

Strong edge-colouring of sparse planar graphs

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Abstract

A strong edge-colouring of a graph is a proper edge-colouring where each colour class induces a matching. It is known that every planar graph with maximum degree Δ has a strong edge-colouring with at most $4\Delta + 4$ colours. We show that $3\Delta + 1$ colours suffice if the graph has girth 6, and 4Δ colours suffice if $\Delta \geq 7$ or the girth is at least 5. In the last part of the paper, we raise some questions related to a long-standing conjecture of Vizing on proper edge-colouring of planar graphs.

Keywords: planar graphs, girth, proper edge-colouring, strong edge-colouring

1. Introduction

A *proper edge-colouring* of a graph $G = (V, E)$ is an assignment of colours to the edges of the graph such that two adjacent edges do not use the same colour. We use the standard notation, $\chi'(G)$, to denote the chromatic index of G . A *strong edge-colouring* (called also distance 2 edge-colouring) of a graph G is a proper edge-colouring of G , such that the every set of edges using the same colour induces a matching. We denote by $\chi'_s(G)$ the *strong chromatic index* of G which is the smallest integer k such that G can be strongly edge-coloured with k colours. Strong edge-colouring has been studied extensively in the literature by different authors (see [4, 5, 10, 6, 1, 8, 7, 2, 3, 9]).

The *girth* of a graph G is the length of a shortest cycle in G . We denote by Δ the maximum degree of a graph.

Perhaps the most challenging question for strong edge-colouring is the following conjecture:

Conjecture 1 (Erdős and Nešetřil [5]). *For every graph G , $\chi'_s(G) \leq \frac{5}{4}\Delta^2$ for Δ even and $\frac{1}{4}(5\Delta^2 - 2\Delta + 1)$ for Δ odd.*

Andersen [1] and Horák *et al.* [8] showed this conjecture for the case when $\Delta = 3$. When Δ is large enough, Molloy and Reed showed that $\chi'_s(G) \leq 1.998\Delta^2$ [10].

In this note, we study the strong chromatic index of planar graphs. The work in this area started with the paper of Faudree *et al.* [6], who proved the following theorem.

Theorem 1 (Faudree *et al.* [6]). *If G is a planar graph then $\chi'_s(G) \leq 4\Delta + 4$, for $\Delta \geq 3$.*

The proof of Theorem 1 uses the Four Colour Theorem. The authors also provided a construction of planar graphs of girth 4 which satisfy $\chi'_s(G) = 4\Delta - 4$ colours. Hence, the bound of Theorem 1 is optimal up to an additive factor.

The same authors also conjectured that for $\Delta = 3$ the bound can be improved.

Conjecture 2 (Faudree *et al.* [6]). *If G is a planar subcubic graph then $\chi'_s(G) \leq 9$.*

Hocquard *et al.* obtained the following weakening of Conjecture 2.

Theorem 2 (Hocquard *et al.* [7]). *If G is a planar graph with $\Delta \leq 3$ containing neither induced 4-cycles, nor induced 5-cycles, then $\chi'_s(G) \leq 9$.*

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An interesting question is to see how the strong chromatic index behaves for sparse planar graphs. For instance, when the girth is large enough the strong chromatic index decreases to the near optimal lower bound, as showed in the following theorems:

Theorem 3 (Borodin and Ivanova [2]). *If G is a planar graph with maximum degree $\Delta \geq 3$ and girth $g \geq 40\lfloor \frac{\Delta}{2} \rfloor$, then $\chi'_s(G) \leq 2\Delta - 1$.*

Recently this result was improved for $\Delta \geq 6$:

Theorem 4 (Chang et al. [3]). *If G is a planar graph with maximum degree $\Delta \geq 4$ and girth $g \geq 10\Delta + 46$, then $\chi'_s(G) \leq 2\Delta - 1$.*

For smaller values of the girth, Hudák et al. [9] improved the bound in Theorem 1.

