

Fixed-Parameter Tractability and Logic

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Abstract

We exhibit a close connection between parameterized complexity theory and logic. Our approach is that of descriptive complexity theory. We study the definability of parameterized problems and try to obtain information about the parameterized complexity of the problems through the syntactical structure of the defining sentences. On the one hand, we use this approach to prove that certain problems are fixed-parameter tractable because they can be defined by syntactically simple formulas. On the other hand, we characterize classes of intractable problems by syntactical means.

1 Introduction

Parameterized complexity theory gives a framework for a refined complexity analysis of hard algorithmic problems. The basic idea can best be explained by an example: Consider the problem of evaluating a query in a relational database. This problem usually has a high complexity (depending on the query language, of course, but the problem is NP-complete even for the very basic conjunctive queries [6]). The main factor contributing to this complexity is the length of the query. In practice, however, queries are usually short, certainly much shorter than the size of the database. Thus when analyzing the complexity of the problem we should put much more emphasis on the size of the database than on the length of the query. An algorithm evaluating a query of length k in a database of size m in time $O(2^k \cdot m)$ is therefore much better than one performing the same task in time $O(m^{k/2})$, although both are exponential.

Parameterized complexity theory studies problems whose instances are *parameterized* by some function of the input, such as the length of the query in our example. The idea is to choose the parameterization in such a way that it can be assumed to take small values for the instances one is interested in. Then the complexity of an algorithm is measured not only in the size of the input, but also in terms of the parameter. A parameterized problem is *fixed-parameter tractable* if there is an algorithm solving it in time $f(k) \cdot n^c$, where n denotes the size of the input, k the parameter, and $f : \mathbb{N} \rightarrow \mathbb{N}$ is a computable function and $c > 0$ a constant.

Parameterized complexity theory provides methods for proving problems to be fixed-parameter tractable, but also gives a framework for dealing with apparently intractable problems in a similar way that the theory of NP-completeness does in classical complexity theory.

The purpose of this article is to establish a very fruitful connection between parameterized complexity theory and logic. Our approach is that of descriptive complexity theory. We study the definability of parameterized problems and try to obtain information about the parameterized complexity of the problems through the syntactical structure of the defining sentences. On the one hand, we use this approach to prove that certain problems are tractable because they can be defined by syntactically simple formulas. On the other hand, we characterize classes of intractable problems by syntactical means.

Central to our approach are *parameterized model-checking* problems of the following form. For a class Φ of formulas, we let $MC(\Phi)$ be the problem

$MC(\Phi)$	<p><i>Input:</i> A finite structure \mathcal{A}.</p> <p><i>Parameter:</i> A sentence $\varphi \in \Phi$.</p> <p><i>Question:</i> Does \mathcal{A} satisfy φ?</p>
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In most cases, Φ will be a fragment of first-order logic.

After a preliminary section, we discuss some basic facts about parameterized model-checking problems in Section 3. In Section 4 we introduce two notions of definability of parameterized problems, which we call *slicewise definability* and *Fagin definability*, and relate them to model-checking. We then show how Fagin definability can be used to establish the fixed-parameter tractability of various problems.

In Sections 5 and 6 we study the parameterized complexity of the model-checking problem for Σ_1 -formulas (i.e. existential first-order formulas in prenex normal form). We associate a graph with each such formula and use it to establish a surprisingly close connection between this model-checking problem, the homomorphism problem, and the subgraph isomorphism problem. As an application of our result we show that for Σ_1 -sentences whose graph has bounded tree-width the model-checking problem is fixed-parameter tractable.

So far we have only looked for tractable cases of the model-checking problem $MC(\Phi)$ that are obtained by restricting the class of formulas Φ . A different approach, that has received more attention in the literature [7, 24, 13], is to restrict the class of structures where the input structure \mathcal{A} is taken from. We prove a far reaching result: For any class C of graphs with an excluded minor, the model-checking problem for first-order logic is fixed-parameter tractable if the inputs are taken from C . This implies, for example, that parameterized versions of the dominating set problem or the (induced) subgraph isomorphism problem are fixed-parameter tractable when restricted to such classes of graphs.

Our last result on fixed-parameter tractability is a descriptive characterization of the complexity class FPT of all fixed-parameter tractable problems in terms of slicewise definability in finite variable fragments of least-fixed point logic. This simple result can be seen as a parameterized analogue of the well-known Immerman-Vardi Theorem [16, 25] characterizing the class PTIME in terms of definability in least-fixed-point logic.

The final section is devoted to fixed-parameter intractability. We define a hierarchy $A[t]$ of parameterized complexity classes in terms of alternating Turing machine acceptance (t is the number of alternations). This hierarchy can be seen as a parameterized analogue of the polynomial hierarchy. We prove that for all $t \geq 1$, the model-checking problem for Σ_t -formulas is complete for the t th level of this hierarchy. Then we study the relation between our A-hierarchy and Downey and Fellows' W-hierarchy. It is known that the first levels of the respective hierarchies, $A[1]$ and $W[1]$ coincide [5]. We slightly improve a result of Downey, Fellows, and Regan [9] relating $W[t]$, the t th level of the W-hierarchy, to the model-checking problems for a certain fragment of Σ_t . However, the questions whether $A[t]$ and $W[t]$ coincide for $t \geq 2$ remains open.

2 Preliminaries

2.1 Logic We assume that the reader is familiar with first-order logic, we just recall a few basic notions to fix our notation.

In this article, a vocabulary is a finite set of relation symbols. Associated with every relation symbol is a natural number, its *arity*. The *arity* of a vocabulary is the maximal arity of the relation symbols it contains. Usually, vocabularies are also permitted to contain function and constant symbols. All results of this article, with the single exception of Theorem 37, would remain true if function and constant symbols were allowed, but adding them would not give us any new insights. So, for convenience, we restrict our attention to relational vocabularies. In the following, τ always denotes a vocabulary.

A τ -*structure* \mathcal{A} consists of a set A , called the *universe* of \mathcal{A} , and a relation $R^{\mathcal{A}} \subseteq A^r$ for each r -ary relation symbol $R \in \tau$. We synonymously write $\bar{a} \in R^{\mathcal{A}}$ or $R^{\mathcal{A}}\bar{a}$ to denote that the tuple $\bar{a} \in A^r$ belongs to the relation $R^{\mathcal{A}}$. For $\tau \subseteq \tau'$, a τ -structure \mathcal{A} is the τ -*reduct* of a τ' -structure \mathcal{A}' if $A = A'$ and $R^{\mathcal{A}} = R^{\mathcal{A}'}$ for all $R \in \tau$. A τ' -structure \mathcal{A}' is a τ' -*expansion* of a τ -structure \mathcal{A} if \mathcal{A} is the τ -*reduct* of \mathcal{A}' .

We only consider finite structures. When we consider classes of structures, they are always assumed to be closed under isomorphism. STR denotes the class of all (finite) structures. If C is a class of structures, $C[\tau]$ denotes the subclass of all τ -structures in C . We consider graphs as $\{E\}$ -structures $\mathcal{G} = (G, E^{\mathcal{G}})$, where $E^{\mathcal{G}}$ is an anti-reflexive and symmetric binary relation (i.e. graphs are loop-free and undirected). $GRAPH$ denotes the class of all graphs.

If L is a class of formulas of some logic, then $L[\tau]$ denotes the class of all formulas of vocabulary τ in L .

The class of all first-order formulas is denoted by FO. Recall that *atomic formulas* are formulas of the form $x = y$ or $Rx_1 \dots x_r$, where x, y, x_1, \dots, x_r are variables and R is a r -ary relation symbol. *Literals* are atomic or negated atomic formulas. A first-order formula φ is in *negation normal form* if negation symbols only occur directly in front of atomic subformulas. φ is *existential* (*universal*) if it is in negation normal form and contains no universal quantifiers (no existential quantifiers, respectively). φ is in *prenex normal form* if it is of the form $Q_1 x_1 \dots Q_k x_k \theta$, where $Q_1, \dots, Q_k \in \{\exists, \forall\}$ and θ is quantifier-free.

EFO (AFO) denotes the class of all existential (universal, respectively) first-

order formulas. For $t \geq 1$, Σ_t denotes the class of all FO-formulas of the form

$$\exists x_{11} \dots \exists x_{1k_1} \forall x_{21} \dots \forall x_{2k_2} \dots Q x_{t1} \dots Q x_{tk_t} \theta,$$

where $Q = \forall$ if t is even and $Q = \exists$ otherwise and θ is quantifier-free. Π_t -formulas are defined analogously starting with a block of universal quantifiers.

2.2 Coding issues We use random access machines (RAMs) with the uniform cost measure as our underlying model of computation (cf. [1]).

Very often, the objects of our computations are structures. Therefore, we have to fix a way of representing structures on a RAM. The two most common ways of doing this are the *array representation* and the *list representation*. For both representations we assume that the universes of our structures are initial segments of the natural numbers; of course this is no real restriction because every structure is isomorphic to one with such a universe.

Both representations start with an encoding of the vocabulary and a natural number representing the size of the universe of the structure. The difference between the two representations is in how relations are stored. In the array representation, a k -ary relation is stored as a k -dimensional array with 0,1-entries. For graphs, this is just the adjacency matrix. The advantage of this representation is that for each tuple it can be checked in constant time whether it belongs to the relation or not. However, for sparse relations this representation wastes a lot of space.

In the more concise list representation, a relation is represented as a list of all tuples it contains. Clearly, the list representation of a structure can be computed from the array representation in linear time, but not vice versa. For graphs \mathcal{G} , it is easy to construct the common adjacency list representation from the list representation (in time linear in $|G| + (\text{size of the representation})$, where $|G|$ denotes the number of elements in G). In this article, we always assume that structures are given in the list representation, but all results also hold for the array representation. The *size* of a structure \mathcal{A} , denoted by $||\mathcal{A}||$, is defined to be $|A| + (\text{size of the list representation of } \mathcal{A})$. If not stated otherwise, the complexity of algorithms on structures is measured in this size.

2.3 Parameterized complexity theory We only recall those notions of the theory needed in this article. For a comprehensive treatment we refer the reader to Downey and Fellows' recent monograph [8]. A *parameterized problem* is a set $P \subseteq \Sigma^* \times \Pi^*$, where Σ and Π are finite alphabets. Following [8], we usually represent a parameterized problem P in the following form:

P	<p style="margin: 0;"><i>Input:</i> $x \in \Sigma^*$.</p> <p style="margin: 0;"><i>Parameter:</i> $y \in \Pi^*$.</p> <p style="margin: 0;"><i>Question:</i> Is $(x, y) \in P$?</p>
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In most cases, we have $\Pi = \{0, 1\}$ and consider the parameters $y \in \Pi^*$ as natural numbers (in binary). A natural example is the parameterized version of the well-known *VERTEX COVER* problem:

<i>VC</i>	<p><i>Input:</i> Graph \mathcal{G}.</p> <p><i>Parameter:</i> $k \in \mathbb{N}$.</p> <p><i>Question:</i> Does \mathcal{G} have a vertex cover of size k?</p>
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Recall that a *vertex cover* of a graph is a set X of vertices such that every edge is incident to one of the vertices in X . Similarly, we can define parameterized versions of *DOMINATING SET (DS)* and *CLIQUE*. (A *dominating set* of a graph is a set X of vertices such that every vertex not contained in X is adjacent to a vertex in X . A *clique* is a set of pairwise adjacent vertices.)

An example where the set of parameters is not \mathbb{N} , but the class *GRAPH* of all finite graphs is the following parameterized *SUBGRAPH ISOMORPHISM* problem:

<i>SI</i>	<p><i>Input:</i> Graph \mathcal{G}.</p> <p><i>Parameter:</i> Graph \mathcal{H}.</p> <p><i>Question:</i> Is \mathcal{H} isomorphic to a subgraph of \mathcal{G}?</p>
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Similarly, we can define parameterized versions of the *INDUCED SUBGRAPH ISOMORPHISM* problem and the *GRAPH HOMOMORPHISM* problem.

Definition 1 A parameterized problem $P \subseteq \Sigma^* \times \Pi^*$ is *fixed-parameter tractable* if there is a computable function $f : \mathbb{N}^* \rightarrow \mathbb{N}$, a constant $c \in \mathbb{N}$, and an algorithm that, given a pair $(x, y) \in \Sigma^* \times \Pi^*$, decides if $(x, y) \in P$ in time $f(|y|) \cdot |x|^c$.¹

We denote the class of all fixed-parameter tractable problems by FPT.

Of course we can always consider parameterized problems as classical problems and determine their complexity in the classical sense. Clearly, every parameterized problem in PTIME is also in FPT.

The best currently known algorithm for vertex cover *VC* has running time $O(k \cdot n + \max\{1.255^k \cdot k^2, 1.291^k \cdot k\})$ [12], where n denotes the size of the input graph. Thus $VC \in \text{FPT}$. On the other hand, it is conjectured that none of the problems *DS*, *CLIQUE*, *SI* is in FPT. As it is often the case in complexity theory, we can not actually prove this, but only prove that the problems are hard for certain complexity classes that are conjectured to contain FPT strictly. To do this we need a suitable concept of reduction. We actually introduce three different types of reduction:

Definition 2 Let $P \subseteq \Sigma^* \times \Pi^*$ and $P' \subseteq (\Sigma')^* \times (\Pi')^*$ be parameterized problems.

¹This is what Downey and Fellows call *strongly uniformly fixed-parameter tractable*. For variants of this definition, and also of Definition 2 and the definition of the W-hierarchy, the reader should consult [8].

- (1) A *parameterized T-reduction* from P to P' is an algorithm with an oracle for P' that solves any instance (x, y) of P in time $f(|y|) \cdot |x|^c$ in such a way that for all questions $(x', y') \in P'?$ to the oracle we have $|y'| \leq g(|y|)$ (for computable functions $f, g : \mathbb{N} \rightarrow \mathbb{N}$ and a constant $c \in \mathbb{N}$).

P is *fixed-parameter T-reducible* to P' (we write $P \leq_{\text{T}}^{\text{fp}} P'$), if there is a parameterized T-reduction from P to P' .

- (2) A *parameterized m-reduction* from P to P' is an algorithm that computes for every instance (x, y) of P an instance (x', y') of P' in time $f(|y|) \cdot |x|^c$ such that $|y'| \leq g(|y|)$ and

$$(x, y) \in P \iff (x', y') \in P'$$

(for computable functions $f, g : \mathbb{N} \rightarrow \mathbb{N}$ and a constant $c \in \mathbb{N}$).

P is *fixed-parameter m-reducible* to P' (we write $P \leq_{\text{m}}^{\text{fp}} P'$), if there is a parameterized m-reduction from P to P' .

