**Slide 1**

## Plan

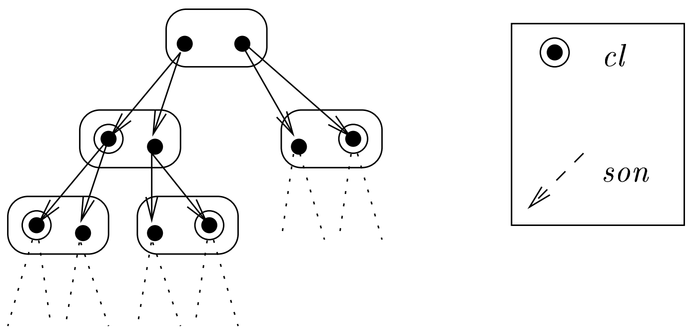
**An extension of Rabin's tree theorem to tree-like structures**

- Tree-like structures
- Automata on tree-like structures
- Closure properties of automata
- Muchnik's Thm
- MSOL over trees= FPL
- Bisimulation invariant MSOL=$\mu$-calculus

**Slide 2**

## Tree-like structure: an example

- $M = \langle \{l, r\} \rangle$,
- Consider: $M^+ = \langle \{l, r\}^+, son, cl \rangle$



**Slide 3**

## Binary tree is definable in $M^+$

- Let $\theta(S) =$ ( $S$ consists of all descendants of one of the roots)
- Instead of checking $\{l,r\}^* \models \alpha$ we can check

$$M^+ \models \exists S.\ \theta(S) \wedge \alpha_S$$

$\alpha_S$ is obtained by relativising all the quantifiers to $S$:

$$\forall X. \quad \longmapsto \quad \forall X \subseteq S.$$

**Slide 4**

## Tree-like structures

- Let $Sig = \{r_1, \ldots\}$ be a relational signature, let $M = \langle D_M, r_1, \ldots\rangle$ be a structure of this signature.
- Define the structure $M^+ = \langle D_M^+, son, cl, r_1^+, \ldots\rangle$:

$$\begin{array}{rl} D_M^+ & \text{nonempty sequences over } D_M \\ r^+(wd_1, \ldots, wd_k) & \text{iff} \quad r(d_1, \ldots, d_k) \\ son(w, wd) & \text{for every } w \in D_{\mathcal{M}}^+,\ d \in D_{\mathcal{M}} \\ cl(wdd) & \text{for every } w \in D_{\mathcal{M}}^*,\ d \in D_{\mathcal{M}} \end{array}$$

## Example: $\omega$-sequence

**Slide 5**

- Consider $M = \langle \{0\} \rangle$
- $M^+ = \langle \{0\}^+, son, cl \rangle$

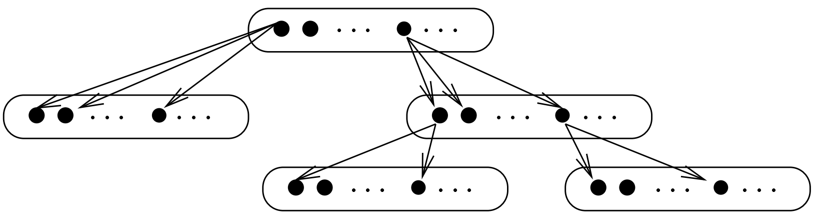
$$\begin{array}{ll} son(0^n, 0^{n+1}) & \text{for } n > 0 \\ cl(0^n) & \text{for } n > 1 \end{array}$$

## Example: Tree

**Slide 6**

- Consider $M = \langle D_M \rangle$ for some set $D_M$.
- $M^+ = \langle D_M^+, son, cl \rangle$

$$\begin{array}{ll} son(w, wd) & \text{for } w \in D_M^+,\ d \in D_M \\ cl(wdd) & \text{for } w \in D_M^*,\ d \in D_M \end{array}$$



