

# Incidence coloring of planar graphs without adjacent small cycles

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## Abstract

An incidence of an undirected graph  $G$  is a pair  $(v, e)$  where  $v$  is a vertex of  $G$  and  $e$  an edge of  $G$  incident with  $v$ . Two incidences  $(v, e)$  and  $(w, f)$  are adjacent if one of the following holds: (i)  $v = w$ , (ii)  $e = f$  or (iii)  $vw = e$  or  $f$ . An incidence coloring of  $G$  assigns a color to each incidence of  $G$  in such a way that adjacent incidences get distinct colors. In 2012, Yang [15] proved that every planar graph has an incidence coloring with at most  $\Delta + 5$  colors, where  $\Delta$  denotes the maximum degree of the graph. In this paper, we show that  $\Delta + 4$  colors suffice if the graph is planar and without a  $C_3$  adjacent to a  $C_4$ . Moreover, we prove that every planar without  $C_4$  and  $C_5$  and maximum degree at least 5 admits an incidence coloring with at most  $\Delta + 3$  colors.

*Keywords:* Graph coloring - Incidence coloring - Planar graphs - Incidence chromatic number

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## 1. Introduction

Let  $G$  be a graph without loops and multiple edges. Let  $V(G)$  and  $E(G)$  be its vertex and edge set respectively. We denote by  $\Delta(G)$  the maximum degree of  $G$ .

A *proper edge-coloring* of a graph  $G = (V, E)$  is an assignment of colors to the edges of the graph such that no two adjacent edges use the same color. A *strong edge-coloring* (called also distance 2 edge-coloring) of  $G$  is a proper edge-coloring where each color class induces a matching. We denote by  $\chi'_s(G)$  the strong chromatic index of  $G$  which is the smallest integer  $k$  such that  $G$  can be strong edge-colored with  $k$  colors.

An *incidence* in  $G$  is a pair  $(v, e)$  with  $v \in V(G)$  and  $e \in E(G)$ , such that  $v$  and  $e$  are incident. The set of all incidences in  $G$  is denoted by  $I(G)$ , where

$$I(G) = \{(v, e) \in V(G) \times E(G) : \text{edge } e \text{ is incident to } v\}.$$

Two incidences  $(u, e)$  and  $(v, f)$  are adjacent if one of the following holds :

$$i) u = v, ii) e = f \text{ and } iii) \text{ the edge } uv = e \text{ or } uv = f.$$

A  *$k$ -incidence coloring* of a graph  $G$  is defined as a function  $\phi$  on  $I(G)$  into a set of colors  $C = \{1, 2, \dots, k\}$ , such that adjacent incidence are assigned with distinct colors. The minimum cardinality  $k$  for which  $G$  has a  $k$ -incidence coloring is the incidence chromatic number  $\chi_i(G)$  of  $G$ .

An alternate way of looking at the incidence chromatic number of a graph  $G$  is to consider the bipartite graph  $G' = (X \cup Y, E)$ , obtained from  $G$  such that,  $X = V(G)$ ,  $Y = E(G)$  and  $E(G') = \{(v, e), v \in V(G), e \in E(G), v \text{ is incident with } e\}$ . Each edge of  $G'$  corresponds to an incidence of  $G$  ; therefore, any incidence coloring of  $G$  corresponds to a strong edge coloring of  $G'$ .

$$\chi_i(G) = \chi'_s(G').$$

The notion of incidence coloring was introduced by Brualdi and Massey [3] in 1993. They proved the following theorem :

**Theorem 1 (Brualdi and Massey [3]).** *For every graph  $G$ ,  $\Delta(G) + 1 \leq \chi_i(G) \leq 2\Delta(G)$ .*

And they posed the Incidence Coloring Conjecture, which states that :

**Conjecture 1 (Brualdi and Massey [3]).** *For every graph  $G$ ,  $\chi_i(G) \leq \Delta(G) + 2$ .*

However, in 1997, by observing that the concept of incidence coloring is a particular case of directed star arboricity introduced by Algor and Alon [1], Guiduli [6] disproved the Incidence Coloring Conjecture showing that Paley graphs have an incidence chromatic number at least  $\Delta + \Omega(\log \Delta)$ . He also improved the upper bound proposed by Brualdy and Massey in Theorem 1.

**Theorem 2 (Guiduli [6]).** *For every graph  $G$ ,  $\chi_i(G) \leq \Delta(G) + O(\log \Delta(G))$ .*

The incidence coloring of graphs has been extensively studied. Most of authors consider the values of  $\chi_i(G)$  on particular classes of graphs (tree [3], cubic graphs [7, 10, 14], Halin graphs [12],  $k$ -degenerated graphs [4],  $K_4$ -minor free graph [4], outerplanar graphs [13], regular graphs and complement graphs [11], pseudo-Halin graphs [8], the powers of cycles [9], graphs with maximum degree 3 [7]).

In [4], Dolama, Sopena and Zhu give an upper bound of  $\chi_i(G)$  for planar graph.

**Theorem 3 (Dolama et al. [4]).** *For every planar graph  $G$ ,  $\chi_i(G) \leq \Delta(G) + 7$ .*

This last result was improved by Yang in [15], (paper written in 2007) by using the link between the incidence chromatic number, the star arboricity and the chromatic index of a graph :

**Theorem 4 (Yang [15]).** *For every planar graph  $G$ ,  $\chi_i(G) \leq \Delta(G) + 5$ , if  $\Delta(G) \neq 6$  and  $\chi_i(G) \leq 12$ , if  $\Delta(G) = 6$*

An interesting question is to see how the incidence chromatic number behaves for sparse planar graphs. Recall that the girth of a graph is the length of a shortest cycle in this graph. For instance, we collect results concerning the incidence chromatic number of planar graphs in the following lemma :

**Lemma 1.**

1.  $\chi_i(G) \leq \Delta(G) + 4$  for every triangle free planar graph  $G$ . [5]
2.  $\chi_i(G) \leq \Delta(G) + 3$  for every planar graph  $G$  with girth  $g \geq 6$ . [5]
3.  $\chi_i(G) \leq \Delta(G) + 2$  for every planar graph  $G$  with girth  $g \geq 6$  and  $\Delta(G) \geq 5$ . [5]
4.  $\chi_i(G) \leq \Delta(G) + 2$  for every planar graph  $G$  with girth  $g \geq 11$ . [5]
5.  $\chi_i(G) = \Delta(G) + 1$  for every planar graph  $G$  with girth  $g \geq 14$  and  $\Delta(G) \geq 4$ . [2]

Our mains results in this paper improve the upper bound in Theorem 3 and in Theorem 4. We denote by  $C_k$  a cycle of length  $k$  ( $k \in \mathbb{N}, k \geq 3$ ). In particular, we show the following.

**Theorem 5.**

1.  $\chi_i(G) \leq \Delta(G) + 4$  for every planar graph  $G$  without a  $C_3$  adjacent to a  $C_4$ .
2.  $\chi_i(G) \leq \Delta(G) + 3$  for every planar graph  $G$  without  $C_4$  and  $C_5$ , if  $\Delta(G) \neq 4$  and  $\chi_i(G) \leq 8$ , if  $\Delta(G) = 4$ .

From this last result we deduce the following easy Corollary :

**Corollary 1.**  $\chi_i(G) \leq \Delta(G) + 4$  for every planar graph  $G$  without a  $C_4$ .

Before proving our results we introduce some notation.

