

On strong edge-colouring of subcubic graphs

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Abstract

A strong edge-colouring of a graph G is a proper edge-colouring such that every path of length 3 uses three different colours. In this paper we improve some previous results on the strong edge-colouring of subcubic graphs by showing that every subcubic graph with maximum average degree strictly less than $\frac{7}{3}$ (resp. $\frac{5}{2}$, $\frac{8}{3}$, $\frac{20}{7}$) can be strongly edge-coloured with six (resp. seven, eight, nine) colours. These upper bounds are optimal except the one of $\frac{8}{3}$. Also, we prove that every subcubic planar graph without 4-cycles and 5-cycles can be strongly edge-coloured with nine colours.

Key words: Strong edge-colouring, subcubic graphs, planar graphs, maximum average degree

1. Introduction

In this paper the graphs considered are finite, simple and without loops. A proper edge-colouring of a graph $G = (V, E)$ is an assignment of colours to the edges of the graph such that two adjacent edges do not use the same colour. A *strong edge-colouring* (called also distance 2 edge-colouring) of a graph G is a proper edge-colouring of G , such that the edges of any path of length 3 use three different colours. We denote by $\chi'_s(G)$ the *strong chromatic index* of G which is the smallest integer k such that G can be strongly edge-coloured with k colours.

Strong edge-colouring was introduced by Fouquet and Jolivet in 1983 [8, 9]. Strong edge-colouring can be used to model the conflict-free channel assignment in radio networks [2, 13–15].

Let $\Delta(G)$ be the maximum degree of a graph G (we will use Δ if no ambiguity). The following conjecture was posed by Erdős and Nešetřil [5, 6] and revised by Faudree *et al.* [7] and Chung *et al.* [3]:

Conjecture 1 (Erdős and Nešetřil [5, 6]). *For every graph G ,*

$$\chi'_s(G) \leq \begin{cases} \frac{5}{4}\Delta^2, & \text{if } \Delta \text{ is even;} \\ \frac{1}{4}(5\Delta^2 - 2\Delta + 1), & \text{if } \Delta \text{ is odd.} \end{cases}$$

If this conjecture is true, then the given upper bounds for the strong chromatic index are tight as the authors gave constructions of graphs with strong chromatic index reaching these bounds. The conjecture was verified for graphs having $\Delta \leq 3$ [1, 12]. When $\Delta > 3$, the only case on which some progress was made is when $\Delta = 4$ and the best upper bound stated is $\chi'_s(G) \leq 22$ [4].

An upper bound for the strong chromatic index of subcubic graphs in terms of the maximum average degree $\text{mad}(G) = \max \left\{ \frac{2|E(H)|}{|V(H)|}, H \subseteq G \right\}$, was given in [11]. More precisely, it was proved the following.

Theorem 1 (Hocquard and Valicov [11]). *Let G be a subcubic graph (a graph with $\Delta \leq 3$).*

1. *If $\text{mad}(G) < \frac{15}{7}$, then $\chi'_s(G) \leq 6$.*
2. *If $\text{mad}(G) < \frac{27}{11}$, then $\chi'_s(G) \leq 7$.*
3. *If $\text{mad}(G) < \frac{13}{5}$, then $\chi'_s(G) \leq 8$.*

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4. If $\text{mad}(G) < \frac{36}{13}$, then $\chi'_s(G) \leq 9$.

Recall that the girth of a graph is the length of a shortest cycle in this graph. As every planar graph with girth g satisfies $\text{mad}(G) < \frac{2g}{g-2}$, the following corollary, can be easily derived from Theorem 1:

Corollary 1 (Hocquard and Valicov [11]). *Let G be a planar subcubic graph with girth g .*

1. If $g \geq 30$, then $\chi'_s(G) \leq 6$.
2. If $g \geq 11$, then $\chi'_s(G) \leq 7$.
3. If $g \geq 9$, then $\chi'_s(G) \leq 8$.
4. If $g \geq 8$, then $\chi'_s(G) \leq 9$.

In this paper, we strengthen Theorem 1 by proving that:

Theorem 2. *Let G be a subcubic graph.*

1. If $\text{mad}(G) < \frac{7}{3}$, then $\chi'_s(G) \leq 6$.
2. If $\text{mad}(G) < \frac{5}{2}$, then $\chi'_s(G) \leq 7$.
3. If $\text{mad}(G) < \frac{8}{3}$, then $\chi'_s(G) \leq 8$.
4. If $\text{mad}(G) < \frac{20}{7}$, then $\chi'_s(G) \leq 9$.

For cases 1, 2 and 4, the given upper bounds on the maximum average degree are sharp: there exist graphs with $\text{mad}(G) = \frac{7}{3}$ (resp. $\frac{5}{2}$, $\frac{20}{7}$) which are not strong edge-colourable with six (resp. 7, 9) colours. Examples of such graphs are given in Section 2.2.

For planar graphs it follows:

Corollary 2. *Let G be a planar subcubic graph with girth g :*

1. If $g \geq 14$, then $\chi'_s(G) \leq 6$.
2. If $g \geq 10$, then $\chi'_s(G) \leq 7$.
3. If $g \geq 8$, then $\chi'_s(G) \leq 8$.
4. If $g \geq 7$, then $\chi'_s(G) \leq 9$.

In this paper we are also interested in finding a bound for the strong chromatic index of planar subcubic graphs. The interest for this class of graphs is motivated by the following conjecture:

Conjecture 2 (Faudree *et al.* [7]). *If G is a planar subcubic graph, then $\chi'_s(G) \leq 9$.*

If the conjecture is true, then it is the best possible bound since the prism P has $\chi'_s(P) = 9$ (see Figure 1).

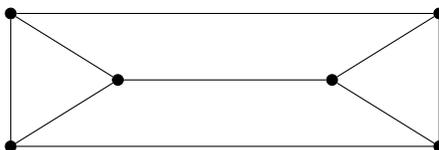


Figure 1: The prism P has $\chi'_s(P) = 9$

An interesting fact about Conjecture 2 is that from the algorithmic aspect the problem of computing the strong chromatic index was proved to be NP-complete for the class of subcubic planar graphs with an arbitrarily large girth [10].

For general planar graphs, an upper bound in terms of Δ was proved by Faudree *et al.* [7]:

Theorem 3 (Faudree *et al.* [7]). *If G is a planar graph, then $\chi'_s(G) \leq 4\Delta + 4$ for $\Delta \geq 3$.*

In case of planar graphs we improve the fourth part of Corollary 2 and give a partial answer to Conjecture 2, by showing the following:

Theorem 4. *Let G be a planar subcubic graph containing neither induced 4-cycles, nor induced 5-cycles. Then $\chi'_s(G) \leq 9$.*

The paper is organized as follows. In Section 2 we prove Theorem 2 and discuss the optimality of the upper bounds on the maximum average degree. In Section 3 we provide the proof of Theorem 4.

Let us introduce some notations.

Definitions and notations. Two edges are at *distance 1* if they share one of their ends and they are at *distance 2* if they are not at distance 1 and there exists an edge adjacent to both of them. Let $d_G(v)$ (or $d(v)$ if it is clear from the context) denote the degree of a vertex v in a graph G . A vertex of degree k (resp. at most k) is called a k -*vertex* (resp. k^- -*vertex*). A 3_k -*vertex* is a 3-vertex adjacent to exactly k 2-vertices. A *bad 2-vertex* is a 2-vertex adjacent to another 2-vertex and a *good 2-vertex* is a 2-vertex adjacent to two 3-vertices. We define $N_2[uv]$ as the set of edges at distance at most 2 from the edge uv and $N_2(uv) = N_2[uv] - uv$. We denote by $SC(N_2(uv))$ ($SC(N_2[uv])$ respectively) the set of colours used by edges in $N_2(uv)$ ($N_2[uv]$ respectively). Denote by $N(v)$ the neighbourhood of the vertex v , *i.e.* the set of its adjacent vertices. Finally, we use $\llbracket n \rrbracket$ to denote the set of integers $\{1, 2, \dots, n\}$.

2. Bounds using maximum average degree

2.1. Proof of Theorem 2

The proof is done by induction. Let H be a minimum counterexample. In each of the cases, first of all we prove the non-existence of some set of subgraphs \mathcal{S} in H . In the next step, we use the discharging technique in order to obtain a contradiction. For this, we define a weight function $\omega : V(H) \rightarrow \mathbb{R}$ with $\omega(x) = d(x) - m$ (where $m \in \mathbb{R}$ is the value of the upper bound on the maximum average degree given by Theorem 2). An important observation is that by hypothesis on the maximum average degree, the total sum of weights must be strictly negative. Next, we define discharging rules to redistribute weights and once the discharging process is finished, a new weight function ω^* will be produced. During the discharging process the total sum of weights is kept fixed. Nevertheless, by the non-existence of \mathcal{S} , it will follow that $\omega^*(x) \geq 0$ for all $x \in V(H)$. This will lead to the following contradiction:

$$0 \leq \sum_{x \in V(H)} \omega^*(x) = \sum_{x \in V(H)} \omega(x) < 0$$

Therefore, such a counterexample cannot exist.