Whereas every parameterized problem that is in PTIME (when considered as a classical problem) is in FPT, it is not the case that every PTIME many-one reduction between two parameterized problems is also a parameterized m-reduction. To capture both concepts we occasionally use the following third kind of reduction:

Definition 3 Let $P \subseteq \Sigma^* \times \Pi^*$ and $P' \subseteq (\Sigma')^* \times (\Pi')^*$ be parameterized problems.

A *pp m-reduction* from P to P' is a parameterized m-reduction from P to P' that is also a polynomial time many-one reduction from P to P' in the classical sense, i.e. the function f in Definition 2(2) is polynomial.

P is *pp m-reducible* to P' (we write $P \leq_{\text{m}}^{\text{fpp}} P'$), if there is a pp m-reduction from P to P' .

For example, $\text{CLIQUE} \leq_{\text{m}}^{\text{fp}} \text{SI}$ by the simple parameterized m-reduction that reduces the instance (\mathcal{G}, k) of CLIQUE to the instance $(\mathcal{G}, \mathcal{K}_k)$ of SI . Here \mathcal{K}_k denotes the complete graph with k vertices. Note that if we represent integers in binary, this reduction is not a pp m-reduction.

Observe that $\leq_{\text{T}}^{\text{fp}}$, $\leq_{\text{m}}^{\text{fp}}$, and $\leq_{\text{m}}^{\text{fpp}}$ are transitive and that for all P, P' we have

$$P \leq_{\text{m}}^{\text{fpp}} P' \implies P \leq_{\text{m}}^{\text{fp}} P' \quad \text{and} \quad P \leq_{\text{m}}^{\text{fp}} P' \implies P \leq_{\text{T}}^{\text{fp}} P'.$$

Furthermore, if $P \leq_{\text{T}}^{\text{fp}} P'$ and $P' \in \text{FPT}$ then $P \in \text{FPT}$. For any of the reductions $\leq_{\text{T}}^{\text{fp}}$, $\leq_{\text{m}}^{\text{fp}}$, $\leq_{\text{m}}^{\text{fpp}}$ we let $\equiv \dots$ denote the corresponding equivalence relation.

We define *hardness* and *completeness* of parameterized problems for a parameterized complexity class (under parameterized m- or T-reductions) in the usual way.

To classify problems that are not in FPT, Downey and Fellows (cf. [8]) have introduced a hierarchy

$$\text{W}[1] \subseteq \text{W}[2] \subseteq \dots$$

of classes above FPT. They conjecture that $W[1]$ strictly contains FPT and that the hierarchy is strict. They prove that *CLIQUE* and *SI* are complete for $W[1]$ and that *DS* is complete for $W[2]$.

We will further discuss the W -hierarchy and other seemingly intractable classes in Section 9.

Remark 4 Very often, it is natural to think of a parameterized problem P as derived from a (classical) problem $L \subseteq \Sigma^*$ by a *parameterization* $p : \Sigma^* \rightarrow \mathbb{N}$ in such a way that $P = \{(x, k) \mid x \in L, k = p(x)\}$.

Slightly abusing notation, we represent such a P in the form

P	<p><i>Input:</i> $x \in \Sigma^*$.</p> <p><i>Parameter:</i> $p(x)$.</p> <p><i>Question:</i> Decide if $x \in L$?</p>
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As an example, let us reconsider the subgraph isomorphism problem. Instead of taking graph \mathcal{H} as the parameter, we may also consider pairs of graphs $(\mathcal{G}, \mathcal{H})$ as inputs and parameterize the problem by the size of \mathcal{H} . In our new notation, this would be the problem

SI'	<p><i>Input:</i> Graphs \mathcal{G}, \mathcal{H}.</p> <p><i>Parameter:</i> \mathcal{H}.</p> <p><i>Question:</i> Is \mathcal{H} isomorphic to a subgraph of \mathcal{G}?</p>
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It is easy to see, however, that $SI \equiv_m^{\text{fp}} SI'$.

3 Model-checking

In this article we are mainly concerned with the complexity of various *parameterized model-checking problems*. For a set Φ of formulas, we let

$$MC(\Phi) := \{(\mathcal{A}, \varphi) \mid \mathcal{A} \in STR, \varphi \text{ sentence in } \Phi, \mathcal{A} \models \varphi\},$$

or more intuitively,

$MC(\Phi)$	<p><i>Input:</i> $\mathcal{A} \in STR$.</p> <p><i>Parameter:</i> $\varphi \in \Phi$.</p> <p><i>Question:</i> Does $\mathcal{A} \models \varphi$?</p>
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In this section we collect a few basic facts about parameterized model-checking problems. For every Φ we consider, we assume that we have fixed an encoding $\gamma : \Phi \rightarrow \{0, 1\}^*$, and we let $||\varphi||$ be the length of $\gamma(\varphi)$.

Taking sentences as parameters seems a little unusual. The following parameterization of the model checking problem looks more natural:

$MC'(\Phi)$	<p><i>Input:</i> $\mathcal{A} \in STR, \varphi \in \Phi.$</p> <p><i>Parameter:</i> $\ \varphi\ .$</p> <p><i>Question:</i> Does $\mathcal{A} \models \varphi?$</p>
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However, it is easy to see that $MC(\Phi) \equiv_{\text{m}}^{\text{fp}} MC'(\Phi).$

The following two lemmas are not really about fixed-parameter tractability, but about polynomial time reducibility. Although their proofs only use standard techniques, they are subtle and require some care. Therefore we decided to give the proofs in some detail. We will apply these lemmas several times later.

Lemma 5 *There are polynomial time transformations that associate with every structure $\mathcal{A} \in STR$ a graph $\mathcal{H}(\mathcal{A})$ and with every sentence $\varphi \in \text{FO}$ a sentence $\varphi_{\text{GRAPH}} \in \text{FO}$, respectively, such that*

$$\mathcal{A} \models \varphi \iff \mathcal{H}(\mathcal{A}) \models \varphi_{\text{GRAPH}}.$$

Furthermore for every $t \geq 1$, if $\varphi \in \Sigma_t$ then $\varphi_{\text{GRAPH}} \in \Sigma_{t+1}$ and if $\varphi \in \Pi_t$ then $\varphi_{\text{GRAPH}} \in \Pi_{t+1}.$

Proof: Let $\mathcal{A} \in STR[\tau]$ and $\varphi \in \text{FO}[\tau].$ Without loss of generality we can assume that φ is in prenex and in negation normal form.

Step 1. In the first step we translate \mathcal{A} to a structure $\mathcal{B}(\mathcal{A})$ of a vocabulary $\beta(\tau)$ that only consists of unary and binary relation symbols. φ is translated to a corresponding sentence φ_B of vocabulary $\beta(\tau).$

$\beta(\tau)$ contains unary relation symbols U and U_R for each symbol $R \in \tau$ and binary relation symbols $E_1, \dots, E_s,$ where s is the arity of $\tau.$

The universe of $\mathcal{B}(\mathcal{A})$ is

$$B(\mathcal{A}) := A \cup \{b(R, \bar{a}) \mid R \in \tau, \bar{a} \in R^{\mathcal{A}}\}.$$

We assume that the elements $b(R, \bar{a})$ are all pairwise distinct and distinct from those in $A.$ Note that the cardinality of $B(\mathcal{A})$ is essentially $\|\mathcal{A}\|,$ up to an additive term depending on $\tau.$ The unary relations are defined in the obvious way: We let $U^{B(\mathcal{A})} := A$ and $U_R^{B(\mathcal{A})} := \{b(R, \bar{a}) \mid \bar{a} \in R^{\mathcal{A}}\}$ for every $R \in \tau.$ The binary relations E_1, \dots, E_s are defined by

$$E_i^{B(\mathcal{A})} := \{(a_i, b(R, \bar{a})), (b(R, \bar{a}), a_i) \mid R \in \tau, \bar{a} = (a_1, \dots, a_r) \in R^{\mathcal{A}}, 1 \leq i \leq r\}.$$

Note that $E_i^{B(\mathcal{A})}$ is symmetric, this will be useful later.

To define $\varphi_B,$ we first relativize all quantifiers to $U,$ i.e. we inductively replace all subformulas $\exists x\psi$ by $\exists x(Ux \wedge \psi)$ and all subformulas $\forall x\psi$ by $\forall x(Ux \rightarrow \psi).$ We obtain a formula $\varphi'.$

φ_B is obtained from φ' by replacing every atomic subformula $R\bar{x},$ for r -ary $R \in \tau,$ by

$$\exists z(U_R z \wedge \bigwedge_{i=1}^r E_i x_i z) \tag{1}$$

where z is a new variable. Then we have

$$\mathcal{A} \models \varphi \iff \mathcal{B}(\mathcal{A}) \models \varphi_B. \quad (2)$$

Furthermore, $\mathcal{B}(\mathcal{A})$ can be computed from \mathcal{A} in time $O(\|\tau\| \cdot \|\mathcal{A}\|)$, where $\|\tau\|$ denotes the length of the encoding of τ , and φ_B can be computed from φ in linear time.

Step 2. In this step we replace the binary relations E_1, \dots, E_s by a single new binary relation E . We let

$$\gamma(\tau) := (\beta(\tau) \setminus \{E_1, \dots, E_s\}) \cup \{E, P_1, \dots, P_s\},$$

where P_1, \dots, P_s are new unary relation symbols.

We transform $\mathcal{B}(\mathcal{A})$ to a $\gamma(\tau)$ -structure $\mathcal{C}(\mathcal{A})$ as follows: For $1 \leq i \leq s$ and for all $a, b \in B(\mathcal{A})$ such that $E_i^{B(\mathcal{A})} ab$ we introduce a new element $c(i, a, b)$, add the pairs $(a, c(i, a, b))$, $(c(i, a, b), a)$, $(c(i, a, b), b)$, $(b, c(i, a, b))$ to $E^{C(\mathcal{A})}$ and add $c(i, a, b)$ to $P_i^{C(\mathcal{A})}$.

We define a sentence φ_C by replacing each subformula of φ_B of the form $E_i xy$ by $\exists z(Exz \wedge Ezy \wedge P_i z)$. Then we have the analogue of (2) and the subsequent remarks for $\mathcal{C}(\mathcal{A}), \varphi_C$ instead of $\mathcal{B}(\mathcal{A}), \varphi_B$.

Step 3. The restriction of $\mathcal{C}(\mathcal{A})$ to E is already a graph, i.e. $E^{C(\mathcal{A})}$ is symmetric and anti-reflexive, so all we have to do is to eliminate the unary relations $U, (U_R)_{R \in \tau}, (P_i)_{1 \leq i \leq s}$. Say, Q_1, \dots, Q_l is an enumeration of all these relations. Note that every $a \in C(\mathcal{A})$ is either isolated or of valence two or adjacent to a vertex of valence two. We use this to define certain trees $\mathcal{T}_1, \dots, \mathcal{T}_l$ and corresponding existential first-order formulas $\xi_1(x), \dots, \xi_l(x)$ and attach a copy of \mathcal{T}_i to each vertex in $Q_i^{C(\mathcal{A})}$ in such a way that in the resulting graph $\mathcal{H}(\mathcal{A})$ we have for all vertices a :

$$\mathcal{H}(\mathcal{A}) \models \xi_i(a) \iff a \in Q_i^{C(\mathcal{A})}.$$

Furthermore, the \mathcal{T}_i and thus the ξ_i can be chosen of size polynomial in l . We omit the details.

Then we let φ_{GRAPH} be the formula obtained from φ_C by replacing every subformula of the form $Q_i x$ by $\xi_i(x)$, for $1 \leq i \leq l$, and transforming the resulting formula into prenex normal form in the usual manner.

Clearly the transformations $\mathcal{A} \mapsto \mathcal{H}(\mathcal{A})$ and $\varphi \mapsto \varphi_{GRAPH}$ are polynomial, and we have

$$\mathcal{A} \models \varphi \iff \mathcal{H}(\mathcal{A}) \models \varphi_{GRAPH}.$$

It remains to prove that if $\varphi \in \Sigma_t$ ($\varphi \in \Pi_t$) then $\varphi_{GRAPH} \in \Sigma_{t+1}$ ($\varphi_{GRAPH} \in \Pi_{t+1}$, respectively). This follows easily from the way we defined the sentences φ_B, φ_C , and φ_{GRAPH} in Steps 1–3, noting that each positive (negative) occurrence of a relation symbol $R \in \tau$ only gives rise to positive (negative, respectively) occurrences of U_R , the E_i , and the P_i . Thus for the positive occurrences

of the $R \in \tau$ we get a new block of existential quantifiers and for the negative occurrences a new block of universal quantifiers. This increases the alternation depth by at most one. \square

If we bound the arity of the vocabularies, we can stay within Σ_t (or Π_t). For all $s \geq 1$ and every class C of structures, we let $C[s]$ denote the class of all structures in C whose vocabulary is at most s -ary. Similarly, we use the notation $\Phi[s]$ for a class Φ of formulas.

Lemma 6 *Let $s \geq 1$. There is a polynomial time transformation that associates with every structure $\mathcal{A} \in STR[s]$ a graph $\mathcal{H}'(\mathcal{A})$ and a linear time function that associates with every sentence $\varphi \in FO$ a sentence $\varphi'_{GRAPH} \in FO$, such that*

$$\mathcal{A} \models \varphi \iff \mathcal{H}'(\mathcal{A}) \models \varphi'_{GRAPH}.$$

Furthermore, for all $t \geq 1$, if $\varphi \in \Sigma_t$ then $\varphi'_{GRAPH} \in \Sigma_t$ and if $\varphi \in \Pi_t$ then $\varphi'_{GRAPH} \in \Pi_t$.

Proof: Let us first look back at the proof of the last lemma and note that if all relation symbols only occur positively in φ then the transformation $\varphi \mapsto \varphi_{GRAPH}$ only generates a new block of existential quantifiers. If all relation symbols only occur negatively then we only get a new block of universal quantifiers. Thus if $\varphi \in \Sigma_{2t-1}$ ($\varphi \in \Pi_{2t}$) and all relation symbols only occur positively in φ then also $\varphi_{GRAPH} \in \Sigma_{2t-1}$ ($\varphi_{GRAPH} \in \Pi_{2t}$, respectively). Similarly, if $\varphi \in \Sigma_{2t}$ ($\varphi \in \Pi_{2t-1}$) and all relation symbols only occur negatively in φ then also $\varphi_{GRAPH} \in \Sigma_{2t}$ ($\varphi_{GRAPH} \in \Pi_{2t-1}$, respectively).

