

Adjacent vertex-distinguishing edge coloring of graphs with maximum degree Δ *

Hervé Hocquard and Mickaël Montassier
LaBRI, Université Bordeaux I, 33405 Talence Cedex, France

November 27, 2011

Abstract

An adjacent vertex-distinguishing edge coloring, or avd-coloring, of a graph G is a proper edge coloring of G such that no pair of adjacent vertices meets the same set of colors. Let $\text{mad}(G)$ and $\Delta(G)$ denote the maximum average degree and the maximum degree of a graph G , respectively. In this paper, we prove that every graph G with $\Delta(G) \geq 5$ and $\text{mad}(G) < 3 - \frac{2}{\Delta}$ can be avd-colored with $\Delta(G) + 1$ colors. This completes a result of Wang and Wang [10].

1 Introduction

A proper edge coloring of a graph $G = (V, E)$ is an assignment of colors to the edges of the graph such that two adjacent edges do not use the same color. An adjacent vertex-distinguishing k -edge coloring, or k -avd-coloring for short, of a graph G is a proper edge coloring of G using at most k colors such that, for every pair of adjacent vertices u, v , the set of colors of the edges incident to u differs from the set of colors of the edges incident to v . We denote by $\chi'_{avd}(G)$ the *avd-chromatic number* of G which is the smallest integer k such that G can be k -avd-colored. In the following we consider graphs with no isolated edges. Adjacent vertex-distinguishing colorings are also known as *adjacent strong edge coloring* [11] and *1-strong edge coloring* [1]. Zhang *et al.* completely determined the avd-chromatic number for paths, cycles, trees, complete graphs, and complete bipartite graphs [11]. For example, they proved that:

Theorem 1 [11] *For cycle C_p , we have :*

$$\chi'_{avd}(C_p) = \begin{cases} 3 & \text{if } p \equiv 0 \pmod{3} \\ 4 & \text{if } p \not\equiv 0 \pmod{3} \text{ and } p \neq 5 \\ 5 & \text{if } p = 5 \end{cases}$$

Moreover, they proposed the following conjecture [11]:

Conjecture 1 *If G is a connected graph with at least 6 vertices, then $\chi'_{avd}(G) \leq \Delta(G) + 2$.*

where $\Delta(G)$ denotes the maximum degree of G .

In [2], Balister *et al.* proved Conjecture 1 for graphs with maximum degree three and for bipartite graphs.

Note that the notion of avd-coloring is an extension of the notion of *vertex-distinguishing proper edge coloring*, which is a proper edge coloring such that for any vertices u, v ($u \neq v$), the set of colors assigned to the edges incident to u differs from the set of colors assigned to the edges incident to v . The smallest integer k such that G can be vertex-distinguishing proper edge colored

*This research is supported by ANR Project GRATOS - GRAPhs through Topological Structures ANR-09-JCJC-0041-01 (2009-2012)

with at most k colors is called the *observability* of G and was studied for different families of graphs [3, 4, 5, 6, 7, 8, 9].

Let $\text{mad}(G) = \max \left\{ \frac{2|E(H)|}{|V(H)|}, H \subseteq G \right\}$ be the maximum average degree of the graph G , where $V(H)$ and $E(H)$ are the sets of vertices and edges of H , respectively.

In [10], Wang and Wang made the link between maximum average degree and avd-colorings and proved:

Theorem 2 [10] *Let G be a connected graph with maximum degree $\Delta(G)$ and maximum average degree $\text{mad}(G)$.*

1. *If $\text{mad}(G) < 3$ and $\Delta(G) \geq 3$, then $\chi'_{\text{avd}}(G) \leq \Delta(G) + 2$.*
2. *If $\text{mad}(G) < \frac{5}{2}$ and $\Delta(G) \geq 4$, or $\text{mad}(G) < \frac{7}{3}$ and $\Delta(G) = 3$, then $\chi'_{\text{avd}}(G) \leq \Delta(G) + 1$.*
3. *If $\text{mad}(G) < \frac{5}{2}$ and $\Delta(G) \geq 5$, then $\chi'_{\text{avd}}(G) = \Delta(G) + 1$ if and only if G contains adjacent vertices of maximum degree.*

In this paper, we generalize Theorem 2.2 proving that:

Theorem 3 *Let G be a graph of maximum degree $\Delta(G) \geq 5$ and $\text{mad}(G) < 3 - \frac{2}{\Delta(G)}$, then $\chi'_{\text{avd}}(G) \leq \Delta(G) + 1$.*

One can observe that Theorem 3 holds for $\Delta(G) = 3$ and $\Delta(G) = 4$ [10].