Theorem 5 (Hudák et al. [9]). *If G is a planar graph with with girth $g \geq 6$ then $\chi'_s(G) \leq 3\Delta + 6$.*

Our main result in this paper improves the upper bound in Theorem 5. In particular, we show the following.

Theorem 6. *If G is a planar graph with with girth $g \geq 6$ then $\chi'_s(G) \leq 3\Delta + 1$.*

Moreover, in Section 3, by a more careful analysis of the proof of Theorem 1 given in [6] and by using some results on proper edge-colouring, we obtain the following strengthening.

Theorem 7. *Let G be a planar graph with maximum degree Δ and girth g . If G satisfies one of the following conditions below, then $\chi'_s(G) \leq 4\Delta$*

- $\Delta \geq 7$,
- $\Delta \geq 5$ and $g \geq 4$,
- $g \geq 5$.

Before proving our results we introduce some notation.

Notation. Let G be a graph. Let $d(v)$ denote the degree of a vertex v in G . A vertex of degree k is called a k -vertex. A k^+ -vertex (respectively, k^- -vertex) is a vertex of degree at least k (respectively, at most k). A k_l -vertex is a k -vertex adjacent to exactly l 2-vertices. A *bad 2-vertex* is a 2-vertex adjacent to another 2-vertex. Two edges are at distance 1 if they share one of their ends and they are at distance 2 if they are not at distance 1 and there exists an edge adjacent to both of them. We define $N_2[uv]$ as the set of edges at distance at most 2 from the edge uv and $N_2(uv) = N_2[uv] - uv$. Given an edge-colouring of G , we denote by $SC(N_2(uv))$ ($SC(N_2[uv])$ respectively) the set of colours used by edges in $N_2(uv)$ ($N_2[uv]$ respectively). We denote by $N(v)$ the neighbourhood of the vertex v , i.e., the set of its adjacent vertices. Finally, we use $\llbracket n \rrbracket$ to denote the set of integers $\{1, 2, \dots, n\}$.

2. Proof of Theorem 6

2.1. Structural properties

We proceed by contradiction. Let H be a counterexample to the theorem that minimizes $|E(H)| + |V(H)|$. By minimality of H we can assume that it is connected and that by Theorem 2 it has $\Delta(H) \geq 4$.

Claim 1. *H satisfies the following properties:*

1. *H does not contain a 1-vertex adjacent to a 4^- -vertex.*
2. *H does not contain a 2-vertex adjacent to two 3^- -vertices.*

3. H does not contain a 2-vertex adjacent to a 4_2 -vertex and to a 3^- -vertex.
4. H does not contain a 2-vertex adjacent to a 4_3 -vertex and to a 4_2 -vertex.
5. If $k \geq 4$, then H does not contain a k -vertex adjacent to $k - 2$ 1-vertices; if the k -vertex is adjacent to $k - 3$ 1-vertices, then it has no other 2^- -neighbour.
6. If $k \geq 4$, then H does not contain a k -vertex adjacent to k 2^- -vertices.
7. If $k \geq 5$, then H does not contain a k -vertex u with $N(u) = \{u_1, u_2, \dots, u_{k-1}, x\}$, such that each u_i with $i \in \llbracket k - 1 \rrbracket$ is a 2^- -vertex and u_1 is either a 1-vertex or a 2-vertex adjacent to either a 3^- -vertex or a 4_3 -vertex.
8. If $k \geq 5$, then H does not contain a k -vertex adjacent to $k - 2$ vertices of degree 2, u_1, \dots, u_{k-2} , such that for $i \in \llbracket k - 3 \rrbracket$, each u_i is adjacent to either a 3^- -vertex or a 4_3 -vertex.
9. If $k \geq 5$ and $1 \leq \alpha \leq k - 4$, then H does not contain a k -vertex adjacent to α 1-vertices and to $k - 2 - \alpha$ vertices of degree 2, $u_1, \dots, u_{k-2-\alpha}$, such that for $i \in \llbracket k - 3 - \alpha \rrbracket$ each u_i is adjacent to either a 3^- -vertex or a 4_3 -vertex.