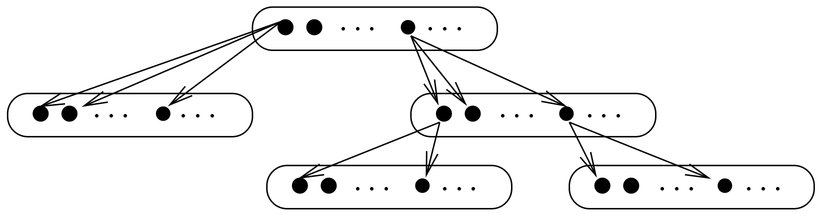
Slide 7

## Example: Tree with order on successors

- Consider a linear order $M = \langle D_M, \leq \rangle$
- $M^+ = \langle D_M^+, son, cl, \leq^+ \rangle$

$$wd \leq^+ wd' \quad \text{iff} \quad d \leq d'$$

$$son(w, wd) \quad cl(wdd) \quad \text{for } w \in D_M^+,\ d \in D_M$$



Slide 8

## Example: Unwinding of a graph

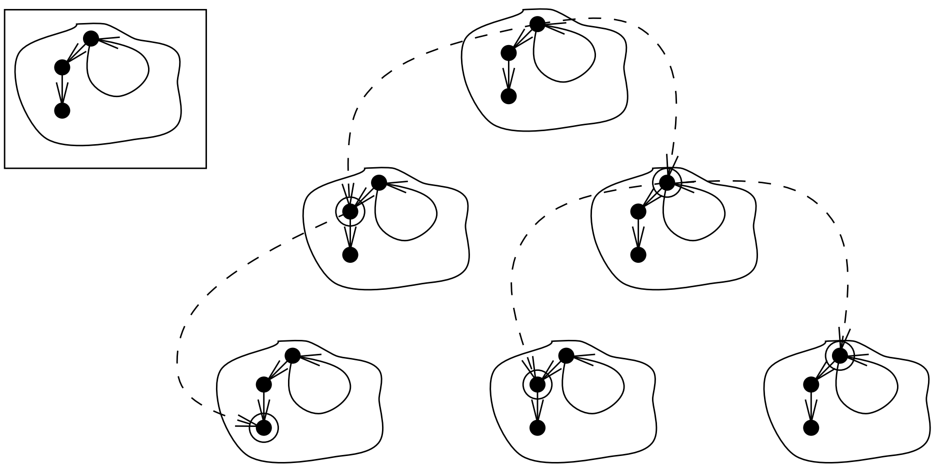
- Consider a graph $M = \langle D_M, E \subseteq D_M \times D_M, sr \in D_M \rangle$.
- An unwinding $Un(M)$ is a tree consisting of paths of $M$ starting in $sr$.
- $M^+ = \langle D_M^+, son, cl, E^+, sr^+ \rangle$

$$E^+(wd, wd') \quad \text{iff} \quad E(d, d')$$

$$sr^+(wd) \quad \text{iff} \quad sr(d)$$

$$son(w, wd) \quad cl(wdd) \quad \text{for } w \in D_M^+, d \in D_M$$

Slide 9



Slide 10

**Goal:**

Automata on tree-like structures.

Closure under sum, projection and complementation

This would give us decidability results and an understanding of the expressive power of MSOL on tree-like structures.

**Slide 11**

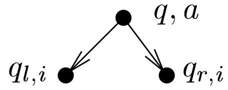
## History

- Stupp [manuscript, 75]
- Shelah [MSOL theory of Order, 75]
- Semenov [MFCS '84]
- Muchnik [Daghstuhl '8?]

**Slide 12**

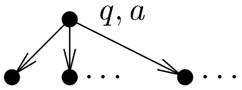
## Transition function

- Automata on binary trees: $\delta(q, a) = \{(q_{l,1}, q_{r,1}), \ldots, (q_{l,k}, q_{r,k})\}$




- Automata on $M^+$ for $M = \langle D_M = \{0, 1, \ldots\}, S\rangle$

$m : D_M \to Q$



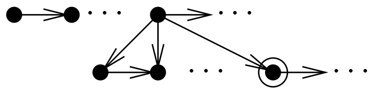

- We use formulas to define sets of markings:

$$M, [m^{-1}(q_1)/Q_1], \ldots, [m^{-1}(q_n)/Q_n] \models \varphi(Q_1, \ldots, Q_n)$$

**Slide 13**

## Expressing clone relation

- How to express the clone relation?



