

Vertex-distinguishing edge-coloring

M. Bonamy
N. Bousquet
H. Hocquard
J. Przybyło

May 12, 2017



Definition - vd-coloring

A **vertex-distinguishing edge-coloring** (vd-coloring for short) is an edge-coloring

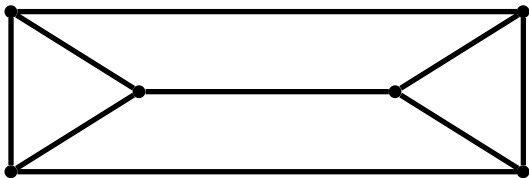
- 1 that is **proper**: two adjacent edges have distinct colors
- 2 such that, for every pair of vertices u, v , we have $S(u) \neq S(v)$, where $S(u)$ denotes the set of colors of the edges incident to u .

Definition - vd-coloring

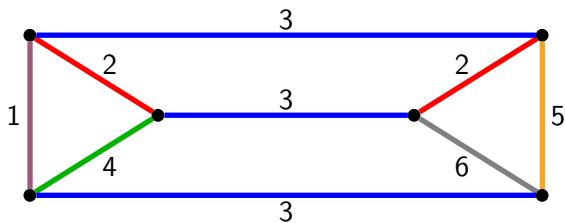
A **vertex-distinguishing edge-coloring** (vd-coloring for short) is an edge-coloring

- 1 that is **proper**: two adjacent edges have distinct colors
- 2 such that, for every pair of vertices u, v , we have $S(u) \neq S(v)$, where $S(u)$ denotes the set of colors of the edges incident to u .

In the following, we consider graphs with at most one isolated vertex and no isolated edges.

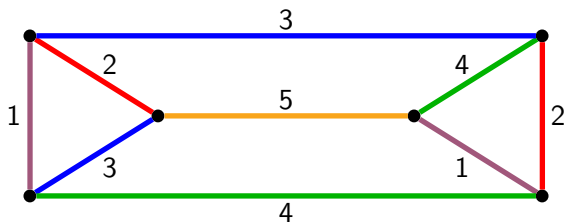


Observability



Definition - observability

The minimum number of colors such that the graph G admits a vd-coloring is called the **observability** of G and is denoted by $\text{Obs}(G)$.



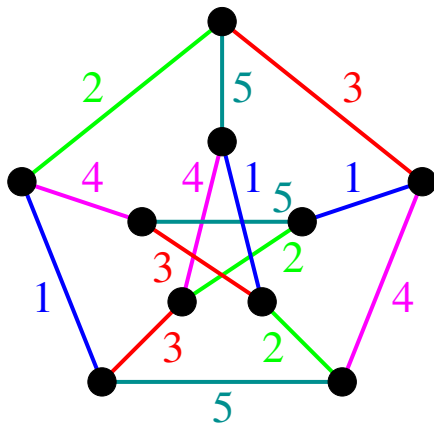
$$\text{Obs}(G) = 5$$

Conjecture (Burris and Schelp, 1997)

Let G be a graph which contains at most one isolated vertex and which does not contain isolated edges. Consider j the smallest integer such that $\binom{j}{k} \geq n_k$ (for $1 \leq k \leq \Delta$), where n_k denotes the number of vertices of degree k , then $\text{Obs}(G) = j$ or $j + 1$.

TRUE for different families of graphs.

Example



$$\text{Obs}(P_{10}) = 5$$

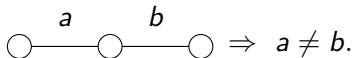
Definition - AVD coloring

An **adjacent vertex-distinguishing edge k -coloring** (AVD k -coloring) of a graph G is

Definition - AVD coloring

An **adjacent vertex-distinguishing edge k -coloring** (AVD k -coloring) of a graph G is

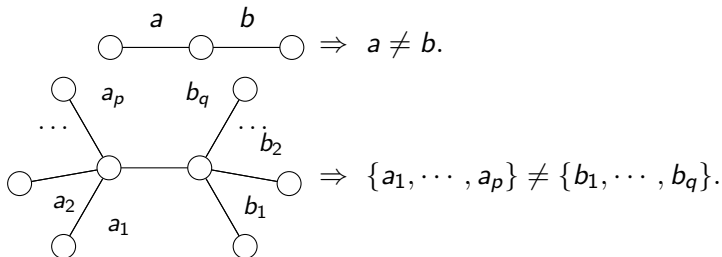
- 1 a proper edge coloring of G using at most k colors and



Definition - AVD coloring

An **adjacent vertex-distinguishing edge k -coloring** (AVD k -coloring) of a graph G is

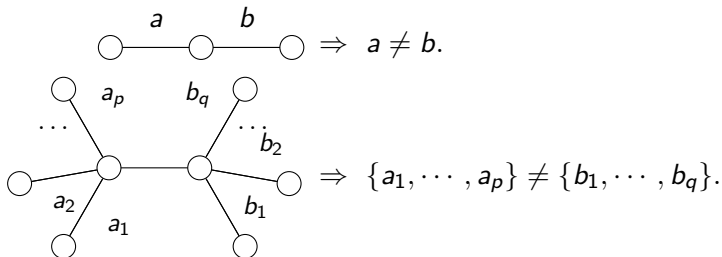
- 1 a proper edge coloring of G using at most k colors and
- 2 for every pair of **adjacent** vertices u, v , $S(u) \neq S(v)$.