Proof

Let L be the set of colours $\llbracket 3\Delta + 1 \rrbracket$. For each of the parts of the claim, we will suppose by contradiction that the described configuration exists in H . Then we will build a graph H' from H by removing a certain number of vertices and edges. By minimality of H we will have $\chi'_s(H') \leq 3\Delta + 1$. Finally, for each of these cases, we will show a contradiction by showing how to extend a strong $(3\Delta + 1)$ -edge-colouring ϕ of H' to a strong edge-colouring of H without using an extra colour.

1. Suppose H contains a 1-vertex u adjacent to a 4^- -vertex v . Then let $H' = H - \{uv\}$. We can extend ϕ to H by colouring uv because $|L \setminus SC_\phi(N_2(uv))| \geq 1$.
2. Suppose H contains a 2-vertex u adjacent to two 3^- -vertices v and w . Then let $H' = H - \{uv, uw\}$. Since $|L \setminus SC_\phi(N_2(uv))| \geq \Delta - 1 \geq 3$ and $|L \setminus SC_\phi(N_2(uw))| \geq \Delta - 1 \geq 3$, we can extend ϕ to H by colouring uv and uw .
3. Suppose H contains a 2-vertex u adjacent to a 4_2 -vertex v and to a 3^- -vertex w . Then let $H' = H - \{uv, uw\}$. One can observe that $|L \setminus SC_\phi(N_2(uv))| \geq \Delta - 3 \geq 1$ and $|L \setminus SC_\phi(N_2(uw))| \geq \Delta - 2 \geq 2$. We can extend ϕ to H by colouring uv and uw in this order.
4. Suppose H contains a 2-vertex u adjacent to a 4_3 -vertex v and to a 4_2 -vertex w . We assume that $N(w) = \{u, w_1, x, y\}$, $N(v) = \{u, v_1, v_2, z\}$, where w_1, v_1 and v_2 are 2-vertices. Then take $H' = H - \{uv, uw\}$. In order to extend ϕ to H we proceed as follows. We uncolour the edges vv_1 and vv_2 . One can observe that $|L \setminus SC_\phi(N_2(uv))| \geq 2\Delta - 4 \geq 4$ and $|L \setminus SC_\phi(N_2(uw))| \geq \Delta - 2 \geq 2$. Hence, we colour uv and uw . Observe now that $|L \setminus SC_\phi(N_2(vv_1))| \geq \Delta - 2 \geq 2$ and $|L \setminus SC_\phi(N_2(vv_2))| \geq \Delta - 2 \geq 2$. We can extend ϕ to H by colouring vv_1 and vv_2 .
5. Suppose H contains a k -vertex u adjacent to $k - 3$ 1-vertices u_1, u_2, \dots, u_{k-3} with $4 \leq k \leq \Delta$. Let $H' = H - \{uu_1\}$. We can extend ϕ to H by colouring uu_1 which is possible because $|L \setminus SC_\phi(N_2(uu_1))| \geq \Delta - k + 3 \geq 3$.
6. Suppose H contains a k -vertex u adjacent to k 2^- -vertices u_1, u_2, \dots, u_k with $4 \leq k \leq \Delta$. Then let $H' = H - \{uu_1, uu_2, \dots, uu_k\}$. We extend ϕ to H by colouring the edges uu_1, \dots, uu_k in this order. Observe that for all $i \in \llbracket k \rrbracket$, $|L \setminus SC_\phi(N_2(uu_i))| \geq 2\Delta - 2k + 3 \geq 3$. Therefore, ϕ can be extended.
7. Let $5 \leq k \leq \Delta$. Suppose H contains a k -vertex u with neighbours $u_1, u_2, \dots, u_{k-1}, x$ such that each u_i with $i \in \llbracket k - 1 \rrbracket$ is a 2^- -vertex and u_1 is either a 1-vertex or a 2-vertex adjacent to either a 3^- -vertex or a 4_3 -vertex. Let $H' = H - \{u_1\}$. If u_1 is a 1-vertex then ϕ obviously can be extended to H . Therefore, u_1 is a 2-vertex. Let v_1 be the neighbour of u_1 other than u . By contradiction we assumed that v_1 is either a 3^- -vertex or a 4_3 -vertex. In order to show how to extend ϕ to H , we consider two cases:

- If v_1 is a 3^- -vertex, then $|L \setminus SC_\phi(N_2(uu_1))| \geq 2\Delta - 2k + 3 \geq 3$ and $|L \setminus SC_\phi(N_2(u_1v_1))| \geq \Delta - k + 2 \geq 2$.
- If v_1 is a 4_3 -vertex, then $|L \setminus SC_\phi(N_2(uu_1))| \geq 2\Delta - 2k + 2 \geq 2$ and $|L \setminus SC_\phi(N_2(u_1v_1))| \geq 2\Delta - k - 2 \geq 2$.

Therefore, in both cases ϕ can be extended.

- Let u be a k -vertex in H with $5 \leq k \leq \Delta$ such that there exists $k - 2$ paths in H , uu_jv_j with $j \in \llbracket k - 2 \rrbracket$ and such that $d_H(u_j) = 2$ and $d_H(v_j) \geq 2$ (by Claim 1.1). By contradiction we assume that each v_i , for $i \in \llbracket k - 3 \rrbracket$, is either a 3^- -vertex or a 4_3 -vertex. Then let $H' = H - \{uu_1, uu_2, \dots, uu_{k-3}, u_1v_1, u_2v_2, \dots, u_{k-3}v_{k-3}\}$. In order to extend ϕ to H , we distinguish the following two cases:

- Assume that there exists a vertex v_i with $i \in \llbracket k - 3 \rrbracket$ such that v_i is a 3^- -vertex. Without loss of generality assume this vertex is v_{k-3} . Then we colour each edge uu_i for $i \in \llbracket k - 4 \rrbracket$ (this is possible since $|L \setminus SC_\phi(N_2(uu_i))| \geq \Delta - 4 \geq k - 4$). We continue by colouring uu_{k-3} and $u_{k-3}v_{k-3}$ in this order, so that at each step there is at least one colour left. Now, for each edge u_iv_i with $i \in \llbracket k - 4 \rrbracket$ we have $|L \setminus SC_\phi(N_2(u_iv_i))| \geq 1$ and we can colour them independently.
- Each vertex v_i , with $i \in \llbracket k - 3 \rrbracket$, is a 4_3 -vertex. Let v be a 2-vertex adjacent to v_{k-3} and distinct from u_{k-3} . We uncolour the edge vv_{k-3} . Now, similarly to the previous case, we colour each edge uu_i for $i \in \llbracket k - 4 \rrbracket$ and this is possible since for all i , $|L \setminus SC_\phi(N_2(uu_i))| \geq \Delta - 4 \geq k - 4$. Now, we colour uu_{k-3} , vv_{k-3} and $u_{k-3}v_{k-3}$ in this order (at each step we have at least one colour left for the current edge). It remains to colour the edges u_iv_i , with $i \in \llbracket k - 4 \rrbracket$, and since $|L \setminus SC_\phi(N_2(u_iv_i))| \geq 1$ we can colour them independently.

- Let u be a k -vertex in H with $5 \leq k \leq \Delta$ and suppose by contradiction that u is adjacent to α 1-vertices and to $k - 2 - \alpha$ 2-vertices $u_1, \dots, u_{k-2-\alpha}$, such that for each $i \in \llbracket k - 3 - \alpha \rrbracket$, u_i is adjacent to either a 3^- -vertex or a 4_3 -vertex v_i .

Let $H' = H - \{uu_1, uu_2, \dots, uu_{k-3-\alpha}, u_1v_1, u_2v_2, \dots, u_{k-3-\alpha}v_{k-3-\alpha}\}$. Then we proceed exactly as in the proof of the previous claim.