We first transform \mathcal{A} to an \mathcal{A}' and φ to a φ' in the same prefix class such that

$$\mathcal{A} \models \varphi \iff \mathcal{A}' \models \varphi',$$

and either all relation symbols in φ' occur positively or negatively, whatever we need to apply the previous remark. For this purpose, let τ be an at most s -ary vocabulary. We let $\tau' := \tau \cup \{\overline{R} \mid R \in \tau\}$, where \overline{R} is a new relation symbol that has the same arity as R . For every $\mathcal{A} \in STR[\tau]$, we let \mathcal{A}' be the τ' -expansion of \mathcal{A} with $\overline{R}^{\mathcal{A}'} = A^r \setminus R^{\mathcal{A}}$ for r -ary $R \in \tau$. Note that \mathcal{A}' can be computed from \mathcal{A} in time $O(|\mathcal{A}|^s)$. To define φ' we either replace each negative literal $\neg R\bar{x}$ by $\overline{R}\bar{x}$ or each positive literal $R\bar{x}$ by $\neg \overline{R}\bar{x}$. \square

Remark 7 Note that if we represent structures by the array representation, then the transformation of Lemma 6 is actually polynomial even if we do not fix the arity of the vocabulary in advance. This follows from the fact that the array representation of the structure \mathcal{A}' (in the proof of the lemma) can be computed from the array representation of \mathcal{A} in linear time, uniformly over all vocabularies.

For a parameterized problem $P \subseteq STR \times \Pi^*$ and a class C of structures we let $P|_C$ denote the *restriction* of P to C . In particular,

$$MC(\Phi)|_{GRAPH} = \{(\mathcal{G}, \varphi) \mid \mathcal{G} \in GRAPH, \varphi \in \Phi, \mathcal{G} \models \varphi\}.$$

Note that for every class Φ of formulas and vocabulary τ the two problems $MC(\Phi[\tau])$ and $MC(\Phi)|_{STR[\tau]}$, though formally different, are essentially the same. Therefore we do not distinguish between them.

Corollary 8 (1) $MC(\text{FO}) \equiv_m^{\text{fpp}} MC(\text{FO})|_{GRAPH}$.

(2) For all $t \geq 1$ we have

$$MC(\Sigma_t) \leq_m^{\text{fpp}} MC(\Sigma_{t+1})|_{GRAPH} \quad \text{and} \quad MC(\Pi_t) \leq_m^{\text{fpp}} MC(\Pi_{t+1})|_{GRAPH}.$$

(3) For all $s \geq 2, t \geq 1$ we have

$$MC(\Sigma_t[s]) \equiv_m^{\text{fpp}} MC(\Sigma_t)|_{GRAPH} \quad \text{and} \quad MC(\Pi_t[s]) \equiv_m^{\text{fpp}} MC(\Pi_t)|_{GRAPH}.$$

We do not know whether $MC(\Sigma_t) \leq_m^{\text{fp}} MC(\Sigma_t)|_{GRAPH}$ and $MC(\Pi_t) \leq_m^{\text{fp}} MC(\Pi_t)|_{GRAPH}$ for $t \geq 1$.

4 Defining parameterized problems

Definability is the connection between arbitrary parameterized problems and our logical analysis that focuses on model-checking problems. In [9], Downey, Fellows, and Regan distinguish between two forms of definability, which we call *slicewise definability* and *Fagin definability*.

For a parameterized problem $P \subseteq \Sigma^* \times \Pi^*$ and $y \in \Pi^*$, we call $P \cap (\Sigma^* \times \{y\})$ the y th *slice* of P .

Definition 9 Let $P \subseteq STR \times \Pi^*$ be a parameterized problem and Φ a class of formulas. P is *slicewise Φ -definable* if there is a computable function $\delta : \Pi^* \rightarrow \Phi$ such that for all $\mathcal{A} \in STR$ and $y \in \Pi^*$ we have $(\mathcal{A}, y) \in P \iff \mathcal{A} \models \delta(y)$.

For example, the parameterized subgraph isomorphism problem SI is slice-wise Σ_1 -definable via the function $\delta : GRAPH \rightarrow \Sigma_1$ defined as follows: For a graph \mathcal{H} with vertex set $H = \{h_1, \dots, h_n\}$ of cardinality n , $\delta(\mathcal{H})$ is the sentence

$$\exists x_1 \dots \exists x_n \left(\bigwedge_{1 \leq i < j \leq n} x_i \neq x_j \wedge \bigwedge_{\substack{1 \leq i, j \leq n \\ E^{\mathcal{H}}_{h_i h_j}}} E x_i x_j \right).$$

Slicewise Φ -definability is closely related to the model-checking problem for Φ : If $P \subseteq STR \times \Pi^*$ is slicewise Φ -definable, then $P \leq_m^{\text{fp}} MC(\Phi)$. On the other hand, if Φ is a decidable set of formulas, then the problem $MC(\Phi)$ is slicewise Φ -definable for trivial reasons.

Definition 10 Let τ be a vocabulary, $P \subseteq STR[\tau] \times \mathbb{N}$ a parameterized problem, and Φ a class of formulas. P is Φ -Fagin-definable if there is a relation symbol $X \notin \tau$ (say, r -ary) and a sentence $\varphi \in \Phi[\tau \cup \{X\}]$ such that for all $\mathcal{A} \in STR[\tau]$ and $k \in \mathbb{N}$ we have $(\mathcal{A}, k) \in P$ if and only if there is a $B \subseteq A^r$ such that $|B| = k$ and $(\mathcal{A}, B) \models \varphi$. (Here (\mathcal{A}, B) denotes the $\tau \cup \{X\}$ -expansion of \mathcal{A} that interprets X by B .) Then, φ Fagin-defines P .

We often consider X as a relation *variable* and thus write $\varphi(X) \in \Phi[\tau]$ instead of $\varphi \in \Phi[\tau \cup \{X\}]$ and $\mathcal{A} \models \varphi(B)$ instead of $(\mathcal{A}, B) \models \varphi$.

For example, parameterized vertex cover VC is Fagin-defined by the formula

$$\varphi_{VC} := \forall y \forall z (Eyz \rightarrow (Xy \vee Xz)).$$

It is easy to see that every problem that is FO-Fagin-definable is also FO-slicewise definable. Indeed, if $P \subseteq STR[\tau] \times \mathbb{N}$ is Fagin-defined by a formula $\varphi(X) \in FO[\tau]$, where X is r -ary, then it is slicewise FO-defined via the function $\delta : \mathbb{N} \rightarrow FO[\tau]$ with $\delta(k) = \exists \bar{x}_1 \dots \exists \bar{x}_k \varphi_k$, where $\bar{x}_1, \dots, \bar{x}_k$ are r -tuples of distinct variables not occurring in φ and φ_k is the sentence obtained from φ by replacing each subformula of the form $X\bar{y}$ by $\bigvee_{i=1}^k \bar{x}_i = \bar{y}$. Fagin-definability implies slicewise definability also for other reasonable classes Φ of formulas, e.g., for the class Σ_1^1 of formulas of second-order logic of the form $\exists X_1 \dots \exists X_l \psi$, where X_1, \dots, X_l are relation variables and ψ is first-order.

The converse is certainly not true, not even for problems of the specific form $P \subseteq STR[\tau] \times \mathbb{N}$: It is obvious that there are slicewise FO-definable problems of arbitrarily high classical complexity. On the other hand we have the following characterization of Σ_1^1 -Fagin-definable problems.

Proposition 11 *Let $P \subseteq STR[\tau] \times \mathbb{N}$. Then, (1) and (2) are equivalent, where*

- (1) P is Σ_1^1 -Fagin-definable.
- (2) P is in NP (when considered as a classical problem) and for some $r \geq 1$, $(\mathcal{A}, k) \in P$ implies $k \leq |A|^r$.

Proof: The implication of (1) \Rightarrow (2) being clear, we turn to a proof of (2) \Rightarrow (1). Choose r according to (2). Then,

$$\{(\mathcal{A}, B) \mid B \subseteq A^r \text{ and } (\mathcal{A}, |B|) \in P\}$$

is a class of $\tau \cup \{X\}$ -structures in NP, where X is r -ary. By Fagin's Theorem [11], there is a Σ_1^1 -formula $\varphi(X)$ of vocabulary $\tau \cup \{X\}$ axiomatizing this class. Then, $\varphi(X)$ Σ_1^1 -Fagin-defines P . \square

Thus, slicewise definability is the more general notion. Nevertheless, Fagin definability can be very useful. We illustrate this by the following slight generalization of a result due to Cai and Chen [4], which is based on a result due to Kolaitis and Thakur [19] that syntactically characterizes certain minimization problems. It is motivated by comparing the formula φ_{VC} defining

the fixed-parameter tractable problem VC with the following formulas φ_{DS} and φ_{CLIQUE} defining the $W[1]$ -hard problems DS and $CLIQUE$, respectively:

$$\begin{aligned}\varphi_{DS} &:= \forall y \exists x (Xx \wedge (x = y \vee Exy)), \\ \varphi_{CLIQUE} &:= \forall y \forall z ((Xy \wedge Xz) \rightarrow (y = z \vee Eyz)).\end{aligned}$$

Observe that in φ_{DS} the relation variable X is in the scope of an existential quantifier and in φ_{CLIQUE} it occurs negatively.

Theorem 12 *Let τ be a vocabulary and $P \subseteq STR[\tau] \times \mathbb{N}$ a parameterized problem that is Fagin-defined by a $FO[\tau]$ -formula $\varphi(X)$ in which X does not occur in the scope of an existential quantifier or negation symbol. Then P is in FPT.*

Proof: For simplicity, let us assume that X is unary. Without loss of generality we can assume that $\varphi = \forall y_1 \dots \forall y_l \bigvee_{i=1}^m \bigwedge_{j=1}^p \psi_{ij}$, where each ψ_{ij} either is Xy_q for some $q \in \{1, \dots, l\}$, or a first-order formula with free variables in $\{y_1, \dots, y_l\}$ in which X does not occur. In a preprocessing phase we replace the latter ones by atomic formulas: For each such ψ_{ij} we introduce a new relation symbol R_{ij} whose arity matches the number of free variables of ψ_{ij} and let τ^* be the set of all these relation symbols. We let φ^* be the formula obtained from φ by replacing each subformula $\psi_{ij}(\bar{z})$ by $R_{ij}\bar{z}$. Then $\varphi^* = \forall y_1 \dots \forall y_l \bigvee_{i=1}^m \chi_i$, where each χ_i is a conjunction of atomic formulas.

For a structure $\mathcal{A} \in STR[\tau]$ we let \mathcal{A}^* be the τ^* -structure with universe A and with

$$R_{ij}^{\mathcal{A}^*} := \{\bar{a} \mid \mathcal{A} \models \psi_{ij}(\bar{a})\}.$$

Then we have for $B \subseteq A$,

$$\mathcal{A} \models \varphi(B) \iff \mathcal{A}^* \models \varphi^*(B).$$

Given \mathcal{A} , each $R_{ij}^{\mathcal{A}^*}$ can be computed in time $O(|A|^{\|\psi_{ij}\|})$, thus \mathcal{A}^* can certainly be computed in time $O(|A|^{\|\varphi\|})$.

For $1 \leq i \leq m$, $\bar{a} = a_1 \dots a_l \in A^l$, and $B \subseteq A$ we let

$$\beta(B, \bar{a}, i) := B \cup \{a_j \mid Xy_j \text{ is a conjunct of } \chi_i\}.$$

Since $\chi_i(X, \bar{y})$ is positive in X , the following two statements are equivalent for every B' with $B \subseteq B' \subseteq A$:

- $\mathcal{A}^* \models \chi_i(B', \bar{a})$.
- $\beta(B, \bar{a}, i) \subseteq B'$ and $\mathcal{A}^* \models \chi_i(\beta(B, \bar{a}, i), \bar{a})$.

This equivalence is used by Algorithm 1 to decide P .

Recall that, given a τ -structure \mathcal{A} and a parameter $k \in \mathbb{N}$, the algorithm is supposed to decide whether there is a $B \subseteq A$ with $|B| = k$ such that for all $\bar{a} \in A^l$ there is an i such that $\mathcal{A}^* \models \chi_i(B, \bar{a})$.

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CHECK- $\varphi(A \in STR[\tau], k \in \mathbb{N})$ 