Notations Let G be a graph. Let $d_G(v)$ denote the degree of a vertex v in G . A vertex of degree k (resp. at least k , at most k) is called a k -vertex (resp. k^+ -vertex, k^- -vertex). A 2-vertex is called *bad* if it is adjacent to a 2-vertex, otherwise we call it *good*. Finally, we use $[n]$ to denote the set of integers $\{1, 2, \dots, n\}$.

Let ϕ be a k -avd-coloring of a graph G . We denote by $C_\phi(v) = \{\phi(uv) | uv \in E(G)\}$ the set of colors assigned to edges incident to the vertex v . We recall that a proper edge coloring of a graph is adjacent vertex-distinguishing if $C_\phi(u) \neq C_\phi(v)$ for any pair of adjacent vertices u and v .

In Section 2, we give the proof of Theorem 3 by using the method of reducible configurations and the discharging technique, that is inspired from the proof of [10].

2 Proof of Theorem 3

Let G be a counterexample to Theorem 3 minimizing $|E(G)| + |V(G)|$. Set $\Delta = \Delta(G)$.

First, we prove that G is connected. By contradiction, consider G_1 and G_2 two connected components of G ($\text{mad}(G_1) < 3 - \frac{2}{\Delta}$ and $\text{mad}(G_2) < 3 - \frac{2}{\Delta}$). Without loss of generality suppose there exists one vertex, say x , in $V(G_1)$ such that $d_G(x) = \Delta$. By minimality of G , G_1 admits a $(\Delta + 1)$ -avd-coloring. If there exists a vertex y in G_2 such that $d_G(y) = \Delta$, then by minimality of G , G_2 admits also a $(\Delta + 1)$ -avd-coloring. Otherwise, suppose that every vertex of G_2 has a degree strictly less than Δ . If $\Delta(G_2) \geq 3$ then by Theorem 2.1 (recall that $\text{mad}(G_2) < 3 - \frac{2}{\Delta} < 3$), $\chi'_{\text{avd}}(G_2) \leq \Delta(G_2) + 2 \leq \Delta(G) + 1$. If $\Delta(G_2) = 2$, then by Theorem 1, $\chi'_{\text{avd}}(G_2) \leq 5 \leq \Delta(G) + 1$. In each case we obtain a contradiction, thus G is connected.

Let H be the graph obtained from G by removing all 1-vertices of G , i.e. $H = G \setminus \{v \in V(G), d_G(v) = 1\}$. Clearly, H is connected. Moreover $\text{mad}(H) < 3 - \frac{2}{\Delta}$.

2.1 Structural properties of H

In this section, we give some structural properties of H .

Claim 1 *H has the following properties:*

1. $\delta(H) \geq 2$, where $\delta(H)$ is the minimum degree of H .
2. Let $v \in V(H)$ such that $d_H(v) = 2$, then $d_G(v) = 2$.
3. Let uvw be a path in H such that $d_H(v) = d_H(w) = 2$, then $d_G(u) = d_H(u)$ and $d_G(x) = d_H(x)$.

Proof

The proofs of Claims 1.1 and 1.2 are based on Claims 1 and 2 of [10].

1. By contradiction suppose $\delta(H) \leq 1$. We have two cases:

If $\delta(H) = 0$, then G is the star $K_{1, \Delta(G)}$ and $\chi'_{avd}(G) = \Delta(G)$, a contradiction.

Suppose now $\delta(H) = 1$. Let u be a 1-vertex in H adjacent to a vertex v . One can observe that $d_G(u) = k \geq 2$. In G , call u_1, \dots, u_{k-1} the $(k-1)$ 1-vertices adjacent to u distinct from v . Consider $G' = G \setminus \{u_1\}$.

If $\Delta(G') < \Delta(G)$, then G' admits a $(\Delta(G') + 2)$ -avd-coloring ϕ by Theorem 2.1 (recall that $\text{mad}(G') \leq \text{mad}(G) < 3 - \frac{2}{\Delta} < 3$). This coloring ϕ is a partial $(\Delta(G) + 1)$ -avd-coloring of G . Coloring uu_1 properly (i.e. with a color distinct from those assigned to the adjacent edges) extends ϕ to G (since $|C_\phi(u)| = \Delta(G)$, $|C_\phi(v)| < \Delta(G)$, and $|C_\phi(u_1)| = 1$), a contradiction.