2.1.1. First part

Let H be a counterexample to Theorem 2.1 minimizing $|E(H)| + |V(H)|$: H is not strong edge-colourable with six colours, $\text{mad}(H) < \frac{7}{3}$ and for any edge e , $\chi'_s(H - e) \leq 6$. One can assume that H is connected; otherwise, by minimality of H , we can colour each connected component independently. In this subsection, a 3-vertex adjacent to a 1-vertex is a *light 3-vertex*. Otherwise it is a *heavy 3-vertex*.

In order to prove our result we need the following claim proved in [11]:

Claim 1. (*Hocquard and Valicov [11]*) *The minimal counterexample H to Theorem 2.1 satisfies the following properties:*

1. H does not contain a 1-vertex adjacent to a 2-vertex.
2. H does not contain a 3-vertex adjacent to a 1-vertex and a 2-vertex.
3. H does not contain a 3-vertex adjacent to two 1-vertices.

4. H does not contain a path uvw where u, v and w are 2-vertices.
5. H does not contain a path uvw where u, v and w are three light 3-vertices.

Claim 2. *The minimal counterexample H to Theorem 2.1 satisfies the following properties:*

1. H does not contain a triangle xyz , where x is a light 3-vertex.
2. H does not contain a path $stuvw$ where s, t, v and w are four light 3-vertices, u is a 3-vertex adjacent to a light 3-vertex x .

Proof

Suppose H contains a triangle xyz , where x is a light 3-vertex and let x_1 be the 1-vertex neighbour of x . By minimality of H , the graph $H - xx_1$ can be strongly edge-coloured with at most six colours. Since $|N_2(xx_1)| \leq 5$, every colouring of $H - xx_1$ using the minimum number of colours can be extended to H .

We prove the second item.

Suppose H contains a path $stuvw$ where s, t, v and w are four light 3-vertices, u is a 3-vertex adjacent to a light 3-vertex x distinct from s, t, v, w . Call s_1 (resp. t_1, v_1, w_1, x_1) the neighbour of s (resp. t, v, w, x) of degree 1. Let r (resp. y, z) be the third neighbour of s (resp. x, w). Also, for $i = 1, 2$, let r_i (resp. y_i, z_i) be the neighbours of r (resp. y, z), other than s (resp. x, w). By Claims 1.2 and 1.3, r, y and z are 3-vertices. By Claims 1.5 and 2.1, we can assume that Figure 2 illustrates the given configuration (with r, y, z possibly not distinct).

Let us consider $H' = H \setminus \{ss_1, tt_1, vv_1, ww_1, xx_1, st, tu, uv, vw, ux\}$. By minimality of H , there exists a strong edge-colouring ϕ of H' , using six colours. We show how to extend this colouring to H .

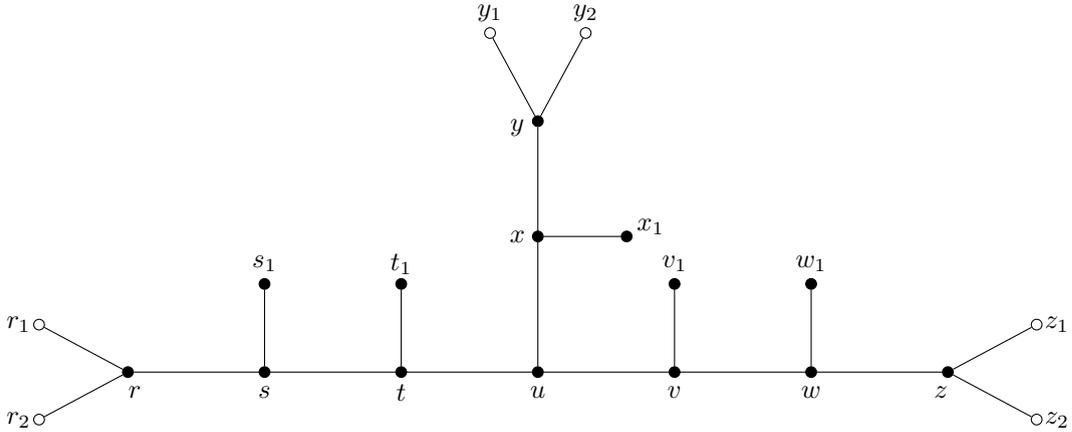


Figure 2: The configuration of Claim 2.2

Without loss of generality we can suppose that $\phi(xy) = 1$, $\phi(yy_1) = 2$ and $\phi(yy_2) = 3$. First, we colour edge ux and we distinguish two cases:

1. Suppose that rs is not coloured in $\{1, 2, 3\}$, say $\phi(rs) = 4$. Colour uv with a colour in $\{\phi(yy_1), \phi(yy_2)\}$ that does not appear on wz . Finally, we consider the remaining edges in the following order: $vw, ww_1, vv_1, tu, st, ss_1, tt_1$ and xx_1 . At each step, there exists an available colour for the corresponding edge. Similarly, we get the result if wz is not coloured in $\{1, 2, 3\}$.
2. Suppose now that $\phi(rs), \phi(wz) \in \{1, 2, 3\}$. Then it is easy to observe that there exists a colour, say 4, such that ux and ss_1 , can be coloured with 4. We set $\phi(ux) = \phi(ss_1) = 4$.
Next, we distinguish the following cases for $\phi(rs)$ and $\phi(wz)$:

- Suppose $\phi(rs) \in \{\phi(yy_1), \phi(yy_2)\} = \{2, 3\}$ and $\phi(wz) \neq \phi(rs)$. Without loss of generality we can suppose that $\phi(rs) = \phi(yy_1) = 2$. Then we assign $\phi(uv) = 2$. We colour the remaining edges in the following order: $vw, ww_1, vv_1, tu, st, tt_1$ and xx_1 . Note that at each step, there exists a colour left for the corresponding edge.
- Suppose $\phi(rs) \in \{\phi(yy_1), \phi(yy_2)\} = \{2, 3\}$ and $\phi(wz) = \phi(rs)$. Without loss of generality we can suppose that $\phi(wz) = \phi(rs) = 2$ and we assign $\phi(uv) = 3$.
Suppose we can assign $\phi(ww_1) = 4$. Next, we colour first st and then tu (note that at each step there is at least one colour left). If there is a colour left for vw , then the colouring of H can be finished easily as edges tt_1, vv_1 and xx_1 are pairwise at distance 3 and for each of these edges there would be a colour left. Therefore, there is no colour left for vw which implies that $\{\phi(tu), \phi(zz_1), \phi(zz_2)\} = \{1, 5, 6\}$ and since tu cannot be coloured 1, without loss of generality we can assume $\phi(tu) = 5, \phi(zz_1) = 1$ and $\phi(zz_2) = 6$. Similarly, by uncolouring st , recolouring tu with 6 and assigning to vw colour 5, we conclude that $\{\phi(rr_1), \phi(rr_2)\} = \{1, 5\}$. But then we do the following reassignment of colours $\phi(ss_1) = 6, \phi(st) = 4, \phi(tu) = 5, \phi(ux) = 6, \phi(vw) = 4, \phi(ww_1) = 5$. Finally there is a colour left for each of the edges tt_1, vv_1 and xx_1 , thus we are done.
Suppose ww_1 cannot be coloured with 4. Therefore, without loss of generality $\phi(zz_1) = 4$. On the other hand, recall that $\phi(ss_1) = \phi(ux) = 4$ and by previous paragraphs, it is not possible to colour ss_1, ux and ww_1 with the same colour (4, 5 or 6). Hence we must have $\{5, 6\} \subseteq \{\phi(zz_2), \phi(rr_1), \phi(rr_2)\}$. Obviously one of the colours 5 or 6, say 5, is not assigned to zz_2 and is assigned to either rr_1 or rr_2 (we can suppose $\phi(rr_1) = 5$). Hence we can assign $\phi(ww_1) = \phi(tu) = 5$ and we colour the remaining edges in the following order: vw, vv_1, st, tt_1, xx_1 . Observe that at each step there exists at least one colour left for every edge.
- Suppose $\{\phi(rs), \phi(wz)\} \cap \{\phi(yy_1), \phi(yy_2)\} = \emptyset$. Hence, $\phi(rs) = \phi(wz) = 1$. We assign $\phi(tu) = 2$ and then we colour the following edges in the given order: st, vw, uv, vv_1, tt_1 and xx_1 . It remains to colour ww_1 . If we have a colour left for ww_1 , then we are done. Otherwise, $\{\phi(uv), \phi(vv_1), \phi(vw), \phi(zz_1), \phi(zz_2)\} = \{2, 3, 4, 5, 6\}$. Therefore, $\{\phi(uv), \phi(vv_1), \phi(vw)\} = \{3, 5, 6\}$ and $\{\phi(zz_1), \phi(zz_2)\} = \{2, 4\}$. But then we permute the colours of tu and uv and we obtain a free colour for ww_1 (which is the same as the colour of tu), a contradiction.

□

As a corollary from the proof of Claim 2 we derive the following:

Corollary 3. *The minimal counterexample H to Theorem 2.1 does not contain a path $stuvw$ where s, t, v and w are either light 3-vertices or 2-vertices and u is a 3-vertex adjacent to a vertex x which is either a light 3-vertex x or a 2-vertex.*

Let H'' be the graph obtained from H by removing all 1-vertices of H , i.e. $H'' = H \setminus \{v \in V(H), d_H(v) = 1\}$. Clearly, H'' is connected and $\text{mad}(H'') < \frac{7}{3}$.