1  compute  $\mathcal{A}^*$ 
2  initialize set  $\mathcal{S} \subseteq \text{Pow}(A)$  by  $\mathcal{S} := \{\emptyset\}$ 
3  for all  $\bar{a} \in A^l$  do
4      for all  $B \in \mathcal{S}$  do
5          if  $\mathcal{A}^* \not\models \bigvee_{i=1}^m \chi_i(B, \bar{a})$  then
6               $\mathcal{S} := \mathcal{S} \setminus \{B\}$ 
7              for  $i = 1$  to  $m$  do
8                  compute  $B' := \beta(B, \bar{a}, i)$ 
9                  if  $|B'| \leq k$  and  $\mathcal{A}^* \models \chi_i(B', \bar{a})$ 
10                     then  $\mathcal{S} := \mathcal{S} \cup \{B'\}$ 
11 if  $\mathcal{S} \neq \emptyset$ 
12     then accept
13     else reject.

```

Algorithm 1

The crucial observation to see that the algorithm is correct is that whenever the main loop in Lines 3–10 is entered, \mathcal{S} is a set of subsets $B \subseteq A$ such that $|B| \leq k$ and for all \bar{a} considered so far (in earlier runs through the loop) we have $\mathcal{A}^* \models \bigvee_{i=1}^m \chi_i(B, \bar{a})$.

To get a bound on the running time, we note that whenever a new set is added to \mathcal{S} (in Line 10) then it is an extension of a strictly smaller set that has just been removed from \mathcal{S} (in Line 6). Furthermore, for each set removed (in Line 6) at most m such extensions can be added. Thus an upper bound for the number of sets that can be in \mathcal{S} at any time is m^k . The main loop (in Lines 3–10) is called n^l times, where $n := |A|$. This gives an overall bound on the running time of $O(m^k \cdot n^l)$ plus the time needed to compute \mathcal{A}^* .

Since l does not depend on the instance, but just on the formula φ , this yields the fixed-parameter tractability of P . \square

Besides VC , many other parameterized problems can be shown to be fixed-parameter tractable by a simple application of this theorem. Let us consider one example in detail: The *valence* of a graph is the maximal number of neighbors a vertex in the graph has. For an $l \geq 1$, we consider the restriction of *DOMINATING SET* to graphs of valence at most l , i.e. the problem

<i>Input:</i>	Graph \mathcal{G} .
<i>Parameter:</i>	$k \in \mathbb{N}$.
<i>Question:</i>	Is the valence of \mathcal{G} at most l and does \mathcal{G} have a dominating set of size at most k ?

This problem is Fagin-defined by the following first-order formula:

$$\forall y \exists^{\leq l} z \text{ } E x z \wedge \forall y_0 \forall y_1 \dots \forall y_l (\forall z (E y_0 z \rightarrow \bigvee_{i=1}^l z = y_i) \rightarrow \bigvee_{i=0}^l X y_i).$$

($\exists^{\leq m} x \psi(x)$ abbreviates $\exists y_1 \dots \exists y_m \forall x (\psi(x) \rightarrow \bigvee_{i=1}^m x = y_i)$.)

Other examples of problems that can be shown to be in FPT by Theorem 11 are *HITTING SET FOR SIZE THREE SETS*, *MATRIX DOMINATION*, or *SHORT 3DIMENSIONAL MATCHING* of [8].

5 Homomorphisms, embeddings, and model-checking

In this section we establish a close relationship between the homomorphism problem, the embedding problem, and model-checking problems for Σ_1 -formulas.

A *homomorphism* from a τ -structure \mathcal{B} into a τ -structure \mathcal{A} is a mapping $h : B \rightarrow A$ such that for all $R \in \tau$ and tuples $\bar{b} \in R^{\mathcal{B}}$ we have $h(\bar{b}) \in R^{\mathcal{A}}$. The parameterized *HOMOMORPHISM PROBLEM (HOM)* is defined as follows:

HOM	<p><i>Input:</i> $A \in STR$.</p> <p><i>Parameter:</i> $B \in STR$.</p> <p><i>Question:</i> Is there a homomorphism from \mathcal{B} to \mathcal{A}?</p>
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A (*weak*) *embedding* of \mathcal{B} into \mathcal{A} is an injective homomorphism from \mathcal{B} to \mathcal{A} . Note that a graph \mathcal{H} is isomorphic to a subgraph of a graph \mathcal{G} in the usual graph theoretic sense, if there is an embedding of \mathcal{H} into \mathcal{G} . Thus the following parameterized *EMBEDDING PROBLEM (EMB)* is a generalization of the subgraph isomorphism problem *SI*:

EMB	<p><i>Input:</i> $A \in STR$.</p> <p><i>Parameter:</i> $B \in STR$.</p> <p><i>Question:</i> Is there an embedding of \mathcal{B} into \mathcal{A}?</p>
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The *Gaifman graph* of a τ -structure \mathcal{A} is the graph $\mathcal{G}(\mathcal{A})$ with universe A in which two elements $a \neq b$ are adjacent if there is an $R \in \tau$ and a tuple $\bar{a} \in R^{\mathcal{A}}$ such that both a and b occur in the tuple \bar{a} . For a class C of graphs we let $STR[C]$ denote the class of all structures whose Gaifman graph is in C . Note that $STR[GRAPH] = STR$. Furthermore, we let $STR[\tau, C] := STR[\tau] \cap STR[C]$ for every vocabulary τ and $STR[s, C] := STR[s] \cap STR[C]$ for every $s \geq 1$. We define restrictions $HOM[\dots]$ and $EMB[\dots]$ of the respective problems, where for every possible restriction ' \dots ' in the square brackets we require the *parameter* \mathcal{B} to belong to the class $STR[\dots]$. For example, we let

$$EMB[s, C] := EMB \cap (STR \times STR[s, C]).$$

Lemma 13 For all classes C of graphs and $s \geq 1$ we have $HOM[C] \leq_m^{\text{fpp}}$ $EMB[C]$ and $HOM[s, C] \leq_m^{\text{fpp}}$ $EMB[s, C]$.

Proof: Suppose we are given an instance $(\mathcal{A}, \mathcal{B})$ of $HOM[C]$. Let τ be the vocabulary of \mathcal{A} . Let \mathcal{A}_B be the τ -structure which for every element of \mathcal{A} contains $|B|$ duplicates, i.e. $\mathcal{A}_B := \mathcal{A} \times B$ and, for every r -ary $R \in \tau$,

$$R^{\mathcal{A}_B} := \{((a_1, b_1), \dots, (a_r, b_r)) \mid R^{\mathcal{A}} a_1 \dots a_r\}.$$

Then, every homomorphism $h : \mathcal{B} \rightarrow \mathcal{A}$ gives rise to an embedding $f : \mathcal{B} \rightarrow \mathcal{A}_B$ defined by $f(b) = (h(b), b)$ and every embedding $f : \mathcal{B} \rightarrow \mathcal{A}_B$ induces a homomorphism $h : \mathcal{B} \rightarrow \mathcal{A}$ defined by letting $h(b)$ be the projection on the first component of $f(b)$. \square

Note that, unless $\text{PTIME} = \text{NP}$, there is no polynomial-time reduction from $EMB[C]$ to $HOM[C]$ for the class C of all paths, because $HOM[C]$ can easily be seen to be in PTIME by a dynamic programming algorithm, whereas $EMB[2, C]$ is NP -complete by a reduction from $HAMILTONIAN PATH$. Thus we do not have $EMB[s, C] \leq_m^{\text{fpp}}$ $HOM[s, C]$ for any $s \geq 2$. However, we will see that for all $s \geq 2$ and C we actually have $HOM[s, C] \equiv_T^{\text{fpp}}$ $EMB[s, C]$.

Before we show this, we introduce two related model-checking problems. With each first-order formula φ we associate a graph $\mathcal{G}(\varphi)$. Its universe is $\text{var}(\varphi)$, the set of all variables in φ , and there is an edge between distinct $x, y \in \text{var}(\varphi)$ in $\mathcal{G}(\varphi)$ if φ has an atomic subformula in which both x, y occur.

Let φ^\neq be the formula obtained from φ by deleting all inequalities, i.e. all atomic subformulas of the form $x = y$ that occur in the scope of an odd number of negation symbols. We are also interested in $\mathcal{G}(\varphi^\neq)$. Let us see an example:

$$\varphi := \exists x_1 \dots \exists x_k \left(\bigwedge_{1 \leq i < j \leq k} \neg x_i = x_j \wedge \bigwedge_{i=1}^{k-1} E x_i x_{i+1} \right).$$

$\mathcal{G}(\varphi)$ is the complete graph with vertex set $\{x_1, \dots, x_k\}$, whereas $\mathcal{G}(\varphi^\neq)$ is the path $x_1 \dots x_k$. Note that φ says that a graph has a subgraph isomorphic to a path of length k , whereas φ^\neq says that a graph contains a homomorphic image of a path of length k . This generalizes to the following simple lemma, whose proof we omit.

Lemma 14 For every structure $\mathcal{B} \in \text{STR}$ there is a Σ_1 -sentence $\varphi_{\mathcal{B}}$ (whose quantifier-free part is a conjunction of literals) such that $\mathcal{G}(\varphi_{\mathcal{B}}^\neq) = \mathcal{G}(\mathcal{B})$, and for every structure \mathcal{A} we have:

$$\begin{aligned} \mathcal{A} \models \varphi_{\mathcal{B}} &\iff \text{There is an embedding of } \mathcal{B} \text{ into } \mathcal{A}. \\ \mathcal{A} \models \varphi_{\mathcal{B}}^\neq &\iff \text{There is a homomorphism from } \mathcal{B} \text{ to } \mathcal{A}. \end{aligned}$$

Furthermore, the mapping $\mathcal{B} \mapsto \varphi_{\mathcal{B}}$ is computable in linear time.

For $s \in \mathbb{N}$ and a class C graphs we let

$$\begin{aligned}\Sigma_1[s, C] &:= \{\varphi \in \Sigma_1[\tau] \mid \tau \text{ } s\text{-ary vocabulary, } \mathcal{G}(\varphi) \in C\}, \\ \Sigma_1^\neq[s, C] &:= \{\varphi \in \Sigma_1[\tau] \mid \tau \text{ } s\text{-ary vocabulary, } \mathcal{G}(\varphi^\neq) \in C\}.\end{aligned}$$

Furthermore, we let $\Sigma_1[C] := \bigcup_{s \geq 1} \Sigma_1[s, C]$ and $\Sigma_1^\neq[C] := \bigcup_{s \geq 1} \Sigma_1^\neq[s, C]$.

Lemma 14 implies that for every $s \geq 1$ and for every class C of graphs we have $HOM[s, C] \leq_m^{\text{fpp}} MC(\Sigma_1[s, C])$ and $EMB[s, C] \leq_m^{\text{fpp}} MC(\Sigma_1^\neq[s, C])$. Unless $\text{PTIME} \neq \text{NP}$, the converse of these statements is wrong. To see this, let C be the class of all graphs that only consist of isolated vertices, i.e. all graphs \mathcal{G} with $E^\mathcal{G} = \emptyset$. Then clearly $HOM[C]$ and $EMB[C]$ are in PTIME , but we can reduce the satisfiability problem for propositional formulas to $MC(\Sigma_1[1, C])$ and $MC(\Sigma_1^\neq[1, C])$.

Theorem 15 *Let C be a class of graphs and $s \geq 2$. Then*

$$HOM[s, C] \equiv_T^{\text{fpp}} EMB[s, C] \equiv_T^{\text{fpp}} MC(\Sigma_1^\neq[s, C]) \equiv_T^{\text{fpp}} MC(\Sigma_1[s, C]).$$

Proof: We have already seen that $HOM[s, C] \leq_T^{\text{fpp}} EMB[s, C]$ (Lemma 13) and that $EMB[s, C] \leq_T^{\text{fpp}} MC(\Sigma_1^\neq[s, C])$ (Lemma 14). To complete the cycle we shall prove that $MC(\Sigma_1^\neq[s, C]) \leq_T^{\text{fpp}} MC(\Sigma_1[s, C])$ and that $MC(\Sigma_1[s, C]) \leq_T^{\text{fpp}} HOM[s, C]$.

We first prove that $MC(\Sigma_1[s, C]) \leq_T^{\text{fpp}} HOM[s, C]$. Let $\varphi \in \Sigma_1[s, C]$, say, of vocabulary τ , and \mathcal{A} a τ -structure. We shall describe an algorithm that decides whether $\mathcal{A} \models \varphi$ using $HOM[s, C]$ as an oracle.

Let S be a binary relation symbol not contained in τ and $\tau' := \tau \cup \{S\}$. Furthermore, let \mathcal{A}' be the τ' -expansion of \mathcal{A} with $S^{\mathcal{A}'} = \emptyset$. Our algorithm first computes a sentence $\varphi' := \bigvee_{i=1}^m \exists \bar{x}_i \psi_i$ of vocabulary τ' such that

- (1) $\mathcal{A} \models \varphi$ if, and only if, $\mathcal{A}' \models \varphi'$.
- (2) For $1 \leq i \leq m$, the formula ψ_i is a conjunction of literals, and we have $\mathcal{G}(\psi_i) = \mathcal{G}(\varphi)$.

This can be achieved by first translating φ to a sentence whose quantifier-free part is in disjunctive normal form, then swapping existential quantifiers and the disjunction, and then adding dummy literals of the form $\neg Rxy$ until $G(\psi_i) = G(\varphi)$.

Let $\tau'' := \tau' \cup \{\overline{R} \mid R \in \tau'\} \cup \{E, \overline{E}\}$, where for all $R \in \tau'$ the symbol \overline{R} is a new relation symbol of the same arity as R and E, \overline{E} are new binary relation symbols. Let \mathcal{A}'' be the τ'' -expansion of \mathcal{A}' in which \overline{R} is interpreted as the complement of $R^{\mathcal{A}'}$ and E, \overline{E} are interpreted as equality and inequality, respectively. For $1 \leq i \leq m$, we define a τ'' -structure \mathcal{B}_i with $\mathcal{G}(\mathcal{B}_i) = \mathcal{G}(\varphi)$ such that $\mathcal{A}' \models \exists \bar{x}_i \psi_i$ if, and only if, there is a homomorphism from \mathcal{B}_i into \mathcal{A}'' .

We let \mathcal{B}_i be the τ'' -structure with universe $\text{var}(\psi_i)$ and

$$\begin{aligned} R^{\mathcal{B}_i} &:= \{\bar{y} \mid R\bar{y} \text{ is a literal of } \psi_i\} && (\text{for } R \in \tau'), \\ \overline{R}^{\mathcal{B}_i} &:= \{\bar{y} \mid \neg R\bar{y} \text{ is a literal of } \psi_i\} && (\text{for } R \in \tau'), \\ E^{\mathcal{B}_i} &:= \{yz \mid y = z \text{ is a literal of } \psi_i\}, \\ \overline{E}^{\mathcal{B}_i} &:= \{yz \mid \neg y = z \text{ is a literal of } \psi_i\}. \end{aligned}$$

It is obvious that \mathcal{B}_i does indeed have the desired property. Altogether, our construction yields a parameterized T-reduction.

It remains to prove that $MC(\Sigma_1^\neq[s, C]) \leq_{\text{T}}^{\text{fp}} MC(\Sigma_1[s, C])$. We use the so called *color coding* technique of Alon, Yuster, and Zwick [2].

Let $l \geq 1$ and X a set. An *l-perfect family of hash functions on X* is a family F of functions $f : X \rightarrow \{1, \dots, l\}$ such that for all subsets $Y \subseteq X$ of size l there is an $f \in F$ such that $f(Y) = \{1, \dots, l\}$ (i.e. on Y , f is one-to-one). Alon, Yuster, and Zwick [2] show that given $n, l \geq 1$, a l -perfect family of hash functions on $\{1, \dots, n\}$ of size $2^{O(l)} \cdot \log n$ can be computed in time $2^{O(l)} \cdot n \cdot \log n$.

For a similar reason as outlined above, without loss of generality we can restrict our attention to sentences $\varphi \in MC(\Sigma_1^\neq[s, C])$ whose quantifier-free part is a conjunction of literals. Given such a sentence $\varphi = \exists x_1 \dots \exists x_k \psi$, say of vocabulary τ , and a τ -structure \mathcal{A} , we define a family of sentences $\varphi_\gamma \in MC(\Sigma_1[s, C])$ and a family of structure \mathcal{A}_f such that $\mathcal{A} \models \varphi$ if, and only if, there is a γ and f such that $\mathcal{A}_f \models \varphi_\gamma$.