If $\Delta(G') = \Delta(G)$, then, by minimality of G , G' admits a $(\Delta(G) + 1)$ -avd-coloring ϕ . Without loss of generality suppose that $\phi(uv) = 1$ and $\phi(uu_i) = i$ for $2 \leq i \leq k-1$. We color uu_1 with the color k : either we are done ($C_\phi(u) \neq C_\phi(v)$) or v verifies $C_\phi(v) = \{1, 2, \dots, k-1, k\}$. In that case, we color uu_1 with the color $k+1$. This extends the coloring to G , a contradiction.

2. Let v be a 2-vertex in H adjacent to two vertices x and y . By contradiction suppose $d_G(v) = k > 2$. In G , call v_1, \dots, v_{k-2} the $(k-2)$ 1-vertices adjacent to v distinct from x and y . Consider $G' = G \setminus \{v_1\}$.

If $\Delta(G') < \Delta(G)$, then G' admits a $(\Delta(G') + 2)$ -avd-coloring ϕ by Theorem 2.1. This coloring ϕ is a partial $(\Delta(G) + 1)$ -avd-coloring of G that we can extend to G by coloring vv_1 properly ($|C_\phi(v)| = \Delta(G)$, $|C_\phi(x)| < \Delta(G)$, $|C_\phi(y)| < \Delta(G)$ and $|C_\phi(v_1)| = 1$), a contradiction.

If $\Delta(G') = \Delta(G)$, then, by minimality of G , G' admits a $(\Delta(G) + 1)$ -avd-coloring ϕ . Without loss of generality suppose that $\phi(vx) = 1$, $\phi(vy) = 2$, and $\phi(vv_i) = i+1$ for $2 \leq i \leq k-2$ when $k > 3$. Suppose first $k = 3$. We color vv_1 with the color 3: either we are done ($C_\phi(v) \neq C_\phi(x)$ and $C_\phi(v) \neq C_\phi(y)$) or a neighbor of v , say x , verifies $C_\phi(x) = \{1, 2, 3\}$. We recolor vv_1 with the color 4: again, either we are done or y verifies $C_\phi(y) = \{1, 2, 4\}$. Finally we recolor vv_1 with 6 (recall that $\Delta(G) \geq 5$); this extends the coloring to G , a contradiction. Assume now $k \geq 4$. We color vv_1 with the color k . Either the obtained coloring is an avd-coloring of G , or a neighbor of v , say x , verifies $C_\phi(x) = \{1, 2, 3, \dots, k-1, k\}$. Again we recolor vv_1 with $k+1$. Either we are done or $C_\phi(y) = \{1, 2, 3, \dots, k-1, k+1\}$. In that case, we recolor vv_2 with k (v_2 exists because $k \geq 4$). This extends the coloring to G , a contradiction.

3. Let uvw be a path in H such that $d_H(v) = d_H(w) = 2$. By Claim 1.2, $d_G(v) = d_G(w) = 2$. By contradiction suppose $d_G(u) \neq d_H(u)$ (it follows from Claims 1.1, 1.2 and construction of H that $d_G(u) \geq 3$). Hence there exists at least one 1-vertex adjacent to u in G , say u_1 . Consider $G' = G \setminus \{u_1\}$. By minimality of G , G' admits a $(\Delta(G) + 1)$ -avd-coloring ϕ . If $\phi(uv) \neq \phi(uw)$, then we color uv (1) properly if $d_G(x) \geq 3$, or (2) with a color distinct from those assigned to the edges incident to v and x if $d_G(x) = 2$. The obtained coloring is an avd-coloring of G ((1) $|C_\phi(u)| \geq 3$, $|C_\phi(x)| \geq 3$, $|C_\phi(v)| = |C_\phi(w)| = 2$ and $C_\phi(v) \neq C_\phi(w)$, (2) $|C_\phi(u)| \geq 3$, $|C_\phi(v)| = |C_\phi(w)| = |C_\phi(x)| = 2$, $C_\phi(v) \neq C_\phi(w)$, and $C_\phi(w) \neq C_\phi(x)$). Otherwise, we permute the colors assigned to uu_1 and uv . The obtained coloring is still an avd-coloring of G' . We then extend the coloring to G as previously.