One can derive the following structural properties of H'' :

Claim 3. *Due to Claim 1 and to Corollary 3, H'' has the following properties:*

1. $\delta(H'') \geq 2$, where $\delta(H'')$ is the minimum degree of H'' (from Claim 1.1 and 1.3).
2. H'' does not contain a path uvw where u, v, w are 2-vertices (from Claims 1.2, 1.4 and 1.5).
3. H'' does not contain a 3₃-vertex adjacent to two bad 2-vertices (from Corollary 3).

For each vertex x of H'' , we assign a charge $w(x)$ equal to $d(x) - \frac{7}{3}$. We apply now a discharging procedure on H'' with the following rules:

- (R1) Every 3-vertex gives $\frac{1}{3}$ to each adjacent bad 2-vertex.
- (R2) Every 3-vertex gives $\frac{1}{6}$ to each adjacent good 2-vertex.

Let $v \in V(H'')$ be a k -vertex. By Claim 3.1, $k \geq 2$. Consider the following cases:

Case $k = 2$. Observe that $\omega(v) = -\frac{1}{3}$. Suppose v is a bad 2-vertex. By Claim 3.2, v is adjacent to a 3-vertex. Hence, by (R1), $\omega^*(v) = -\frac{1}{3} + \frac{1}{3} = 0$. If v is a good 2-vertex, then $\omega^*(v) = -\frac{1}{3} + 2 \times \frac{1}{6} = 0$ by (R2).

Case $k = 3$. Observe that $\omega(v) = \frac{2}{3}$. Suppose v is adjacent to a bad 2-vertex. By Claim 3.3, v is not adjacent to another bad 2-vertex. Hence, by (R1) and (R2), $\omega^*(v) \geq \frac{2}{3} - 1 \times \frac{1}{3} - 2 \times \frac{1}{6} = 0$. If v is not adjacent to a bad 2-vertex, then $\omega^*(v) \geq \frac{2}{3} - 3 \times \frac{1}{6} > 0$ by (R2).

Therefore, H'' cannot exist and consequently H does not exist neither. This completes the proof.

2.1.2. Second part

Let H be a counterexample to Theorem 2.2 minimizing $|E(H)| + |V(H)|$: H is not strong edge-colourable with seven colours, $\text{mad}(H) < \frac{5}{2}$, and for any edge e , $\chi'_s(H - e) \leq 7$. Recall that $\omega(x) = d(x) - \frac{5}{2}$. One can assume that H is connected; otherwise, by minimality of H , we can colour each connected component independently. In this subsection a 3-vertex adjacent to a bad 2-vertex is a *light 3-vertex*. Otherwise it is a *heavy 3-vertex*.

Claim 4. *The minimal counterexample H to Theorem 2.2 satisfies the following properties:*

1. H does not contain 1^- -vertices.
2. H does not contain a path uvw where u , v and w are 2-vertices.
3. H does not contain a 3-vertex adjacent to two 2-vertices one of them being bad.
4. H does not contain two 3_3 -vertices having a 2-vertex as a common neighbour.
5. H does not contain a 3_3 -vertex u with one of the neighbours, say v , adjacent to a 3_2 -vertex w having as neighbours a 2-vertex w_1 and a 3-vertex w_2 , such that:
 - (a) either w_1 is adjacent to a 3_3 -vertex.
 - (b) or w_2 is a 3_2 -vertex.
 - (c) or w_2 is a light 3-vertex.

Proof

Claims 4.1 to 4.4 are proved in [11]. We now prove Claim 4.5.

Suppose H contains a path uvw where u is a 3_3 -vertex, v is a 2-vertex, w is a 3_2 -vertex and w is adjacent to a 2-vertex w_1 (distinct from v) and to a 3-vertex w_2 . Let $H' = H \setminus \{u_1u, u_2u, uv, vw\}$. By minimality of H , $\chi'_s(H') \leq 7$ and there exists a strong edge-colouring ϕ of H' , which uses seven colours. We will extend this colouring to H . We colour the edges vw , uv and u_2u in this order. Note that at each step there exists at least one colour left for the corresponding edge. In order to complete the strong edge-colouring of H , we must assign a colour to u_1u . If $|\llbracket 7 \rrbracket \setminus SC(N_2(u_1u))| \geq 1$, then we are done. Hence $|\llbracket 7 \rrbracket \setminus SC(N_2(u_1u))| = 0$ and since $|N_2(u_1u)| = 7$, the edges of $N_2(u_1u)$ must be assigned distinct colours. Next, observe that it is possible to colour u_1u with the colour of u_2u , uncolour u_2u and then apply the same argument as previously to show that all the edges of $N_2(u_2u)$ must be assigned distinct colours. And then similarly, it is possible to colour u_1u with the colour of uv , to uncolour uv and if there is no colour left for uv , then all the colours of $N_2(uv)$ must be distinct. We conclude that $w_1 \neq u_1, u_2$ and $w_2 \neq x, y, x_1, x_2, y_1, y_2$. Hence the configuration and its fixed precolouring of edges is as depicted in Figure 3.

5.a Suppose w_1 is adjacent to a 3_3 -vertex t as in Figure 4.

Let us consider the edge ww_1 . Observe that $SC(N_2(ww_1))$ contains the colours 1, 4 and 5. Otherwise, we can recolour ww_1 with 1 (or 4, or 5), vw with 6 and u_1u with 3. This extends the colouring to whole H , a contradiction.

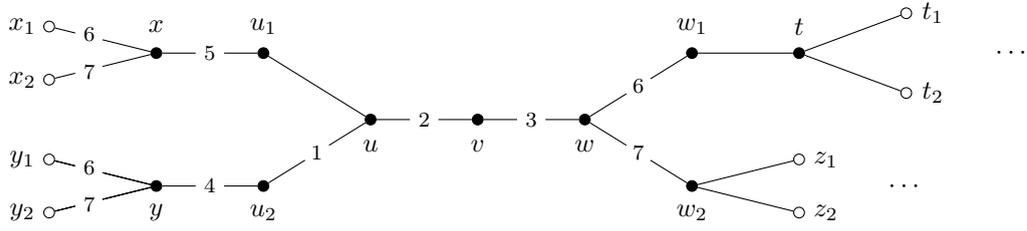


Figure 3: A fixed precolouring of the configuration of Claim 4.5

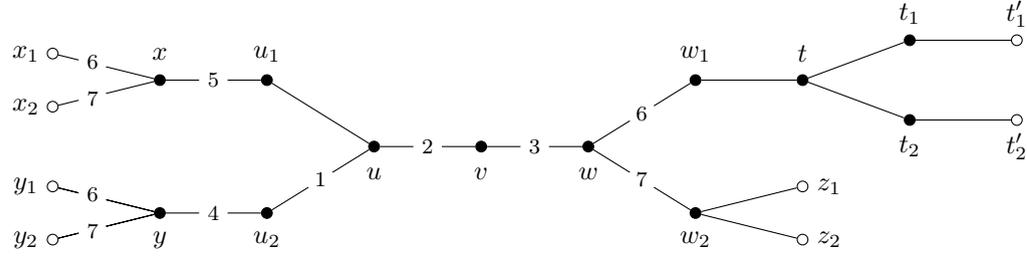


Figure 4: The configuration of Claim 4.5.a

Observe that $3 \in \{\phi(tt_1), \phi(tt_2)\}$. Otherwise, we can permute the colours of vw and ww_1 , and assign colour 3 to u_1u . Similarly, observe that $2 \in \{\phi(w_1t), \phi(tt_1), \phi(tt_2), \phi(w_2z_1), \phi(w_2z_2)\}$. Otherwise, we can permute the colours of ww_1 and uv , and colour u_1u with 2.

From the above, we conclude that $\{\phi(w_1t), \phi(tt_1), \phi(tt_2), \phi(w_2z_1), \phi(w_2z_2)\} = \{1, 2, 3, 4, 5\}$ and $3 \in \{\phi(tt_1), \phi(tt_2)\}$ ($w_1t, tt_1, tt_2, w_2z_1, w_2z_2$ are assigned pairwise distinct colours).

Suppose that $\{\phi(t_1t'_1), \phi(t_2t'_2)\} \neq \{\phi(w_2z_1), \phi(w_2z_2)\}$. Let $\alpha \in \{\phi(w_2z_1), \phi(w_2z_2)\} \setminus \{\phi(t_1t'_1), \phi(t_2t'_2)\}$, $\alpha \in \{1, 2, 4, 5\}$ (because $3 \in \{\phi(tt_1), \phi(tt_2)\}$).

We do the following assignment of colours (in the given order): $\phi(u_1u) = 2$, $\phi(uv) = 3$, $\phi(vw) = 6$, $\phi(ww_1) = \phi(w_1t)$, $\phi(w_1t) = \alpha$.

