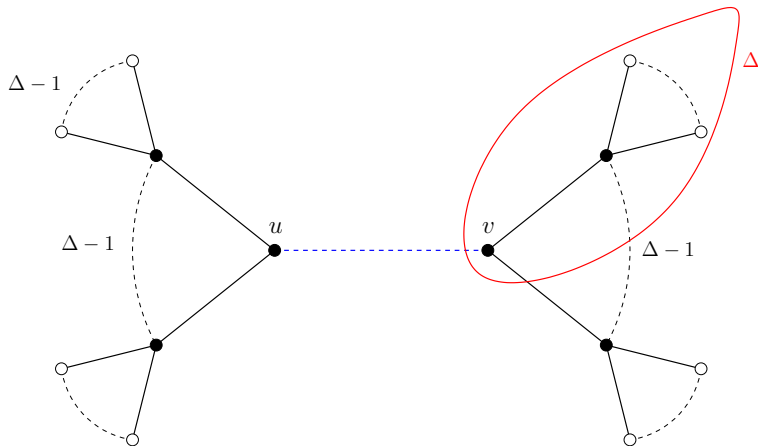


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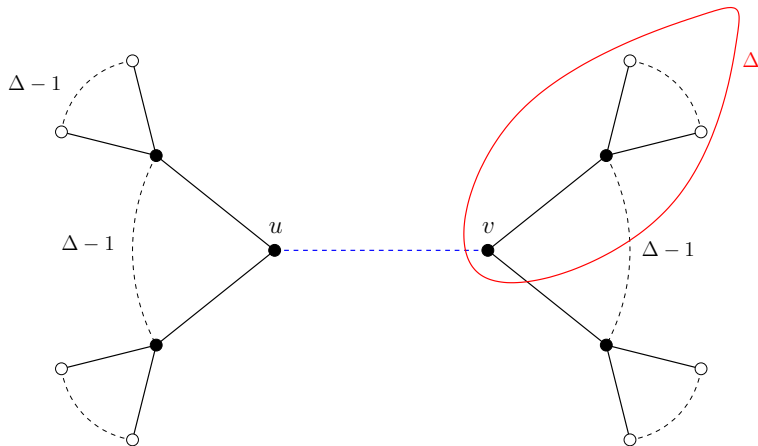


$$\chi'_s(G) \geq 2(\Delta - 1) + 1$$

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$$\chi'_s(G) \leq 2\Delta(\Delta - 1) + 1$$

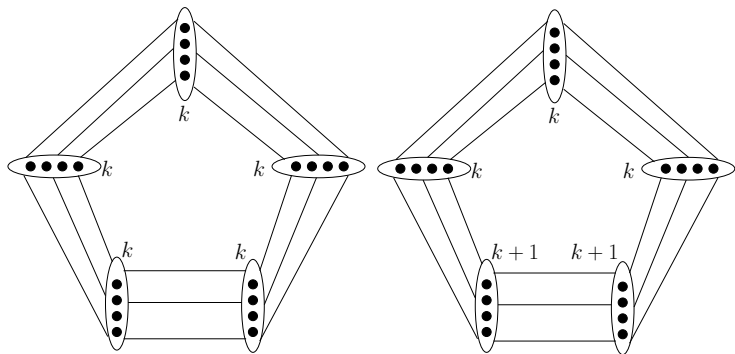
Conjecture [Erdős and Nešetřil, 1985]

For a graph G , $\chi'_s(G) \leq \begin{cases} \frac{5}{4}\Delta^2, \Delta \text{ even} \\ \frac{1}{4}(5\Delta^2 - 2\Delta + 1), \Delta \text{ odd} \end{cases}$

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$$\Delta = 2k, \chi'_s = 5k^2$$

$$\Delta = 2k + 1, \chi'_s = 5k^2 + 4k + 1$$

To summarize

- Greedy algorithm : $\chi'_s(G) \simeq 2\Delta^2$.
- Erdős and Nešetřil Conjecture : $\chi'_s(G) \simeq \frac{5}{4}\Delta^2$.

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Theorem [Molloy and Reed, 1997]

If Δ is large enough, then $\chi'_s(G) \leq 1.998\Delta^2$.

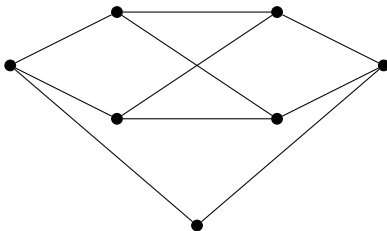
Theorem [Andersen, Horák *et al.*, 1992]

If G is a subcubic graph, $\chi'_s(G) \leq 10$.