*Notation.* Let  $G$  be a planar graph. We use  $V(G)$ ,  $E(G)$ , and  $F(G)$  to denote, respectively, the set of vertices, edges, and faces of  $G$ . Let  $d(v)$  denote the degree of a vertex  $v$  in  $G$  and  $r(f)$  the degree of a face  $f$  in  $G$ . A vertex of degree  $k$  is called a  $k$ -vertex. A  $k^+$ -vertex (respectively,  $k^-$ -vertex) is a vertex of degree at least  $k$  (respectively, at most  $k$ ). We use the same notations for faces : a  $k$ -face (respectively,  $k^+$ -face,  $k^-$ -face) is a face of degree  $k$  (respectively, at least  $k$ , at most  $k$ ). A  $k$ -face having the boundary vertices  $x_1, x_2, \dots, x_k$  in the cyclic order is denoted by  $[x_1x_2\dots x_k]$ . A  $(k_1, k_2, k_3)$ -triangle is a 3-face  $[xyz]$  with  $d(x) = k_1$ ,  $d(y) = k_2$  and  $d(z) = k_3$ . For a vertex  $v \in V(G)$ , let  $n_i(v)$  denote the number of  $i$ -vertices adjacent to  $v$  for  $i \geq 1$ , and  $m_i(v)$  the number of  $i$ -faces incident to  $v$  for  $i \geq 1$ .

**Definition 1 (Dolama et al. [4]).** Let  $G$  be a graph.

1. A partial incidence coloring  $\phi'$  of  $G$ , is an incidence coloring only defined on some subset  $I$  of  $I(G)$ . For every uncolored incidence  $(u, uv) \in I(G) \setminus I$ ,  $F_G^{\phi'}(u, uv)$  is defined by the set of forbidden colors of  $(u, uv)$ , that is :

$$F_G^{\phi'}(u, uv) = \phi'(A_u) \cup \phi'(I_u) \cup \phi'(I_v),$$

where  $I_u$  is the set of incidences of the form  $(u, uv)$  and  $A_u$  is the set of incidences of the form  $(v, vu)$ .

2. A  $(k, l)$ -incidence coloring of a graph  $G$  is a  $k$ -incidence coloring  $\phi$  of  $G$  such that for every vertex  $v \in V(G)$ ,  $|\phi(A_v)| \leq l$ .

**Remark 1.** It is easy to see that every  $(k, l)$ -incidence coloring is also a  $(k', l)$ -incidence coloring for any  $k' > k$ .

The following observation will be used implicitly throughout.

**Observation 1.** For every graph  $G$  with maximum degree  $\Delta(G)$ , by Theorem 1 and Definition 1.2,  $G$  admits a  $(2\Delta(G), \Delta(G))$ -incidence coloring.

## 2. Proof of Theorem 5.1

We will prove the following stronger version of Theorem 5.1 :

**Theorem 6.** Every planar graph  $G$  without a  $C_3$  adjacent to a  $C_4$  admits a  $(k + 4, 4)$ -incidence coloring for every  $k \geq \Delta(G)$ ,  $k \in \mathbb{N}$ . Therefore,  $\chi_i(G) \leq \Delta(G) + 4$ .

**Observation 2.** We necessarily have  $k \geq \Delta(G) \geq 5$  since otherwise we obtain by Theorem 1  $\chi_i(G) \leq 2\Delta(G) \leq \Delta(G) + 4 \leq k + 4$ .

### 2.1. Structural properties

We proceed by contradiction. Let  $H$  be a counterexample to the theorem that minimizes  $|E(H)| + |V(H)|$ . By hypothesis there exists  $k \geq \max\{\Delta(G), 5\}$  such that  $H$  does not admit a  $(k + 4, 4)$ -incidence coloring. Let  $k \geq \max\{\Delta(G), 5\}$  be the smallest integer such that  $H$  does not admit a  $(k + 4, 4)$ -incidence coloring. By using Remark 1, we must have  $k = \max\{\Delta(G), 5\}$ . Moreover by minimality it is easy to see that  $H$  is connected.  $H$  satisfies the following properties :

**Lemma 2.**  $H$  does not contain :

1. 1-vertices,
2. 2-vertices,
3. a 3-vertex adjacent to a  $4^-$ -vertex,
4. a  $(4^-, 4^-, \Delta^-)$ -triangle,

5. a  $(3, 3, 3, 3, \Delta^-)$ -vertex.

For each of the parts of Lemma 2, we will suppose that the described configuration exists in  $H$ . Then we construct a graph  $H'$  obtained from  $H$  by deleting a certain number of vertices and edges. Due to the minimality of  $H$ , the graph  $H'$  admits a  $(k' + 4, 4)$ -incidence coloring  $\phi'$  for any  $k' \geq \max\{\Delta(H'), 5\}$ . Since  $\Delta(H) \geq \Delta(H')$ , the set of integers  $k'$  contains the set of integers  $k$ . Hence for the value  $k' = k$ ,  $H'$  admits a  $(k + 4, 4)$ -incidence coloring  $\phi'$ . Finally, for each cases, we will prove a contradiction by extending  $\phi'$  to a  $(k + 4, 4)$ -incidence coloring  $\phi$  of  $H$ .

**Proof**

We recall that  $k \geq 5$ , its implies that the minimum number of colors we can use is 9.

1. Suppose  $H$  contains a 1-vertex  $u$  and let  $v$  be its unique neighbor in  $H$ . Consider  $H' = H - \{u\}$ . Then by minimality of  $H$ ,  $H'$  admits a  $(k + 4, 4)$ -incidence coloring  $\phi'$ . We will extend  $\phi'$  to a  $(k + 4, 4)$ -incidence coloring  $\phi$  of  $H$  as follows.

Since for all  $v \in V(H)$ ,  $|\phi'(A_v)| \leq 4$ , we have  $|F_H^\phi(v, vu)| = |\phi'(I_v) \cup \phi'(A_v) \cup \phi'(I_u)| \leq \Delta(H) - 1 + 4 + 0 = \Delta(H) + 3 \leq k + 3$ , then there exists at least one color, say  $\alpha$ , such that  $\alpha \notin F_H^\phi(v, vu)$ . Hence, we set  $\phi(v, vu) = \alpha$  and one can observe that  $|\phi'(A_u)| = 1 \leq 4$ . According to  $|\phi'(A_v)| \leq 4$ , it suffices to set  $\phi(u, uv) = \beta$  for any color  $\beta$  in  $\phi'(A_v)$  and we are done. We have extended the coloring, a contradiction.

2. Suppose  $H$  contains a 2-vertex  $v$  and let  $u, w$  be the two neighbors of  $v$  in  $H$ . Consider  $H' = H - \{uv\}$ . Then by minimality of  $H$ ,  $H'$  admits a  $(k + 4, 4)$ -incidence coloring  $\phi'$ . We will extend  $\phi'$  to a  $(k + 4, 4)$ -incidence coloring  $\phi$  of  $H$  as follows. First, we uncolor the incidence  $(v, vw)$  and assume that  $\phi(w, vw) = \beta$ . By counting argument, there exists at least one color  $\alpha \notin F_H^\phi(u, uv)$ . Then we color  $(u, uv)$  with  $\alpha$  and  $|\phi'(A_v)| = 2 \leq 4$ . For coloring the incidence  $(v, vu)$ , we consider the following cases :

- (a) If  $|\phi'(A_u)| = 4$  then we color  $(v, vu)$  with a color  $\gamma \in \phi(A_u) \setminus \{\beta\}$  (because  $\alpha \notin \phi(A_u)$ ).
- (b) If  $|\phi'(A_u)| \leq 3$  then we color  $(v, vu)$  with a color  $\gamma \notin F_H^\phi(v, vu)$  (note that we have three choices). One can observe that  $|\phi'(A_u)| \leq 4$ .