It follows that $\{\phi(t_1t'_1), \phi(t_2t'_2)\} = \{\phi(w_2z_1), \phi(w_2z_2)\}$ and $6 \notin \{\phi(t_1t'_1), \phi(t_2t'_2)\}$. But then we permute the colours of w_1t and ww_1 , recolour uv with colour 6 and assign to u_1u colour 2. We obtain a strong edge-colouring of H , a contradiction.

5.b Suppose w_2 is a 3_2 -vertex as depicted in Figure 5.

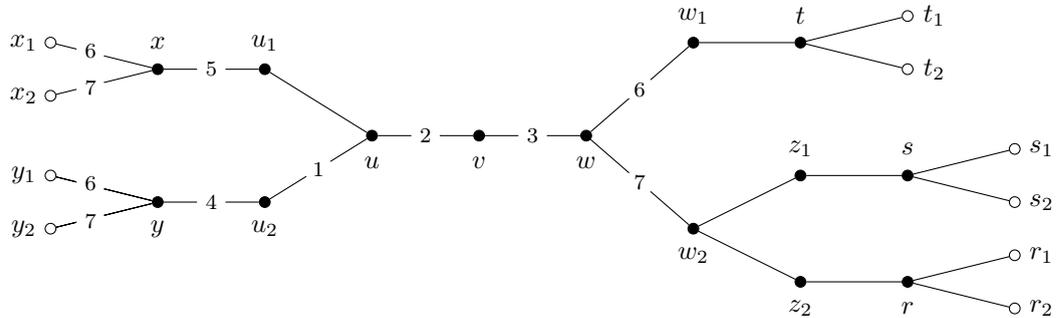


Figure 5: The configuration of Claim 4.5.b

Consider in this case the edge ww_2 .

Observe that $3 \in \{\phi(z_1s), \phi(z_2r)\}$. Otherwise we can permute the colours of vw and ww_2 , and assign 3 to u_1u . Observe that $1, 2, 4, 5 \in \{\phi(w_1t), \phi(w_2z_1), \phi(w_2z_2), \phi(z_1s), \phi(z_2r)\}$. Otherwise we can recolour ww_2 with 1 or 2 or 4 or 5, uv with 7, and assign colour 2 to u_1u . Hence $\{\phi(w_1t), \phi(w_2z_1), \phi(w_2z_2), \phi(z_1s), \phi(z_2r)\} = \{1, 2, 3, 4, 5\}$. Observe that $3 \in \{\phi(tt_1), \phi(tt_2)\}$.

Otherwise we can permute the colours of vw and ww_1 , and assign colour 3 to u_1u . Hence, without loss of generality we can assume $\phi(tt_1) = \phi(z_1s) = 3$. Moreover, we prove that $\phi(tt_2) = \phi(z_2r)$. By contradiction, assume that $\phi(z_2r) = \alpha \neq \phi(tt_2)$ ($\alpha \in \{1, 2, 4, 5\}$). We recolour ww_1 with α , uv with 6, and assign 2 to u_1u .

Now let us uncolour uv and assign colour 2 to u_1u . Observe that $7 \in \{\phi(ss_1), \phi(ss_2)\}$. Otherwise we can permute the colours of ww_2 and w_2z_1 , and assign colour 7 to uv . Observe that $\phi(w_1t) \in \{\phi(ss_1), \phi(ss_2)\}$. Otherwise we use $\phi(w_2z_1)$ to recolour ww_1 , we recolour w_2z_1 with $\phi(w_1t)$ (recall that $\{\phi(tt_1), \phi(tt_2)\} = \{\phi(z_1s), \phi(z_2r)\}$), and assign colour 6 to uv . It follows that $\{7, \phi(w_1t)\} = \{\phi(ss_1), \phi(ss_2)\}$ ($\phi(w_1t) \neq 6$).

Finally we permute the colours of w_2z_1 and ww_1 and assign 6 to uv . A contradiction.

5.c Suppose w_2 is a light 3-vertex as depicted in Figure 6.

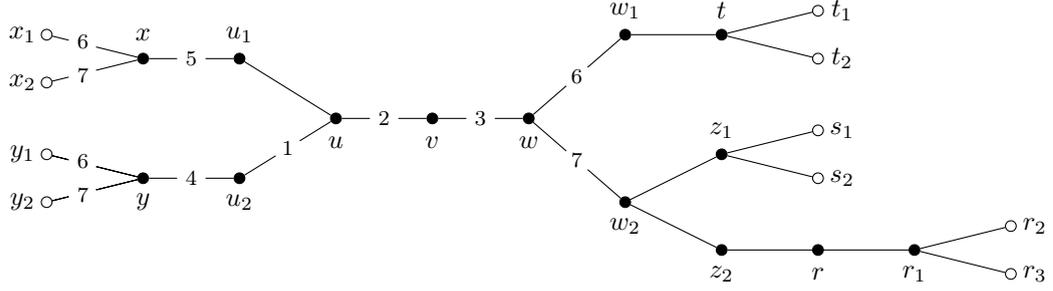


Figure 6: The configuration of Claim 4.5.c

Exactly as the first part of the proof of Claim 4.5.a we have:

$$\{\phi(w_1t), \phi(tt_1), \phi(tt_2), \phi(w_2z_1), \phi(w_2z_2)\} = \{1, 2, 3, 4, 5\}, 3 \in \{\phi(tt_1), \phi(tt_2)\}.$$

Let us uncolour uv and assign to u_1u colour 2. If the permutation of the colours of ww_2 and w_2z_2 is possible, then uv can be recoloured with 7. Hence $\phi(rr_1) = 7$.

Now we uncolour vw and assign colour 3 to uv . Observe that:

$$\{\phi(w_1t), \phi(tt_1), \phi(tt_2)\} = \{\phi(z_1s_1), \phi(z_1s_2), \phi(z_2r)\}$$

By contradiction, let us suppose that there exists an $\alpha \in \{\phi(w_1t), \phi(tt_1), \phi(tt_2)\} \setminus \{\phi(z_1s_1), \phi(z_1s_2), \phi(z_2r)\}$. Recall $\alpha \in \{1, 2, 3, 4, 5\}$ and $\phi(w_2z_2) \neq 3$. We colour vw with 7, assign colour $\phi(w_2z_2)$ to ww_2 , and recolour w_2z_2 with α .

Finally we permute the colours of ww_1 and w_2z_2 , assign colour 6 to uv and colour 3 to vw .

□

The discharging rules are defined as follows:

- (R1) Every 3_3 -vertex gives $\frac{1}{6}$ to each adjacent good 2-vertex.
- (R2) Every 3_2 -vertex and 3_1 -vertex gives $\frac{1}{4}$ to each adjacent good 2-vertex if this 2-vertex is not adjacent to a 3_3 -vertex.
- (R3) Every 3_0 -vertex gives $\frac{1}{12}$ to each adjacent 3_2 -vertex if any.
- (R4) Every 3_1 -vertex u gives $\frac{1}{12}$ to each adjacent 3_2 -vertex v if v has a 2-neighbour w adjacent to a 3_3 -vertex.
- (R5) Every 3-vertex gives $\frac{1}{3}$ to each adjacent 2-vertex which is a neighbour of 3_3 -vertex.
- (R6) Every 3-vertex gives $\frac{1}{2}$ to each adjacent bad 2-vertex.

Let $v \in V(H)$ be a k -vertex. By Claim 4.1, $k \geq 2$.

Case $k = 2$. Observe that $\omega(v) = -\frac{1}{2}$. Suppose v is a good 2-vertex. If v is adjacent to a 3_3 -vertex, then v cannot be adjacent to another 3_3 -vertex by Claim 4.4. Hence, $\omega^*(v) \geq -\frac{1}{2} + 1 \times \frac{1}{6} + 1 \times \frac{1}{3} = 0$ by (R1) and (R5). If v is not adjacent to a 3_3 -vertex, then $\omega^*(v) \geq -\frac{1}{2} + 1 \times \frac{1}{4} + 1 \times \frac{1}{4} = 0$ by (R2). Suppose v is bad. Vertex v is adjacent to one 3-vertex by Claim 4.2. Hence, $\omega^*(v) = -\frac{1}{2} + 1 \times \frac{1}{2} = 0$ by (R6).