For small values of Δ

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Theorem [Cranston, 2006]

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$\chi'_s(G) \leq 20$ by Erdős and Nešetřil Conjecture.

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If G is bipartite, then $\chi'_s(G) \leq \Delta^2$.

Reached for any complete bipartite graph $K_{\Delta, \Delta}$.

Other graph classes

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Theorem [Bensmail, Lagoutte and Valicov, 2014]

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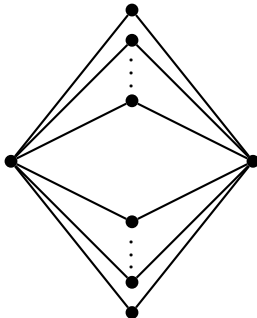
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Other graph classes - planar graphs

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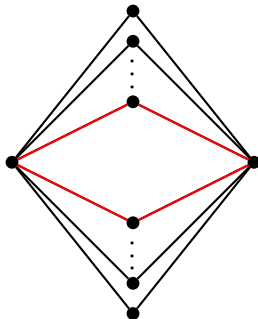
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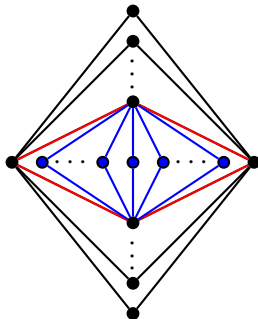
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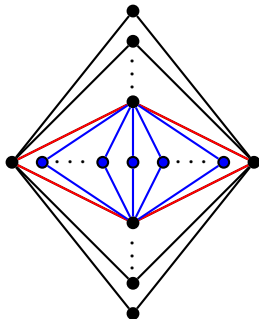
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proved for $\Delta = 7$ (Sanders and Zhao, 2001)

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Question

If $\Delta = g = 4$, then can G be **Class 1**?

Summary

	$\Delta \geq 7$	$\Delta \in \{5, 6\}$	$\Delta = 4$	$\Delta = 3$
no girth restriction	4Δ	$4\Delta + 4$	$4\Delta + 4$	$3\Delta + 1$
$g \geq 4$	4Δ	4Δ	$4\Delta + 4$	$3\Delta + 1$
$g \geq 5$	4Δ	4Δ	4Δ	$3\Delta + 1$
$g \geq 6$	$3\Delta + 1$	$3\Delta + 1$	$3\Delta + 1$	3Δ
$g \geq 7$	3Δ	3Δ	3Δ	3Δ

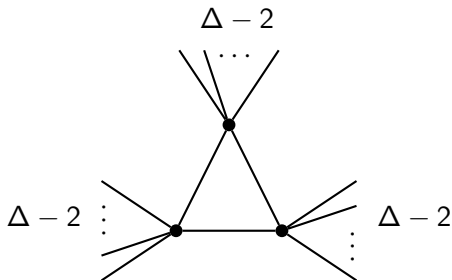
Theorem [H., Ochem and Valicov, 2011]

If G is an outerplanar graph with maximum degree Δ , then
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Other graph classes - planar graphs