Now, we color the incidence  $(v, vw)$  as follow:

- (a) If  $|\phi'(A_w)| = 4$  then we color  $(v, vw)$  with a color  $\zeta \in \phi(A_w) \setminus \{\alpha, \gamma\}$  and we have  $|\phi'(A_w)| = 4$ .
- (b) If  $|\phi'(A_w)| \leq 3$  then we color  $(v, vw)$  with a color  $\zeta \notin F_H^\phi(v, vw)$  and we have  $|\phi'(A_w)| \leq 3 \leq 4$ .

So, we have extended the coloring, a contradiction.

3. Suppose  $H$  contains a 3-vertex  $u$  adjacent to a  $4^-$ -vertex  $v$ . Consider  $H' = H - \{uv\}$ . By minimality of  $H$ ,  $H'$  admits a  $(k + 4, 4)$ -incidence coloring  $\phi'$ . We will extend  $\phi'$  to a  $(k + 4, 4)$ -incidence coloring  $\phi$  of  $H$  as follows. As above, by counting argument, it is easy to see that there exists at least one color  $\alpha \notin F_H^\phi(v, vu)$ . Then we color the incidence  $(v, vu)$  with  $\alpha$  and  $|\phi'(A_u)| \leq 3 \leq 4$ . Now we color the incidence  $(u, uv)$  with a color  $\beta \notin F_H^\phi(u, uv)$  ( $|F_H^\phi(u, uv)| \leq 7$ ), we have  $|\phi'(A_u)| \leq 4$  and we are done. We have extended the coloring, a contradiction.

4. Suppose  $H$  contains a 3-face  $[uvw]$  such that  $d(u) \leq 4$ ,  $d(v) \leq 4$  and  $d(w) \leq \Delta$ . By minimality of  $H$ , the graph  $H' = H - \{uv\}$  admits a  $(k + 4, 4)$ -incidence coloring. We will extend  $\phi'$  to a  $(k + 4, 4)$ -incidence coloring  $\phi$  of  $H$  as follows.

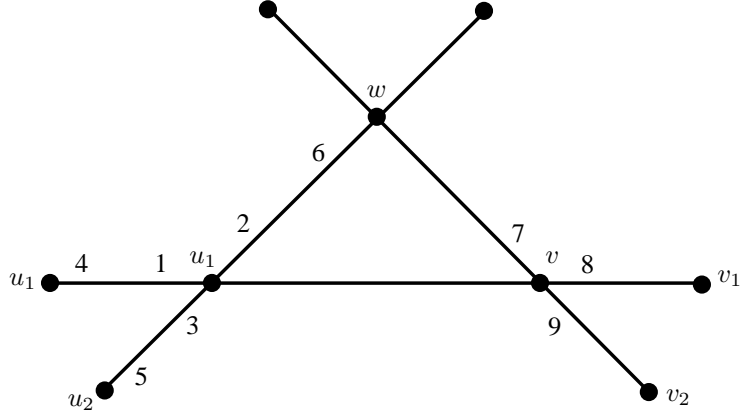


Figure 1:  $|F_H^\phi(u, uv)| = 9$

First we color  $(u, uv)$ .  
It is easy to see that

$$|F_H^{\phi'}(u, uv)| = |\phi'(I_u) \cup \phi'(A_u) \cup \phi'(I_v)| \leq 3 + 3 + 3 = 9$$

- Assume that  $|F_H^\phi(u, uv)| = 9$ , then we are in the situation described in Figure 1. We replace 2 by 7 and we color  $(u, uv)$  with 2 ( $7 \notin \phi(I_w)$ ).  
Now we consider  $\phi'(A_u)$  and  $\phi'(A_v)$ . If there is a color  $a$  in  $\phi'(A_u) \setminus \phi'(A_v)$ . We color  $(v, vu)$  with  $a$ . If  $\phi'(A_u) = \phi'(A_v)$ , let  $b$  be the color of  $(w, wv)$ , we interchange the colors of  $(w, wu)$  and  $(w, wv)$ . Then we color  $(v, vu)$  with the color  $b$ .
- We assume now that  $|F_H^\phi(u, uv)| < 9$  and  $|F_H^\phi(v, vu)| < 9$ . If there are two free colors for one of the two incidences, it is done. So we assume that  $|F_H^\phi(u, uv)| = 8$  and  $|F_H^\phi(v, vu)| = 8$  and the two incidences have the same free color.

We have two cases :

**Case 1:**  $|\phi'(A_u)| = 2$

**Case 1.1** Without loss of generality. assume that  $\phi'(u_1, u_1u) = \phi'(u_2, u_2u) = 4$ , and that the only free color is 5.

- If we can replace 7 by 2 ( $2 \notin \phi(I_w)$ ), then we color  $(u, uv)$  with 7 and  $(v, vu)$  by 5, it is done. Hence we cannot replace 7 by 2, it means that  $2 \in \phi'(A_v \setminus (w, wv))$ .  
Without loss of generality let  $\phi'(v_1, v_1v) = 2$ .
- If we can color  $(v, vu)$  with 6, then we color  $(u, uv)$  with 5 and it is done. Hence  $6 \in \phi'(A_v)$ , the only possibility is  $\phi'(v_2, v_2v) = 6$ .
- If we can color  $(v, vu)$  with 4, then we color  $(u, uv)$  with 5 and it is done. Hence  $4 \in \phi'(A_v)$ , then the only possibility is  $\phi'(w, wv) = 4$ .  
So now we permute 6 and 4 around  $w$  and we color  $(v, vu)$  with 4 and  $(u, uv)$  with 5. It follows that we extend the coloring, a contradiction.

**Case 1.2** If  $\phi'(u_1, u_1u) \neq \phi'(u_2, u_2u)$ , Without loss of generality. we can assume that  $\phi'(w, wu) = 4$  and that the only free color is 6.

- If we can replace 7 by 2 ( $2 \notin \phi(I_w)$ ), then we color  $(u, uv)$  with 7 and  $(v, vu)$  by 6, it is done. Hence we cannot replace 7 by 2, it means that  $2 \in \phi'(A_v)$ .  
Without loss of generality let  $\phi'(v_1, v_1v) = 2$ .
- If we can color  $(v, vu)$  with 4, then we color  $(u, uv)$  with 6 and it is done. Hence  $4 \in \phi'(A_v)$ , let  $\phi'(v_2, v_2v) = 4$ .
- If we can color  $(v, vu)$  with 5, then we color  $(u, uv)$  with 6 and it is done. Hence  $5 \in \phi'(A_v)$ , let  $\phi'(w, wv) = 5$ .

So now we permute 4 and 5 around  $w$  and we color  $(v, vu)$  with 5 and  $(u, uv)$  with 6. It follows that we extend the coloring, a contradiction.

**Case 2:**  $|\phi'(A_u)| = 3$

In this case one of the colors of  $(u_1, u_1u)$  or  $(u_2, u_2u)$  must be in  $\phi'(I_v)$  or the color of  $(w, wu)$  is in  $\phi'(I_v)$ .

**Case 2.1** One of the colors of  $(u_1, u_1u)$  or  $(u_2, u_2u)$  is in  $\phi'(I_v)$ . In our figure assume that 4 is the color of  $(v, vv_1)$ . Then the free color is 8.

- (a) If we can replace 7 by 2, then we color  $(u, uv)$  with 7 and  $(v, vu)$  by 8, it is done. Hence we cannot replace 7 by 2, it means that  $2 \in \phi'(A_v)$ . Without loss of generality let  $\phi'(v_1, v_1v) = 2$ .
- (b) If we can color  $(v, vu)$  with 6, then we color  $(u, uv)$  with 8 and it is done. Hence  $6 \in \phi'(A_v)$ , let  $\phi'(v_2, v_2v) = 6$ .
- (c) If we can color  $(v, vu)$  with 5, then we color  $(u, uv)$  with 8 and it is done. Hence  $5 \in \phi'(A_v)$ , let  $\phi'(w, wv) = 5$ .