Definition - AVD coloring

An **adjacent vertex-distinguishing edge k -coloring** (AVD k -coloring) of a graph G is

- 1 a proper edge coloring of G using at most k colors and
- 2 for every pair of **adjacent** vertices u, v , $S(u) \neq S(v)$.



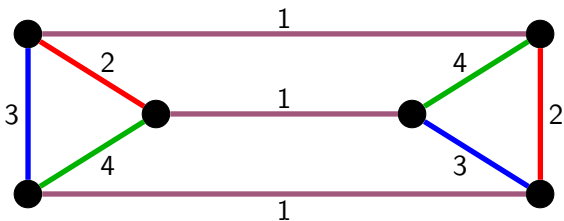
Definition - avd-chromatic index

$\chi'_{avd}(G)$: smallest integer k s.t. G admits an AVD k -coloring.

Definition - AVD coloring

An **adjacent vertex-distinguishing edge k -coloring** (AVD k -coloring) of a graph G is

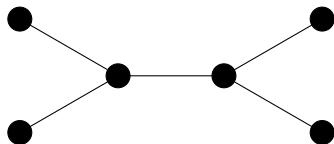
- 1 a proper edge coloring of G using at most k colors and
- 2 for every pair of **adjacent** vertices u, v , $S(u) \neq S(v)$.



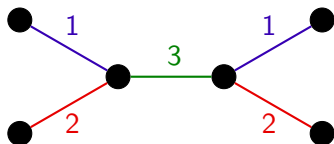
$$\chi'_{avd}(G) = 4$$

Definition - avd-chromatic index

$\chi'_{avd}(G)$: smallest integer k s.t. G admits an AVD k -coloring.



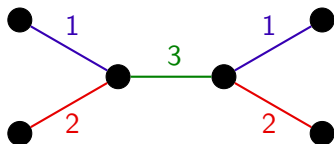
Remarks



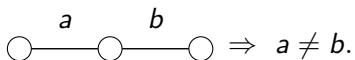
χ' : Minimum number of colors to ensure that

$$\text{---} \overset{a}{\circ} \text{---} \overset{b}{\circ} \text{---} \circ \Rightarrow a \neq b.$$

Remarks



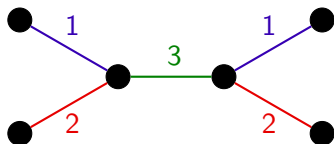
χ' : Minimum number of colors to ensure that



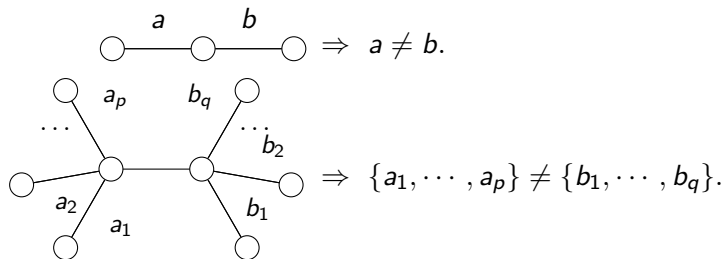
Δ : Maximum degree of the graph.

$$\Delta \leq \chi'$$

Remarks



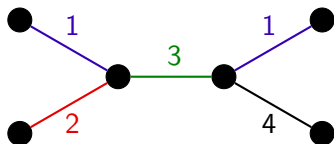
χ'_{avd} : Minimum number of colors to ensure that



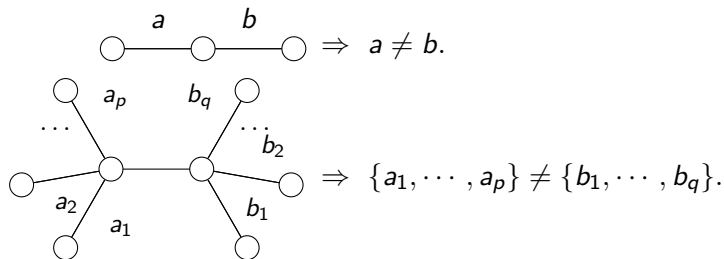
Δ : Maximum degree of the graph.

$$\Delta \leq \chi'$$

Remarks



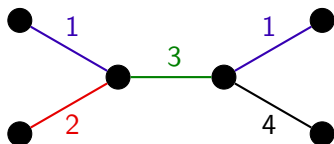
χ'_{avd} : Minimum number of colors to ensure that