□

2.2. Discharging procedure

Euler's formula $|V(H)| - |E(H)| + |F(H)| = 2$ can be rewritten as $(4|E(H)| - 6|V(H)|) + (2|E(H)| - 6|F(H)|) = -12$. Using the relation $\sum_{v \in V(H)} d(v) = \sum_{f \in F(H)} r(f) = 2|E(H)|$ we get that:

$$\sum_{v \in V(G)} (2d(v) - 6) + \sum_{f \in F(G)} (r(f) - 6) = -12 \quad (1)$$

We define the weight function $\omega : V(H) \cup F(H) \rightarrow \mathbb{R}$ by $\omega(x) = 2d(x) - 6$ if $x \in V(H)$ and $\omega(x) = r(x) - 6$ if $x \in F(H)$. It follows from Equation (1) that the total sum of weights is equal to -12. In what follows, we will define discharging rules (R1) to (R6) and redistribute weights accordingly. Once the discharging is finished, a new weight function ω^* is produced. However, the total sum of weights is kept fixed when the discharging is finished. Nevertheless, we will show that $\omega^*(x) \geq 0$ for all $x \in V(H) \cup F(H)$. This will lead us to the following contradiction:

$$0 \leq \sum_{x \in V(H) \cup F(H)} \omega^*(x) = \sum_{x \in V(H) \cup F(H)} \omega(x) = -12 < 0$$

and hence will demonstrate that such a counterexample cannot exist.

The discharging rules are defined as follows:

- (R1) Every face gives 2 to each incident 1-vertex.

(R2) Every k -vertex, for $k \geq 5$, gives 2 to each adjacent 1-vertex.

(R3) Every 4_3 -vertex gives $\frac{2}{3}$ to each adjacent 2-vertex.

(R4) Every 4_2 -vertex gives 1 to each adjacent 2-vertex.

(R5) Every 4_1 -vertex gives 2 to the adjacent 2-vertex.

(R6) Every k -vertex, for $k \geq 5$, gives:

(R6.1) 2 to each adjacent 2-vertex if this 2-vertex is adjacent to a 3^- -vertex.

(R6.2) $\frac{4}{3}$ to each adjacent 2-vertex if this 2-vertex is adjacent to a 4_3 -vertex.

(R6.3) 1 to each adjacent 2-vertex if this 2-vertex is adjacent to a 4^+ -vertex distinct from a 4_3 -vertex.

Let $v \in V(H)$ be a k -vertex. Consider the following cases:

Case $k = 1$. Observe that $\omega(v) = -4$. By Claim 1.1, v is adjacent to a 5^+ -vertex. By (R1) v receives 2 from its incident face and by (R2) v receives 2 from its adjacent vertex. Hence, $\omega^*(v) = -4 + 2 + 2 = 0$.

Case $k = 2$. Observe that $\omega(v) = -2$. By Claim 1.1, v has two neighbours u and w both of degree at least 2. Consider the following cases:

- (a) Suppose one of the neighbours of v , say u , is a 3^- -vertex. Then by Claim 1.2, w is a 4^+ -vertex. If $d(w) = 4$ then by Claim 1.3, w is a 4_1 -vertex and by (R5) we have $\omega^*(v) = -2 + 2 = 0$. If $d(w) \geq 5$ then by (R6.1) we have $\omega^*(v) = -2 + 2 = 0$.
- (b) Assume now that $d(u) = d(w) = 4$. Suppose first that w is a 4_1 -vertex. Then by (R3), (R4) and (R5), $\omega^*(v) = -2 + 1 \times 2 + 1 \times \min\{2, 1, \frac{2}{3}\} > 0$. Assume now that w is a 4_2 -vertex. Then by Claim 1.4, u is not a 4_3 -vertex. Hence, by (R4) and (R5), $\omega^*(v) = -2 + \min\{1 \times 1 + 1 \times 2, 2 \times 1\} \geq 0$. Finally, suppose that w is a 4_3 -vertex. Then by Claim 1.4, u is a 4_1 -vertex. Hence, by (R3) and (R5), $\omega^*(v) = -2 + 1 \times \frac{2}{3} + 1 \times 2 > 0$.
- (c) Suppose $d(u) \geq 5$ and $d(w) = 4$ (the case when $d(u) = 4$ and $d(w) \geq 5$ is symmetric). If w is a 4_1 -vertex then by (R5) and (R6.3), $\omega^*(v) = -2 + 1 \times 2 + 1 \times 1 > 0$. Assume now, w is a 4_2 -vertex hence, by (R4) and (R6.3), $\omega^*(v) = -2 + 1 \times 1 + 1 \times 1 = 0$. Suppose now, w is a 4_3 -vertex then by (R3) and (R6.2), $\omega^*(v) = -2 + 1 \times \frac{2}{3} + 1 \times \frac{4}{3} = 0$.
- (d) Assume $d(u) \geq 5$ and $d(w) \geq 5$. Hence, by (R6.3), $\omega^*(v) = -2 + 2 \times 1 = 0$.