So now we permute 6 and 5 around  $w$  and we color  $(v, vu)$  with 5 and  $(u, uv)$  with 8. It follows that we extend the coloring, a contradiction.

**Case 2.2** The color of  $(w, wu)$  is in  $\phi'(I_v)$ . In our figure assume that 6 is the color of  $(v, vv_1)$ . Then the free color is 8.

- (a) If we can replace 7 by 2, then we color  $(u, uv)$  with 7 and  $(v, vu)$  by 8, it is done. Hence we cannot replace 7 by 2, it means that  $2 \in \phi'(A_v)$ . Without loss of generality let  $\phi'(v_1, v_1v) = 2$ .
- (b) If we can color  $(v, vu)$  with 4, then we color  $(u, uv)$  with 8 and it is done. Hence  $4 \in \phi'(A_v)$ , let  $\phi'(v_2, v_2v) = 4$ .
- (c) If we can color  $(v, vu)$  with 5, then we color  $(u, uv)$  with 8 and it is done. Hence  $5 \in \phi'(A_v)$ , let  $\phi'(w, wv) = 5$ .

Now we replace 2 by 7 ( $7 \notin \phi(I_w)$ ) then we color  $(u, uv)$  with 2 and  $(v, vu)$  by 8, it is done. It completes the proof.

5. Suppose  $H$  contains a  $(3, 3, 3, 3, \Delta^-)$ -vertex  $u$ . Let  $u_i$  for  $i \in \{1, 2, 3, 4\}$  be the neighbors of  $u$  having a degree equal to 3 and  $v$  be the neighbors such that  $d(v) \leq \Delta$ . By minimality of  $H$ , the graph  $H' = H - \{u\}$  admits a  $(k+4, 4)$ -incidence coloring  $\phi'$ . We will extend  $\phi'$  to a  $(k+4, 4)$ -incidence coloring  $\phi$  of  $H$  as follows.

- We have  $|F_H^\phi(v, vu)| = |\phi'(I_v) \cup \phi'(A_v) \cup \phi'(I_u)| \leq \Delta(H) - 1 + 4 = \Delta(H) + 3 \leq k + 3$ . There is one free color for  $(v, vu)$ . Without loss of generality. We set  $\phi(v, vu) = 1$ . For  $(u, uv)$  we have 4 free colors ( $\phi'(A_v)$ ).
- We denote by  $L_i$  the list of available colors of  $(u_i, u_iu)$  for  $i \in \{1, 2, 3, 4\}$  and by  $L'_i$  the list of available colors of  $(u, uu_i)$  for  $i \in \{1, 2, 3, 4\}$ . We denote by  $L_v$  the list available colors of  $(u, uv)$ . By a computation as above it is easy to see that  $|L'_i| \geq k + 2 \geq 7$  and  $|L_i| \geq k \geq 5$ .
- Using a counting argument it is easy to see that there exists a color  $\alpha$  belonging to at least 3 lists among the lists  $L_i$ ,  $i \in \{1, 2, 3, 4\}$ . Without loss of generality. we assume that  $\alpha$  belongs to  $L_i$   $i \in \{1, 2, 3\}$ .
  - (a) If  $\alpha = 1$ , first we set  $\phi(u_i, u_iu) = 1$  for  $i \in \{1, 2, 3\}$ . Next we color  $(u, uv)$  (we have the 4 colors of  $\phi(A_v)$ ) and  $(u_4, u_4u)$  from the list  $L_4$ . Now we color  $(u, uu_i)$  for  $i \in \{1, 2, 3, 4\}$  in this order one after the other. We have extended the coloring, a contradiction.
  - (b) If  $\alpha \neq 1$ . Without loss of generality. we assume that  $\alpha = 2$ , we color  $(u_i, u_iu)$  for  $i \in \{1, 2, 3\}$  by 2. Next we color  $(u, uv)$  with a color different from 2, we set  $\phi(u, uv) = 3$ . Then we color  $(u, uu_i)$  for  $i \in \{1, 2, 3, 4\}$  in this order one after the other. We have enough colors in each list of each incidence  $(u, uu_i)$ ,  $i \in \{1, 2, 3, 4\}$ . Without loss of generality. we set  $\phi(u, uu_i) = i + 3$  for  $i \in \{1, 2, 3, 4\}$ . If we can

color properly the incidence  $(u_4, u_4u)$  we are done. So we cannot color  $(u_4, u_4u)$ . It means that  $L_4 = \{3, 4, 5, 6, 7\}$ . Hence we can assume Without loss of generality.  $\phi'(A_{u_4}) = \{1, 2\}$  and  $\phi'(I_{u_4}) = \{8, 9\}$ . Assume that we can replace one of the colors of  $(u, uu_i)$ ,  $i \in \{1, 2, 3\}$  by 8 or 9 without destroying the incidence coloring (let say 8), then we color the corresponding incidence by 8 (let say  $\phi(u, uu_1) = 8$ ), and we color  $(u_4, u_4u)$  with 4. We are done. Hence  $\phi(I_{u_i}) = \{2, 8, 9\}$ . We recall that  $|L_i| \geq 5$ , by the previous argument,  $L_i \subset \{1, 2, 3, 4, 5, 6, 7\}$ , for  $i \in \{1, 2, 3\}$ .

- If there exists an other color  $\beta \notin \{1, 2\}$  belonging to  $\cap_{i=1}^3 L_i$ , this color belongs also to  $L_4$ . Then we color  $(u_i, u_iu)$  for  $i \in \{1, 2, 3, 4\}$  with  $\beta$ ,  $(u, uv)$  with a color different from  $\beta$ . Next we color  $(u, uu_i)$ ,  $i \in \{1, 2, 3, 4\}$  one after the other, by the way we extend the coloring a contradiction.
- If it is not the case, then it is easy to see that each element of  $\{1, 3, 4, 5, 6, 7\}$  belongs to exactly two list  $L_i$ . Without loss of generality. we assume that  $1 \in L_1 \cap L_2$ . We color  $(u_1, u_1u)$  and  $(u_2, u_2u)$  with 1. We recall that  $|L_i| \geq 5$ , we take any color of  $L_3 \cap L_4$  to color  $(u_3, u_3u)$  and  $(u_4, u_4u)$ , let say 3. Then we color  $(u, uv)$  with a color different from 3. Next we color  $(u, uu_i)$ ,  $i \in \{1, 2, 3, 4\}$  one after the other, by the way we extend the coloring, a contradiction. This completes the proof.

□

## 2.2. Discharging procedure

Euler's formula  $|V(H)| - |E(H)| + |F(H)| = 2$  can be rewritten as  $(6|E(H)| - 10|V(H)|) + (4|E(H)| - 10|F(H)|) = -20$ . Using the relation  $\sum_{v \in V(H)} d(v) = \sum_{f \in F(H)} r(f) = 2|E(H)|$ , we get that:

$$\sum_{v \in V(H)} (3d(v) - 10) + \sum_{f \in F(H)} (2r(f) - 10) = -20 \quad (1)$$

We define the weight function  $\omega : V(H) \cup F(H) \rightarrow \mathbb{R}$  by  $\omega(x) = 3d(x) - 10$  if  $x \in V(H)$  and  $\omega(x) = 2r(x) - 10$  if  $x \in F(H)$ . It follows from Equation (1) that the total sum of weights is equal to  $-20$ . In what follows, we will define discharging rules (R1) to (R8) and redistribute weights accordingly. Once the discharging is finished, a new weight function  $\omega^*$  is produced. However, the total sum of weights is kept fixed when the discharging is finished. Nevertheless, we will show that  $\omega^*(x) \geq 0$  for all  $x \in V(H) \cup F(H)$ . This will lead us to the following contradiction :

$$0 \leq \sum_{x \in V(H) \cup F(H)} \omega^*(x) = \sum_{x \in V(H) \cup F(H)} \omega(x) = -20 < 0$$

and hence will demonstrate that such a counterexample cannot exist.