Case $k = 3$. Observe that $\omega(v) = \frac{1}{2}$. We have the following cases for v :

- Vertex v is adjacent to three 2-vertices. By Claim 4.3 these 2-vertices are good. Moreover, by Claim 4.4 none of these 2-vertices is adjacent to another 3_3 -vertex. Hence, $\omega^*(v) = \frac{1}{2} - 3 \times \frac{1}{6} = 0$ by (R1).
- Vertex v is adjacent to exactly two 2-vertices. By Claim 4.3, none of these 2-vertices is bad. Suppose that none of these 2-vertices are adjacent to a 3_3 -vertex. Hence, $\omega^*(v) \geq \frac{1}{2} - 2 \times \frac{1}{4} = 0$ by (R2). Assume now that one of the 2-vertices adjacent to v is adjacent to a 3_3 -vertex (note that among the 2-vertices adjacent to v , at most one can be adjacent to a 3_3 -vertex by Claim 4.5.a). Hence v cannot be adjacent to a 3_2 -vertex by Claim 4.5.b. Then, v must have either a 3_1 -vertex or a 3_0 -vertex as a neighbour. Hence, $\omega^*(v) \geq \frac{1}{2} - 1 \times \frac{1}{4} + 1 \times \frac{1}{12} - 1 \times \frac{1}{3} = 0$ by (R2), (R3) (or (R4)) and (R5).
- Vertex v is adjacent to exactly one 2-vertex u . If u is a bad 2-vertex, then by Claim 4.5.c, v cannot be adjacent to a 3_2 -vertex w which has a 2-neighbour y adjacent to a 3_3 -vertex. Hence, $\omega^*(v) \geq \frac{1}{2} - 1 \times \frac{1}{2} = 0$ by (R6). Suppose u is a good 2-vertex. Let w be the other neighbour of u ($d(w) = 3$). If w is a 3_3 -vertex, then $\omega^*(v) \geq \frac{1}{2} - 2 \times \frac{1}{12} - 1 \times \frac{1}{3} = 0$ by (R4) and (R5). If w is not a 3_3 -vertex, then $\omega^*(v) \geq \frac{1}{2} - 2 \times \frac{1}{12} - 1 \times \frac{1}{4} > 0$ by (R2) and (R4).
- Vertex v is a 3_0 -vertex. Hence, $\omega^*(v) \geq \frac{1}{2} - 3 \times \frac{1}{12} > 0$ by (R3).

This completes the proof.

2.1.3. Third part

Let H be a counterexample to Theorem 2.3 minimizing $|E(H)| + |V(H)|$: H is not strong edge-colourable with eight colours, $\text{mad}(H) < \frac{8}{3}$ and for any edge e , $\chi'_s(H - e) \leq 8$. One can assume that H is connected; otherwise, by minimality of H , we can colour each connected component independently. Recall that $\omega(x) = d(x) - \frac{8}{3}$.

Claim 5. *The minimal counterexample H to Theorem 2.3 satisfies the following properties:*

1. H does not contain 1^- -vertices.
2. H does not contain two adjacent 2-vertices.
3. H does not contain a 3-vertex adjacent to three 2-vertices.
4. H does not contain a 2-vertex adjacent to two 3_2 -vertices.
5. H does not contain a 3-vertex adjacent to a 3_2 -vertex and a 2-vertex.
6. H does not contain a 3-vertex adjacent to two 3_2 -vertices.

Claims 5.1 to 5.4 are proved in [11].

Before proving Claims 5.5 and 5.6, we need to introduce some definitions and notations. Let ϕ be a partial strong 8-edge-colouring of H . For an edge uv , we denote by $PC_\phi(uv)$ the set of permissible colours that would extend ϕ to uv . Let $SC(N_1(uv))$ be the set of colours used by edges at distance 1 from uv .

Proposition 1. *Suppose H contains a 3_2 -vertex x . Let u and r be its adjacent 2-neighbours, and let y be its adjacent 3-neighbour. Also let v and s be the other neighbours (distinct from x) of u and r respectively. Finally let z and t be the other neighbours of y (distinct from x).*

Consider ϕ a strong 8-edge-colouring of $H' = (V(H), E(H) \setminus \{xy, xu, uv, xr, rs\})$. Then ϕ satisfies the following:

- P1.* $PC_\phi(uv) \cap PC_\phi(rs) = \emptyset$.
- P2.* $\{\phi(zy), \phi(yt)\} \cap (PC_\phi(uv) \cup PC_\phi(rs)) = \emptyset$.
- P3.* $PC_\phi(xy) \subseteq PC_\phi(uv) \cup PC_\phi(rs)$.
- P4.* $SC(N_1(uv)) = SC(N_1(rs))$.
- P5.* $|PC_\phi(uv)| = 2 = |PC_\phi(rs)|$.
- P6.* $SC(N_1(uv)) \cap \{\phi(zy), \phi(yt)\} = \emptyset$.

Proof

In the following we prove the proposition for each of the items.

- P1.* Suppose $PC_\phi(uv) \cap PC_\phi(rs) \neq \emptyset$. Let α be a colour of the intersection. First we colour uv and rs with α , then we colour xy ($|PC_\phi(xy)| \geq 2$; hence it remains at least one colour). Finally we colour ux ($|SC(N_2(ux))| \leq 7$) and xr ($|SC(N_2(ux))| \leq 8$, but colour α is repeated twice).
- P2.* By contradiction, assume that $\phi(zy) \in PC_\phi(uv)$. We colour uv with $\phi(zy)$. Then we colour xy ($|PC_\phi(xy)| \geq 2$; hence it remains at least one colour). We colour sequentially rs (at least one available colour), xr (at least two available colours, since zy and uv have the same colour), and ux (at least one available colour, again zy and uv have the same colour).
- P3.* By contradiction. Observe that $|PC_\phi(uv)| \geq 2$. Let $\alpha, \beta \in PC_\phi(uv)$. Similarly, let $\gamma, \lambda \in PC_\phi(rs)$. Finally let $\zeta \in PC_\phi(xy) \setminus (PC_\phi(uv) \cup PC_\phi(rs))$. Assign ζ to xy , α to uv , β to ux , γ to rs . Finally, by *P1* and *P2* we can assign colour λ to xr .
- P4.* By contradiction. Colour first xy , then uv and rs . Count the number of available colours for ux and xr . If one of them has two available colours, then we colour it the last. So each has one available colour. Suppose these two colours are the same. Then we have $SC(N_1(uv)) = SC(N_1(rs))$.
- P5.* By contradiction suppose $|PC_\phi(uv)| \geq 3$ and $\alpha, \beta, \gamma \in PC_\phi(uv)$. Suppose $PC_\phi(xy) \not\subseteq PC_\phi(uv)$. Colour first xy with a colour that does not appear in $PC_\phi(uv)$, then rs . Assign α to uv , β to ux , and γ to xr (possible by *P1*, *P2* and *P4*). Now suppose that $PC_\phi(xy) \subseteq PC_\phi(uv)$ and $PC_\phi(xy)$ contains α, β . Colour xy with α , ux with β , uv with γ , xr and rs with the colours of $PC_\phi(rs)$ (that is possible by *P1* and *P2*).
- P6.* If not true one can colour sequentially xy , uv , rs , xr , ux .

□

To summarize Proposition 1 one can assume without loss of generality that: $PC_\phi(uv) = \{1, 2\}$, $PC_\phi(rs) = \{3, 4\}$, $\phi(zy) = 5$, $\phi(yt) = 6$, $SC(N_1(uv)) = SC(N_1(rs)) = \{7, 8\}$, $PC_\phi(xy) \subseteq \{1, 2, 3, 4\}$.

Now we prove the remaining parts of the claim.

PROOF OF CLAIM 5.5. This follows from the previous discussion in Proposition 1. By contradiction suppose t is a 2-vertex. Observe that the edge yt is coloured with α and has also another available colour, say β (at most six other coloured edges at distance at most 2). Now $\beta \notin PC_\phi(uv) \cup PC_\phi(rs)$. Otherwise we permute α and β , and this contradicts *P2*. It suffices then to colour xy with β , the edges uv and ux with the colours of $PC_\phi(uv)$, and the edges rs and xr with the colours of $PC_\phi(rs)$. This extends ϕ to whole H . □

PROOF OF CLAIM 5.6. By contradiction, suppose H contains a 3-vertex y adjacent to two 3₂-vertices x_1 and x_2 . Let u_i and r_i be the two 2-neighbours of x_i ($i = 1, 2$). Finally let v_i and s_i be the two other neighbours of u_i and r_i respectively ($i = 1, 2$). Consider $H' = (V(H), E(H) \setminus \{yx_1, x_1u_1, u_1v_1, x_1r_1, r_1s_1\})$. By minimality of H , H' admits a strong 8-edge-colouring ϕ . By the previous discussion and without loss of generality one

can assume that $PC_\phi(u_1v_1) = \{1, 2\}$, $PC_\phi(r_1s_1) = \{3, 4\}$, $\phi(zv) = 5$, $\phi(yx_2) = 6$, $SC(N_1(u_1v_1)) = SC(N_1(r_1s_1)) = \{7, 8\}$, $PC_\phi(yx_1) \subseteq \{1, 2, 3, 4\}$.

Hence observe that if we can change the colour of yx_2 , then we will be able to extend the colouring (by $P2$ or $P6$). To do this uncolour x_2r_2 . Recolour yx_2 with an available colour distinct from 6. Colour x_2r_2 . We are done. \square

We apply now a discharging procedure with the following rules:

(R) Every 3-vertex gives $\frac{1}{3}$ to each adjacent 2-vertex and to each adjacent 3_2 -vertex.

Let $v \in V(H)$ be a k -vertex. By Claim 5.1, $k \geq 2$.

Case $k = 2$. Observe that $\omega(v) = -\frac{2}{3}$. By Claim 5.2, the neighbours of v have degree 3. Hence v receives twice $\frac{1}{3}$ by (R), and so $\omega^*(v) = -\frac{2}{3} + 2 \times \frac{1}{3} = 0$.