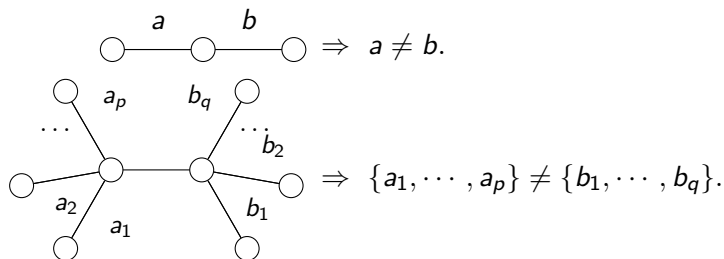
Δ : Maximum degree of the graph.

$$\Delta \leq \chi'$$

Remarks

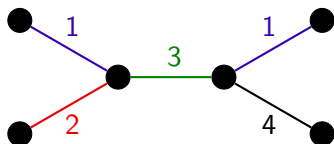


χ'_{avd} : Minimum number of colors to ensure that

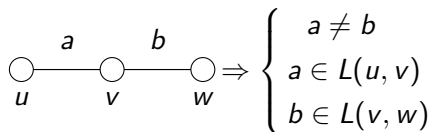


Δ : Maximum degree of the graph.

$$\Delta \leq \chi' \leq \chi'_{avd}$$



χ'_e : Minimum size of every $L(e)$ such that



Δ : Maximum degree of the graph.

$$\Delta \leq \chi' \leq \chi'_e$$

Theorem (Vizing 1964)

$$\chi' \leq \Delta + 1.$$

Theorem (Vizing 1964)

$$\chi' \leq \Delta + 1.$$

Conjecture (weak List Coloring Conjecture 1976)

$$\chi'_\ell \leq \Delta + 1.$$

Main conjectures

Theorem (Vizing 1964)

$$\chi' \leq \Delta + 1.$$

Conjecture (weak List Coloring Conjecture 1976)

$$\chi'_\ell \leq \Delta + 1.$$

Conjecture (Zhang, Liu and Wang, 2002)

For connected graphs on at least 6 vertices,
$$\chi'_{avd} \leq \Delta + 2.$$

Theorem (Zhang, Liu and Wang, 2002)

For cycle C_p , we have :

$$\chi'_{avd}(C_p) = \begin{cases} 3 & \text{if } p \equiv 0 \pmod{3} \\ 4 & \text{if } p \not\equiv 0 \pmod{3} \text{ and } p \neq 5 \\ 5 & \text{if } p = 5 \end{cases}$$

Theorem (Hatami, 2005)

Every graph with maximum degree Δ and with *no isolated edges* has an *avd-coloring* with at most $\Delta + 300$ colors, provided that $\Delta > 10^{20}$.

Definition - mad

The **maximum average degree** of a graph G , denoted $\text{mad}(G)$, is the maximum of the average degrees of all the subgraphs of G :

$$\text{mad}(G) = \max \left\{ \frac{2|E(H)|}{|V(H)|}, H \subseteq G \right\}$$

Theorem (Wang and Wang, 2010)

Let G be a graph with maximum degree $\Delta(G)$ and maximum average degree $\text{mad}(G)$.

- 1 If $\text{mad}(G) < 3$ and $\Delta(G) \geq 3$, then $\chi'_{\text{avd}}(G) \leq \Delta(G) + 2$.
- 2 If $\text{mad}(G) < \frac{5}{2}$ and $\Delta(G) \geq 4$, or $\text{mad}(G) < \frac{7}{3}$ and $\Delta(G) = 3$, then $\chi'_{\text{avd}}(G) \leq \Delta(G) + 1$.

[Wang and Wang, 2010]

- If $\text{mad}(G) < \frac{7}{3}$ and $\Delta(G) = 3$, then $\chi'_{avd}(G) \leq \Delta(G) + 1$.
- If $\text{mad}(G) < \frac{5}{2}$ and $\Delta(G) \geq 4$, then $\chi'_{avd}(G) \leq \Delta(G) + 1$.

Theorem (H. and Montassier, 2011)

If $\text{mad}(G) < 3 - \frac{2}{\Delta(G)}$ and $\Delta(G) \geq 5$, then $\chi'_{avd}(G) \leq \Delta(G) + 1$.

Theorem (Wang and Wang, 2010)

$$\text{mad} < 3, \Delta \geq 3 \Rightarrow \chi'_{\text{avd}} \leq \Delta + 2.$$

In summary...