The discharging rules are defined as follows:

- (R1) Every  $k$ -vertex, for  $k \geq 5$ , gives  $\frac{1}{3}$  to each adjacent 3-vertex.
- (R2) Every 4-vertex gives 1 to each incident 3-face.
- (R3) Every  $k$ -vertex, for  $k \geq 5$ , gives 2 to each incident 3-face.
- (R4) Every 4-vertex gives  $\frac{1}{2}$  to each incident 4-face.
- (R5) Every  $k$ -vertex, for  $k \geq 5$ , gives 1 to each incident 4-face :  $(5^+, 3, 5^+, 3)$ .
- (R6) Every  $k$ -vertex, for  $k \geq 5$ , gives  $\frac{3}{4}$  to each incident 4-face :  $(5^+, 3, 5^+, 4)$ .
- (R7) Every  $k$ -vertex, for  $k \geq 5$ , gives  $\frac{2}{3}$  to each incident 4-face :  $(5^+, 5^+, 5^+, 3)$ .
- (R8) Every  $k$ -vertex, for  $k \geq 5$ , gives  $\frac{1}{2}$  to each incident 4-face :  $(5^+, 4^+, 4^+, 4^+)$ .

Since  $H$  does not contain a  $C_4$  adjacent to a  $C_3$ , by hypothesis, the following fact is easy to observe and will be frequently used throughout the proof without further notice.

**Observation 3.**  $H$  does not contain the following structures:

1. adjacent 3-cycles,
2. a 4-cycle adjacent to a 3-cycle.

One can easily derive the following observation.

**Observation 4.** Let  $v$  be a  $k$ -vertex with  $k \geq 3$  then  $m_3(v) \leq \lfloor \frac{k}{2} \rfloor$  and we have the following cases by Observation 3:

- If  $m_3(v) = \lfloor \frac{k}{2} \rfloor$  then  $m_4(v) = 0$ .
- If  $1 \leq m_3(v) < \lfloor \frac{k}{2} \rfloor$  then  $m_4(v) \leq d(v) - 2 \times m_3(v) - 1$ .
- If  $m_3(v) = 0$  then  $m_4(v) \leq d(v)$ .

Let  $v \in V(H)$  be a  $k$ -vertex. By Lemma 2.1 and Lemma 2.2,  $k \geq 3$ . Consider the following cases:

**Case  $k = 3$ .** Observe that  $\omega(v) = -1$ . By Lemma 2.3,  $v$  has three neighbors both of degree at least 5. Then, by (R1), we have:  $\omega^*(v) = -1 + 3 \times \frac{1}{3} = 0$ .

**Case  $k = 4$ .** Observe that  $\omega(v) = 2$ . By Lemma 2.3,  $v$  has four neighbors both of degree at least 4. By Observation 4, we have the following cases :

- If  $m_3(v) = 2$  then  $m_4(v) = 0$ . Hence, by (R2), we have:  $\omega^*(v) \geq 2 - 2 \times 1 = 0$ .
- If  $m_3(v) = 1$  then  $m_4(v) \leq 1$ . Hence, by (R2) and (R4), we have:  $\omega^*(v) \geq 2 - 1 \times 1 - 1 \times \frac{1}{2} > 0$ .
- If  $m_3(v) = 0$  then  $m_4(v) \leq 4$ . Hence, by (R4), we have:  $\omega^*(v) \geq 2 - 4 \times \frac{1}{2} = 0$ .

**Case  $k = 5$ .** Observe that  $\omega(v) = 5$ . By Lemma 2.5,  $v$  has at most 3 neighbors both of degree 3. By Observation 4, we have the following cases:

- If  $m_3(v) = 2$  then  $m_4(v) = 0$ . Hence, by (R1) and (R3), we have:  $\omega^*(v) \geq 5 - 2 \times 2 - 3 \times \frac{1}{3} = 0$ .
- If  $m_3(v) = 1$  then  $m_4(v) \leq 2$ . In the worst-case  $v$  is incident to two  $(5, 3, 5^+, 3)$ -faces Hence, by (R1), (R3) and (R5), we have:  $\omega^*(v) \geq 5 - 1 \times 2 - 2 \times 1 - 3 \times \frac{1}{3} = 0$ .
- If  $m_3(v) = 0$  then  $m_4(v) \leq 5$ . We have to consider several cases:

(a) If  $n_3(v) = 3$ , then we have three cases :

- (i)  $v$  is incident to two  $(5, 3, 5^+, 3)$ -faces, two  $(5, 3, 5^+, 4^+)$ -faces and one  $(5, 4^+, 4^+, 4^+)$ -face. Hence, by (R1), (R5), (R6) and (R8), we have:  $\omega^*(v) \geq 5 - 2 \times 1 - 2 \times \frac{3}{4} - 3 \times \frac{1}{3} - \frac{1}{2} = 0$ .
- (ii)  $v$  is incident to two  $(5, 3, 5^+, 3)$ -faces and three  $(5^+, 5^+, 5^+, 3)$ -faces. Hence, by (R1), (R5) and (R7), we have:  $\omega^*(v) \geq 5 - 1 \times 2 - 3 \times \frac{2}{3} - 3 \times \frac{1}{3} = 0$ .
- (iii)  $v$  is incident to one  $(5, 3, 5^+, 3)$ -face and four  $(5, 3, 5^+, 4^+)$ -faces. Hence, by (R1), (R5) and (R6), we have:  $\omega^*(v) \geq 5 - 1 \times 1 - 4 \times \frac{3}{4} - 3 \times \frac{1}{3} = 0$ .

(b) If  $n_3(v) = 2$ , then we have five cases:

- (i)  $v$  is incident to one  $(5, 3, 5^+, 3)$ -face, two  $(5, 3, 5^+, 4^+)$ -faces and two  $(5, 4^+, 4^+, 4^+)$ -faces. Hence, by (R1), (R5), (R6) and (R8), we have:  $\omega^*(v) \geq 5 - 1 \times 1 - 2 \times \frac{3}{4} - 2 \times \frac{1}{2} - 2 \times \frac{1}{3} = \frac{5}{6} \geq 0$ .
- (ii)  $v$  is incident to four  $(5, 3, 5^+, 4^+)$ -faces and one  $(5, 4^+, 4^+, 4^+)$ -face. Hence, by (R1), (R6) and (R8), we have:  $\omega^*(v) \geq 5 - 4 \times \frac{3}{4} - 1 \times \frac{1}{2} - 2 \times \frac{1}{3} = \frac{5}{6} \geq 0$ .