- We use free first order variable in transition formulas

$$M, [d/x], [m^{-1}(q_1)/Q_1], \ldots, [m^{-1}(q_n)/Q_n] \vDash \varphi(x, Q_1, \ldots, Q_n)$$

- For example:

$$M, [d/x], [m^{-1}(q_1)/Q_1], \ldots, [m^{-1}(q_n)/Q_n] \vDash Q_1(x)$$

**Slide 14**

## Automata on tree-like structures

- For $n \in \mathbb{N}$ let $\mathrm{BF}(n)$ be a set of MSOL formulas with free variables among $x, Q_1, \ldots, Q_n$, Let $I(n)$ be a set of MSOL formulas with free variables among $Q_1, \ldots, Q_n$.

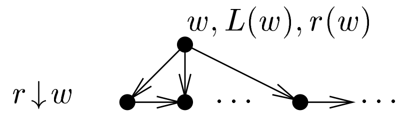
$$\mathcal{A} = \langle Q, \Sigma, \iota \in I(|Q|), \delta : Q \times \Sigma \to \mathrm{BF}(|Q|), \Omega : Q \to \mathbb{N} \rangle$$

- A *run* of $\mathcal{A}$ on $M^+$ with a labelling $L : D_M^+ \to \Sigma$:

  a function $r : D_M^+ \to Q$ respecting the transition relation

Slide 15

## Respecting the transition relation




- Let $r{\downarrow}w : D_M \to Q$ be the restriction of $r$ to sons of $w$.
- Respecting the transition relation:

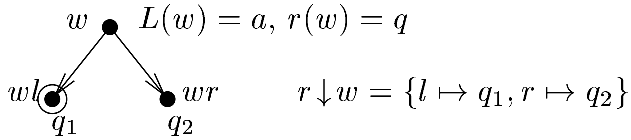
$$M, r{\downarrow}\varepsilon \models \iota$$
$$M, d, r{\downarrow}w \models \delta(r(w), L(w))$$

where $d$ is the last element of $w$.

Slide 16

## Example

- $M = \{l, r\}$
- $\delta(q, a) = \exists y.\ Q_1(x) \wedge Q_2(y)$

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- *Successful run:* For every $(d_0, d_1, \dots) \in D_M^\omega$ the smallest number appearing i.o. in

$$\Omega(r(d_0)),\ \Omega(r(d_0 d_1)),\ \Omega(r(d_0 d_1 d_2)), \dots$$

is even

Slide 17

## Game interpretation

- Take $M = \langle D_M, r_1, \dots \rangle$ and $L : D_M^+ \to \Sigma$,
- Game $G(\mathcal{A}, M, L) = \langle V_0 \cup V_1, V_0, E, \Omega_G \rangle$

\+ Positions for player 0 are $V_0 = Q \times D_M^+$.

\+ Positions for player 1 are $V_1 = (D_M \to Q) \times (D_M^+ \cup \{\varepsilon\})$.

\+ $(q, wd) \to (m, wd)$ iff $M, d, m \vDash \varphi(x, Q_1, \dots, Q_n)$

$\varphi = (q, L(wd))$

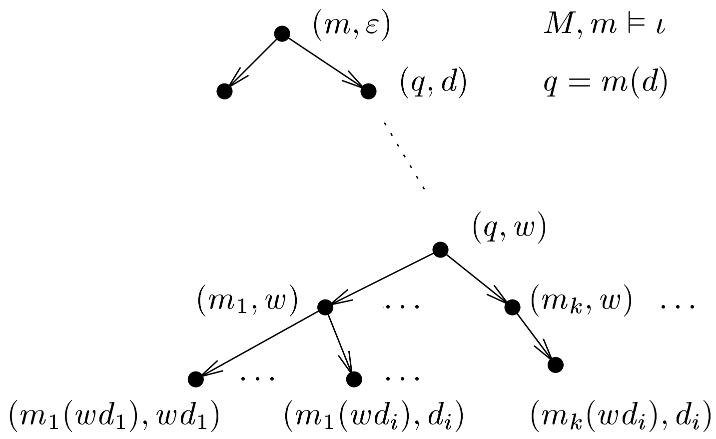
\+ $(m, w) \to (q, wd)$ iff $m(d) = q$.