A *coloring of φ* is a function $\gamma : \{1, \dots, k\} \rightarrow \{1, \dots, k\}$ such that $\gamma(i) \neq \gamma(j)$ if $\neg x_i = x_j$ occurs in φ . For a coloring γ we let ψ_γ be the formula obtained from ψ by replacing all literals $\neg x_i = x_j$ by $C_{\gamma(i)}x_i \wedge C_{\gamma(j)}x_j$ (here, C_1, \dots, C_k are new unary ‘‘color’’ relation symbols) and let $\varphi_\gamma := \exists x_1 \dots \exists x_k \psi_\gamma$. Note that $\mathcal{G}(\varphi_\gamma) = \mathcal{G}(\varphi^\neq)$. Thus $\varphi_\gamma \in MC(\Sigma_1[s, C])$.

With every $f : A \rightarrow \{1, \dots, k\}$, which we call a *coloring of \mathcal{A}* , we let \mathcal{A}_f be the $\tau \cup \{C_1, \dots, C_k\}$ -expansion of \mathcal{A} with $C_i^{\mathcal{A}_f} := f^{-1}(i)$ for $1 \leq i \leq k$.

Observe that

$$\mathcal{A} \models \varphi \iff \text{there is a coloring } \gamma \text{ of } \varphi \text{ and a coloring } f \text{ of } \mathcal{A} \text{ such} \quad (3) \\ \text{that } \mathcal{A}_f \models \varphi_\gamma.$$

The problem is that there are $k^{|A|}$ colorings of \mathcal{A} , so (3) does not yet give rise to a parameterized reduction. The crucial trick is that to achieve this equivalence we do not have to consider all possible colorings f of \mathcal{A} . For $1 \leq l \leq k$, let F_l be an l -perfect family of hash-function on A and $F := \bigcup_{l=1}^k F_l$. We claim that

$$\mathcal{A} \models \varphi \iff \text{there is a coloring } \gamma \text{ of } \varphi \text{ and an } f \in F \text{ such that} \quad (4) \\ \mathcal{A}_f \models \varphi_\gamma.$$

The backward direction follows immediately from (3). For the forward direction, suppose that $\mathcal{A} \models \varphi$. Let $\bar{a} \in A^k$ such that $\mathcal{A} \models \psi(\bar{a})$. There is a function $f \in F$ whose restriction to $\{a_1, \dots, a_k\}$ is one-to-one. Define γ by $\gamma(i) := f(a_i)$. Then, γ is a coloring of φ , $\mathcal{A}_f \models \psi_\gamma(\bar{a})$ and hence, $\mathcal{A}_f \models \varphi_\gamma$.

Since the family F can be chosen sufficiently small and computed sufficiently fast, the equivalence (4) gives rise to a parameterized reduction. \square

Remark 16 We do not know if the parameterized T-reductions in Theorem 15 can be replaced by parameterized m-reductions. However, for many interesting classes C they can be replaced. One such example is the class of all graphs, as the following Proposition 17 shows. Similar techniques work for all classes C of graphs for which there exist an algorithm that, given a graph $\mathcal{H} \in C$, computes a connected $\mathcal{H}' \in C$ such that \mathcal{H} is a subgraph of \mathcal{H}' . For all such classes C we can show that

$$HOM[s, C] \equiv_m^{fp} EMB[s, C] \equiv_m^{fp} MC(\Sigma_1^\neq[s, C]) \equiv_m^{fp} MC(\Sigma_1[s, C]).$$

(for all $s \geq 1$).

The following result follows from the known fact that the problems *CLIQUE* and $MC(\Sigma_1[s])$ are complete for the class $W[1]$ (cf. [8, 21]), but we find it illustrative to give a direct proof.

Proposition 17 ([21]) *For all $s \geq 2$ we have $CLIQUE \equiv_m^{fp} MC(\Sigma_1[s])$.*

Proof: $CLIQUE \leq_m^{fp} MC(\Sigma_1[s])$ follows from the fact that *CLIQUE* is slicewise Σ_1 -definable, so we only have to prove the converse. By Corollary 8(3) it suffices to prove that $MC(\Sigma_1)|_{GRAPH} \leq_m^{fp} CLIQUE$.

An *atomic k-type* (in the theory of graphs) is a sentence $\theta(x_1, \dots, x_k)$ of the form $\bigwedge_{1 \leq i < j \leq k} \alpha_{ij}(x_i, x_j)$, where $\alpha_{ij}(x_i, x_j)$ is either $x_i = x_j$ or $E x_i x_j$ or $(\neg E x_i x_j \wedge \neg x_i = x_j)$ (for $1 \leq i < j \leq k$).

It is easy to see that there is a computable mapping f that associates with every EFO-sentence φ a sentence $\tilde{\varphi}$ of the form

$$\bigvee_{i=1}^l \exists x_1 \dots \exists x_k \theta_i(x_1, \dots, x_k), \quad (5)$$

where each θ_i is an atomic k -type, such that for all graphs \mathcal{G} we have $\mathcal{G} \models \varphi \iff \mathcal{G} \models \tilde{\varphi}$.

For each graph \mathcal{G} and each atomic k -type $\theta(\bar{x}) = \bigwedge_{1 \leq i < j \leq k} \alpha_{ij}(x_i, x_j)$ we define a graph $h(\mathcal{G}, \theta)$ as follows:

- The universe of $h(\mathcal{G}, \theta)$ is $\{1, \dots, k\} \times G$.
- There is an edge between (i, v) and (j, w) , for $1 \leq i < j \leq k$ and $v, w \in G$, if $\mathcal{G} \models \alpha_{ij}(v, w)$.

Then $h(\mathcal{G}, \theta)$ contains a k -clique if, and only if, $\mathcal{G} \models \exists \bar{x} \theta(\bar{x})$.

Now we are ready to define the reduction from $MC(\Sigma_1)|_{GRAPH}$ to *CLIQUE*. Given an instance (\mathcal{G}, φ) of $MC(\Sigma_1)|_{GRAPH}$, we first compute the sentence $\tilde{\varphi} = \bigvee_{i=1}^l \exists x_1 \dots \exists x_k \theta_i$. We let \mathcal{G}' be the disjoint union of the graphs $h(\mathcal{G}, \theta_i)$ for $1 \leq i \leq l$. Then \mathcal{G}' has a k -clique if, and only if, $\mathcal{G} \models \varphi$. \square

6 Sentences of bounded tree-width

Our main application of Theorem 15 will be to classes of graphs of bounded tree-width. Before we present it, we need to give a few basic facts on tree-decompositions of structures.

We think of a *tree* \mathcal{T} as directed from its root, which we denote by $r^{\mathcal{T}}$, to the leaves and thus can speak of a *child* and of the *parent* of a vertex.

A *tree-decomposition* of a τ -structure \mathcal{A} is a pair $(\mathcal{T}, (A_t)_{t \in T})$, where \mathcal{T} is a tree and $(A_t)_{t \in T}$ a family of subsets of A such that

- (1) For every $a \in A$, the set $\{t \in T \mid a \in A_t\}$ is non-empty and induces a subtree of \mathcal{T} (that is, is connected).
- (2) For every k -ary relation symbol $R \in \tau$ and all $a_1, \dots, a_k \in A$ such that $R^{\mathcal{A}}a_1, \dots, a_k$ there exists a $t \in T$ such that $a_1, \dots, a_k \in A_t$.

The *width* of a tree-decomposition $(\mathcal{T}, (A_t)_{t \in T})$ is $\max\{|A_t| \mid t \in T\} - 1$. The *tree-width* $\text{tw}(\mathcal{A})$ of \mathcal{A} is the minimal width of a tree-decomposition of \mathcal{A} .

Note that $\text{tw}(\mathcal{A}) \leq n - 1$, if $|A| = n$. Cycles have tree-width 2 (thus, their tree-width is independent of their length). The tree-width of a clique of size k is $k - 1$. Hence, it is maximal.

Often, we tacitly use the following simple lemma:

Lemma 18 *Let $(\mathcal{T}, (B_t)_{t \in T})$ be a tree-decomposition of a graph \mathcal{G} and $X \subseteq G$ a clique. Then there exists a node $t \in T$ such that $X \subseteq B_t$.*

Bodlaender proved the following deep result.

Theorem 19 ([3]) *There is an algorithm that, given a graph \mathcal{G} and $k \in \mathbb{N}$, computes a tree-decomposition of \mathcal{G} of width at most k if $\text{tw}(\mathcal{G}) \leq k$ or rejects if $\text{tw}(\mathcal{G}) > k$. The algorithm needs time $O(2^{ck^3} \cdot \|\mathcal{G}\|)$ for some (huge) constant c .*

To see that the analogous result holds for arbitrary structures, we observe that every tree-decomposition of a structure \mathcal{A} is also a tree-decomposition of its Gaifman graph $\mathcal{G}(\mathcal{A})$ and vice versa. This follows almost immediately from Lemma 18. It implies the following Corollary, which is of course well-known.

Corollary 20 *There is an algorithm that, given a structure \mathcal{A} and $k \in \mathbb{N}$, computes a tree-decomposition of \mathcal{A} of width at most k if $\text{tw}(\mathcal{A}) \leq k$ or rejects if $\text{tw}(\mathcal{A}) > k$. The algorithm needs time $O(2^{ck^3} \cdot \|\mathcal{A}\|)$ for some constant c .*

For $s \geq 1$, let W_s denote the class of all structures of tree-width at most s and $GW_s := \text{GRAPH} \cap W_s$. Note that $W_s \subseteq \text{STR}[s + 1]$, because a graph of tree-width s has clique number at most $s + 1$.² (The clique number of a graph \mathcal{G} is the maximal cardinality of a set of pairwise adjacent vertices of \mathcal{G} .)

²This is slightly imprecise, because a structure $\mathcal{A} \in W_s$ might have a vocabulary of arbitrarily high arity, as long as no tuple contained in a relation of \mathcal{A} consists of more than $s + 1$ distinct elements. But since any structure with this property can easily be transformed to an $(s + 1)$ -ary structure that is essentially the same, we decided to accept this imprecision in exchange for a simpler notation.

Plehn and Voigt [22] proved that for every $s \geq 1$ the following restriction of the subgraph isomorphism problem is fixed-parameter tractable:

Input: $\mathcal{G} \in \text{GRAPH}$.
Parameter: Graph $\mathcal{H} \in \text{GW}_s$.
Question: Is \mathcal{H} isomorphic to a subgraph of \mathcal{G} ?

Although we could base our following investigation on Plehn and Voigt's proof of this result, we prefer a different and somewhat simpler approach based on logic. Kolaitis and Vardi proved that, for every $s \geq 1$, the problem $\text{HOM}[\text{GW}_s]$ is fixed-parameter tractable. We sketch their proof and then apply Theorem 15 to the classes GW_s , for $s \geq 1$.

The starting point is the following result due to Vardi. For all $s \geq 1$, we let FO^s denote the class of first-order formulas with at most s variables.

Theorem 21 ([26]) *Let $s \geq 1$. Then $\text{MC}(\text{FO}^s)$ is solvable in time $O(|\varphi| \cdot |\mathcal{A}|^s)$.*

An algorithm solving $\text{MC}(\text{FO}^s)$ in the desired time proceeds as follows: Given an instance (\mathcal{A}, φ) , starting with the atomic subformulas the algorithm inductively computes the set $\{\bar{a} \mid \mathcal{A} \models \psi(\bar{a})\}$ for every subformula $\psi(\bar{x})$ of φ . For every subformula $\psi(\bar{x})$ this requires time $O(|\mathcal{A}|^s)$, because $\psi(\bar{x})$ has at most s free variables.

The following Lemma due to Kolaitis and Vardi relates the tree-width and the number of variables of a formula. Recall that $\Sigma_1[\text{GW}_s]$ denotes the class of all Σ_1 -formulas φ such that the graph $\mathcal{G}(\varphi)$ has tree-width at most s .

Lemma 22 ([20]) *Let $s \geq 1$. There is a linear time algorithm that, given a $\Sigma_1[\text{GW}_s]$ -sentence whose quantifier-free part is in disjunctive normal form, computes an equivalent existential FO^{s+1} -sentence.*

Proof: As $\exists x_1 \dots \exists x_k \bigvee_{i=1}^l \bigwedge_{j=1}^m \psi_{ij}$ is equivalent to $\bigvee_{i=1}^l \exists x_1 \dots \exists x_k \bigwedge_{j=1}^m \psi_{ij}$, we can assume that $l = 1$ and thus that $\varphi = \exists x_1 \dots \exists x_k \bigwedge_{j=1}^m \psi_j$ with literals ψ_j .

By Theorem 19 we can compute a tree-decomposition $(\mathcal{T}, (V_t)_{t \in T})$ of width s of $\mathcal{G}(\varphi)$ in linear time. For $V \subseteq \text{var}(\varphi) = \{x_1, \dots, x_k\}$, let α_V be the conjunction of the literals in φ with all variables in V ($\alpha_V := \text{TRUE}$ in case there is no such literal). Now, by induction from the leaves to the root $r := r^{\mathcal{T}}$ we define for $t \in T$ a formula φ_t with the properties (1) and (2), where

- (1) if s is the parent of t then the free variables of φ_t are in $V_t \cap V_s$;
- (2) φ_r is a sentence equivalent to φ ,

as follows:

If $t \in T$ has the children t_1, \dots, t_m and y_1, \dots, y_l are the variables in $V_t \setminus V_s$ for the parent s of t , then

$$\varphi_t := \exists y_1 \dots \exists y_l (\alpha_{V_t} \wedge \varphi_{t_1} \wedge \dots \wedge \varphi_{t_m})$$

(Thus, if t is a leaf $\varphi_t = \exists y_1 \dots \exists y_l \alpha_{V_t}$.)

By (1), every subformula of φ_r has at most $(s+1)$ free variables and hence (cf. [15]), φ_r is equivalent to an existential sentence in FO^{s+1} . \square

Remark 23 Let us make two remarks ruling out possible extensions of this lemma:

(1) Unless $\text{PTIME} = \text{NP}$, the result cannot be extended to arbitrary Σ_1 -sentences, because the existence of a polynomial time transformation that associates an equivalent FO^2 -sentence with every $\Sigma_1[GW_1]$ -sentence would imply $MC(\Sigma_1[GW_1]) \leq_m^{\text{P}} MC(\text{FO}^2)$. Recall that for the class C of all graphs consisting only of isolated vertices, which happens to be the class GW_0 , the problem $MC(\Sigma_1[C])$ is NP-complete. Thus for all $s \geq 0$, $MC(\Sigma_1[GW_s])$ is NP-complete. On the other hand, $MC(\text{FO}^{s+1})$ is in PTIME.

(2) However, since every Σ_1 -sentence φ is equivalent to a Σ_1 -sentence φ' whose quantifier-free part is in disjunctive normal form such that $\mathcal{G}(\varphi) = \mathcal{G}(\varphi')$, the lemma implies that every $\Sigma_1[GW_s]$ -sentence is equivalent to an existential FO^{s+1} -sentence. This result cannot be extended to arbitrary EFO-sentences. This follows from the simple fact that, for all $s \geq 1$, there is no FO^s -sentence saying that a graph has an $(s+1)$ -clique. However, the following EFO-sentence φ , whose graph $\mathcal{G}(\varphi)$ has tree-width 2, says that a graph has an $(s+1)$ -clique

$$\exists x_0 \dots \exists x_s \bigwedge_{0 \leq i < j \leq s} \exists y \exists z (y = x_i \wedge z = x_j \wedge Eyz).$$