□

Claim 2 *The graph H does not contain*

1. *a path uvw such that $d_H(u) = d_H(v) = d_H(w) = 2$ [10],*
2. *a 3-vertex adjacent to a 2-vertex,*
3. *a 4-vertex adjacent to three 2-vertices,*
4. *a k -vertex adjacent to a bad 2-vertex for $3 \leq k \leq \lceil \frac{\Delta}{2} \rceil$,*
5. *a k -vertex adjacent to $(k - 2)$ bad 2-vertices for $\lceil \frac{\Delta}{2} \rceil + 1 \leq k \leq \Delta - 1$,*
6. *a k -vertex adjacent to k 2-vertices for $k \geq 5$.*

Proof

1. This case was proven in [10].
2. Let u be a 3-vertex in H . In H , call u_1, u_2 and u_3 the three neighbors of u . Moreover, suppose that $d_H(u_1) = 2$. By Claim 1.2, $d_H(u_1) = 2 = d_G(u_1)$. Call v the second neighbor of u_1 distinct from u . We consider two cases:

- (1) Suppose first $d_H(v) = 2$. By Claims 1.2 and 1.3, $d_H(v) = 2 = d_G(v)$ and $d_H(u) = 3 = d_G(u)$. Call w the second neighbor of v distinct from u_1 . Consider $G' = G \setminus \{u_1v\}$. By minimality of G , G' admits an $(\Delta(G) + 1)$ -avd-coloring ϕ . If $\phi(uu_1) \neq \phi(vw)$, then we color u_1v properly, and we obtain an extension of ϕ to G which is a contradiction. Note that if $u = w$, we have immediately $\phi(uu_1) \neq \phi(vw)$. Without loss of generality suppose now $u \neq w$, $\phi(uu_1) = \phi(vw) = 1$, $\phi(uu_2) = 2$ and $\phi(uu_3) = 3$. We recolor uu_1 with 4. If the obtained coloring is still an avd-coloring, then we extend the coloring to G as previously. Otherwise this means that a neighbor of u , say u_2 , verifies $C_\phi(u_2) = \{2, 3, 4\}$. We recolor uu_1 with 5: either we are done, or u_3 verifies $C_\phi(u_3) = \{2, 3, 5\}$. Finally, we can color uu_1 with 6 (recall that $\Delta(G) \geq 5$) and we can extend the coloring to G , a contradiction.
- (2) Suppose now $d_H(v) > 2$. If $d_G(u) = d_H(u) = 3$, then we consider $G' = G \setminus \{uu_1\}$. By minimality of G , G' admits a $(\Delta(G) + 1)$ -avd-coloring ϕ . Without loss of generality suppose that $\phi(u_1v) = \alpha \leq 3$, $\phi(uu_2) = 2$ and $\phi(uu_3) = 3$. We color uu_1 with 4, either we are done or a neighbor of u , say u_2 , verifies $C_\phi(u_2) = \{2, 3, 4\}$. We color uu_1 with 5: either we are done, or u_3 verifies $C_\phi(u_3) = \{2, 3, 5\}$. Finally, we can color uu_1 with 6 and we can extend the coloring to G , a contradiction.

Assume $d_G(u) = k > 3$ ($k \leq \Delta(G)$). In G , call u_4, \dots, u_k the $(k - 3)$ 1-vertices adjacent to u . Consider $G' = G \setminus \{uu_k\}$. If $\Delta(G') < \Delta(G)$, then G' admits a $(\Delta(G') + 2)$ -avd-coloring ϕ by Theorem 2.1 (recall that $\text{mad}(G') \leq \text{mad}(G) < 3 - \frac{2}{\Delta} < 3$). This coloring ϕ is a partial $(\Delta(G) + 1)$ -avd-coloring of G that we can extend to G by coloring uu_k properly ($|C_\phi(u)| = \Delta(G)$, $|C_\phi(u_2)| < \Delta(G)$, $|C_\phi(u_3)| < \Delta(G)$, $|C_\phi(u_1)| = 2$ and $|C_\phi(u_k)| = 1$), a contradiction.

If $\Delta(G') = \Delta(G)$, then, by minimality of G , G' admits a $(\Delta(G) + 1)$ -avd-coloring ϕ . Without loss of generality suppose that $\phi(uu_i) = i$ for all $i \in [k - 1]$ (with $4 \leq k \leq \Delta(G)$). Suppose first $k > 4$. We color uu_k with k . If the obtained coloring is an avd-coloring of G' , then this extends the coloring to G . Otherwise this means that a neighbor of u , say u_2 , verifies $C_\phi(u_2) = \{1, 2, 3, \dots, k - 1, k\}$. We color uu_k with $k + 1$. Either we are done, or $C_\phi(u_3) = \{1, 2, 3, \dots, k - 1, k + 1\}$. In that case, we recolor uu_{k-1} with k . This extends the coloring to G , a contradiction. Assume now $k = 4$. We color uu_4 with 4, either we are done or a neighbor of u (distinct from u_1), say u_2 , verifies $C_\phi(u_2) = \{1, 2, 3, 4\}$. We color uu_4 with 5, either we are done or u_3 verifies $C_\phi(u_3) = \{1, 2, 3, 5\}$. Then we color uu_4 with 6 (recall that $\Delta(G) \geq 5$), this extends the coloring to G , a contradiction.