Case $k = 3$. The initial charge of v is $\omega(v) = 0$ and it remains unchanged during the discharging process. Hence $\omega(v) = \omega^*(v) = 0$.

Case $k = 4$. Observe that $\omega(v) = 2$. By Claim 1.1 v is not adjacent to a 1-vertex. By Claim 1.6, v is adjacent to at most three 2-vertices. If v is a 4_1 -vertex, then by (R5), $\omega^*(v) = 2 - 1 \times 2 = 0$. If v is a 4_2 -vertex, then by (R4), $\omega^*(v) = 2 - 2 \times 1 = 0$. Suppose now v is a 4_3 -vertex. Hence, by (R3), $\omega^*(v) = 2 - 3 \times \frac{2}{3} = 0$.

Case $k \geq 5$. Observe that $\omega(v) = 2k - 6$. Consider the following cases:

- (a) Assume v is not adjacent to a 1-vertex. By Claim 1.6, v is adjacent to at most $k - 1$ 2-vertices. If v is adjacent to at most $k - 3$ 2-vertices then by (R6), $\omega^*(v) \geq 2k - 6 - (k - 3) \times 2 = 0$. If the number of 2-neighbours of v is $k - 2$, then by Claim 1.8 at most $k - 4$ of them have a 3^- -neighbour or a 4_3 -neighbour. Hence, by (R6.1) and (R6.3), $\omega^*(v) \geq 2k - 6 - (k - 4) \times 2 - 2 \times 1 = 0$. Suppose now that the number of 2-neighbours of v is exactly $k - 1$. Then by Claim 1.7, none of these 2-neighbours is adjacent to a 3^- -vertex or to a 4_3 -vertex. Therefore, by (R6.3), we have $\omega^*(v) = 2k - 6 - (k - 1) \times 1 = k - 5 \geq 0$.

- (b) Suppose v is adjacent to α 1-vertices with $\alpha \geq 1$. By Claim 1.5 we have $\alpha \leq k - 3$. Moreover, if $\alpha = k - 3$ then, by the same claim, v cannot be adjacent to a 2-vertex and thus, by (R2), $\omega^*(v) = 2k - 6 - (k - 3) \times 2 = 0$. So we may suppose that $\alpha \leq k - 4$. If the number of 2-neighbours of v is at most $k - 3 - \alpha$, then by (R6) $\omega^*(v) \geq 2k - 6 - \alpha \times 2 - (k - 3 - \alpha) \times 2 = 0$. Suppose the number of 2-neighbours of v is at least $k - 2 - \alpha$. This number cannot be $k - \alpha$ according to Claim 1.6, and, since $\alpha \geq 1$, by Claim 1.7 this number cannot be $k - 1 - \alpha$ neither. So v has exactly $k - 2 - \alpha$ neighbours of degree 2. Then by Claim 1.9, at most $k - 4 - \alpha$ of the 2-neighbours of v are adjacent to either a 3-vertex or a 4-vertex. Therefore, by (R2) and (R6), $\omega^*(v) \geq 2k - 6 - \alpha \times 2 - (k - 4 - \alpha) \times 2 - 2 \times 1 = 0$.