- (iii)  $v$  is incident to two  $(5, 3, 5^+, 4^+)$ -faces and three  $(5, 5^+, 5^+, 3)$ -faces. Hence, by (R1), (R6) and (R7), we have:  $\omega^*(v) \geq 5 - 2 \times \frac{3}{4} - 3 \times \frac{2}{3} - 2 \times \frac{1}{3} = \frac{5}{6} \geq 0$ .
  - (iv)  $v$  is incident to two  $(5, 3, 5^+, 5^+)$ -faces, one  $(5, 3, 5^+, 3)$ -face, one  $(5, 5^+, 4^+, 4^+)$ -face and  $(5, 3, 5^+, 4^+)$ -face. Hence, by (R1), (R5), (R6), (R7) and (R8), we have:  $\omega^*(v) \geq 5 - 2 \times \frac{1}{3} - 2 \times \frac{2}{3} - 1 \times 1 - 1 \times \frac{1}{2} - 1 \times \frac{3}{4} = \frac{3}{4} \geq 0$ .
  - (v)  $v$  is incident to four  $(5, 3, 5^+, 5^+)$ -faces, one  $(5, 3, 5^+, 3)$ -face. Hence, by (R1), (R5) and (R7), we have:  $\omega^*(v) \geq 5 - 2 \times \frac{1}{3} - 4 \times \frac{2}{3} - 1 \times 1 = \frac{2}{3} \geq 0$ .
- (c) If  $n_3(v) = 1$ , then we have four cases:
- (i)  $v$  is incident to five  $(5, 5^+, 5^+, 3)$ -faces. Hence by (R1) and (R7), we have:  $\omega^*(v) \geq 5 - 5 \times \frac{2}{3} - 1 \times \frac{1}{3} = \frac{4}{3} \geq 0$ .
  - (ii)  $v$  is incident to two  $(5, 3, 5^+, 4^+)$ -faces, two  $(5, 4^+, 5^+, 4)$  and one  $(5, 5^+, 5^+, 3)$ -face. Hence by (R1), (R6), (R7) and (R8), we have:  $\omega^*(v) \geq 5 - 2 \times \frac{3}{4} - 2 \times \frac{1}{2} - 1 \times \frac{2}{3} - 1 \times \frac{1}{3} = \frac{3}{2} \geq 0$ .
  - (iii)  $v$  is incident to three  $(5, 4^+, 4^+, 4^+)$ -faces and two  $(5, 3, 5^+, 4^+)$ -faces. Hence by (R1), (R6) and (R8), we have:  $\omega^*(v) \geq 5 - 2 \times \frac{3}{4} - 3 \times \frac{1}{2} - 1 \times \frac{1}{3} = \frac{5}{3} \geq 0$ .
  - (iv)  $v$  is incident to three  $(5, 5^+, 5^+, 3)$ -faces, one  $(5, 4^+, 4^+, 4^+)$ -face and one  $(5, 3, 5^+, 4)$ -face. Hence by (R1), (R6), (R7) and (R8), we have:  $\omega^*(v) \geq 5 - 3 \times \frac{2}{3} - 1 \times \frac{1}{2} - 1 \times \frac{3}{4} - 1 \times \frac{1}{3} = \frac{17}{12} \geq 0$ .
- (d) If  $n_3(v) = 0$ , in the worst-case  $v$  can be incident to five  $(5, 5^+, 5^+, 3)$ -faces. Then by (R7), we have:  $\omega^*(v) \geq 5 - 5 \times \frac{2}{3} = \frac{5}{3} \geq 0$ .

**Case  $k = 6$ .** Observe that  $\omega(v) = 8$ . By Observation 4, we have the following cases:

- If  $m_3(v) = 3$  then  $m_4(v) = 0$ . By Lemma 2.3 and Lemma 2.4,  $v$  is adjacent to at most three 3-vertices. Hence, by (R1) and (R3), we have:  $\omega^*(v) \geq 8 - 3 \times 2 - 3 \times \frac{1}{3} = 1 > 0$ .
- If  $m_3(v) = 2$  then  $m_4(v) \leq 1$ . By Lemma 2.3 and Lemma 2.4,  $v$  is adjacent to at most four 3-vertices and incident to at most one  $(5^+, 3, 5^+, 3)$ -face. Hence, by (R1), (R3) and (R5), we have:  $\omega^*(v) \geq 8 - 2 \times 2 - 1 \times 1 - 4 \times \frac{1}{3} = \frac{5}{3} > 0$ .
- If  $m_3(v) = 1$  then  $m_4(v) \leq 3$ . By Lemma 2.3 and Lemma 2.4,  $v$  is adjacent to at most five 3-vertices and incident to at most three  $(5^+, 3, 5^+, 3)$ -faces. Hence, by (R1), (R3) and (R5), we have:  $\omega^*(v) \geq 8 - 1 \times 2 - 3 \times 1 - 5 \times \frac{1}{3} = \frac{4}{3} > 0$ .
- If  $m_3(v) = 0$  then  $m_4(v) \leq 6$ .  $v$  can be incident to six  $(5^+, 3, 5^+, 3)$ -faces. Hence, by (R1) and (R5), we have:  $\omega^*(v) \geq 8 - 6 \times 1 - 6 \times \frac{1}{3} = 0$ .

**Case  $k = 7$ .** Observe that  $\omega(v) = 11$ . By Observation 4, we have the following cases:

- If  $m_3(v) = 3$  then  $m_4(v) = 0$ . By Lemma 2.3 and Lemma 2.4,  $v$  is adjacent to at most four 3-vertices. Hence, by (R1) and (R3), we have:  $\omega^*(v) \geq 11 - 3 \times 2 - 4 \times \frac{1}{3} = \frac{11}{3} > 0$ .
- If  $m_3(v) = 2$  then  $m_4(v) \leq 2$ . By Lemma 2.3 and Lemma 2.4,  $v$  is adjacent to at most five 3-vertices and incident to at most two  $(5^+, 3, 5^+, 3)$ -face. Hence, by (R1), (R3) and (R5), we have:  $\omega^*(v) \geq 11 - 2 \times 2 - 2 \times 1 - 5 \times \frac{1}{3} = \frac{10}{3} > 0$ .
- If  $m_3(v) = 1$  then  $m_4(v) \leq 4$ . By Lemma 2.3 and Lemma 2.4,  $v$  is adjacent to at most six 3-vertices and incident to at most four  $(5^+, 3, 5^+, 3)$ -faces. Hence, by (R1), (R3) and (R5), we have:  $\omega^*(v) \geq 11 - 1 \times 2 - 4 \times 1 - 6 \times \frac{1}{3} = 3 > 0$ .
- If  $m_3(v) = 0$  then  $m_4(v) \leq 7$ .  $v$  can be incident to seven  $(5^+, 3, 5^+, 3)$ -faces and seven 3-vertices. Hence, by (R1) and (R5), we have:  $\omega^*(v) \geq 11 - 7 \times 1 - 7 \times \frac{1}{3} = \frac{5}{3} > 0$ .

**Case  $k \geq 8$ .** Observe that  $\omega(v) = 3k - 10$ . By Observation 4, we have the following cases:

- If  $m_3(v) = \lfloor \frac{k}{2} \rfloor$  then  $m_4(v) = 0$ . Hence, by (R1) and (R3):

$$\begin{aligned}
\omega^*(v) &= 3k - 10 - 2 \times m_3(v) - \frac{1}{3} \times n_3(v) \\
&\geq 3k - 10 - 2 \times \left\lfloor \frac{k}{2} \right\rfloor - \frac{1}{3} \times k \\
&\geq \frac{5}{3}k - 10 > 0
\end{aligned}$$

- If  $1 \leq m_3(v) \leq \lfloor \frac{k}{2} \rfloor - 1$  then  $m_4(v) \leq k - 3$ . Hence, by (R1), (R3) and (R5), we have:

$$\begin{aligned}\omega^*(v) &= 3k - 10 - 2 \times m_3(v) - 1 \times m_4(v) - \frac{1}{3} \times n_3(v) \\ &\geq 3k - 10 - 2 \times (\lfloor \frac{k}{2} \rfloor - 1) - (k - 3) \times 1 - \frac{1}{3} \times k \\ &\geq \frac{2}{3}k - 5 > 0\end{aligned}$$

- If  $m_3(v) = 0$  then  $m_4(v) \leq k$ . Hence, by (R1) and (R5), we have:  $\omega^*(v) \geq 3k - 10 - k \times 1 - k \times \frac{1}{3} = \frac{5}{3}k - 10 > 0$ .

Let  $f \in F(H)$  be a  $k$ -face.