3. Let u be a 4-vertex in H . In H , call u_1, u_2, u_3 and u_4 the four neighbors of u . Moreover, suppose that $d_H(u_1) = d_H(u_2) = d_H(u_3) = 2$. By Claim 1.2, $d_G(u_1) = d_G(u_2) = d_G(u_3) = 2$. For $i \in \{1, 2, 3\}$, call v_i the second neighbor of u_i (distinct from u). We consider two cases:

- (1) Suppose one of the u_i 's is a bad 2-vertex, say u_1 . Call w the second neighbor of v_1 distinct from u_1 . Then, by Claim 1.3, $d_H(w) = d_G(w)$ and $d_H(u) = d_G(u) = 4$. Moreover, by Claim 2.1, $d_H(w) > 2$. Consider $G' = G \setminus \{u_1v_1\}$. By minimality of G , G' admits an $(\Delta(G) + 1)$ -avd-coloring ϕ . Without loss of generality suppose that $\phi(uu_i) = i$ for all $i \in [4]$. If $\phi(uu_1) \neq \phi(v_1w)$, then we color u_1v_1 properly. This extends ϕ to G , a contradiction. Suppose now $\phi(uu_1) = \phi(v_1w) = 1$. We recolor uu_1 with 5, either we are done or u_4 verifies $C_\phi(u_4) = \{2, 3, 4, 5\}$. Then we recolor uu_1 with 6 and we color u_1v_1 properly, this extends the coloring to G , a contradiction.
- (2) Assume that u_1, \dots, u_3 are three good 2-vertices. If $d_G(u) = d_H(u) = 4$, then we consider $G' = G \setminus \{uu_1\}$. By minimality of G , G' admits a $(\Delta(G) + 1)$ -avd-coloring ϕ . Without loss of generality suppose that $\phi(u_1v_1) = \alpha \leq 4$ and $\phi(uu_{i+1}) = i + 1$ for all $i \in [3]$. We color uu_1 with 5, either we are done or u_4 verifies $C_\phi(u_4) = \{2, 3, 4, 5\}$. Then we recolor uu_1 with 6, this extends the coloring to G , a contradiction. Assume $d_G(u) = k > 4$ ($k \leq \Delta(G)$). In G , call u_5, \dots, u_k the $(k - 4)$ 1-vertices adjacent to u . Consider $G' = G \setminus \{uu_k\}$. If $\Delta(G') < \Delta(G)$, then G' admits a $(\Delta(G') + 2)$ -avd-coloring ϕ by Theorem 2.1 (recall that $\text{mad}(G') \leq \text{mad}(G) < 3 - \frac{2}{\Delta} < 3$). This coloring ϕ is a partial $(\Delta(G) + 1)$ -avd-coloring of G that we can extend to G by coloring uu_k properly ($|C_\phi(u)| = \Delta(G)$, $|C_\phi(u_4)| < \Delta(G)$, $|C_\phi(u_1)| = |C_\phi(u_2)| = |C_\phi(u_3)| = 2$ and $|C_\phi(u_k)| = 1$), a contradiction. If $\Delta(G') = \Delta(G)$, then, by minimality of G , $G' = G \setminus \{uu_k\}$ admits a $(\Delta(G) + 1)$ -avd-coloring ϕ . Without loss of generality suppose that $\phi(uu_i) = i$ for all $i \in [k - 1]$ (with $5 \leq k \leq \Delta(G)$). We color uu_k with k . If the obtained coloring is an avd-coloring of G , then we are done. Otherwise this means that u_4 verifies $C_\phi(u_4) = \{1, 2, 3, \dots, k - 1, k\}$, then we color uu_k with $k + 1$. This extends the coloring to G , a contradiction.