\+ $\Omega_G((q, w)) = \Omega(q) \qquad \Omega_G((m, w)) = \max\{\Omega_G(q) : q \in Q\}$.

- *Initial positions*: $(m, \varepsilon)$ of $V_1$ with $M, m \vDash \iota$.
- $M^+$ with $L$ is *accepted* by $\mathcal{A}$ if there is a winning strategy for player 0 from some initial vertex.

Slide 18

## Game example



Slide 19

## Closure requirements for $\{\mathbf{BF}(n)\}_{n\in\mathbb{N}}$

- Fix *Sig* and a class of structures *Struct*.

**C1)** There is $tt \in \mathrm{BF}(n)$, s.t., $\mathit{Struct} \models tt$.

**C2)** Disjunction. If $\varphi(x, Q_1, \ldots, Q_n)$, $\psi(x, Q_1, \ldots, Q_n)$ in $\mathrm{BF}(n)$ then for some $\theta(x, Q_1, \ldots, Q_n)$:

$$\mathit{Struct} \models \theta(x, Q_1, \ldots, Q_n) \Leftrightarrow [\varphi(x, Q_1, \ldots, Q_n) \vee \psi(x, Q_1, \ldots, Q_n)]$$

**C3** Renaming. Given a 1-1 function $\rho : \{1, \ldots, m\} \to \{1, \ldots, n\}$ and $\varphi(x, Q_1, \ldots, Q_m) \in \mathrm{BF}(m)$, there is $\psi(x, Q_1, \ldots, Q_n) \in \mathrm{BF}(n)$ with

$$\mathit{Struct} \models \psi(x, Q_1, \ldots, Q_n) \Leftrightarrow \varphi(x, Q_{\rho(1)}, \ldots, Q_{\rho(m)})$$

Slide 20

## Closure requirements for $\{\mathbf{BF}(n)\}_{n\in\mathbb{N}}$

**C4** Product. If there are $\varphi(x, P_1, \ldots, P_m) \in \mathrm{BF}(m)$ and $\psi(x, Q_1, \ldots, Q_n) \in \mathrm{BF}(n)$ then there is $\theta(x, R_{1,1}, \ldots, R_{m,n}) \in \mathrm{BF}(mn)$ s.t.:

$$\mathit{Struct} \models \theta(x, R_{1,1}, \ldots, R_{m,n}) \Leftrightarrow$$
$$[\varphi(x, \bigcup_{j=1}^{n} R_{1,j}, \ldots, \bigcup_{j=1}^{n} R_{m,j}) \wedge \psi(x, \bigcup_{i=1}^{m} R_{i,1}, \ldots, \bigcup_{i=1}^{m} R_{i,n})]$$

- $\theta$ allows markings $m' : D_M \to Q \times Q$, s.t., $m'\!\downarrow_1$ satiafies $\varphi$ and $m'\!\downarrow_2$ satisfies $\psi$.

Slide 21

## $\boxed{\textbf{Closure requirements for } \{\mathbf{BF}(n)\}_{n\in\mathbb{N}}}$

**C5** Choice operation. If $\varphi(x, Q_1, \ldots, Q_n) \in \mathrm{BF}(n)$ then there is $\overline{\varphi}(x, R_{\{1\}}, \ldots, R_{\{1,\ldots,n\}}) \in \mathrm{BF}(2^n - 1)$ with:

$$Struct \vDash \overline{\varphi}(x, R_{\{1\}}, \ldots, R_{\{1,\ldots,n\}}) \Leftrightarrow \big[\forall Z_1, \ldots, Z_n.\ \varphi(x, Z_1, \ldots, Z_n) \Rightarrow \bigvee_{i=1,\ldots,n} \bigcup \{R_S : i \in S\} \cap Z_i \neq \emptyset\big]$$

• $\overline{\varphi}$ allows markings $\overline{m} : D_M \to \mathcal{P}(Q)$, s.t., for every $m : D_M \to Q$ with $m \vDash \varphi$ there is $d \in D_m$ with $m(d) \in \overline{m}(d)$.