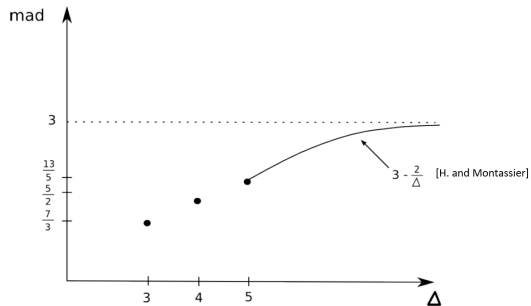
Theorem (Wang and Wang, 2010)

$$\text{mad} < 3, \Delta \geq 3 \Rightarrow \chi'_{avd} \leq \Delta + 2.$$

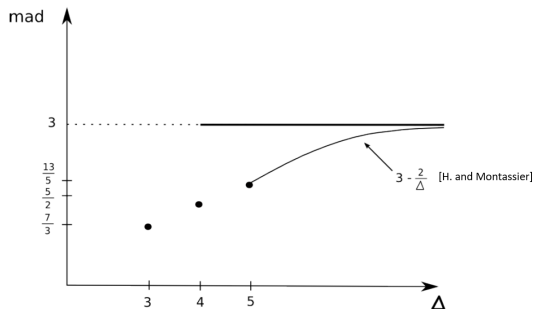
Theorem (H. and Montassier, 2011)

$$\text{mad} < 3 - \frac{2}{\Delta}, \Delta \geq 5 \Rightarrow \chi'_{avd} \leq \Delta + 1.$$

Our results with M. Bonamy and N. Bousquet



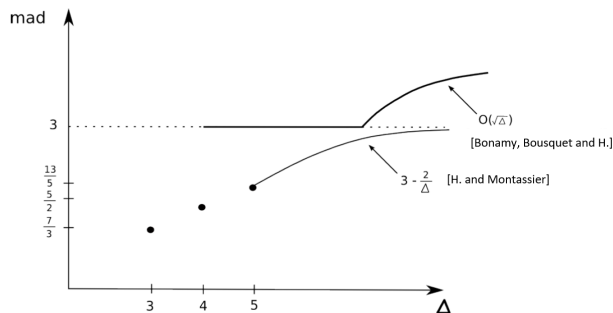
Our results with M. Bonamy and N. Bousquet



Theorem (Bonamy, Bousquet and H., 2013)

$$\text{mad} < 3, \Delta \geq 4 \Rightarrow \chi'_{\text{avd}} \leq \Delta + 1.$$

Our results with M. Bonamy and N. Bousquet



Theorem (Bonamy, Bousquet and H., 2013)

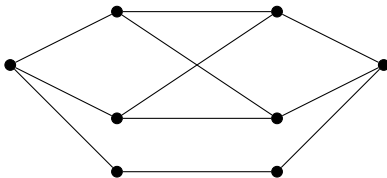
$$\text{mad} < 3, \Delta \geq 4 \Rightarrow \chi'_{\text{avd}} \leq \Delta + 1.$$

Theorem (Bonamy, Bousquet and H., 2013)

$$m > \frac{5}{2}, \text{mad} < m, \Delta \geq 3 \times m^2 \Rightarrow \chi'_{\text{avd}} \leq \Delta + 1.$$

Theorem (Bonamy, Bousquet and H., 2013)

$\text{mad} < 3$, $\Delta \geq 4 \Rightarrow \chi'_{avd} \leq \Delta + 1$.



A graph G with $\Delta = 3$ and $\text{mad} = \frac{11}{4} < 3$ such that $\chi'_{avd} = 5$.

Theorem (Edwards, Horňák and Woźniak, 2006)

For every *bipartite planar* graph with $\Delta \geq 12$, $\chi'_{avd} \leq \Delta + 1$.

Theorem (Horňák, Huang and Wang, 2013)

For every *planar* graph with $\Delta \geq 12$, $\chi'_{avd} \leq \Delta + 2$.

Theorem (Bonamy, Bousquet and H., 2013)

For every *planar* graph with $\Delta \geq 12$, $\chi'_{avd} \leq \Delta + 1$.

Theorem (Bonamy, Bousquet and H., 2013)

For every *planar* graph with $\Delta \geq 12$, $\chi'_{avd} \leq \Delta + 1$.

Theorem (Bonamy, Bousquet and H., 2013)

Let $k \geq 12$.

For every *planar* graph with $\Delta \leq k$, $\chi'_{avd} \leq k + 1$.

Let G be a minimum counterexample to the theorem.

- 1 Study the structural properties of G
- 2 Reach a contradiction using a discharging method on G

Key Claim

In G , a vertex v with $d(v) \leq 6$ has at most one neighbor w with $d(w) \leq 6$, and then $d(w) = d(v)$.

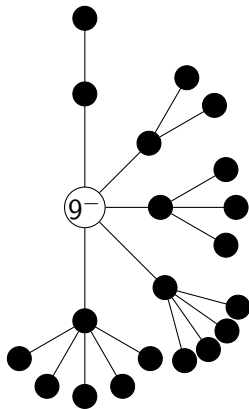
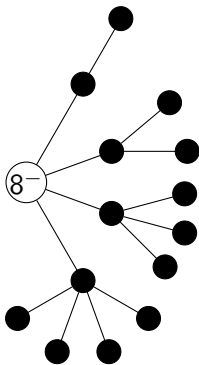
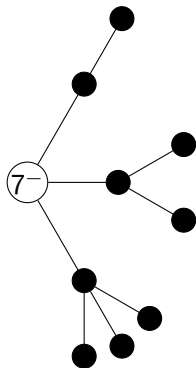
Key Claim

In G , a vertex v with $d(v) \leq 6$ has at most one neighbor w with $d(w) \leq 6$, and then $d(w) = d(v)$.