**Case  $k = 3$ .** Observe that  $\omega(f) = -4$ . Suppose  $f = [rst]$ . Consider the following cases:

- (a) Suppose  $d(r) = 3$ . Then, by Lemma 2.3,  $r$  is the unique 3-vertex and  $d(s) \geq 5$  and  $d(t) \geq 5$ . Hence, by (R3), we have:  $\omega^*(f) = -4 + 2 \times 2 = 0$
- (b) Suppose now  $d(r) \geq 4$ ,  $d(s) \geq 4$  and  $d(t) \geq 4$ . By Lemma 2.4, at least two of the three vertices  $r$ ,  $s$  and  $t$  is a  $5^+$ -vertex. Assume that  $d(s) \geq 5$  and  $d(t) \geq 5$ . Then, by (R2) and (R3), we have:  $\omega^*(f) \geq -4 + 2 \times 2 + 1 \times 1 = 1 \geq 0$ .

**Case  $k = 4$ .** The initial charge of  $f$  is  $\omega(f) = -2$ . By Lemma 2.3, at most two 3-vertices are incident to the 4-face. Suppose  $f = [rstu]$ . Consider the following cases:

- (a) Suppose  $d(r) = d(t) = 3$ . Then, by Lemma 2.3,  $d(s) \geq 5$  and  $d(u) \geq 5$ . Hence, by (R5), we have:  $\omega^*(f) = -2 + 2 \times 1 = 0$
- (b) Suppose now  $d(r) = 3$ . Then, by Lemma 2.3,  $d(s) \geq 5$  and  $d(u) \geq 5$ . Moreover, assume  $d(t) = 4$ . Then, by (R4) and (R6), we have:  $\omega^*(f) \geq -2 + 2 \times \frac{3}{4} + 1 \times \frac{1}{2} = 0$ . If  $d(t) \geq 5$ , by (R7) we have  $\omega^*(f) \geq -2 + 3 \times \frac{2}{3} = 0$ .
- (c) Assume  $d(r) \geq 4$ ,  $d(s) \geq 4$ ,  $d(t) \geq 4$  and  $d(u) \geq 4$ . Then, by (R4), we have:  $\omega^*(f) \geq -2 + 4 \times \frac{1}{2} = 0$ .

**Case  $k \geq 5$ .** The initial charge of  $f$  is  $\omega(f) = 2k - 10 \geq 0$  and it remains unchanged during the discharging process. Hence,  $\omega(v) = \omega^*(v) = 2k - 10 \geq 0$ .

After performing the discharging procedure the new weights of all faces and vertices are positive and therefore,  $H$  cannot exist. This completes the proof of Theorem 5.1.

### 3. Proof of Theorem 5.2

We will prove the following stronger version of Theorem 5.2 :

**Theorem 7.** *Every planar graph  $G$  without  $C_4$  and  $C_5$  admits a  $(k + 3, 3)$ -incidence coloring for every  $k \geq \Delta(G) \geq 5$ ,  $k \in \mathbb{N}$ . Therefore,  $\chi_i(G) \leq \Delta(G) + 3$ .*

#### 3.1. Structural properties

We proceed by contradiction. Let  $H$  be a counterexample to the theorem that minimizes  $|E(H)| + |V(H)|$ . By hypothesis there exists  $k \geq \max\{\Delta(G), 5\}$  such that  $H$  does not admit a  $(k + 3, 3)$ -incidence coloring. Let  $k \geq \max\{\Delta(G), 5\}$  be the smallest integer such that  $H$  does not admit a  $(k + 3, 3)$ -incidence coloring. By using Remark 1, we must have  $k = \max\{\Delta(G), 5\}$ . Moreover by minimality it is easy to see that  $H$  is connected.

$H$  satisfies the following properties:

**Lemma 3.**  *$H$  does not contain:*

1. 1-vertices,

2. 2-vertices,
3. a 3-vertex adjacent to a 3-vertex,
4. a (3, 4, 4)-triangle.

**Proof**

First, we will suppose by contradiction that the described configuration exists in  $H$ . Then we consider a graph  $H'$  obtained from  $H$  by deleting an edge or a vertex from  $H$ . The graph  $H'$  does not contain a  $C_4$  neither a  $C_3$ . Due to the minimality of  $H$ , the graph  $H'$  admits a  $(k' + 3, 3)$ -incidence coloring for any  $k' \geq \max\{\Delta(H'), 5\}$ . Since  $\Delta(H) \geq \Delta(H')$ , the set of integers  $k'$  contains the set of integers  $k$ . Hence for the value  $k' = k$ ,  $H'$  admits a  $(k + 3, 3)$ -incidence coloring  $\phi'$ . Finally, for each cases, we will prove a contradiction by extending  $\phi'$  to a  $(k + 3, 3)$ -incidence coloring  $\phi$  of  $H$ .

1. Lemma 3.1 can be easily checked by using the proof of Lemma 2.1.
2. Lemma 3.2 can be easily checked by using the proof of Lemma 2.2.
3. Suppose  $H$  contains a 3-vertex  $u$  adjacent to a 3-vertex  $v$ . By minimality of  $H$ , the graph  $H' = H \setminus \{uv\}$  has a  $(k + 3, 3)$ -incidence coloring  $\phi'$ . We recall that we have at least 8-colors and  $|\phi'(A_u)| \leq 2$  and  $|\phi'(A_v)| \leq 2$ . Note that

$$|F_H^{\phi'}(u, uv)| = |\phi'(I_u) \cup \phi'(A_u) \cup \phi'(I_v)| \leq 2 + 2 + 2 = 6$$

We have at least  $k - 3 \geq 2$  free colors for  $(u, uv)$ . Choose a color for  $(u, uv)$ , then for  $(v, vu)$  by using the same calculation at most 7 colors are forbidden for this incidence. We have at least  $k - 4 \geq 1$  free colors. So we can extend the coloring, a contradiction.

4. Suppose that  $H$  contains a (3, 4, 4)-triangle. Let  $u$  be the vertex of degree equal to 3 and  $v, w$  the two vertices of degree equal to 4. Let  $t$  be the neighbor of  $u$  different from  $v$  and  $w$ . By minimality of  $H$ , the graph  $H' = H - \{uv, uw\}$  admits a  $(k + 3, 3)$ -incidence coloring. We will extend  $\phi'$  to a  $(k + 3, 3)$ -incidence coloring  $\phi$  of  $H$  as follows. We recall that we have at least 8-colors. We have:

- $|F_H^{\phi'}(v, vu)| = |\phi'(I_v) \cup \phi'(A_v) \cup \phi'(I_u)| \leq 3 + 3 + 1 = 7$ . Hence we have one available color for  $(v, vu)$ .
- In the same way, we have one available for  $(w, wu)$ .
- We have 3 available colors for  $(u, uv)$  belonging to  $\phi'(A_v)$  and 3 available colors for  $(u, uw)$  belonging to  $\phi'(A_w)$ .

Without loss of generality, we assume that we are in the situation described in Figure 2.

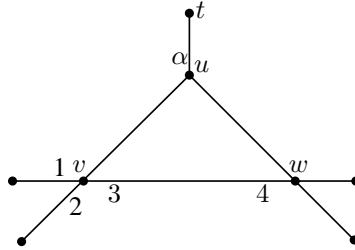


Figure 2:  $|F_H^{\phi'}(u, uv)| \leq 7$

First, if  $\phi'(u, ut) = 3$  or  $\phi'(u, ut) = 4$ , then we recolor  $(u, ut)$  with a color  $\alpha$  different from 3 and 4 (it is possible because we have 3 available colors for  $(u, ut)$  belonging to  $\phi'(A_t)$ ).

Now, we color  $(v, vu)$  with the available color (say  $a$ ) and  $(w, wu)$  with one available color (say  $b$ ). We consider two cases.