4. Let u be a k -vertex in H adjacent to a bad 2-vertex v with $3 \leq k \leq \lceil \frac{\Delta}{2} \rceil$. Let w be the second neighbor of v distinct from u and call x the second neighbor of w distinct from v . Call u_1, \dots, u_{k-1} the $(k - 1)$ neighbors of u distinct from v . By Claims 1.2 and 1.3, $d_H(w) = d_G(w) = 2 = d_G(v) = d_H(v)$ and $d_G(u) = d_H(u) = k$. By Claim 2.1, $d_H(x) > 2$. Consider $G' = G \setminus \{vw\}$. By minimality of G , G' admits an $(\Delta(G) + 1)$ -avd-coloring ϕ . Without loss of generality suppose that $\phi(uu_i) = i$ for all $i \in [k - 1]$ and $\phi(uv) = k$. If $\phi(uv) \neq \phi(wx)$, then we color vw properly. This extends ϕ to G , a contradiction. Suppose now $\phi(uv) = \phi(wx) = k$. We try to recolor uv with each color $c \in \{k + 1, \dots, 2k - 1\}$. If we succeed, then we are done; otherwise this means that for all $c \in \{k + 1, \dots, 2k - 1\}$ there exists $i \in [k - 1]$ such that $C_\phi(u) \setminus \{k\} \cup \{c\} = C_\phi(u_i)$. In that case we can recolor uv with $2k$ which is possible because $3 \leq k \leq \lceil \frac{\Delta}{2} \rceil$. The obtained coloring is still an avd-coloring of G' . We then extend the coloring to G as previously.

5. Let u be a k -vertex in H . Consider $(k - 2)$ paths in H , $uv_j^1v_j^2v_j^3$ with $j \in [k - 2]$, such that $d_H(v_j^1) = d_H(v_j^2) = 2$ and $d_H(v_j^3) > 2$ (by Claim 2.1). By Claim 1.2, $d_G(v_j^1) = d_G(v_j^2) = 2$ and by Claim 1.3, $d_G(u) = d_H(u) = k$. Call x and y the two neighbors of u distinct from the v_j^1 's. Consider $G' = G \setminus \{v_1^1v_1^2\}$. By minimality of G , G' admits an $(\Delta(G) + 1)$ -avd-coloring ϕ . We consider two cases:

- (1) $\phi(uv_1^1) \neq \phi(v_1^2v_1^3)$. We color $v_1^1v_1^2$ properly. This extends ϕ to G , a contradiction.
- (2) Without loss of generality assume $\phi(uv_1^1) = 1 = \phi(v_1^2v_1^3)$, for all $j \in [k - 2]$, $\phi(uv_j^1) = j$, $\phi(ux) = k - 1$ and $\phi(uy) = k$. If one of the $v_j^2v_j^3$ ($j \in [k - 2]$), say $v_2^2v_2^3$, is such that $\phi(v_2^2v_2^3) \neq 1$, then we uncolor $v_2^1v_2^2$, we permute the colors of uv_1^1 and uv_2^1 , and finally we color properly $v_1^1v_1^2$ and $v_2^1v_2^2$. The obtained coloring is an $(\Delta(G) + 1)$ -avd-coloring

of G . Consider now that for all $j \in [k-2]$, $\phi(v_j^2 v_j^3) = 1$. We recolor uv_1^1 with $k+1$. If the obtained coloring is still an avd-coloring of G' , then we extend the coloring to G by coloring $v_1^1 v_1^2$ properly. Otherwise this means that a neighbor of u (x or y), say x , verifies $C_\phi(x) = \{2, 3, 4, \dots, k-1, k, k+1\}$. Then we recolor uv_1^1 with $k+2$ (which is possible because $k \leq \Delta(G) - 1$). Either we are done, or $C_\phi(y) = \{2, 3, \dots, k-1, k, k+2\}$. In that case, we uncolor $v_2^1 v_2^2$, we recolor uv_2^1 with $k+1$. Then we color properly $v_1^1 v_1^2$ and $v_2^1 v_2^2$. This extends the coloring to G , a contradiction.