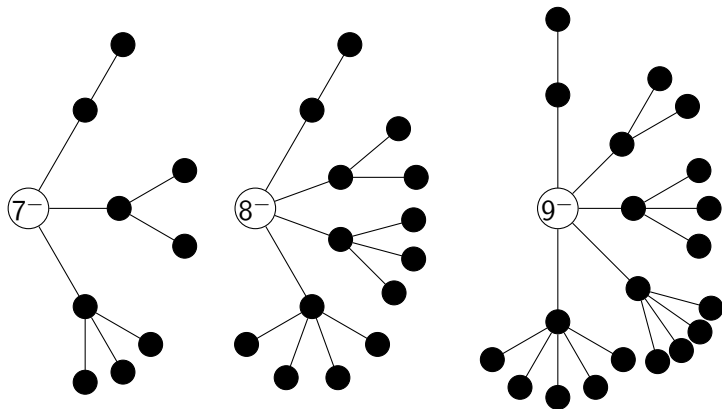
In other words

A small vertex is adjacent to at most one small vertex.

Key Lemma



Key Lemma



A vertex v s.t. $7 \leq d(v) \leq 12$ has few small neighbors.

Discharging procedure

- 1 A weight function ω .
- 2 Discharging rules... [Redistribute weights accordingly. Once the discharging is finished, a new weight function ω^* is produced.]
- 3 Using Euler's formula and the discharging technique we will derive a contradiction.

By Euler's Formula we have:

$$|V| - |E| + |F| = 2$$

By Euler's Formula we have:

$$|V| - |E| + |F| = 2 \times (-6)$$

By Euler's Formula we have:

$$|V| - |E| + |F| = 2 \times (-6)$$

$$6|E| - 6|V| - 6|F| = -12$$

$$2|E| - 6|V| + 4|E| - 6|F| = -12$$

Using

$$\sum_{f \in F(G)} d(f) = \sum_{v \in V(G)} d(v) = 2|E|$$

Using

$$\sum_{f \in F(G)} d(f) = \sum_{v \in V(G)} d(v) = 2|E|$$

$$2|E| - 6|V| + 4|E| - 6|F| = -12$$

Initial charge

Using

$$\sum_{f \in F(G)} d(f) = \sum_{v \in V(G)} d(v) = 2|E|$$

$$2|E| - 6|V| + 4|E| - 6|F| = -12$$

$$\sum_{v \in V(G)} (d(v) - 6) + \sum_{f \in F(G)} (2d(f) - 6) = -12$$

Using

$$\sum_{f \in F(G)} d(f) = \sum_{v \in V(G)} d(v) = 2|E|$$

$$2|E| - 6|V| + 4|E| - 6|F| = -12$$

$$\sum_{v \in V(G)} (d(v) - 6) + \sum_{f \in F(G)} (2d(f) - 6) = -12$$

$$\sum_{v \in V(G)} \omega(v) + \sum_{f \in F(G)} \omega(f) = -12 < 0$$

Initial charge

Using

$$\sum_{f \in F(G)} d(f) = \sum_{v \in V(G)} d(v) = 2|E|$$

$$2|E| - 6|V| + 4|E| - 6|F| = -12$$

$$\sum_{v \in V(G)} (d(v) - 6) + \sum_{f \in F(G)} (2d(f) - 6) = -12$$

$$\sum_{v \in V(G)} \omega(v) + \sum_{f \in F(G)} \omega(f) = -12 < 0$$

Conclusion:

$$\omega(x) = d(x) - 6 \text{ if } x \in V(G)$$

$$\omega(x) = 2d(x) - 6 \text{ if } x \in F(G)$$

$$\omega(f) \geq 0$$

$$\omega(v) = d(v) - 6 < 0 \iff d(v) \leq 5$$

When the discharging procedure is achieved, we proved that:

- 1 The total sum of weights is not changed.
- 2 For all $x \in V(G) \cup F(G)$, $\omega^*(x) \geq 0$.

Hence,

$$0 \leq \sum_{x \in V(G) \cup F(G)} \omega^*(x) = \sum_{x \in V(G) \cup F(G)} \omega(x) = -12 < 0$$

then, no such counterexample can exist.

Sketch of the proof (2): rephrasing Theorem

Theorem (Bonamy, Bousquet and H., 2013)

$$\text{mad} < 3, \Delta \geq 4 \Rightarrow \chi'_{\text{avd}} \leq \Delta + 1.$$

Sketch of the proof (2): rephrasing Theorem

Theorem (Bonamy, Bousquet and H., 2013)

$$\text{mad} < 3, \Delta \geq 4 \Rightarrow \chi'_{avd} \leq \Delta + 1.$$

Theorem (Bonamy, Bousquet and H., 2013)

$$\text{Let } k \geq 4. \text{ Then } \Delta \leq k, \text{ mad} < 3 \Rightarrow \chi'_{avd} \leq k + 1.$$

Sketch of the proof (2)

Theorem (Bonamy, Bousquet and H., 2013)

Let $k \geq 4$. Then $\Delta \leq k$, $\text{mad} < 3 \Rightarrow \chi'_{\text{avd}} \leq k + 1$.