Let $f \in F(H)$ be a k -face. By hypothesis on the girth condition we know that $k \geq 6$. Note that if f has α incident 1-vertices, then $k \geq 6 + 2\alpha$. Since $\omega(f) = k - 6$, by (R1), $\omega^*(f) \geq k - 6 - 2\alpha \geq 0$.

After performing the discharging procedure the new weights of all faces and vertices are positive and therefore, H cannot exist.

3. Proof of Theorem 7

In this section we show how the proof of Theorem 1, given by Faudree *et al.* in [6], can be analysed in order to get a better bound for χ'_s for several subclasses of planar graphs. Below we provide this proof as we will need it for the observations which follow.

PROOF OF THEOREM 1. Decompose first the edges of the planar graph into $\Delta + 1$ distinct matchings (this is possible by Vizing's Theorem). For each matching M build the following graph G_M :

Each vertex of G_M corresponds to an edge of M . Two vertices of G_M are adjacent if the corresponding edges are adjacent in G (do not form an induced matching in G). The graph G_M is planar and hence its vertices can be coloured properly with 4 colours using Four Colour Theorem. This colouring corresponds to a strong edge-colouring of the matching M in G .

Since there are at most $\Delta + 1$ matchings and for each we use 4 colours, we obtain a strong $4(\Delta + 1)$ -edge-colouring of G . \square

The two main tools used in the previous proof are Vizing's Theorem and Four Colour Theorem. Therefore, if one could show that under some restrictions a planar graph is properly Δ -edge-colourable, then the bound given by the proof of Faudree *et al.* would be improved. To this end, we would like to mention the following conjecture:

Conjecture 3 (Vizing's Planar Graph Conjecture [12]). *Every planar graph G with $\Delta \geq 6$ satisfies $\chi'(G) = \Delta$.*

The cases of $\Delta \geq 7$ of this conjecture have been already shown:

Theorem 8 (Vizing [12]). *If G is a planar graph with $\Delta \geq 8$ then $\chi'(G) = \Delta$.*

Theorem 9 (Sanders & Zhao [11]). *If G is a planar graph with $\Delta = 7$ then $\chi'(G) = \Delta$.*

These two theorems applied in the proof of Theorem 1 give an immediate corollary:

Corollary 1. *Every planar graph with $\Delta \geq 7$ is strongly 4Δ -edge-colourable.*

It would be interesting if the above result would hold for all Δ . As the result is known for $\Delta = 3$, the remaining cases are $\Delta \in \{4, 5, 6\}$. In the following we will say that a graph G is *edge- Δ -critical* if $\chi'(G) = \Delta + 1$ and removing any edge of G creates a graph G' with $\chi'(G') = \Delta$.

Theorem 10 (Yap [16]). *Let G be an edge- Δ -critical graph with n vertices and m edges and $\Delta = 6$. Then $m \geq \frac{9}{4}n$.*

The following corollary is an immediate consequence of Euler's formula and Theorem 10 in the case of planar graphs:

Corollary 2 (Yap [16]). *Let G be a triangle-free planar graph with $\Delta = 6$. Then $\chi'(G) = \Delta = 6$.*

Therefore we can apply this result to improve the upper bound for the strong chromatic index for the above mentioned planar graphs:

Corollary 3. *Let G be a triangle-free planar graph with $\Delta \geq 6$. Then $\chi'_s(G) \leq 4\Delta$.*

Conjecture 4 (Vizing [13, 14]). *Let G be an edge- Δ -critical graph with n vertices and m edges. Then $m \geq \frac{1}{2}[(\Delta - 1)n + 3]$.*

Conjecture 4 is known to be true if $\Delta \leq 6$, the lower bound for the number of edges given by the conjecture being improved for small values of Δ (see the list of known results on this topic in [15]). We refer here to the last known result:

Theorem 11 (Woodall [15]). *Let G be an edge- Δ -critical graph with n vertices and m edges. Then the following holds:*

- if $\Delta = 4$ then $m \geq \frac{12}{7}n$
- if $\Delta = 5$ then $m \geq \frac{15}{7}n$