Slide 22

• Given some sets of basic and initial formulas $\{\mathrm{BF}(n)\}_{n\in\mathbb{N}}$, $\{I(n)\}_{n\in\mathbb{N}}$ we have the family of automata:

$$\mathcal{A} = \langle Q, \Sigma, \iota \in I(|Q|), \delta : Q \times \Sigma \to \mathrm{BF}(|Q|), \Omega : Q \to \mathbb{N}\rangle$$

• We want to show:

If $\{\mathrm{BF}(n)\}_{n\in\mathbb{N}}$ and $\{I(n)\}_{n\in\mathbb{N}}$ satisfy the closure conditions over some class of structures *Struct* then the family of automata is closed under sum, projection and complementation over structures $Struct^+$.

**Slide 23**

## Closure under sum

• Fix a signature *Sig*, class of structures *Struct* and families $\{\mathrm{BF}(n)\}_{n\in\mathbb{N}}$, $\{I(n)\}_{n\in\mathbb{N}}$.

$$\begin{aligned}\mathcal{A}_1 =&\langle Q_1, \Sigma, \iota_1 \in I(|Q_1|), \delta_1 : Q_1 \times \Sigma \to \mathrm{BF}(|Q_1|), \Omega_1 : Q_1 \to \mathbb{N}\rangle\\ \mathcal{A}_2 =&\langle Q_2, \Sigma, \iota_2 \in I(|Q_2|), \delta_2 : Q_2 \times \Sigma \to \mathrm{BF}(|Q_2|), \Omega_2 : Q_2 \to \mathbb{N}\rangle\end{aligned}$$

Construct an automaton

$$\mathcal{A}_\vee = \langle Q, \Sigma, \iota \in I(|Q|), \delta : Q \times \Sigma \to \mathrm{BF}(|Q|), \Omega : Q \to \mathbb{N}\rangle$$

accepting $M^+$ with $L : D_M^+ \to \Sigma$ iff it is accepted by $\mathcal{A}_1$ or by $\mathcal{A}_2$.

**Slide 24**

## Construction of $\mathcal{A}_\vee$

• $Q = Q_1 \cup Q_2$; assume $Q_1 \cap Q_2 = \emptyset$

• Given $\varphi(x, Q_1, \ldots, Q_{n_2}) \in \mathrm{BF}(n_2)$, we write $\Rsh(\varphi)$ for the formula $\varphi(x, Q_{n_1+1}, \ldots, Q_{n_1+n_2})$

• $\iota = \iota_1 \vee \Rsh(\iota_2)$

$$\delta(q,a) = \begin{cases} \delta_1(q,a) & q \in Q_1 \\ \Rsh(\delta_2(q,a)) & q \in Q_2 \end{cases}$$

$$\Omega(q) = \begin{cases} \Omega(q) = \Omega_1(q) & q \in Q_1 \\ \Omega(q) = \Omega_2(q) & q \in Q_2 \end{cases}$$

Slide 25

## Closure under projection

For every $\mathcal{A}$ over $\{0,1\} \times \Sigma$ there is $\mathcal{A}_\exists$ over $\Sigma$, s.t., for every $M \in \mathit{Struct}$ and every $L : D_M^+ \to \Sigma$:

$(M^+, L)$ is accepted by $\mathcal{A}_\exists$ iff $(M^+, L')$ is accepted by $\mathcal{A}$, for some $L' : D_M^+ \to \{0,1\} \times \Sigma$, with $L'{\downarrow_2} = L$.

Slide 26

## Closure under complementation

For every $\mathcal{A}$ there is $\overline{\mathcal{A}}$, s.t.
$(M^+, L)$ is accepted by $\overline{\mathcal{A}}$ iff $(M^+, L)$ is not accepted by $\mathcal{A}$.