Case $k = 3$. Observe that $\omega(v) = \frac{1}{3}$. We have the following cases for v :

- If v is not adjacent to any 2-vertices, then v is adjacent to at most one 3_2 -vertex by Claim 5.6, and so gives at most $\frac{1}{3}$ by (R); it follows $\omega^*(v) \geq \frac{1}{3} - \frac{1}{3} = 0$.
- If v is adjacent to exactly one 2-vertex, then its 3-neighbours are not 3_2 -vertices by Claim 5.5. It follows that $\omega^*(v) = \frac{1}{3} - \frac{1}{3} = 0$ by (R).
- If v is a 3_2 -vertex, then it receives $\frac{1}{3}$ from its 3-neighbour (which is not a 3_2 -vertex by Claim 5.5) and gives $\frac{1}{3}$ to each adjacent 2-vertex. Hence $\omega^*(v) = \frac{1}{3} + \frac{1}{3} - 2 \times \frac{1}{3} = 0$.
- The case where v is adjacent to three 2-vertices does not appear by Claim 5.3.

This completes the proof.

2.1.4. Fourth part

Let H be a counterexample to Theorem 2.4 minimizing $|V(H)| + |E(H)|$. We can assume that H is connected as otherwise, by minimality of H , we can colour each connected component independently. Recall that $\omega(x) = d(x) - \frac{20}{7}$. We first prove some structural properties of H .

Claim 6. *The minimum counterexample to Theorem 2.4 does not contain:*

1. 1^- -vertices.
2. two adjacent 2-vertices.
3. a 3-vertex adjacent to two 2-vertices.
4. two adjacent 3_1 -vertices.
5. a triangle.
6. a path of three 3-vertices ztu where z and u are 3_1 -vertices.

Proof

The proofs of first four parts are given in [11].

5. Suppose H contains a triangle xyz .

If $d(x) = 2$, then by minimality of H , the graph $H - xy$ can be strongly edge-coloured with at most nine colours. Since $|N_2(xy)| \leq 6$, there exists at least three colours left for xy . Hence $d(x) = d(y) = d(z) = 3$.

Let u , v and t be the neighbours of x , y and z respectively (u , v and t being outside the triangle). Let $H' = H - x$. By minimality of H , we have $\chi'_s(H') \leq 9$. Consider a strong edge-colouring ϕ of H' using the minimum number. We show how to extend it to H . We colour xu and xy (in each case there exists a free colour). If we have a colour left for xz , then we are done. Therefore, $|N_2(xz)| = 9$ and $|SC(N_2(xz))| = 9$, which implies that all edges in $N_2(xz)$ are assigned pairwise distinct colours. Now, one of the following assignment of colours is possible:

- Assign $\phi(xz) = \phi(yz)$ and recolour yz with a free colour.
- Assign $\phi(xz) = \phi(xy)$ and recolour xy with a free colour.

This is a contradiction.

6. Let ztu be such a path and y and v be the 2-vertices neighbours of z and u respectively. Let x be the neighbour of y distinct from z and w be the neighbour of v distinct from u . By Claims 6.1, 6.2 and 6.3, x and w are 3-vertices. By Claims 6.3 and 6.4, t is a 3_0 -vertex. Let z_1, t_1 and u_1 be the neighbours of z, t and u respectively. Since H has no triangles (by Claim 6.5), we have the configuration depicted in Figure 7. Note that in H there might exist edges z_1t_1, t_1u_1 or z_1u_1 and the representation of the given figure is a general one.

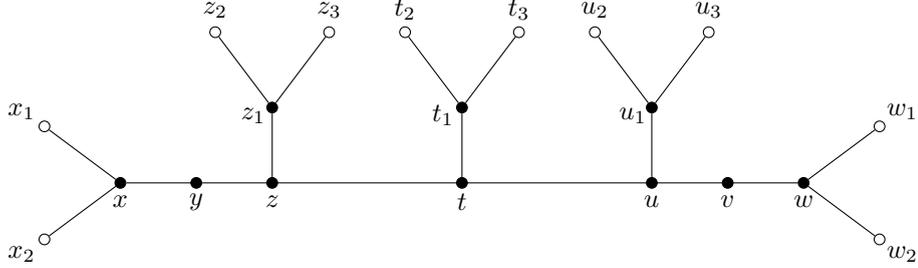


Figure 7: The configuration of Claim 6.6

Let $H' = H - y$. By minimality of H , H' can be strongly edge-coloured with at most nine colours. Let us consider such a colouring ϕ . We show how to extend ϕ to H .

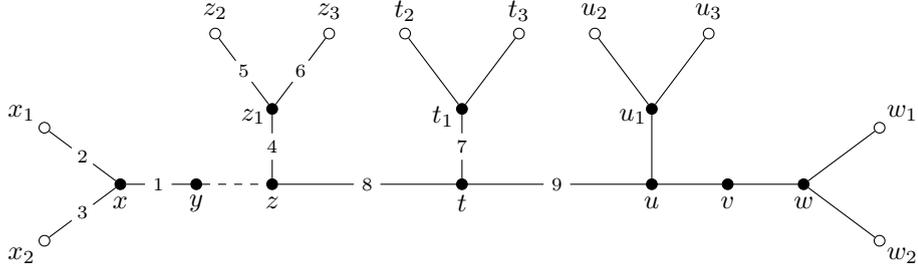


Figure 8: The initial fixed colouring of the edges of the configuration of Claim 6.6. Edge yz is the only non-coloured edge.

In order to complete the colouring of H one need to assign a colour to xy and yz . By counting the number of edges in $N_2(xy)$, it is easy to see that there is at least one colour left for xy , so we assign it to this edge. Now, if there is a colour left for yz , then we are done. Therefore, since $|N_2(yz)| = 9$, all the colours of $[[9]]$ must appear exactly once in $N_2(yz)$ and without loss of generality we can fix the colours of all edges of $N_2(yz)$ as depicted in Figure 8.

Observe that $\{5, 6, 7, 9\} \subseteq SC(N_2(xy))$ as otherwise one could recolour xy with one of these colours and assign $\phi(yz) = 1$. Therefore, all edges incident to x_1 and x_2 for which we did not fix a colour yet, must have distinct colours from the set $\{5, 6, 7, 9\}$. If one could recolour tu with 1, 2, 3, 5 or 6 then yz could be coloured 9 which is impossible. Hence $\{1, 2, 3, 5, 6\} \subseteq SC(N_2(tu))$. Observe that t cannot be neither x_1 nor x_2 as none of the edges incident to t is coloured 2 or 3. Also, since u is adjacent to v which is a 2-vertex, by Claim 6.4 u cannot be neither x_1 nor x_2 . Therefore, if one could permute the colours of tu and zt , then 8 would not belong to $SC(N_2(xy))$ any more, thus xy could be recoloured with 8 and yz could be assigned colour 1. Therefore, $8 \in \{\phi(u_1u_2), \phi(u_1u_3), \phi(vw)\}$.

Observe that recolouring zz_1 with 2 or 3 must not be possible as otherwise colour 4 could be used for yz to complete the colouring of H . Hence out of all the edges incident to z_2 and z_3 , two of them must be coloured 2 and 3 respectively. Let us uncolour edges xy, zz_1, zt and

tu . We claim that it is not possible to assign colour 4 to tu . Indeed, if tu could be coloured with 4, then we assign $\phi(tu) = \phi(xy) = 4$ and by using the fact that two out of all the edges incident to z_2 and z_3 must be coloured 2 and 3 respectively, one of the following assignments of colours would be valid:

- $\phi(yz) = 1, \phi(zz_1) = 9$ and $\phi(zt) = 8$.
- $\phi(yz) = 9, \phi(zz_1) = 1$ and $\phi(zt) = 8$.
- $\phi(yz) = 1, \phi(zz_1) = 8$ and $\phi(zt) = 9$.

Therefore, tu cannot be assigned colour 4 and we must have the following statement:

$$\{\phi(t_1t_2), \phi(t_1t_3), \phi(uu_1), \phi(u_1u_2), \phi(u_1u_3), \phi(uv), \phi(vw)\} = \{1, 2, 3, 4, 5, 6, 8\} \quad (\star)$$

Observe that in (\star) both sets have the same cardinality and hence $\phi(t_1t_2), \phi(t_1t_3), \phi(uu_1), \phi(u_1u_2), \phi(u_1u_3), \phi(uv), \phi(vw)$ are pairwise distinct and this implies that there is no edge between t_1 and u_1 .

Consider the edge uv . Since at the beginning of the proof we have fixed $\phi(tt_1) = 7, \phi(zt) = 8$ and $\phi(tu) = 9$, obviously $\phi(uv) \in \llbracket 6 \rrbracket$.