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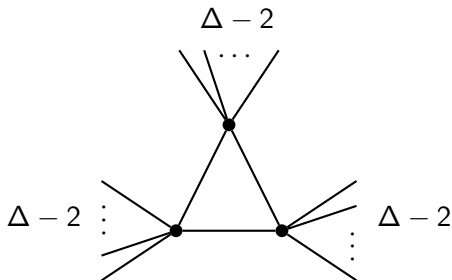
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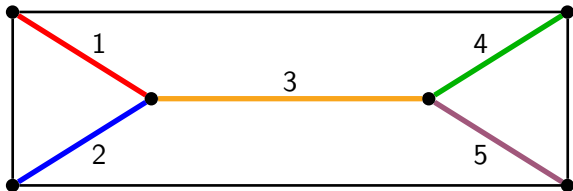
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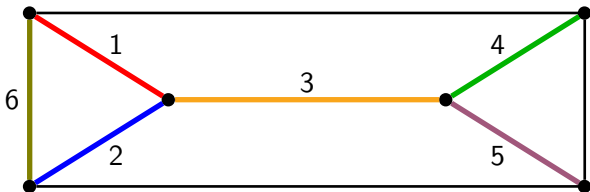
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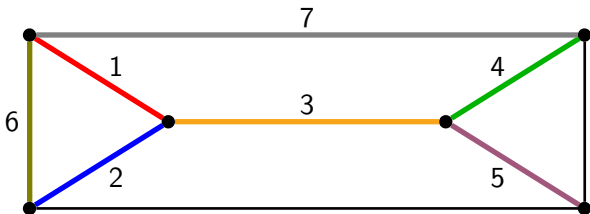
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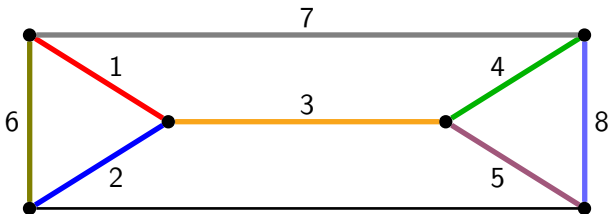
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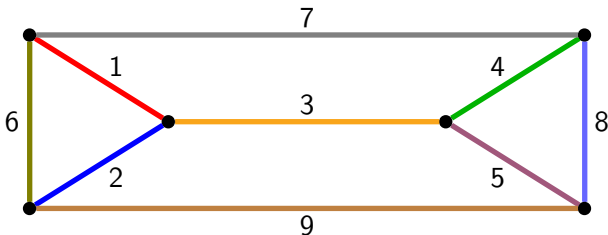
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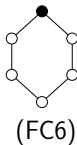
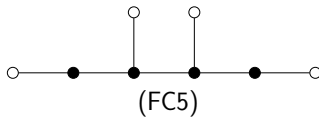
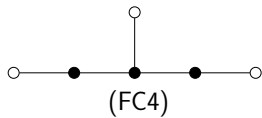
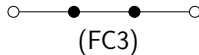
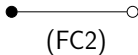
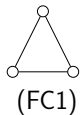
Theorem

Let G be a planar subcubic graph with no induced cycles of length 4 nor 5. Then $\chi'_s(G) \leq 9$.

Proof by minimum counterexample and using planarity.

Sketch of the proof

Let H be a smallest counterexample. H does not contain:



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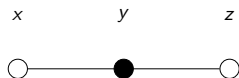
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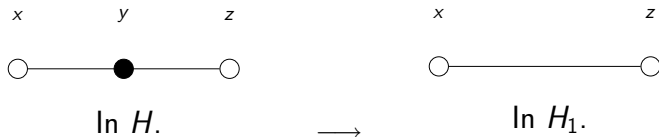
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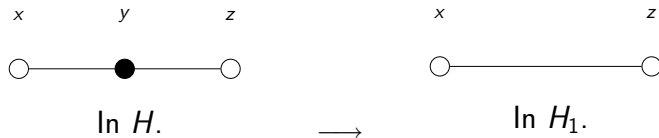


Sketch of the proof

Let H be a smallest counterexample.

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Build a new graph H_1 from H :



Clearly, H_1 is planar.

Sketch of the proof

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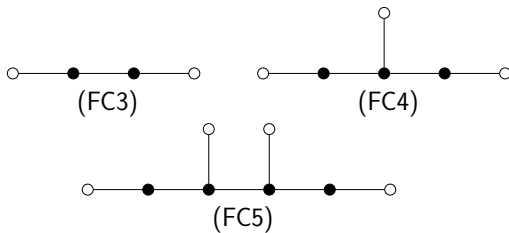
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$$\left. \begin{array}{l} H_1 \text{ is planar.} \\ H_1 \text{ is simple.} \\ H_1 \text{ is 3-regular.} \end{array} \right\} \Rightarrow H_1 \text{ must contain a face } \mathcal{C}' \text{ of length at most 5.}$$

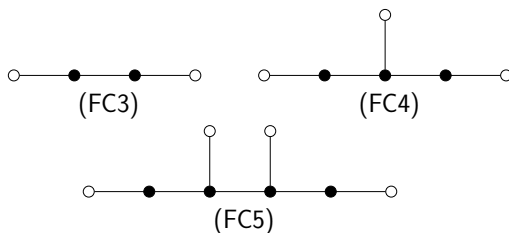
Sketch of the proof

Recall that H does not contain:



Sketch of the proof

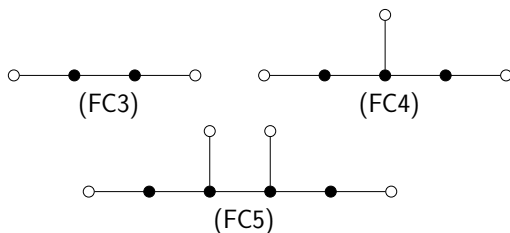
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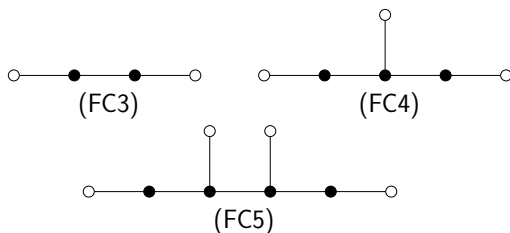


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$\Rightarrow C'$ cannot be obtained from a cycle of H of length $l \geq 7$.

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Recall that H does not contain:



and that $g(H) \geq 6$.

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$\Rightarrow g(H) = 6$.

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In H there exists a cycle \mathcal{C} of length 6 having a vertex of degree 2 on its boundary.

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Impossible by (FC6). } $\Rightarrow H$ cannot exist.

Definition

The **maximal average degree** of a graph G , denoted $\text{mad}(G)$, is the maximal of the average degrees of all the subgraphs of G :

$$\text{mad}(G) = \max \left\{ \frac{2|E(H)|}{|V(H)|}, H \subseteq G \right\}$$

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The mad parameter is computable in polynomial time.

Theorem

Let G be a subcubic graph:

- 1 If $\text{mad}(G) < \frac{15}{7}$, then $\chi'_s(G) \leq 6$.
- 2 If $\text{mad}(G) < \frac{27}{11}$, then $\chi'_s(G) \leq 7$.
- 3 If $\text{mad}(G) < \frac{13}{5}$, then $\chi'_s(G) \leq 8$.
- 4 If $\text{mad}(G) < \frac{20}{7}$, then $\chi'_s(G) \leq 9$.

Lemma (Derived from Euler's Formula)

Every planar graph of girth g satisfies:

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Corollary

Let G be a planar subcubic graph with girth g :

- 1 If $g \geq 30$, then $\chi'_s(G) \leq 6$ (can be improved to $g \geq 16$).
- 2 If $g \geq 11$, then $\chi'_s(G) \leq 7$.
- 3 If $g \geq 9$, then $\chi'_s(G) \leq 8$.
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Discharging methods

Introduced by **Wernicke** in **1904**.

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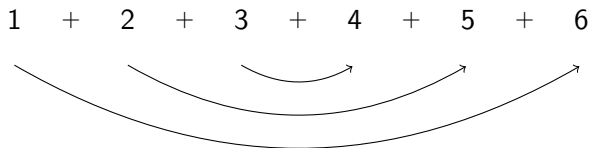
Just a counting argument...

$$1 + 2 + 3 + 4 + 5 + 6$$

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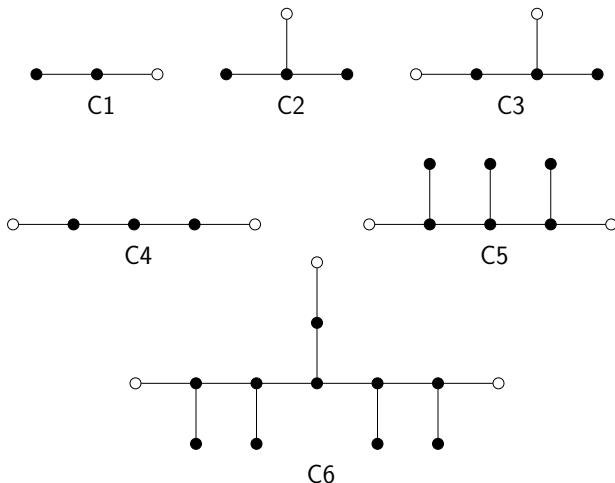
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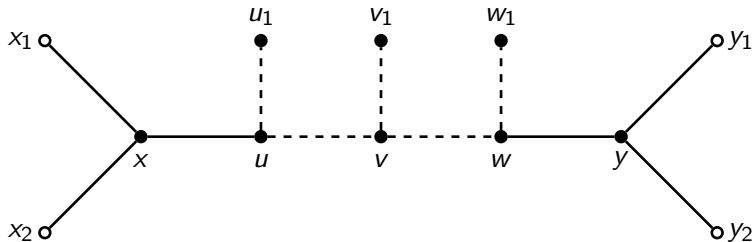
Therefore, the counterexample cannot exist.