By Lemma 14 and Theorem 21, we obtain the following Corollary:

Corollary 24 ([20]) *Let $s \geq 1$. Then the problem $\text{HOM}[GW_s]$ is in PTIME.*

Now we can apply Theorem 15. Just remember that for all $s \geq 1$ we have $\text{STR}[GW_s] = W_s \subseteq \text{STR}[s+1]$.

Corollary 25 *Let $s \geq 1$. Then the problems $\text{EMB}[GW_s]$, $MC(\Sigma_1[GW_s])$, and $MC(\Sigma_1^\neq[GW_s])$ are in FPT.*

Papadimitriou and Yannakakis [21] proved the model-checking results of this corollary for the special case of acyclic conjunctive queries.

7 FO-model-checking on graphs with excluded minors

The fixed-parameter tractability results of the previous section were obtained by putting syntactical restrictions on the sentences, i.e. the *parameter* of the model-checking problem. In this section we put restrictions on the structures, i.e. the *input* of the model-checking problem.

Recall the definition of the parameterized problem $MC(\Phi)|_C$, for a class Φ of formulas and a class C of structures:

$MC(\Phi) _C$	<p style="margin: 0;"><i>Input:</i> $\mathcal{A} \in STR.$</p> <p style="margin: 0;"><i>Parameter:</i> $\varphi \in \Phi.$</p> <p style="margin: 0;"><i>Question:</i> Is $\mathcal{A} \in C$ and $\mathcal{A} \models \varphi?$</p>
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Our starting point is the following theorem due to Courcelle. Remember that *monadic second-order logic* is the extension of first-order logic where one is allowed to quantify not only over individual elements of a structure but also over sets of elements. MSO denotes the class of all formulas of monadic second-order logic. Remember that W_s denotes the class of all structures of tree-width at most s (for $s \geq 0$).

Theorem 26 ([7]) *Let $s \geq 0$. Then $MC(MSO)|_{W_s}$ is in FPT.*

A graph \mathcal{H} is a *minor* of a graph \mathcal{G} (we write $\mathcal{H} \preceq \mathcal{G}$) if \mathcal{H} can be obtained from a subgraph of \mathcal{G} by contracting edges. \mathcal{H} is an *excluded minor* for a class C if \mathcal{H} is not a minor of any graph in C . Note that a class C of graphs has an excluded minor if, and only if, there is an $n \in \mathbb{N}$ such that \mathcal{K}_n is an excluded minor for C .

Examples of classes of graphs with an excluded minor are classes of graphs of bounded tree-width or classes of graphs embeddable in a fixed surface.

Recall that for a class C of graphs, $STR[C]$ denotes the class of all structures whose Gaifman graph is in C .

Theorem 27 *Let C be a PTIME-decidable class of graphs with an excluded minor. Then $MC(FO)|_{STR[C]}$ is in FPT.*

The rest of this section is devoted to the proof of this theorem, which needs some preparation.

A class C of graphs is called *minor closed* if for all $\mathcal{G} \in C$ and $\mathcal{H} \preceq \mathcal{G}$ we have $\mathcal{H} \in C$. Robertson and Seymour proved:

Theorem 28 ([23]) *Every minor-closed class of graphs is PTIME-decidable.*

This immediately yields:

Corollary 29 *Let $C \subsetneq GRAPH$ be minor closed. Then $MC(FO)|_{STR[C]}$ is in FPT.*

Recall the definition of the Gaifman graph $\mathcal{G}(\mathcal{A})$ of a structure \mathcal{A} (cf. Page 15). The distance $d^{\mathcal{A}}(a, b)$ between $a, b \in A$ is the length of the shortest path from a to b in $\mathcal{G}(\mathcal{A})$. For $r \in \mathbb{N}$ and $a \in A$, the r -ball around a is the set $B_r^{\mathcal{A}}(a) := \{b \in A \mid d^{\mathcal{A}}(a, b) \leq r\}$. For an $X \subseteq A$, $\langle X \rangle^{\mathcal{A}}$ denotes the substructure induced by \mathcal{A} on X , i.e. the structure with universe X and $R^{\langle X \rangle^{\mathcal{A}}} = R^{\mathcal{A}} \cap X^r$ for all r -ary relation symbols R in the vocabulary of \mathcal{A} . Furthermore, we let $\mathcal{A} \setminus X := \langle A \setminus X \rangle^{\mathcal{A}}$.

The *local tree-width* of \mathcal{A} is the function $\text{ltw}(\mathcal{A}) : \mathbb{N} \rightarrow \mathbb{N}$ defined by

$$\text{ltw}(\mathcal{A})(r) := \max\{\text{tw}(\langle B_r^{\mathcal{A}}(a) \rangle^{\mathcal{A}}) \mid a \in A\}.$$

For functions $f, g : \mathbb{N} \rightarrow \mathbb{N}$ we write $f \leq g$ if $f(n) \leq g(n)$ for all $n \in \mathbb{N}$. A class C of structures has *bounded local tree-width* if there is a function $\lambda : \mathbb{N} \rightarrow \mathbb{N}$ such that for all $\mathcal{A} \in C$ we have $\text{ltw}(\mathcal{A}) \leq \lambda$.

Theorem 30 ([13]) *Let C be a PTIME-decidable class of structures of bounded local tree-width. Then $MC(\text{FO})|_C$ is in FPT.*

For $\lambda : \mathbb{N} \rightarrow \mathbb{N}$ we let

$$GL(\lambda) := \{\mathcal{G} \in \text{GRAPH} \mid \forall \mathcal{H} \preceq \mathcal{G} : \text{ltw}(\mathcal{H}) \leq \lambda\},$$

and, for $\mu \in \mathbb{N}$,

$$B(\lambda, \mu) := \{\mathcal{A} \in \text{STR} \mid \exists X \subseteq A (|X| \leq \mu \wedge \mathcal{G}(\mathcal{A} \setminus X) \in GL(\lambda))\}.$$

Note that the clique-number of a graph in $GL(\lambda)$ is at most $\lambda(1) + 1$, thus the clique-number of a graph in $B(\lambda, \mu)$ is at most $\mu + \lambda(1) + 1$. This implies that $B(\lambda, \mu) \subseteq \text{STR}[\mu + \lambda(1) + 1]$.³

Lemma 31 *Let $\lambda : \mathbb{N} \rightarrow \mathbb{N}$ and $\mu \in \mathbb{N}$. Then $MC(\text{FO})|_{B(\lambda, \mu)}$ is in FPT.*

Proof: The class $GL(\lambda)$ of graphs is minor closed and hence PTIME-decidable by Theorem 28. This implies that $B(\lambda, \mu)$ is PTIME-decidable.

Then for $\mu = 0$ the statement follows from Theorem 30. The case $\mu > 0$ can be reduced to the case $\mu = 0$ as follows: For every $\mathcal{A} \in B(\lambda, \mu)$ and sentence $\varphi \in \text{FO}$, we define a structure $\mathcal{A}^* \in B(\lambda, 0)$ and a sentence $\varphi^* \in \text{FO}$ in such a way that $\mathcal{A} \models \varphi$ if, and only if, $\mathcal{A}^* \models \varphi^*$, and the mappings $\mathcal{A} \mapsto \mathcal{A}^*$ and $\varphi \mapsto \varphi^*$ are computable in polynomial time.

So suppose we are given $\mathcal{A} \in B(\lambda, \mu)$ and $\varphi \in \text{FO}$. For simplicity, we assume that their vocabulary consists of a single binary relation symbol E .

Let $X \subseteq A$ such that $|X| \leq \mu$ and $\mathcal{A} \setminus X \in B(\lambda, 0)$. Such an X can be computed in polynomial time because the class $B(\lambda, 0)$ is PTIME-decidable. Say, $X = \{a_1, \dots, a_\mu\}$.

³Cf. Footnote ² on Page 20.

Let $\tau := \{E, P_1, \dots, P_\mu, Q_1, \dots, Q_\mu, R_1, \dots, R_\mu\}$, where the P_i , Q_i , and R_i are unary. We let $A^* := A$, $E^{A^*} := E^A \cap (A \setminus X)^2$, and, for $1 \leq i \leq \mu$

$$\begin{aligned} P_i^{A^*} &:= \{a_i\}, \\ Q_i^{A^*} &:= \{b \in A \mid E^A a_i b\}, \\ R_i^{A^*} &:= \{b \in A \mid E^A b a_i\}. \end{aligned}$$

Furthermore, we let φ^* be the sentence obtained from φ by replacing each subformula Exy by

$$Exy \vee \bigvee_{i=1}^{\mu} ((P_i x \wedge Q_i y) \vee (P_i y \wedge R_i x)).$$

Clearly, these definitions lead to the desired result. \square

To complete the proof of Theorem 27 we use a decomposition theorem for non-trivial minor-closed classes of graphs that roughly says that all graphs in such a class are built up in a tree-like manner from graphs in a $B(\lambda, \mu)$. It is based on Robertson and Seymour's deep structure theory for graphs without \mathcal{K}_n -minors. The precise statement requires some new notation. Let $(\mathcal{T}, (A_t)_{t \in \mathcal{T}})$ be a tree-decomposition of a structure \mathcal{A} . The *torso* of this decomposition at $t \in \mathcal{T}$, denoted by $[A_t]$, is the graph with universe A_t and an edge between two distinct vertices $a, b \in A_t$ if either there is an edge between a and b in the Gaifman graph $\mathcal{G}(\mathcal{A})$ or there exists an $s \in \mathcal{T} \setminus \{t\}$ such that $a, b \in A_s$. $(\mathcal{T}, (A_t)_{t \in \mathcal{T}})$ is a tree-decomposition *over* a class C of graphs if all its torsos belong to C .

Theorem 32 ([14]) *Let C be a class of graphs with an excluded minor. Then there exist $\lambda : \mathbb{N} \rightarrow \mathbb{N}$ and $\mu \in \mathbb{N}$ such that every $\mathcal{G} \in C$ has a tree-decomposition over $B(\lambda, \mu)$.*

Furthermore, given \mathcal{G} such a decomposition can be computed in PTIME.

Clearly, this theorem implies the analogous statement for all structures in $STR[C]$.

The *adhesion* of a tree-decomposition $(\mathcal{T}, (A_t)_{t \in \mathcal{T}})$ is $\max\{|A_s \cap A_t| \mid E^{\mathcal{T}} st\}$. The *clique number* of a class of structures is the maximum of the clique numbers of the Gaifman graphs of structures in C , if this maximum exists, or ∞ otherwise. Note that if $(\mathcal{T}, (A_t)_{t \in \mathcal{T}})$ is a decomposition over a class C then the clique-number of C is an upper bound for the adhesion of $(\mathcal{T}, (A_t)_{t \in \mathcal{T}})$. Remembering that the clique-number of $B(\lambda, \mu)$ is $\lambda(1) + \mu + 1$, we see that the adhesion of a tree-decomposition over $B(\lambda, \mu)$ is at most $\lambda(1) + \mu + 1$.

Proof (of Theorem 27): Let C be a PTIME-decidable class of graphs with an excluded minor and λ, μ such that every $\mathcal{G} \in C$ has a tree-decomposition over $B(\lambda, \mu)$. Let $\nu := \lambda(1) + \mu + 1$.

We shall describe an algorithm that, given $\mathcal{A} \in STR[C]$ and $\varphi \in \text{FO}$, decides if $\mathcal{A} \models \varphi$.

So let $\mathcal{A} \in STR[C]$, say, of vocabulary τ and $\varphi \in FO[\tau]$. Our algorithm starts by computing a tree-decomposition $(\mathcal{T}, (A_t)_{t \in T})$ of \mathcal{A} over $B(\lambda, \mu)$. For $t \in T$ we let $A_{\geq t} := \bigcup_{u \geq t} A_u$ ($u \geq t$ if there is a path from t to u in the (directed) tree \mathcal{T}). In particular, $A_{\geq r} = A$ for the root $r := r^{\mathcal{T}}$ of \mathcal{T} .

Furthermore, we let $B_r := \emptyset$ and $B_t := A_t \cap A_s$ for $t \in T \setminus \{r\}$ with parent s . Recall that the adhesion of $(\mathcal{T}, (A_t)_{t \in T})$ is at most ν . Thus $|B_t| \leq \nu$ for $t \in T$.

The *quantifier rank* of a first-order formula is the maximal depth of nested quantifiers in this formula. Let q be the quantifier rank of φ . Simple techniques from logic show that there is an algorithm that, given a vocabulary τ and $q, m \in \mathbb{N}$, computes a finite set $\Phi_{\tau, q, m}$ of first-order formulas of vocabulary τ of quantifier rank $\leq q$ with free variables among v_1, \dots, v_m such that every such formula is equivalent to a formula in $\Phi_{\tau, q, m}$. Without loss of generality we can assume that $\varphi \in \Phi_{\tau, q, m}$ (otherwise we can compute a $\varphi' \in \Phi_{\tau, q, m}$ equivalent to φ and work with φ').

A (τ, q, m) -*type* is a subset of $\Phi_{\tau, q, m}$. Given a τ -structure \mathcal{A}' and $B = \{b_1, \dots, b_m\} \subseteq A'$ let $\text{tp}_q^{\mathcal{A}'}(B)$ – more precisely, $\text{tp}_q^{\mathcal{A}'}(b_1, \dots, b_m)$ – be the (τ, q, m) -type

$$\text{tp}_q^{\mathcal{A}'}(B) := \{\psi(v_1, \dots, v_m) \in \Phi_{\tau, q, m} \mid \mathcal{A}' \models \psi(b_1, \dots, b_m)\}.$$

We come back to our structure \mathcal{A} and the tree-decomposition $(\mathcal{T}, (A_t)_{t \in T})$. By induction from the leaves to the root, for every $t \in T$ we compute $\text{tp}_q^{\langle A_{\geq t} \rangle^{\mathcal{A}}}(B_t)$, which for brevity we denote by $\text{tp}_q^{\geq t}(B_t)$. Since $B_r = \emptyset$, $\text{tp}_q^{\geq r}(B_r)$ is a set of sentences, and we have $\mathcal{A} \models \varphi$ if, and only if, $\varphi \in \text{tp}_q^{\geq r}(B_r)$.

So let t be a vertex of \mathcal{T} and assume that we have already computed $\text{tp}_q^{\geq u}(B_u)$ for all children u of t (if there are any). (Actually, the case that t has no children is much simpler than the following general case, because it is a direct application of Lemma 31.)

For every (τ, q, m) -type Φ we introduce a new $(m+1)$ -ary relation symbol R_{Φ} . Furthermore, we let P_1, \dots, P_{ν} be new unary relation symbols and $\tau' := \tau \cup \{R_{\Phi} \mid m \leq \nu, \Phi \text{ a } (\tau, q, m)\text{-type}\} \cup \{P_1, \dots, P_{\nu}\}$. In four steps, we define a τ' -structure $\tilde{\mathcal{A}}_t$ that contains all the relevant information to compute $\text{tp}_q^{\geq t}(B_t)$. In the first three steps we define “intermediate” structures $\mathcal{A}_t^1, \mathcal{A}_t^2, \mathcal{A}_t^3$.

- (1) \mathcal{A}_t^1 is the induced subgraph of \mathcal{G} with universe A_t .
- (2) Suppose that $B_t = \{b_1, \dots, b_m\}$ for an $m \leq \nu$. Then \mathcal{A}_t^2 is the $\{E, P_1, \dots, P_{\nu}\}$ -expansion of \mathcal{A}_t^1 with $P_i^{\mathcal{A}_t^2} := \{b_i\}$ for $1 \leq i \leq m$ and $P_i^{\mathcal{A}_t^2} := \emptyset$ for $m+1 \leq i \leq \nu$.
- (3) \mathcal{A}_t^3 is obtained from \mathcal{A}_t^2 by adding a new vertex c_u for every child u of t and edges from c_u to all vertices of B_u .
- (4) $\tilde{\mathcal{A}}_t$ is the τ' -expansion of $\tilde{\mathcal{A}}_t^3$ with

$$R_{\Phi}^{\tilde{\mathcal{A}}_t} := \{(d_1, \dots, d_{m_u}, c_u) \mid u \text{ child of } t, \\ B_u = \{d_1, \dots, d_{m_u}\}, \text{tp}_q^{\geq u}(B_u) = \Phi\}.$$

MODELCHECK_C($\mathcal{A} \in STR, \varphi \in FO$)