Corollary 4. *Let G be a planar graph of girth at least g and degree Δ . Then the following holds:*

- if $g \geq 5$ and $\Delta = 4$, then $\chi'(G) = \Delta$
- if $g \geq 4$ and $\Delta = 5$, then $\chi'(G) = \Delta$

Proof

In the following we will denote by $F(G)$ the set of faces of a planar graph G . By contradiction assume that there exists an edge- Δ -critical planar graph G with n vertices and m edges having girth at least g and degree Δ as stated in the hypothesis. We prove each case separately:

- Assume $g \geq 5$ and $\Delta = 4$. Hence we have $5|F(G)| \leq \sum_{f \in F(G)} r(f) = 2m$. Thus, from Euler's formula we get that $m \leq \frac{5}{3}(n-2)$. On the other hand, by Theorem 11 we know that $m \geq \frac{12}{7}n$, a contradiction.
- Assume $g \geq 4$ and $\Delta = 5$. Hence we have $4|F(G)| \leq \sum_{f \in F(G)} r(f) = 2m$. Thus, from Euler's formula we get that $m \leq 2n - 4$. On the other hand, by Theorem 11 we know that $m \geq \frac{15}{7}n$, which gives us a contradiction.

□

Finally, using Faudree *et al.*'s proof of Theorem 1 and Corollaries 1, 3, 4 we get the following:

Corollary 5. *Let G be a planar graph with degree Δ and girth g . Then the following holds:*

- if $g \geq 5$ then $\chi'_s(G) \leq 4\Delta$
- if $\Delta \geq 5$ and $g \geq 4$, then $\chi'_s(G) \leq 4\Delta$

Now, Corollaries 1, 3 and 5 imply Theorem 7.

We summarize all the observations of this section in the following table, where the upper bounds marked in bold are the ones given by Theorem 1 and that have not been improved since then:

The last line of the table is an immediate consequence of Grötzsch's Theorem and Theorem 1 as observed in [3] and [9].

	$\Delta \geq 7$	$\Delta \in \{5, 6\}$	$\Delta = 4$	$\Delta = 3$
no girth restriction	4Δ	$4\Delta + 4$	$4\Delta + 4$	$3\Delta + 1$
$g \geq 4$	4Δ	4Δ	$4\Delta + 4$	$3\Delta + 1$
$g \geq 5$	4Δ	4Δ	4Δ	$3\Delta + 1$
$g \geq 6$	$3\Delta + 1$	$3\Delta + 1$	$3\Delta + 1$	3Δ
$g \geq 7$	3Δ	3Δ	3Δ	3Δ

Table 1: Known upper bounds on the strong chromatic index of discussed subclasses of planar graphs

4. Concluding remarks and open problems

As mentioned in the introduction, for each $\Delta \geq 4$ there exist a planar graph G of girth 4 such that $\chi'_s(G) = 4\Delta - 4$ [6]. Thus, the values in the first three rows of the table might not be optimal.

For planar graphs of girth 6 there exists graphs satisfying $\chi'_s(G) \geq \left\lceil \frac{12(\Delta-1)}{5} \right\rceil$ as shown by Hudák *et al.* [9].

Regarding Conjecture 3, the condition of $\Delta \geq 6$ cannot be improved as Vizing himself showed in [12] that there exist planar graphs with $\Delta \in \{2, 3, 4, 5\}$ and which are not properly Δ -edge-colourable. The graphs proposed in his paper for $\Delta \in \{3, 4, 5\}$ are the graphs of geometric solids having one edge subdivided. For the cases of $\Delta \in \{4, 5\}$ these graphs contain many triangles. Moreover, Corollary 4 shows that planar graphs with $\Delta \geq 5$ having girth at least 4 are properly Δ -edge-colourable and thus this result is tight (the size of the girth cannot be decreased). Therefore, the remaining natural question to which we could not find an answer is the following:

Question 1. *Let G be a planar graph with $\Delta = 4$ and girth at least 4. Is it true that $\chi'(G) = \Delta$?*

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