Reducible configurations (Theorem 1.1)

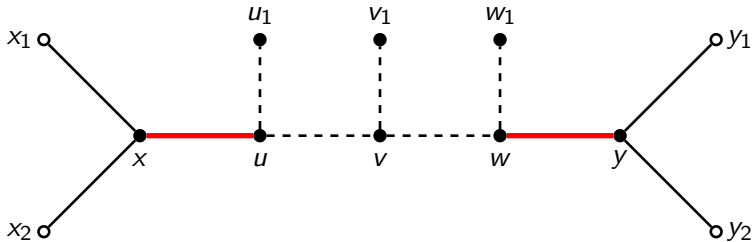
Let H be a smallest counterexample. H does not contain:



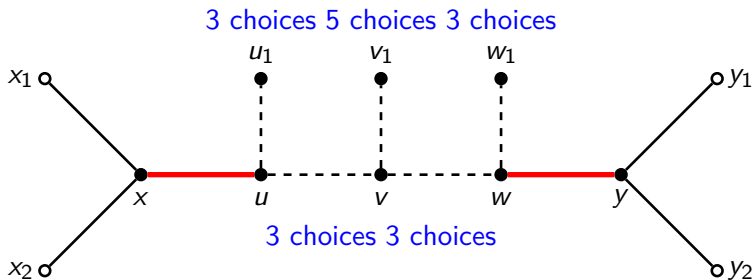
Proof of configuration (C5)