In a *minimal* counter-example,

Sketch of the proof (2)

Theorem (Bonamy, Bousquet and H., 2013)

Let $k \geq 4$. Then $\Delta \leq k$, $\text{mad} < 3 \Rightarrow \chi'_{\text{avd}} \leq k + 1$.

In a *minimal* counter-example, there is no:

- Vertex adjacent to too many vertices of degree 1.

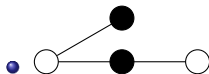
Sketch of the proof (2)

Theorem (Bonamy, Bousquet and H., 2013)

Let $k \geq 4$. Then $\Delta \leq k$, $\text{mad} < 3 \Rightarrow \chi'_{\text{avd}} \leq k + 1$.

In a *minimal* counter-example, there is no:

- Vertex adjacent to too many vertices of degree 1.



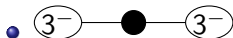
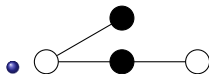
Sketch of the proof (2)

Theorem (Bonamy, Bousquet and H., 2013)

Let $k \geq 4$. Then $\Delta \leq k$, $\text{mad} < 3 \Rightarrow \chi'_{\text{avd}} \leq k + 1$.

In a *minimal* counter-example, there is no:

- Vertex adjacent to too many vertices of degree 1.



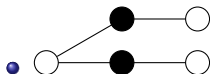
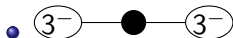
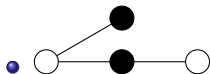
Sketch of the proof (2)

Theorem (Bonamy, Bousquet and H., 2013)

Let $k \geq 4$. Then $\Delta \leq k$, $\text{mad} < 3 \Rightarrow \chi'_{\text{avd}} \leq k + 1$.

In a *minimal* counter-example, there is no:

- Vertex adjacent to too many vertices of degree 1.



Sketch of the proof (3)

Theorem (Bonamy, Bousquet and H., 2013)

$$\forall m > \frac{5}{2}, \text{mad} < m, \Delta \geq 3 \times m^2 \Rightarrow \chi'_{avd} \leq \Delta + 1$$

In a minimal counter-example, there is no:

- Vertex adjacent to many vertices of degree 1.

Sketch of the proof (3)

Theorem (Bonamy, Bousquet and H., 2013)

$$\forall m > \frac{5}{2}, \text{mad} < m, \Delta \geq 3 \times m^2 \Rightarrow \chi'_{avd} \leq \Delta + 1$$

In a minimal counter-example, there is no:

- Vertex adjacent to many vertices of degree 1.



Sketch of the proof (3)

Theorem (Bonamy, Bousquet and H., 2013)

$$\forall m > \frac{5}{2}, \text{mad} < m, \Delta \geq 3 \times m^2 \Rightarrow \chi'_{avd} \leq \Delta + 1$$

In a minimal counter-example, there is no:

- Vertex adjacent to many vertices of degree 1.



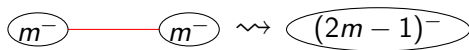
Theorem (Woodall, 2010)

$$\forall m \geq 2, \text{mad} < m, \Delta \geq 2 \times m^2 \Rightarrow \chi'_e = \Delta.$$

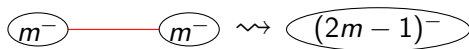
Using list-coloring proofs for avd-coloring



Using list-coloring proofs for avd-coloring



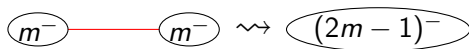
Using list-coloring proofs for avd-coloring



Theorem (Borodin, Kostochka and Woodall, 1997)

Any bipartite multigraph G is L -edge-colorable if
 $\forall (u, v) \in E(G), |L(u, v)| \geq \max(d(u), d(v))$.

Using list-coloring proofs for avd-coloring



Theorem (Borodin, Kostochka and Woodall, 1997)

Any bipartite multigraph G is L -edge-colorable if
 $\forall (u, v) \in E(G), |L(u, v)| \geq \max(d(u), d(v))$.

$$|\{m^-\}| \leq m \times |\{M^+\}|.$$

- (R1) Every vertex of degree at least $\frac{k}{2}$ gives $\sum_{i=2}^{m-1} 2(2i - 3)$ to the bank.
- (R2) Every vertex u such that $\frac{m}{2} \leq d(u) < m$ receives $m - d(u)$ from the bank.
- (R3) Every vertex u such that $d(u) < \frac{m}{2}$ receives $d(u)$ from the bank.