We distinguish the following cases for $\phi(uv)$:

- Suppose $\phi(uv) \in \{1, 2, 3, 5, 6\}$. We will denote this colour a . By (\star) we know that $\phi(t_1t_2) \neq a$ and $\phi(t_1t_3) \neq a$. We uncolour uv and do the following assignment of colours: $\phi(xy) = 1, \phi(yz) = 9, \phi(zt) = 8$ and $\phi(tu) = a$. If we manage to colour uv , then we are done. In order to do this, observe that by $(\star), \{9, \phi(t_1t_2), \phi(t_1t_3)\} \cap \{\phi(uu_1), \phi(u_1u_2), \phi(u_1u_3), \phi(vw)\} = \emptyset$. Therefore we could use one of the three colours 9, $\phi(t_1t_2)$ or $\phi(t_1t_3)$ for uv , distinct from the colours assigned to w_1 and w_2 . A contradiction.
- We have $\phi(uv) = 4$. Let us come back to the fixed colouring of Figure 8. If one could recolour zt with 2 or 3, then yz could be coloured 8. Therefore, two of the three edges t_1t_2, t_1t_3 and uu_1 must be coloured with 2 and 3 respectively. Without loss of generality we can assume that $\phi(t_1t_2) = 2$. Moreover, by $(\star) \phi(uv) \notin \{\phi(t_1t_2), \phi(t_1t_3)\} = \{2, \phi(t_1t_3)\}$, which means that $\{\phi(w_1w_2), \phi(w_1w_3)\} = \{2, \phi(t_1t_3)\}$ and without loss of generality we have $\phi(w_1w_2) = 2$. Now, we uncolour zz_1, zt, uv and assign $\phi(xy) = \phi(tu) = 4$. We obtain a valid partial strong edge-colouring of the configuration as depicted in Figure 9.

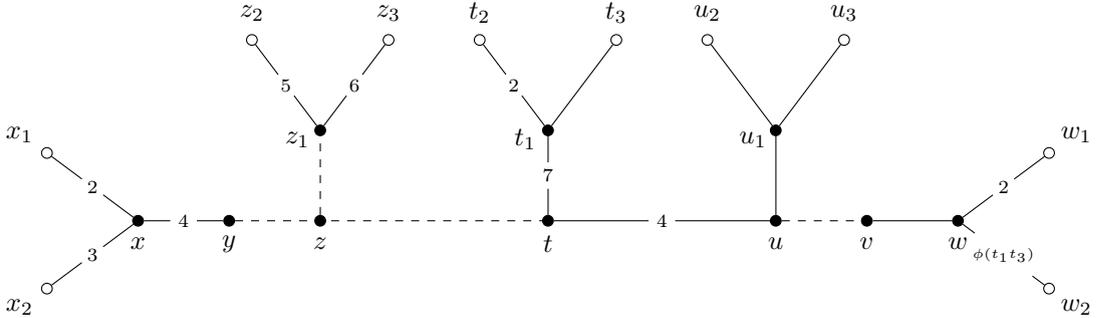


Figure 9: Case (b) of the proof of Claim 6.6. The dashed edges are not coloured.

Since $\phi(t_1t_3) \neq 9$ and none of the other edges of $N_2(uv)$ is coloured 9, we assign $\phi(uv) = 9$.

We claim that out of all edges incident to z_2 and z_3 , two of them must be coloured with 1 and 9. Indeed, if it is not the case then by assigning $\phi(zt) = 8$, we could assign either $\phi(zz_1) = 1$ and $\phi(yz) = 9$ or $\phi(zz_1) = 9$ and $\phi(yz) = 1$ and we would be done. Therefore, four edges incident to z_2 and z_3 except z_1z_2 and z_1z_3 must have distinct

colours which are namely 2, 3, 1 and 9. If none of the edges t_1t_3 and uu_1 is coloured 1 then one of the following assignments of colours would be a valid strong edge-colouring:

- $\phi(yz) = 1$, $\phi(zt) = 8$ and $\phi(zz_1) = 9$.
- $\phi(yz) = 9$, $\phi(zt) = 8$ and $\phi(zz_1) = 1$.
- $\phi(yz) = 9$, $\phi(zt) = 1$ and $\phi(zz_1) = 8$.

Therefore, one of the edges t_1t_3 and uu_1 must be coloured 1. On the other hand, as proved previously, one of these edges must be coloured 3. Therefore, $\{\phi(t_1t_3), \phi(uu_1)\} = \{1, 3\}$. Recall from the previous paragraph that in $N_2(zz_1)$, none of the edges is coloured 7. We recolour zz_1 with 7 and uncolour edges tt_1 and tu . Observe that by (\star) $7 \notin \{\phi(uu_1), \phi(u_1u_2), \phi(u_1u_3), \phi(vw), \phi(wu_1), \phi(wu_2)\}$, so we recolour uv with 7. Moreover, we assign colour 1 to yz and obtain the partial strong edge-colouring of the configuration as depicted in Figure 10.

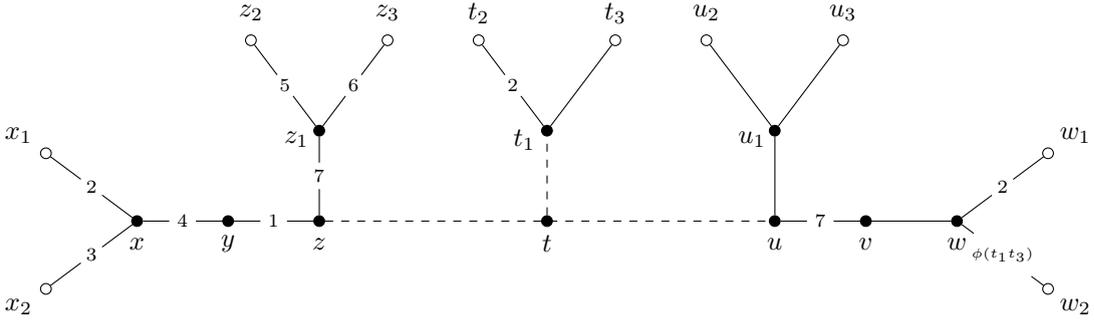


Figure 10: A partial strong edge-colouring of the configuration of Claim 6.6. The dashed edges are not coloured.

In order to finish the colouring of H we must assign colours to zt , tt_1 and tu . We know that $\phi(t_1t_3) \in \{1, 3\}$ and $\phi(t_1t_2) = 2$. Let us consider temporarily the colouring given in Figure 8. If one could recolour edge tt_1 with 5 or 6, then colour 7 could be assigned to yz which implies that out of all the edges incident to t_2 and t_3 , two of them must be coloured 5 and 6 respectively. Applying these to the colouring given in Figure 10, we conclude that one of the following assignments of colours is valid:

- $\phi(zt) = 8$, $\phi(tt_1) = 4$ and $\phi(tu) = 9$.
- $\phi(zt) = 8$, $\phi(tt_1) = 9$ and $\phi(tu) = 4$.
- $\phi(zt) = 9$, $\phi(tt_1) = 8$ and $\phi(tu) = 4$.

This is a contradiction. □

The discharging rules are defined as follows:

- (R1) Every 3_0 -vertex gives $\frac{1}{7}$ to each adjacent 3_1 -vertex.
- (R2) Every 3_1 -vertex gives $\frac{3}{7}$ to its adjacent 2-vertex.

Let $v \in V(H)$ be a k -vertex. By Claim 6.1, $k \geq 2$.

Case $k = 2$. Observe that $\omega(v) = -\frac{6}{7}$. By Claims 6.2 and 6.3, v is adjacent to two 3_1 -vertices. Hence, by (R2), $\omega^*(v) = -\frac{6}{7} + 2 \times \frac{3}{7} = 0$.

Case $k = 3$. Observe that $\omega(v) = \frac{1}{7}$. By Claim 6.3, v can be a 3_1 -vertex or a 3_0 -vertex. Suppose, v is a 3_1 -vertex. By Claim 6.4 and by (R1) and (R2), $\omega'(v) = \frac{1}{7} + 2 \times \frac{1}{7} - \frac{3}{7} = 0$. Suppose now that v is a 3_0 -vertex. By Claim 6.6, v is adjacent to at most one 3_1 -vertex. Then by (R1), $\omega^*(v) \geq \frac{1}{7} - 1 \times \frac{1}{7} = 0$.

This completes the proof.

2.2. Optimality

In [11], the question about the optimality of the bounds on the strong chromatic index of subcubic graphs in terms of maximum average degree was raised. Let $f(r) = \inf\{\text{mad}(G) \mid \chi'_s(G) > r\}$. It was proved that:

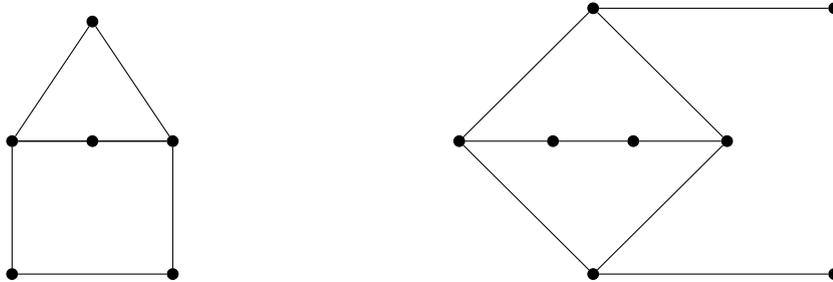
$$f(6) \leq \frac{7}{3}, f(7) \leq \frac{5}{2}, f(8) \leq \frac{20}{7} \text{ and } f(9) \leq \frac{20}{7}.$$

The authors provided graphs reaching the upper bounds for $f(6)$, $f(7)$ and $f(9)$, as depicted in Figure 11.