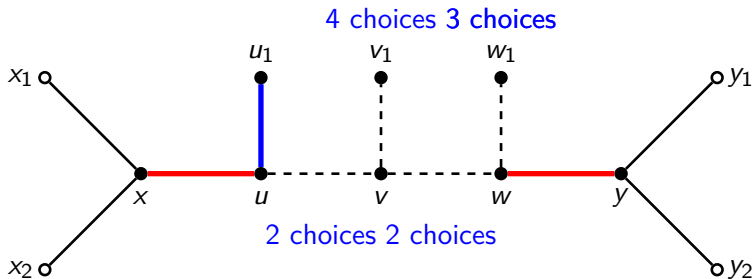
First case : $c(xu) = c(wy)$



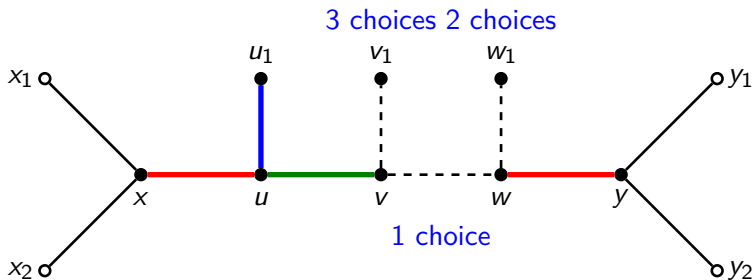
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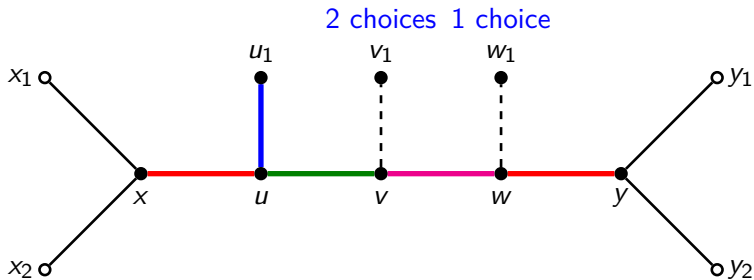
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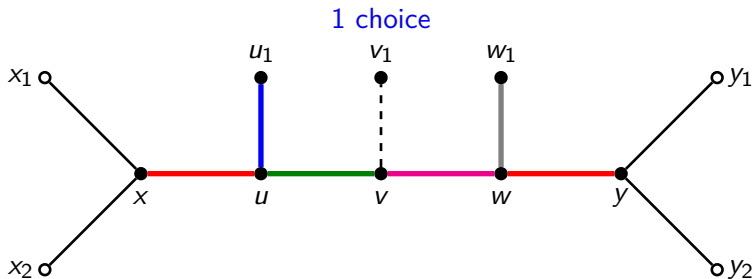
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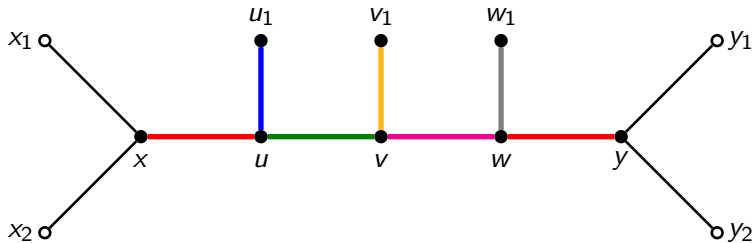
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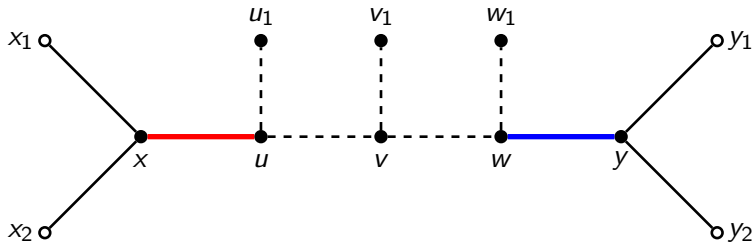
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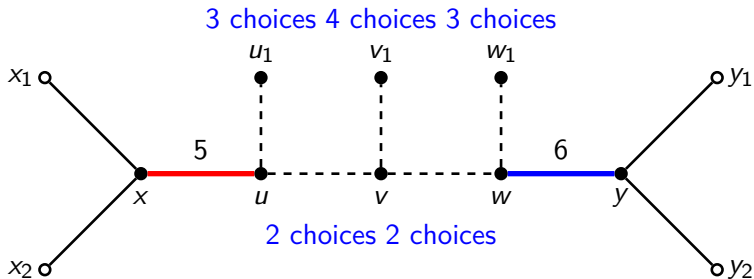
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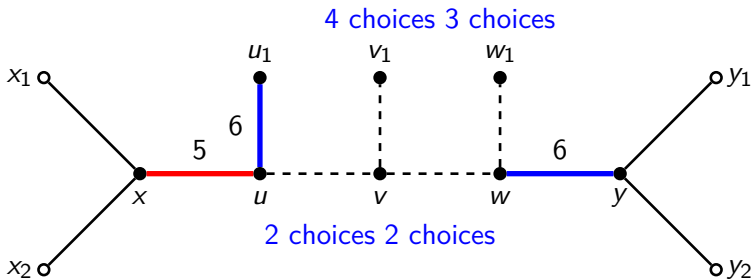
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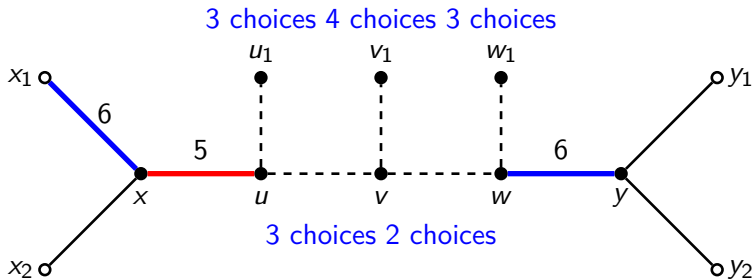
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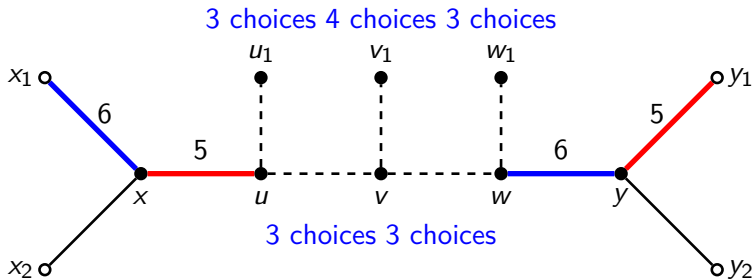
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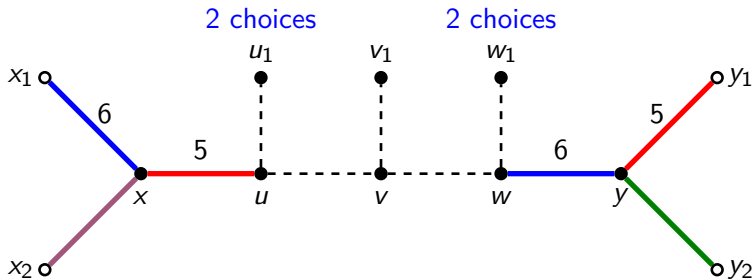
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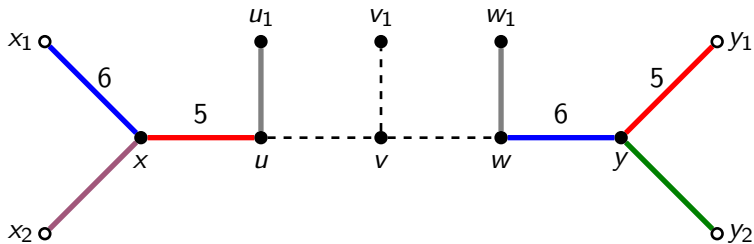
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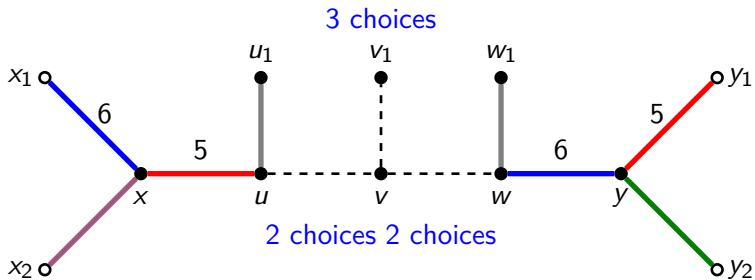
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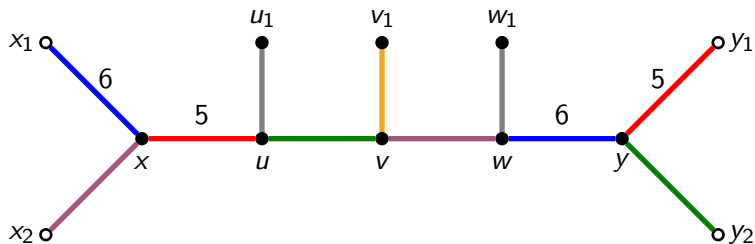
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Discharging procedure