```

1  if  $\mathcal{G}(\mathcal{A}) \notin C$  then reject
2  compute tree-decomposition  $(\mathcal{T}, (A_t)_{t \in T})$  of  $\mathcal{A}$  over  $B(\lambda, \mu)$ 
3   $q := \text{qr}(\varphi)$ ,  $\tau := \text{vocabulary of } \varphi$ 
4  for  $m = 0$  to  $\nu$ 
5    compute  $\Phi_{\tau, q, m}$ 
6  for all  $t \in T$  (from the leaves to the root)
7    compute  $\tilde{\mathcal{A}}_t$ 
8     $\text{tp}_q^{\geq t}(B_t) := \emptyset$ 
9     $m := |B_t|$ 
10   for all  $\psi \in \Phi_{\tau, q, m}$ 
11     compute  $\tilde{\psi}$ 
12     if  $\tilde{\mathcal{A}}_t \models \tilde{\psi}$ 
13       then  $\text{tp}_q^{\geq t}(B_t) := \text{tp}_q^{\geq t}(B_t) \cup \{\psi\}$ 
14 if  $\varphi \in \text{tp}_q^{\geq r}(B_r)$ 
15   then accept
16   else reject.

```

Algorithm 2

Standard Ehrenfeucht-Fraïssé type methods show that there is a computable function that associates with every formula $\psi \in \Phi_{\tau, q, m}$ a sentence $\tilde{\psi} \in FO[\tau']$ such that $\psi \in \text{tp}_q^{\geq t}(B_t)$ if, and only if, $\tilde{\mathcal{A}}_t \models \tilde{\psi}$.

We claim that $\tilde{\mathcal{A}}_t \in B(\lambda + 1, \mu)$. To see this, observe that the Gaifman graph $G(\tilde{\mathcal{A}}_t)$ is the graph obtained from the torso $[A_t]$ by adding the vertices c_u and edges between c_u and every element of B_u . Recall that, by the definition of the torso, each B_u is a clique in $[A_t]$. It is an easy consequence of Lemma 18 that adding vertices and connecting them with cliques can only increase the tree-width of a graph by one. This implies the claim.

Now we can put everything together and obtain an algorithm deciding $MC(FO)_{STR[C]}$, a high-level description of which is given as Algorithm 2. Its correctness is straightforward. Let us just have a look at the running time: Let n be the size of the input structure. Then Lines 1 and 2 require time polynomial in n (independently of φ). The time required in Lines 3–5 only depends on φ . The main loop in Lines 6–13 is called $|T|$ times, which is polynomial in n . Computing $\tilde{\mathcal{A}}_t$ is polynomial in $|A_t|$ and the number of children of t since we have already computed $\text{tp}_q^{\geq u}(B_u)$ for all children u of t (with constants heavily depending on $\|\varphi\|$). The main task is to decide whether $\tilde{\mathcal{A}}_t \models \tilde{\psi}$ in Line 12; by Lemma 31 this is fixed-parameter tractable because $\tilde{\mathcal{A}}_t \in B(\lambda + 1, \mu)$. The time required in Lines 14–16 again only depends on φ . \square

A consequence of our results is that first-order definable parameterized problems are fixed-parameter tractable when restricted to minor non-trivial classes of structures.

Corollary 33 *Let C be a PTIME-decidable class of graphs with an excluded minor and $P \subseteq STR \times \Pi^*$ a parameterized problem that is slice-wise FO-definable. Then $P|_{STR[C]}$ is in FPT.*

8 A logical characterization of fixed-parameter tractability

In this section we give a characterization of FPT in the spirit of descriptive complexity theory.

We briefly review some facts from this area (see [10, 17] for details). It is common in descriptive complexity theory to identify decision problems, usually modeled by languages $L \subseteq \Sigma^*$ for a finite alphabet Σ , with classes of finite structures. More precisely, one identifies problems with classes of *ordered finite structures*. An *ordered structure* is a structure whose vocabulary contains the binary relation symbol \leq , and this symbol is interpreted as a linear order of the universe. ORD denotes the class of all ordered structures. In this section, τ always denotes a vocabulary that contains \leq .

One of the most important results in descriptive complexity theory is the Immerman-Vardi Theorem [16, 25] saying that a class of ordered structures is in PTIME if, and only if, it is definable in least-fixed point logic FO(LFP). More concisely,

$$\text{PTIME} = \text{FO(LFP)}.$$

We prove a similar result characterizing the class FPT in terms of the finite variable least fixed-point logics LFP^s , for $s \geq 1$, which were introduced by Kolaitis and Vardi [18]. Analogously to the classical setting we model parameterized problems by subsets of $ORD[\tau] \times \mathbb{N}$, for some τ .

Theorem 34 *A parameterized problem $P \subseteq ORD[\tau] \times \mathbb{N}$ is in FPT if, and only if, there is an $s \geq 1$ such that P is slice-wise LFP^s -definable. More concisely, we may write*

$$\text{FPT} = \bigcup_{s \geq 1} LFP^s.$$

In the proof of this result we assume that the reader is familiar with descriptive complexity theory, in particular with least fixed-point logic and the proof of the Immerman-Vardi Theorem. Those who are not may safely skip the rest of this section.

We first recall the definition of LFP^s : In the terminology of [10] (p. 174), LFP^s -sentences are FO(LFP)-sentences in the form

$$\exists \bar{y} [\text{S-LFP}_{\bar{x}_1, X, \dots, \bar{x}_m, X_m} \varphi_1, \dots, \varphi_m] \bar{y},$$

<p>DECIDE-$P(\mathcal{A} \in ORD, k \in \mathbb{N})$</p> <ol style="list-style-type: none"> 1 compute $\delta(k) \in \text{LFP}^s$ 2 compute $\mathbb{A}_{\delta(k)}$ (cf. (1)) 3 simulate $\mathbb{A}_{\delta(k)}$ on input \mathcal{A} 4 if $\mathbb{A}_{\delta(k)}$ accepts \mathcal{A} 5 then <u>accept</u> 6 else <u>reject</u>.

Algorithm 3

where $\varphi_1, \dots, \varphi_m$ are first-order formulas with at most s individual variables. (That is, LFP^s -sentences are existential closures of simultaneous fixed-points over FO^s -formulas.)

We use the following two facts, the first implicit in [26] and the second in the proof of the Immerman-Vardi Theorem (cf. [10]). Fix τ and $s \in \mathbb{N}$ and let n always denote the size of the input structure.

- (1) There is a computable function that associates an $O(n^s)$ -algorithm \mathbb{A}_φ with each $\varphi \in \text{LFP}^s[\tau]$ such that \mathbb{A}_φ accepts a structure $\mathcal{A} \in ORD[\tau]$ if, and only if, \mathcal{A} satisfies φ .
- (2) There is a $t \in \mathbb{N}$ and a computable function that associates with every $O(n^s)$ -algorithm \mathbb{A} accepting a class $C \subseteq ORD[\tau]$ a sentence $\varphi_{\mathbb{A}} \in \text{LFP}^t[\tau]$ such that a τ -structure \mathcal{A} satisfies $\varphi_{\mathbb{A}}$ if, and only if, $\mathcal{A} \in C$.

There is a slight twist in (2). When proving it one usually assumes that all structures are sufficiently large, in particular larger than the constant hidden in $O(n^s)$. This way it can be assumed that the algorithm is actually an n^{s+1} -algorithm (without any hidden constants.) Then one argues that small structures are no problem because they can be described up to isomorphism in first-order logic. When restricting the number of variables, one has to be careful with such an argument. Luckily, we are safe here because we only consider ordered structures, and there is a $t \in \mathbb{N}$ (depending on τ) such that every structure $\mathcal{A} \in ORD[\tau]$ can be characterized up to isomorphism by an FO^t -sentence.

Proof (of Theorem 34): For the backward direction, suppose that $P \subseteq ORD[\tau] \times \mathbb{N}$ is slicewise LFP^s -definable via $\delta : \mathbb{N} \rightarrow \text{LFP}^s$. Then Algorithm 3 shows that P is in FPT. The crucial fact is that Lines 1 and 2 do not depend on the input structure \mathcal{A} and Line 3 requires time $O(n^s)$.

For the forward direction, suppose that $P \subseteq ORD[\tau] \times \mathbb{N}$ is in FPT. Choose $f : \mathbb{N} \rightarrow \mathbb{N}$, $c \in \mathbb{N}$ and an algorithm \mathbb{A} deciding P in time $f(k) \cdot n^c$. The algorithm \mathbb{A} gives rise to a sequence \mathbb{A}_k ($k \geq 1$) of algorithms, where \mathbb{A}_k decides the class $\{\mathcal{A} \mid (\mathcal{A}, k) \in P\} \subseteq ORD[\tau]$ in time $O(n^c)$. Then (2) yields the desired slicewise definition of P . \square

9 Beyond FPT

In this last section we deal with problems that are apparently not fixed-parameter tractable. Of course, for some combinatorial problems one can prove that they are not fixed-parameter tractable, but here we consider a class of problems and a hierarchy that are related to FPT as NPTIME and the polynomial hierarchy are to PTIME. These issues have been extensively studied by Downey and Fellows (cf. [8]).

For a parameterized problem P , we let $[P]_m^{\text{fp}} := \{P' \mid P' \leq_m^{\text{fp}} P\}$, and for a class \mathcal{P} of parameterized problems $[\mathcal{P}]_m^{\text{fp}} := \bigcup_{P \in \mathcal{P}} [P]_m^{\text{fp}}$. We mentioned that Downey and Fellows introduced the so-called W-hierarchy. We will not give the original definition of this hierarchy in terms of weighted circuit satisfiability problems. Later, we will present a characterization in terms of certain model checking problems; for now let us just set $\text{W}[1] := [\text{MC}(\Sigma_1)|_{\text{GRAPH}}]_m^{\text{fp}}$. The class $\text{W}[1]$ plays a similar role in parameterized complexity theory as NP does in classical complexity theory. This point of view is certainly reasonable as a pragmatic guideline, because it has turned out that most natural parameterized problems are either in FPT or $\text{W}[1]$ -hard. But there are also “structural” similarities between $\text{W}[1]$ -completeness and NP-completeness. They are nicely illustrated by the following result that Downey and Fellows call an “analog of Cook’s Theorem”:

Theorem 35 ([5]) *The parameterized problem SHORT TURING MACHINE ACCEPTANCE (NM) is $\text{W}[1]$ -complete, where*

NM	<p><i>Input:</i> A non-deterministic Turing machine M.</p> <p><i>Parameter:</i> $k \in \mathbb{N}$.</p> <p><i>Question:</i> Does M accept the empty word in at most k steps?</p>
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In our notation, we may write $[NM]_m^{\text{fp}} = \text{W}[1]$. It is now very natural to define a “parameterized analogue of the polynomial hierarchy”, which we call the *A-hierarchy*, by letting $\text{A}[t] := [AM_t]_m^{\text{fp}}$ for all $t \geq 1$, where

AM_t	<p><i>Input:</i> An alternating Turing machine M whose initial state is existential.</p> <p><i>Parameter:</i> $k \in \mathbb{N}$.</p> <p><i>Question:</i> Does M accept the empty word in at most k steps with at most t alternations?</p>
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Note that $NM = AM_1$, thus $\text{W}[1] = \text{A}[1]$. Recalling that

$$\text{W}[1] = [\text{MC}(\Sigma_1)|_{\text{GRAPH}}]_m^{\text{fp}},$$

our following theorem can be seen as a natural generalization of Theorem 35. Remember that, for every $s \geq 1$, $\Sigma_1[s]$ denotes the class of all Σ_1 -formulas whose vocabulary is at most s -ary.

Theorem 36 *For all $t \geq 1$, the problem $MC(\Sigma_t)|_{GRAPH}$ is $A[t]$ -complete under parameterized m -reductions. Thus*

$$A[t] = [MC(\Sigma_t)|_{GRAPH}]_m^{\text{fp}} = \bigcup_{s \geq 1} [MC(\Sigma_t[s])]_m^{\text{fp}}.$$

Proof: The second equality is just a different formulation of Corollary 8(3).

To prove that $MC(\Sigma_t)|_{GRAPH} \leq_m^{\text{fp}} AM_t$, we first observe that for every graph \mathcal{G} and every quantifier-free formula $\theta(x_1, \dots, x_m)$ there is a deterministic Turing machine $M(\mathcal{G}, \theta)$ with input alphabet G that accepts an input word $a_1 \dots a_m$ over G if, and only if, $\mathcal{G} \models \theta(a_1, \dots, a_m)$. Just to give an example, to check whether $Ex_k x_l$ holds, say with $k < l$, we need states $s(i)$ for $1 \leq i \leq k$ and $s(i, a)$ for $k < i \leq l, a \in G$. The machine starts in state $s(1)$ with head in position 1 in state $s(1)$. It moves its head right until it reaches position k in state $s(k)$, reads a_k and goes to position $(k+1)$ in state $s(k+1, a_k)$. Then it moves right again until it reaches position l in state $s(l, a_k)$. From this state it can reach an accepting state if, and only if, $E^G a_k a_l$. The machine $M(\mathcal{G}, \theta)$ can be constructed in time $p(|\theta|) \cdot |G|^2$, for a suitable polynomial p , and there is a computable function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that $f(|\theta|)$ is an upper bound for the number of steps M has to perform.