6. Let u be a k -vertex in H adjacent to k 2-vertices u_1, \dots, u_k , with $k \geq 5$. For all $i \in [k]$, call v_i the second neighbor of u_i (distinct from u). By Claim 1.2, $d_G(u_i) = 2$ for all $i \in [k]$. We consider two cases:

- (1) Suppose one of the u_i 's is a bad 2-vertex, say u_1 . Call w the second neighbor of v_1 distinct from u_1 . Then, by Claim 1.3, $d_H(w) = d_G(w)$ and $d_H(u) = d_G(u) = k$. Moreover, by Claim 2.1, $d_H(w) > 2$. Consider $G' = G \setminus \{u_1 v_1\}$. By minimality of G , G' admits an $(\Delta(G) + 1)$ -avd-coloring ϕ . Without loss of generality suppose that $\phi(uu_i) = i$ for all $i \in [k]$. If $\phi(uu_1) \neq \phi(v_1 w)$, then we color $u_1 v_1$ properly. This extends ϕ to G , a contradiction. Suppose now $\phi(uu_1) = \phi(v_1 w) = 1$. We recolor uu_1 with $k+1$ and we color $u_1 v_1$ properly, this extends the coloring to G , a contradiction.
- (2) Assume now u_1, \dots, u_k are k good 2-vertices. Suppose first $d_G(u) = d_H(u)$. Consider $G' = G \setminus \{uu_1\}$. By minimality of G or Theorem 2.1, G' admits an $(\Delta(G) + 1)$ -avd-coloring ϕ . Without loss of generality suppose $\phi(uu_{i+1}) = i+1$ for all $i \in [k-1]$ and $\phi(u_1 v_1) = \alpha \leq k-1$. We color uu_1 with a color of $[\Delta(G) + 1] \setminus (\{2, \dots, k\} \cup \{\alpha\})$. This extends the coloring to G , a contradiction. Suppose now $d_G(u) = l > d_H(u)$ such that $l \leq \Delta(G)$. In G , call u_{k+1}, \dots, u_l the $(l-k)$ 1-vertices adjacent to u (with $k < l \leq \Delta(G)$). Consider $G' = G \setminus \{uu_l\}$. If $\Delta(G') < \Delta(G)$, then G' admits a $(\Delta(G') + 2)$ -avd-coloring ϕ by Theorem 2.1 (recall that $\text{mad}(G') \leq \text{mad}(G) < 3 - \frac{2}{\Delta} < 3$). This coloring ϕ is a partial $(\Delta(G) + 1)$ -avd-coloring of G that we can extend to G by coloring uu_l properly, a contradiction. If $\Delta(G') = \Delta(G)$, then by minimality of G , G' admits an $(\Delta(G) + 1)$ -avd-coloring ϕ . Without loss of generality suppose that $\phi(uu_i) = i$ for all $i \in [l-1]$. We color uu_l with l . This extends the coloring to G , a contradiction.

□

2.2 Discharging procedure

In this section we use discharging technique on the vertices of the graph H by defining the weight function $\omega : V(H) \rightarrow \mathbb{R}$ with $\omega(x) = d_H(x)$. It follows from the hypothesis on the maximum average degree ($\text{mad}(H) < 3 - \frac{2}{\Delta}$) that the total sum of weights is strictly less than $(3 - \frac{2}{\Delta})|V(H)|$. Then we define discharging rules to redistribute weights, and once the discharging is finished, a new weight function ω^* will be produced such that during the discharging process the total sum of weights is kept fixed. This leads to the following contradiction:

$$\left(3 - \frac{2}{\Delta}\right) |V(H)| \leq \sum_{x \in V(H)} \omega^*(x) = \sum_{x \in V(H)} \omega(x) < \left(3 - \frac{2}{\Delta}\right) |V(H)|$$

and hence, this counterexample cannot exist.

The discharging rules are defined as follows:

- (R1) Every $(\lceil \frac{\Delta}{2} \rceil + 1)^+$ -vertex gives $1 - \frac{2}{\Delta}$ to each adjacent bad 2-vertex.
- (R2) Every 4^+ -vertex gives $\frac{1}{2} - \frac{1}{\Delta}$ to each adjacent good 2-vertex.

Let $v \in V(H)$ be a k -vertex. By Claim 1.1, $k \geq 2$. Consider the following cases:

Case $k = 2$. Observe that $\omega(v) = 2$. By Claim 2.1, v is adjacent to at most one 2-vertex. Moreover by Claim 2.2, v is not adjacent to a 3-vertex. Suppose v is a good 2-vertex. Hence, by (R2), $\omega^*(v) \geq 2 + 2 \times (\frac{1}{2} - \frac{1}{\Delta}) = 3 - \frac{2}{\Delta}$. Suppose v is bad. By Claim 2.4, v is adjacent to a $(\lceil \frac{\Delta}{2} \rceil + 1)^+$ -vertex. Hence, by (R1), $\omega^*(v) = 2 + 1 \times (1 - \frac{2}{\Delta}) = 3 - \frac{2}{\Delta}$.