Recall that $\text{mad}(H) < \frac{15}{7}$!

We define a weight function $\omega : V(H) \rightarrow \mathbb{R}$ by
 $\omega(x) = d(x) - \frac{15}{7}$.

$$\omega(u) = -\frac{8}{7} \quad \omega(v) = -\frac{1}{7} \quad \omega(w) = \frac{6}{7}$$

$$\begin{array}{c} \downarrow \\ d(u) = 1 \end{array}$$

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- 1 Every 3-vertex not adjacent to a 1-vertex gives $\frac{2}{7}$ to each of its adjacent 3-vertices having a 1-vertex in its neighbourhood and $\frac{1}{7}$ to each of its adjacent 2-vertices.
- 2 By (C3) every 3-vertex has at most one 1-vertex in its neighbourhood and it gives to this vertex $\frac{8}{7}$.

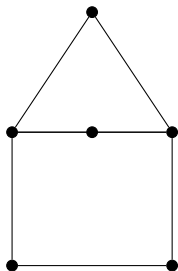
At the end of the discharging procedure, we have:

- 1 The total sum of weights did not change.
- 2 For every $x \in V(H)$, $\omega^*(x) \geq 0$.

$$\text{Hence, } 0 \leq \sum_{x \in V(H)} \omega^*(x) = \sum_{x \in V(H)} \omega(x) < 0$$

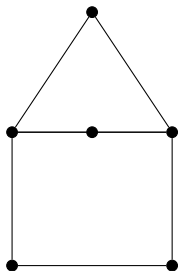
Contradiction!

Tightness?



A graph G with $\text{mad}(G) = \frac{7}{3}$ and $\chi'_s(G) > 6$.

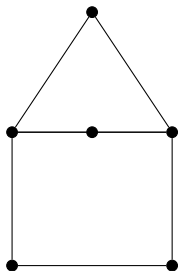
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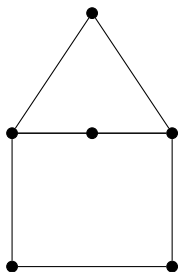
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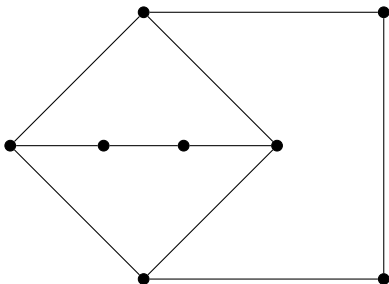
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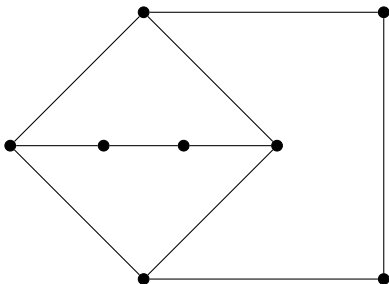
If $\text{mad}(G) < \frac{7}{3}$ then $\chi'_s(G) \leq 6$.