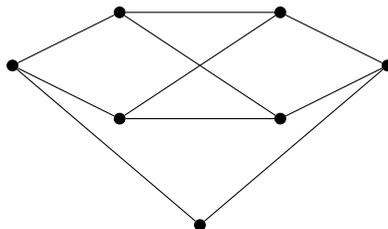
By Theorem 2, we have:

$$\frac{7}{3} \leq f(6), \frac{5}{2} \leq f(7) \text{ and } \frac{20}{7} \leq f(9).$$

Hence the upper bounds on the maximum average degree for $f(6)$, $f(7)$ and $f(9)$ are best possible. Although, we improved the lower bound for $f(8)$ given in [11] from $\frac{13}{5}$ to $\frac{8}{3}$, we did not manage to find a better upper bound. The problem of finding this value is even more challenging since till now we do not have an example of subcubic graph G having $\chi'_s(G) = 9$ and $\text{mad}(G) < \frac{20}{7}$.



(a) A graph G with $\text{mad}(G) = \frac{7}{3}$ and $\chi'_s(G) > 6$. (b) A graph G with $\text{mad}(G) = \frac{5}{2}$ and $\chi'_s(G) > 7$.



(c) A graph G with $\text{mad}(G) = \frac{20}{7}$ having $\chi'_s(G) > 9$.

Figure 11: Graphs proving the optimality of the bounds of parts 1, 2 and 4 of Theorem 2.

3. Proof of Theorem 4

We prove Theorem 4 by contradiction. Suppose the statement is not true and let H be a counterexample minimizing $|V(H)| + |E(H)|$. We will prove some structural properties of H in order to show that H does not exist.

In the following we use Claim 6 of the proof of Theorem 2.4 as the proof of this claim remains valid within the hypothesis of Theorem 4.

Claim 7. H has no 6-cycle $C = xyztvwx$ where y is a 2-vertex.

Proof

Suppose there exists such a cycle C as depicted in Figure 12. Observe that $x, z, t, u, v, x_1, z_1, t_1, v_1$ are 3-vertices by Claims 6.2, 6.3, 6.4 and u_1 is a 3-vertex by Claim 6.6.

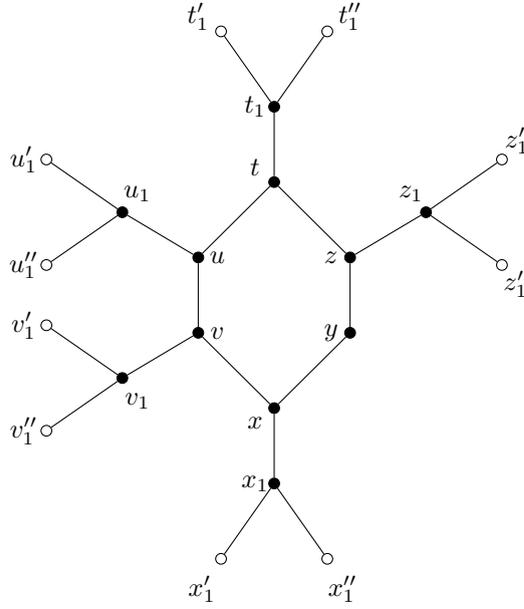


Figure 12: An induced cycle C of length 6 of H having a 2-vertex on its boundary.

Consider the graph $H' = H - y$. Consider a strong edge-colouring ϕ of H' using at most nine colours. We will extend ϕ to H in order to obtain a contradiction. Observe that $|SC(N_2(xy))| \leq 8$, thus there exists a colour left for xy . If we can colour yz , then we are done. Therefore, since $|N_2(yz)| = 9$, we must have $SC(N_2(yz)) = \llbracket 9 \rrbracket$ and every colour is used exactly once in $N_2(yz)$. Therefore, we claim that $|SC(N_2[xy])| = 9$ as otherwise one could recolour xy with another colour and obtain a free colour for yz . Without loss of generality we can assume that $\phi(zt) = 1$, $\phi(zz_1) = 2$, $\phi(xx_1) = 3$, $\phi(vx) = 4$, $\phi(uv) = 5$, $\phi(vv_1) = 6$, $\phi(x_1x'_1) = 7$, $\phi(x_1x''_1) = 8$ and $\phi(xy) = 9$. Since $SC(N_2(yz)) = \llbracket 9 \rrbracket$ we have $\{\phi(tu), \phi(tt_1), \phi(z_1z'_1), \phi(z_1z''_1)\} = \{5, 6, 7, 8\}$. Observe that since $5 \in SC(N_2(tu))$ and $5 \in SC(N_2(tt_1))$, without loss of generality we can assume that $\phi(z_1z'_1) = 5$. Also, $6 \in SC(N_2(tu))$ and therefore $\phi(tu) \in \{7, 8\}$. Since colours 7 and 8 are fixed only on edges $x_1x'_1$ and $x_1x''_1$ respectively, we can assume without loss of generality that $\phi(tu) = 7$ and therefore $\{\phi(tt_1), \phi(z_1z''_1)\} = \{6, 8\}$. Figure 13 shows the unique colouring (up to permutation) of the edges described previously.

We claim that one of the edges $v_1v'_1$ or $v_1v''_1$, say $v_1v'_1$, must have the same colour as the edge zt (colour 1 in Figure 13). Otherwise, one could change the colour of vx to the colour of zt and colour yz with 4. Similarly, $2 \in \{\phi(v_1v'_1), \phi(uu_1)\}$ (we can assign 2 to vx and 4 to yz). Observe that one can use the same argument conversely (by trying to assign to tz the colour of vx) by recalling from the previous paragraph that $\{\phi(tt_1), \phi(z_1z'_1)\} = \{6, 8\}$. Hence, we conclude that one of the edges $t_1t'_1$ or $t_1t''_1$, say $t_1t'_1$, must have the same colour as the edge vx (colour 4 in Figure 13). If it is possible to permute the colours of edges uv and vx , one could obtain a free colour (colour 4) for yz , thus either uu'_1 or uu''_1 must have the same colour as vx (colour 4 in Figure 13). Without loss of generality $\phi(u_1u'_1) = 4$. If it is possible to permute the colours of edges tu and uv (7 and 5 respectively), then one could obtain a free colour for yz . Hence either $\phi(v_1v'_1) = 7$ or $\phi(t_1t'_1) = 5$ (or both).

1. Suppose $\phi(v_1v'_1) = 7$. Hence $\phi(uu_1) = 2$. If one can permute the colours of tu and zt , such that tu is assigned colour 1 and zt is assigned colour 7, then xy could be recoloured with 1 and colour 9 would be free for yz . Hence $\phi(u_1u'_1) = 1$. But now it is possible to permute the colours of xy and uv and to use colour 9 for yz . A contradiction.
2. Suppose $\phi(t_1t'_1) = 5$. If it is possible to change the colour of edge zt (which is 1) to the colour of the edge xx_1 (which is 3), then yz could be coloured with 1. Hence $\phi(uu_1) = 3$ and therefore, $\phi(v_1v'_1) = 2$. By permuting the colours of edges vx and xy (4 and 9 respectively) and by recolouring zt with 9, we can colour yz with 1. A contradiction.

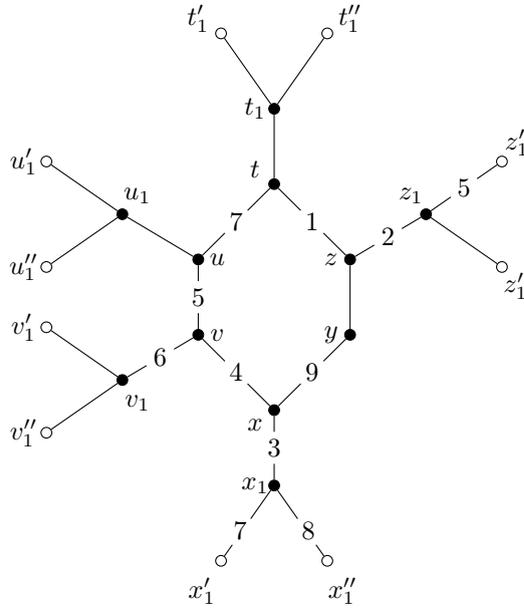


Figure 13: The unique colouring of $C - yz$ in H' .

□

By Claim 6.5 H has no triangle and by hypothesis of the theorem H does not contain induced cycles of length 4 or 5. Hence the counterexample H must have girth $g \geq 6$.

Consider now the graph H_1 obtained from H by replacing each path of two edges xyz , where y is a 2-vertex and x, z are 3-vertices, by an edge xz . Clearly, H_1 is planar. By Claim 6.5 H has no triangle and since it does not contain an induced 4-cycle, H_1 is simple. Moreover, since it has no 1-vertices (Claim 6.1) and no two adjacent 2-vertices (Claim 6.2), H_1 is 3-regular. Therefore, H_1 must contain a face of length at most 5, say C' . Recall that H has girth at least 6, thus by Claim 6.2, Claim 6.3 and Claim 6.4, C' cannot be obtained from a cycle of H of length $l \geq 7$. Therefore, in H there exists a cycle C of length 6 having a vertex of degree 2 on its boundary. But this is impossible by Claim 7. Hence H cannot exist.

This completes the proof of Theorem 4.

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