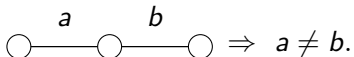
Definition - neighbor sum distinguishing edge coloring

A **NSD k -coloring** of a graph G is

Definition - neighbor sum distinguishing edge coloring

A **NSD k -coloring** of a graph G is

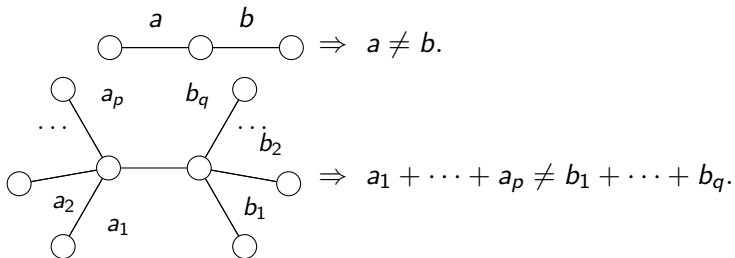
- 1 a proper edge coloring of G using at most k colors and



Definition - neighbor sum distinguishing edge coloring

A **NSD k -coloring** of a graph G is

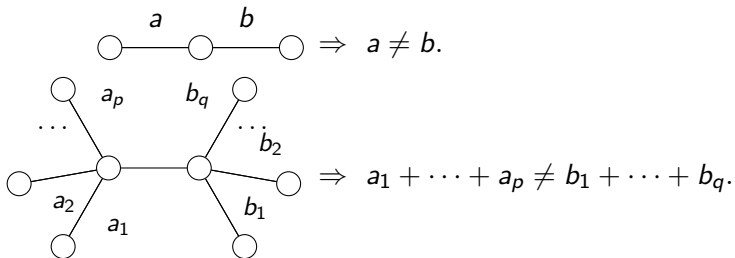
- 1 a proper edge coloring of G using at most k colors and
- 2 for every pair of **adjacent** vertices u, v , $s(u) \neq s(v)$, where $s(u)$ is the sum of colors taken on the edges incident with u .



Definition - neighbor sum distinguishing edge coloring

A **NSD k -coloring** of a graph G is

- 1 a proper edge coloring of G using at most k colors and
- 2 for every pair of **adjacent** vertices u, v , $s(u) \neq s(v)$, where $s(u)$ is the sum of colors taken on the edges incident with u .



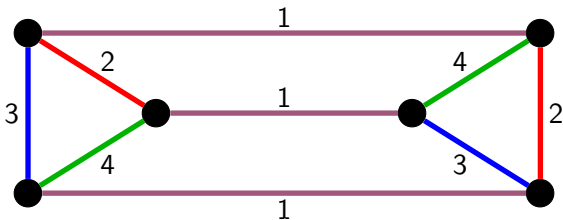
Definition - nsd-chromatic index

$\chi'_{\Sigma}(G)$: smallest integer k s.t. G admits a NSD k -coloring.

Definition - neighbor sum distinguishing edge coloring

A **NSD k -coloring** of a graph G is

- 1 a proper edge coloring of G using at most k colors and
- 2 for every pair of **adjacent** vertices u, v , $s(u) \neq s(v)$, where $s(u)$ is the sum of colors taken on the edges incident with u .



$$\chi'_{\Sigma}(G) = 4$$

Definition - nsd-chromatic index

$\chi'_{\Sigma}(G)$: smallest integer k s.t. G admits a NSD k -coloring.

Conjecture (Flandrin *et al.*, 2013)

If G is a connected graph of order at least three different from the cycle C_5 , then $\chi'_{\Sigma}(G) \leq \Delta(G) + 2$.

Conjecture (Flandrin *et al.*, 2013)

If G is a connected graph of order at least three different from the cycle C_5 , then $\chi'_{\Sigma}(G) \leq \Delta(G) + 2$.

TRUE for different families of graphs...paths, cycles, complete graphs, complete bipartite graphs and trees.

Conjecture (Flandrin *et al.*, 2013)

If G is a connected graph of order at least three different from the cycle C_5 , then $\chi'_{\Sigma}(G) \leq \Delta(G) + 2$.

TRUE for different families of graphs...paths, cycles, complete graphs, complete bipartite graphs and trees.

Theorem (Bonamy and Przybyło, 2014)

Any planar graph G with $\Delta(G) \geq 28$ and with no isolated edges satisfies $\chi'_{\Sigma}(G) \leq \Delta(G) + 1$.

Theorem (Dong *et al.*, 2014)

Any graph G with no isolated edges, $\Delta(G) \geq 6$ and $\text{mad}(G) < \frac{5}{2}$ satisfies $\chi'_{\Sigma}(G) \leq \Delta(G) + 1$.

Theorem (Dong *et al.*, 2014)

Any graph G with no isolated edges, $\Delta(G) \geq 6$ and $\text{mad}(G) < \frac{5}{2}$ satisfies $\chi'_{\Sigma}(G) \leq \Delta(G) + 1$.