Tightness?



A graph G with $\text{mad}(G) = \frac{5}{2}$ and $\chi'_s(G) > 7$.

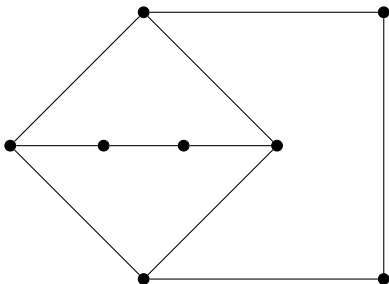
Tightness?



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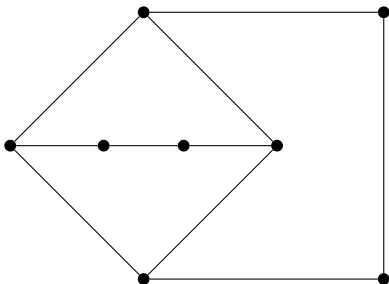
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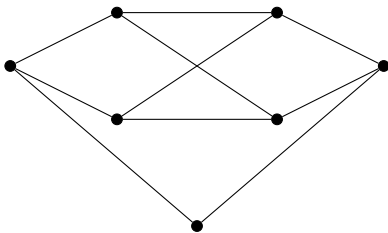
Tightness?



A graph G with $\text{mad}(G) = \frac{5}{2}$ and $\chi'_s(G) > 7$.

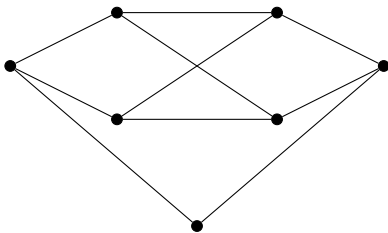
If $\text{mad}(G) < \frac{5}{2}$ then $\chi'_s(G) \leq 7$.

Tightness?



A graph G with $\text{mad}(G) = \frac{20}{7}$ and $\chi'_s(G) > 9$.

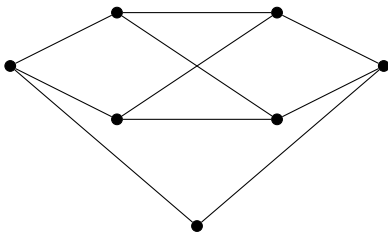
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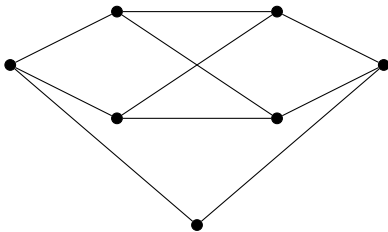
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If $\text{mad}(G) < \frac{20}{7}$ then $\chi'_s(G) \leq 9$.

To summarize

Let $f(n) = \inf\{\text{mad}(G) \mid \chi'_s(G) > n\}$.

$$f(6) = \frac{7}{3}$$

$$f(7) = \frac{5}{2}$$

$$\frac{56}{21} = \frac{8}{3} < f(8) \leq \frac{20}{7} = \frac{60}{21}$$

$$f(9) = \frac{20}{7}$$

- The conjecture of Faudree *et al.*...

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- Does there exist a subcubic graph G with $\text{mad}(G) < \frac{20}{7}$ and having $\chi'_s(G) = 9$?
- Could we reach the bound proposed by the conjecture of Erdős and Nešetřil for graphs with maximum degree 4?
- Bigger challenge: the conjecture of Erdős and Nešetřil.

Thank you for your attention

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- And what about planar graphs?

Theorem

4-SEC is NP-complete even when restricted to planar subcubic bipartite graphs with arbitrarily large girth.

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3-COLOURING is NP-complete even when restricted to planar graphs with degree 4.

3-COLOURING \leq_P 4-SEC.

What else?

By reducing 3-COLOURING of planar graphs of maximum degree 4 we also proved:

- 5-SEC is NP-complete for planar bipartite graphs of maximum degree 3 and girth $g = 8$.
- 6-SEC is NP-complete for planar graphs of maximum degree 3, with girth $g = 4$.