Now suppose we are given an instance of $MC(\Sigma_t)|_{GRAPH}$, i.e. a graph \mathcal{G} and a sentence

$$\varphi = \exists x_{11} \dots \exists x_{1k_1} \forall x_{21} \dots \forall x_{2k_2} \dots Q x_{t1} \dots Q x_{tk_t} \theta,$$

where θ is quantifier-free. Then we can construct an alternating Turing machine $M(\mathcal{G}, \varphi)$ that accepts the empty word if, and only if, $\mathcal{G} \models \varphi$. It first writes a sequence of elements of \mathcal{G} on the tape using existential and universal states appropriately and then simulates $M(\mathcal{G}, \theta)$ on this input.

To finish the proof, by Corollary 8(3) it suffices to show that $AM_t \leq_m^{\text{fp}} MC(\Sigma_t[\tau])$ for a suitable vocabulary τ . To illustrate the idea, we first consider the case $t = 1$. Suppose we are given a nondeterministic Turing machine M with alphabet Σ , set Q of states, initial state q_0 , accepting state q_{acc} and transition relation δ . Let $\Sigma_H := \{a_H \mid a \in \Sigma\}$, a_H coding the information that the head of M scans a cell containing a . Let $\tau := \{ST, AL, H, IN, ACC, R, L, S\}$ with unary ST, AL, H, IN, ACC and 4-ary R, L, S and let \mathcal{A}_M be the τ -structure given by

$$\begin{aligned} A_M &:= Q \dot{\cup} \Sigma \dot{\cup} \Sigma_H; \\ ST^{A_M} &:= Q; \quad AL^{A_M} := \Sigma; \quad H^{A_M} := \Sigma_H; \\ IN^{A_M} &:= \{q_0\}; \quad ACC^{A_M} := \{q_{\text{acc}}\}; \\ R^{A_M} &:= \{(q, a_H, b, q') \mid (q, a, 1, b, q') \in \delta\}; \\ L^{A_M} &:= \{(q, a_H, b, q') \mid (q, a, -1, b, q') \in \delta\}; \\ S^{A_M} &:= \{(q, a_H, b_H, q') \mid (q, a, 0, b, q') \in \delta\} \cup \{(q_{\text{acc}}, a_H, a_H, q_{\text{acc}}) \mid a \in \Sigma\}, \end{aligned}$$

where $(q, a, h, b, q') \in \delta$ means: if M is in state q and its head scans $a \in \Sigma$, then M replaces a by b , moves its head one cell to the right ($h = 1$), to the left ($h = -1$), or does not move its head ($h = 0$); finally, it changes to state q' .

Let k be given as parameter for AM_1 . In k steps, M scans at most the first k cells. The quantifier-free formula (note that the following formulas only depend on k and not on M)

$$\varphi_{\text{config}}(x, y_1, \dots, y_k) := STx \wedge \bigvee_{i=1}^k (Hy_i \wedge \bigwedge_{j \neq i} ALy_j)$$

states that (x, y_1, \dots, y_k) is a configuration with state x , with the head facing y_i , and with y_j being the content of the j th cell ($j \neq i$). Let $\varphi_{\text{start}}(x, y_1, \dots, y_k)$ be a quantifier-free formula stating that (x, y_1, \dots, y_k) is the starting configuration. Similarly, we define a quantifier-free formula $\varphi_{\text{step}}(x, y_1, \dots, y_k, x', y'_1, \dots, y'_k)$ stating that the configuration (x', y'_1, \dots, y'_k) is the successor configuration of (x, y_1, \dots, y_k) (we agree that each accepting configuration is its own successor).

Now, the equivalence

$$M \text{ stops in } \leq k \text{ steps} \iff \mathcal{A}_M \models \varphi_k$$

holds for the existential sentence φ_k :

$$\begin{aligned} & \exists x_1 \exists y_{11} \dots \exists y_{1k} \dots \exists x_k \exists y_{k1} \dots \exists y_{kk} (\varphi_{\text{start}}(x, y_{11}, \dots, y_{1k}) \\ & \wedge \bigwedge_{i=1}^{k-1} \varphi_{\text{step}}(x_i, y_{i1}, \dots, y_{ik}, x_{i+1}, y_{i+11}, \dots, y_{i+1k}) \wedge ACCx_k). \end{aligned}$$

For $t \geq 1$, we can proceed similarly, with the addition that universal quantifiers are needed to take care of universal states of the input machine. Now, τ also contains two further unary symbols F (for the existential states) and U (for the universal states) which get the corresponding interpretations in \mathcal{A}_M . For example, for $t = 2$, we can take a Σ_t -sentence equivalent to:

$$\begin{aligned} & \bigvee_{l \leq k} \exists x_1 \exists y_{11} \dots \exists y_{1k} \dots \exists x_k \exists y_{l1} \dots \exists y_{lk} \left(\right. \\ & \quad \varphi_{\text{start}}(x, y_{11}, \dots, y_{1k}) \\ & \quad \wedge \bigwedge_{i=1}^{l-1} \varphi_{\text{step}}(x_i, y_{i1}, \dots, y_{ik}, x_{i+1}, y_{i+11}, \dots, y_{i+1k}) \\ & \quad \wedge Fx_1 \wedge \dots \wedge Fx_{l-1} \wedge Ux_l \\ & \quad \wedge \forall x_{l+1} \forall y_{l+11} \dots \forall y_{l+1k} \dots \forall x_k \forall y_{k1} \dots \forall y_{kk} \left(\right. \\ & \quad \quad \left(Ux_{l+1} \wedge \dots \wedge Ux_{k-1} \right. \\ & \quad \quad \left. \left. \wedge \bigwedge_{i=l}^{k-1} \varphi_{\text{step}}(x_i, y_{i1}, \dots, y_{ik}, x_{i+1}, y_{i+11}, \dots, y_{i+1k}) \right) \rightarrow ACCx_k \right) \left. \right). \end{aligned}$$

(Without loss of generality we assume that the accepting state is both existential and universal and that at least one transition is always possible in a universal state). \square

The following result due to Downey, Fellows, and Regan allows to compare the W- and the A-hierarchy. Let $t, u \geq 1$. A formula φ is $\Sigma_{t,u}$, if it is Σ_t and all quantifier blocks after the leading existential block have length $\leq u$.

Theorem 37 ([9])

$$W[t] = \bigcup_{\substack{\tau \text{ vocabulary} \\ u \geq 1}} [MC(\Sigma_{t,u}[\tau])]_{\text{m}}^{\text{fp}} = \bigcup_{u \geq 1} [MC(\Sigma_{t,u})|_{\text{GRAPH}}]_{\text{m}}^{\text{fp}}.$$

We take the first equality as definition of the W-hierarchy and encourage the reader to give a purely “logical” proof of the second equality.

Since $\Sigma_{1,u} = \Sigma_1$, we get

$$W[1] = \bigcup_{\tau \text{ vocabulary}} [MC(\Sigma_1[\tau])]_{\text{m}}^{\text{fp}} = [MC(\Sigma_1)|_{\text{GRAPH}}]_{\text{m}}^{\text{fp}} = A[1].$$

By Theorem 36, $W[t] \subseteq A[t]$ for all $t \geq 2$. The question whether $W[t] = A[t]$ for all t is stated as an open problem in [9]. Consider, for example, the following parameterized problem

P	<p><i>Input:</i> Graph G.</p> <p><i>Parameter:</i> $(k, l) \in \mathbb{N}^2$.</p> <p><i>Question:</i> Are there $a_1, \dots, a_k \in G$ such that every cycle of length l contains an a_i?</p>
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Since P is slicewise Σ_2 -definable, we have $P \in A[2]$. But is P in $W[2]$?

In the definition of the W-hierarchy we can restrict the length of the non-leading quantifier-blocks to one:

Proposition 38 *For all $t \geq 1$,*

$$W[t] = \bigcup_{\tau \text{ vocabulary}} [MC(\Sigma_{t,1}[\tau])]_{\text{m}}^{\text{fp}}.$$

Proof: The inclusion \supseteq being trivial we turn to a proof of \subseteq : Fix τ and $u \geq 1$. We show that $MC(\Sigma_{t,u}[\tau]) \leq_{\text{m}}^{\text{fp}} MC(\Sigma_{t,1}[\tau'])$ for suitable τ' . The idea is to replace the blocks of at most u quantifiers by a single quantifier ranging over the set of u -tuples of a structure. To explain this idea we first use a vocabulary containing function symbols (and sketch afterwards how one can do without). Let $\tau' := \tau \cup \{T, p_1, \dots, p_u\}$, where T is a unary relation symbol (for ordered u -tuples) and p_1, \dots, p_u are unary function symbols (the projection functions). Given a τ -structure \mathcal{A} let \mathcal{A}' be a τ' -structure with

$$A' := A \dot{\cup} A^u, \quad T^{\mathcal{A}'} := A^u,$$

where for $(a_1, \dots, a_u) \in A^u$, $p_i(a_1, \dots, a_u) = a_i$ and where the relation symbols of τ are interpreted as in \mathcal{A} . Now, e.g. for

$$\varphi = \exists x_1 \dots \exists x_k \forall y_1 \dots \forall y_u \psi(\bar{x}, \bar{y})$$

with quantifier-free ψ , let

$$\varphi' = \exists x_1 \dots \exists x_k \forall y (\neg T x_1 \wedge \dots \wedge \neg T x_k \wedge (T y \rightarrow \psi(\bar{x}, p_1(y) \dots p_u(y))).$$

Then,

$$\mathcal{A} \models \varphi \iff \mathcal{A}' \models \varphi',$$

which gives the desired parameterized m-reduction.

Let us explain, for the case $t = 2$, how to proceed to avoid function symbols. One has to add to τ , besides T as above, for every relation symbol $R \in \tau$, say r -ary, every subset $M \subseteq \{1, \dots, r\}$, and every function $\rho: M \rightarrow \{1, \dots, u\}$ a new relation symbol $R_{M,\rho}$; e.g., if $M = \{s, \dots, r\}$ then $R_{M,\rho}^{\mathcal{A}'} a_1 \dots a_{s-1} b$ if, and only if, $a_1 \dots a_{s-1} \in A$, $b = (b_1, \dots, b_u) \in A^u$, and $R^{\mathcal{A}} a_1 \dots a_{s-1} b_{\rho(s)} \dots b_{\rho(r)}$; and a subformula $R x_{i_1} \dots x_{i_{s-1}} y_{\rho(s)} \dots y_{\rho(r)}$ of φ is replaced by $R_{M,\rho} x_{i_1} \dots x_{i_{s-1}} y$. \square

We close with a characterization of the W-hierarchy in terms of Fagin-definability. For a class Φ of formulas we let $\text{FD}(\Phi)$ be the class of all problems that are Φ -Fagin-definable. Downey, Fellows, and Regan have proved the following:

Theorem 39 ([9]) *For all $t \geq 1$ we have $\text{W}[t] = [\text{FD}(\Pi_t)]_{\text{m}}^{\text{fp}}$.*

We give a characterization in terms of bounded formulas. A first-order formula $\varphi(X)$ is *bounded to* the r -ary relation variable X , if in $\varphi(X)$ quantifiers appear only in the form $\exists x_1 \dots \exists x_r (X x_1 \dots x_r \wedge \psi)$ or $\forall x_1 \dots \forall x_r (X x_1 \dots x_r \rightarrow \psi)$, which we abbreviate by $\exists \bar{x} \in X \psi$ and $\forall \bar{x} \in X \psi$, respectively. For $t \geq 1$ we let Π_t^b be the class of all formulas $\varphi(X)$ of the form

$$\forall \bar{x}_1 \exists \bar{x}_2 \dots Q \bar{x}_t \theta$$

where $Q = \forall$ if t is odd and $Q = \exists$ otherwise and where θ is bounded to X .

For example, *CLIQUE* is Fagin-defined by the Π_0^b -formula

$$\forall x \in X \forall y \in X (x \neq y \rightarrow Exy)$$

and *DOMINATING SET* by the Π_1^b -formula

$$\forall x \exists y \in X (x = y \vee Exy).$$

Theorem 40 *For $t \geq 1$, $\text{W}[t] = [\text{FD}(\Pi_{t-1}^b)]_{\text{m}}^{\text{fp}}$.*

Proof: First, assume that the problem $P \subseteq \text{STR}[\tau] \times \mathbb{N}$ is Fagin-defined by $\varphi(X) \in \Pi_{t-1}^b$, say

$$\varphi(X) := \forall \bar{y}_1 \exists \bar{y}_2 \forall \bar{y}_3 \dots Q \bar{y}_t \psi,$$

where ψ only contains bounded quantifiers. Let u be the maximum of the lengths of the tuples \bar{y}_i , for $1 \leq i \leq t$. For simplicity, let us assume that X is

unary. Since Xy is equivalent to $\exists z \in X z = y$, we can assume that in ψ , the variable X only occurs in bounded quantifiers. We show that $P \in \mathsf{W}[t]$. Given a parameter k set (with new variables x_1, \dots, x_k)

$$\varphi^k := \exists x_1 \dots \exists x_k \forall \bar{y}_1 \exists \bar{y}_2 \forall \bar{y}_3 \dots Q \bar{y}_t \psi^*,$$

where ψ^* is obtained from ψ by inductively replacing $\forall u \in X \chi(u)$ and $\exists u \in X \chi(u)$ by $\bigwedge_{i=1}^k \chi(x_i)$ and $\bigvee_{i=1}^k \chi(x_i)$, respectively. Note that φ^k is a $\Sigma_{t,u}$ -formula and that for every structure \mathcal{A} ,

$$(\mathcal{A}, k) \in P \iff \mathcal{A} \models \varphi^k.$$

Thus, P is slicewise $\Sigma_{t,u}$ -definable and hence in $\mathsf{W}[t]$.

We turn to a proof of $\mathsf{W}[t] \subseteq [\mathsf{FD}(\Pi_{t-1}^b)]_{\text{m}}^{\text{fp}}$. We already saw that the $\mathsf{W}[1]$ -complete problem *CLIQUE* belongs to $\mathsf{FD}(\Pi_0^b)$, hence the inclusion holds for $t = 1$. We state the proof for $t \geq 2$ for a reader familiar with [9]. Again we encourage the reader to give a direct proof. For odd $t \geq 2$, the problem $\text{AWBES}(t)$, anti-monotonic weighted t -normalized Boolean expression satisfiability, is $\mathsf{W}[t]$ -complete. It is parameterized m-reducible to the problem Fagin-defined by the Π_{t-1}^b -formula

$$\begin{aligned} \varphi(X) := & \forall y_0 \forall y_1 \exists y_2 \forall y_3 \dots \exists y_{t-1} \\ & ((E y_0 y_1 \wedge E y_1 y_2 \wedge \dots \wedge E y_{t-2} y_{t-1}) \rightarrow \forall x \in X \neg E y_{t-1} x). \end{aligned}$$

For even t one uses the completeness of $\text{WBES}(t)$ and argues similarly. \square

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