Theorem (Hu *et al.*, 2015)

Any graph G with no isolated edges, $\Delta(G) \geq 6$ and $\text{mad}(G) < \frac{8}{3}$ satisfies $\chi'_{\Sigma}(G) \leq \Delta(G) + 1$.

Theorem (Dong *et al.*, 2014)

Any graph G with no isolated edges, $\Delta(G) \geq 6$ and $\text{mad}(G) < \frac{5}{2}$ satisfies $\chi'_{\Sigma}(G) \leq \Delta(G) + 1$.

Theorem (Hu *et al.*, 2015)

Any graph G with no isolated edges, $\Delta(G) \geq 6$ and $\text{mad}(G) < \frac{8}{3}$ satisfies $\chi'_{\Sigma}(G) \leq \Delta(G) + 1$.

Theorem (Yu *et al.*, 2016)

Any graph G with no isolated edges, $\Delta(G) \geq 5$ and $\text{mad}(G) < 3$ satisfies $\chi'_{\Sigma}(G) \leq \Delta(G) + 2$.

- Any graph G with no isolated edges, $\Delta(G) \geq 6$ and $\text{mad}(G) < \frac{5}{2}$ satisfies $\chi'_{\Sigma}(G) \leq \Delta(G) + 1$. [Dong *et al.*, 2014]
- Any graph G with no isolated edges, $\Delta(G) \geq 6$ and $\text{mad}(G) < \frac{8}{3}$ satisfies $\chi'_{\Sigma}(G) \leq \Delta(G) + 1$. [Hu *et al.*, 2015]
- Any graph G with no isolated edges, $\Delta(G) \geq 5$ and $\text{mad}(G) < 3$ satisfies $\chi'_{\Sigma}(G) \leq \Delta(G) + 2$. [Yu *et al.*, 2016]

Theorem (H. and Przybyło, 2016)

Any graph G with no isolated edges, $\Delta(G) \geq 6$ and $\text{mad}(G) < 3$ satisfies $\chi'_{\Sigma}(G) \leq \Delta(G) + 1$.

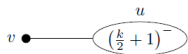
Sketch of the proof

Fix an integer $k \geq 6$.

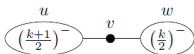
Suppose that H is a minimal graph without isolated edges such that $\Delta(H) \leq k$, $\text{mad}(H) < 3$ and $\chi'_{\Sigma}(H) > k + 1$.

- 1 Study the structural properties of H .
- 2 Reach a contradiction using a discharging method on H .

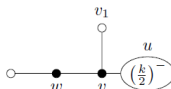
Forbidden configurations in H



(C₁)



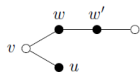
(C₂)



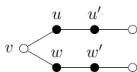
(C₃)



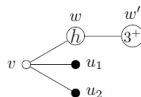
(C₄)



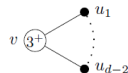
(C₅)



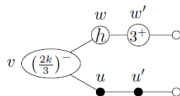
(C₆)



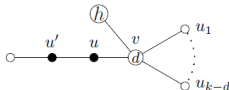
(C₇)



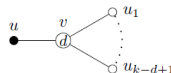
(C₈)



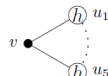
(C₉)



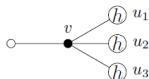
(C₁₀)



(C₁₁)



(C₁₂)



(C₁₃)

Key Lemma

For any finite sets L_1, \dots, L_n of real numbers with $|L_i| \geq n$ for $i = 1, \dots, n$, the set

$$\{x_1 + \dots + x_t : x_1 \in L_1, \dots, x_n \in L_n; x_i \neq x_j \text{ for } i \neq j\}$$

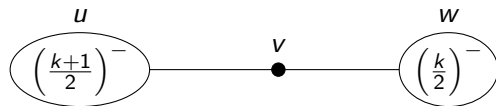
contains at least $\sum_{i=1}^n |L_i| - n^2 + 1$ distinct elements.

Forbidden configurations in H : (C1)



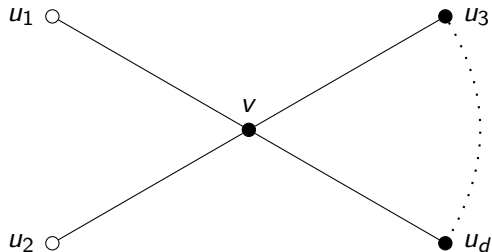
(C1)

Forbidden configurations in H : (C2)



(C₂)

Forbidden configurations in H : (C8)



(C8)

Discharging procedure

- ① A weight function ω .
- ② Discharging rules... [Redistribute weights accordingly. Once the discharging is finished, a new weight function ω^* is produced.]
- ③ Using discharging technique we will derive a contradiction.

- Efficient proofs!
- Settle the conjecture(s) on planar graphs?
- Total vertex-distinguishing. . .

- Efficient proofs!
- Settle the conjecture(s) on planar graphs?
- Total vertex-distinguishing. . .

Thanks for your attention.
Any questions?