- Assume that we can color  $(u, uv)$  with 4. If we can color  $(u, uv)$  with 3, we are done. If we cannot color  $(u, uv)$  with 3. It means that  $\phi'(t, tu) = 3$  (note that  $\alpha \neq 3$ ). If we cannot color  $(u, uv)$  with an other color of  $\phi'(A_w)$  we have to  $\phi'(A_w) = \{3, a, \alpha\}$ . Then we permute 3 and  $a$  in  $\phi'(I_v)$  and we color  $(u, uv)$  with a color  $a$ , we are done.
- Assume that we cannot color  $(u, uv)$  with 4, it means that  $\phi'(t, tu) = 4$  (note that  $\alpha \neq 4$ ). If we cannot color  $(u, uv)$  with an other color of  $\phi'(A_v)$  (if it is the case then we can color  $(u, uv)$  with 3, we are done), it means that  $\phi'(A_v) = \{4, b, \alpha\}$ . Then we permute 4 and  $b$  in  $\phi'(I_w)$  and we color  $(u, uv)$  with a color  $b$  and  $(u, uv)$  with color 3, we are done.

We have extended the coloring for all the cases, a contradiction.

□

### 3.2. Discharging procedure

Euler's formula  $|V(H)| - |E(H)| + |F(H)| = 2$  can be rewritten as  $(4|E(H)| - 6|V(H)|) + (2|E(H)| - 6|F(H)|) = -12$ . Using the relation  $\sum_{v \in V(H)} d(v) = \sum_{f \in F(H)} r(f) = 2|E(H)|$  we get that:

$$\sum_{v \in V(H)} (2d(v) - 6) + \sum_{f \in F(H)} (r(f) - 6) = -12 \quad (2)$$

We define the weight function  $\omega : V(H) \cup F(H) \rightarrow \mathbb{R}$  by  $\omega(x) = 2d(x) - 6$  if  $x \in V(H)$  and  $\omega(x) = r(x) - 6$  if  $x \in F(H)$ . It follows from Equation (2) that the total sum of weights is equal to -12. In what follows, we will define discharging rules (R1) and (R2). Next we redistribute weights accordingly. Once the discharging is finished, a new weight function  $\omega^*$  is produced. However, the total sum of weights is kept fixed when the discharging is finished. Nevertheless, we will show that  $\omega^*(x) \geq 0$  for all  $x \in V(H) \cup F(H)$ . This will lead us to the following contradiction:

$$0 \leq \sum_{x \in V(H) \cup F(H)} \omega^*(x) = \sum_{x \in V(H) \cup F(H)} \omega(x) = -12 < 0$$

and hence will demonstrate that such a counterexample cannot exist.

The discharging rules are defined as follows:

(R1) Every 4-vertex, gives 1 to each incident 3-face.

(R2) Every  $k$ -vertex, for  $k \geq 5$ , gives 2 to each incident 3-face.

Let  $v \in V(H)$  be a  $k$ -vertex.

By Lemma 3.1 and Lemma 3.2,  $k \geq 3$ . We recall that since  $H$  does not contain  $C_4$ , there are no adjacent 3-faces. Consider the following cases:

**Case  $k = 3$ .** Observe that  $\omega(v) = 0$ .  $v$  does not give anything and does not get anything. We have  $\omega^*(v) = \omega(v) = 0$ .

**Case  $k = 4$ .**  $\omega(v) = 2$ . It is easy to see that  $v$  is incident to at most two 3-faces. Then, by (R1) we have  $\omega^*(v) \geq 2 - 2 \times 1 = 0$ .

**Case  $k \geq 5$ .** Observe that  $\omega(v) = 2k - 6$ . It is easy to see that  $m_3(v) \leq \lfloor \frac{k}{2} \rfloor$ . Hence, by (R2), we have:  $\omega^*(v) \geq 2k - 6 - 2 \times \lfloor \frac{k}{2} \rfloor \geq 0$ .

Let  $f \in F(H)$  be a  $k$ -face.

**Case  $k = 3$ .** Observe that  $\omega(f) = -3$ . Suppose  $f = [rst]$ . Consider the following cases:

- Suppose  $d(r) = 3$ . Then, by Lemma 3.3,  $r$  is the unique 3-vertex and by Lemma 3.4,  $d(s) \geq 4$  and  $d(t) \geq 5$ . Hence, by (R1) and (R2), we have  $\omega^*(f) \geq -3 + 1 \times 1 + 1 \times 2 = 0$

(b) Suppose now  $d(r) \geq 4$ ,  $d(s) \geq 4$  and  $d(t) \geq 4$ . Then, by (R1) and (R2),  $\omega^*(f) \geq -3 + 3 \times 1 = 0$ .

**Case  $k \geq 6$ .** The face is not involved in the discharging procedure.  $\omega(f) = \omega^*(f) \geq 0$ .

After performing the discharging procedure the new weights of all faces and vertices are positive and therefore,  $H$  cannot exist. This completes the proof of Theorem 5.2.

**Remark 2.** By Theorem 1, the bound of Theorem 5.2 is true for  $\Delta(G) \leq 3$ . There is a challenging question: prove Theorem 5.2 when  $\Delta(G) = 4$ .

- [1] I. Algor, N. Alon. The star arboricity of graphs, *Discrete Mathematics*, 75 : 11–22, 1989.
- [2] M. Bonamy, B. Lévêque and A. Pinlou. 2-Distance Coloring of Sparse Graphs, *Journal of Graph Theory*, DOI: 10.1002/jgt.21782, 2014.
- [3] R.A. Brualdi and J.J.Q. Massey. Incidence and strong edge colorings of graphs, *Discrete Mathematics*, 122: 51–58, 1993.
- [4] M.H. Dolama, E. Sopena and X. Zhu. Incidence coloring of  $k$ -degenerated graphs, *Discrete Mathematics*, 283: 121–128, 2004.
- [5] M.H. Dolama and E. Sopena. On the maximum average degree and the incidence chromatic number of a graph, *Discrete Mathematics and Theoretical Computer Science*, 7: 203–216, 2005.
- [6] B. Guiduli. On incidence coloring and star arboricity of graphs, *Discrete Mathematics*, 163: 275–278, 1997.
- [7] M. Maydanskiy. The incidence coloring conjecture for graphs of maximum degree 3, *Discrete Mathematics*, 292: 131–141, 2005.
- [8] X. Menga, J. Guo and B. Su. Incidence coloring of pseudo-Halin graphs, *Discrete Mathematics*, 312: 3276–3282, 2012.
- [9] K. Nakprasit and K. Nakprasit. Incidence coloring of the powers of cycles, *International Journal of Pure and Applied Mathematics*, 76: 143–148, 2012.
- [10] W.C. Shiu, P.C.B Lam and D.L Chen . On incidence coloring for some cubic graphs, *Discrete Mathematics*, 252: 259–266, 2002.
- [11] P.K Sun, Incidence coloring of regular graphs and complement graphs, *Taiwanese Journal of Mathematics*, 16: 2289–2295, 2012.
- [12] S.D. Wang, D.L Chen and S.C Pang. The incidence coloring number of Halin graphs and outerplanar graphs, *Discrete Mathematics*, 252: 259–266, 2002
- [13] S. Wang, J. Xu, F. Ma and C. Xu. The  $(\Delta + 2, 2)$ -incidence coloring of outerplanar graphs, *Progress in Natural Science*, 18 : 575–578, 2008.
- [14] J. Wu. Some results on the incidence coloring number of a graph, *Discrete Mathematics*, 309: 3866–3870, 2009.
- [15] D. Yang. Fractional incidence coloring and star arboricity of graphs. *Ars Combin.* 105:213-224 (2012)