Case $k = 3$. By (R1) and (R2), $\omega(v) = 3 = \omega^*(v) > 3 - \frac{2}{\Delta}$.

Case $k = 4$. Observe that $\omega(v) = 4$. Suppose first $5 \leq \Delta(G) \leq 6$. If v is not adjacent to a bad 2-vertex, then by Claim 2.3, v is adjacent to at most two good 2-vertices. Hence, by (R2), $\omega^*(v) \geq 4 - 2 \times (\frac{1}{2} - \frac{1}{\Delta}) > 3 - \frac{2}{\Delta}$. Otherwise, by Claims 2.3 and 2.5, v is adjacent to at most one bad 2-vertex and one good 2-vertex. Hence, by (R1) and (R2), $\omega^*(v) \geq 4 - 1 \times (1 - \frac{2}{\Delta}) - 1 \times (\frac{1}{2} - \frac{1}{\Delta}) > 3 - \frac{2}{\Delta}$. Suppose now $\Delta(G) \geq 7$. Then, by Claims 2.3 and 2.4, v is adjacent to at most two good 2-vertices. Hence, by (R2), $\omega^*(v) \geq 4 - 2 \times (\frac{1}{2} - \frac{1}{\Delta}) > 3 - \frac{2}{\Delta}$.

Case $5 \leq k \leq \lceil \frac{\Delta}{2} \rceil$. Observe that $\omega(v) = k$. By Claims 2.4 and 2.6, v is adjacent to at most $(k - 1)$ good 2-vertices. Hence, by (R2), $\omega^*(v) \geq k - (k - 1) \times (\frac{1}{2} - \frac{1}{\Delta}) > 3 - \frac{2}{\Delta}$ for $k \geq 5$.

Case $\lceil \frac{\Delta}{2} \rceil + 1 \leq k \leq \Delta - 1$. By Claims 2.5 and 2.6, v is adjacent to at most $(k - 3)$ bad 2-vertices and two good 2-vertices. Hence, by (R1) and (R2), $\omega^*(v) \geq k - (k - 3) \times (1 - \frac{2}{\Delta}) - 2 \times (\frac{1}{2} - \frac{1}{\Delta}) \geq 3 - \frac{2}{\Delta}$ for $k \geq \lceil \frac{\Delta}{2} \rceil + 1$.

Case $k = \Delta$. Observe that $\omega(v) = \Delta$. By Claim 2.6, v is adjacent to at most $(\Delta - 1)$ 2-vertices. It follows by (R1), $\omega^*(v) \geq \Delta - (\Delta - 1) \times (1 - \frac{2}{\Delta}) = 3 - \frac{2}{\Delta}$.

This completes the proof.

References

- [1] S. Akbari, H. Bidkhori, and N. Nosrati, r -strong edge colorings of graphs, *Discrete Math.*, 306:3005-3010, 2006.
- [2] P. N. Balister, E. Győri, J. Lehel, and R. H. Schelp, Adjacent vertex distinguishing edge-colorings, *SIAM J. Discrete Math.*, 21:237-250, 2007.
- [3] P.N. Balister, O.M. Riordan, and R.H. Schelp, Vertex-distinguishing edge-colorings of graphs, *Journal of graph theory*, 42:95-109, 2003.
- [4] A.C. Burriss and R.H. Schelp, Vertex-distinguishing proper edge-colorings, *Journal of graph theory*, 26:73-83, 1997.
- [5] J. Cerný, M. Horňák, and R. Soták, Observability of a graph, *Mathematica Slovaca*, 46(1):21-31, 1996.
- [6] E. Dedó, D. Torri, and N. Zagaglia Salvi, The observability of the fibonacci and the lucas cubes, *Discrete Math.*, 255:55-63, 2002.
- [7] O. Favaron, H. Li, and R.H. Schelp, Strong edge coloring of graphs. *Discrete Math.*, 159:103-109, 1996.
- [8] M. Horňák and R. Soták, Observability of complete multipartite graphs with equipotent parts, *Ars Combinatoria*, 41:289-301, 1995.
- [9] M. Horňák, Asymptotic behaviour of the observability of Q_n , *Discrete Math.*, 176:139-148, 1997.
- [10] W. Wang and Y. Wang, Adjacent vertex distinguishing edge-colorings of graphs with smaller maximum average degree, *J. Comb. Optim.*, 19:471-485, 2010.
- [11] Z. Zhang, L. Liu, and J. Wang, Adjacent strong edge coloring of graphs, *Appl. Math. Lett.*, 15:623-626, 2002.