- $\mathcal{A}$ does not accept $M^+$ with $L$ iff there is a winning strategy for player 1 in $G(\mathcal{A}, M, L)$ (determinacy theorem).
- We construct $\overline{\mathcal{A}}$ so that $\overline{\mathcal{A}}$ accepts $(M, L)$ iff there is a winning strategy for player 1 in $G(\mathcal{A}, M, L)$.

**Slide 27**

## Representing a strategy

- (By memoryless determinacy theorem) A strategy for player 1 is

$$\sigma_1 : (D_M \to Q) \times D_M^* \to (Q \times D_M^+)$$

- As $\sigma_1(m, w)$ must be of the form $(m(d), wd)$ we represent $\sigma_1$ by

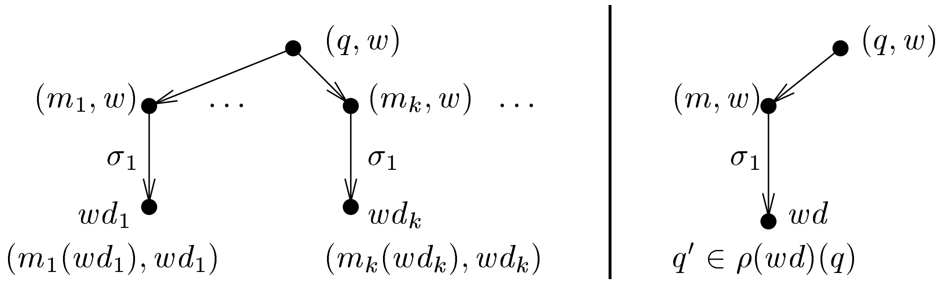
$$\begin{aligned} \sigma_1 :& (D_M \to Q) \times D_M^* \to D_M \\ \sigma_1 :& D_M^* \to ((D_M \to Q) \to D_M) \end{aligned}$$

- We would like to represent $\sigma_1$ as a function $\rho : D_M^+ \to \Lambda$ for some finite set $\Lambda$. We take $\Lambda = Q \to \mathcal{P}(Q)$.

**Slide 28**

## Representing a strategy

$$\sigma_1 : (D_M \to Q) \times D_M^* \to D_M \qquad \rho : D_M^+ \to (Q \to \mathcal{P}(Q))$$



**Slide 29**

## Local condition for representation

Covering: For every $w$, $m$ and $q$:

if $(q, w) \to (m, w)$ then there is $d$ $(q, w) \to (m, w) \xrightarrow{d} m(wd) \in \rho(wd)(q)$

**Slide 30**

## Global condition for representation

Every *trace* in $\rho$ is winning for player 1:

If

$$\begin{array}{ll} w_1 & q_1 \in \rho(w_1)(q_0) \\ \downarrow & \\ w_2 & q_2 \in \rho(w_2)(q_1) \\ \downarrow & \\ w_3 & q_3 \in \rho(w_3)(q_2) \\ \vdots & \end{array}$$

then $q_0, q_1, q_2, \ldots$ is winning for player 1.

**Slide 31**

## The main observation

**Lemma:** There is a winning strategy for player 1 in $\mathcal{G}(\mathcal{A}, M, L)$ iff there is a function $\rho : D_M^+ \to (Q \to \mathcal{P}(Q))$ satisfying local and global conditions for representations.

**Slide 32**

## Construction of $\overline{\mathcal{A}}$

$$\mathcal{A}'' = \langle Q'' = Q \to \mathcal{P}(Q), \Sigma, \iota'' \in I(|Q''|),$$
$$\delta : Q'' \times \Sigma \to \mathrm{BF}(|Q|''), \Omega : Q'' \to \mathbb{N}\rangle$$

• Runs of $\mathcal{A}''$ are functions $\rho : D_M^+ \to (Q \to \mathcal{P}(Q))$ satisfying the local condition.

$$\mathcal{C} = \langle Q_c, \Sigma_c = Q \to \mathcal{P}(Q), q_c, \delta_c : Q_c \times \Sigma_c \to Q_c, \Omega_c : Q_c \to \mathbb{N}\rangle$$

• $\mathcal{C}$ checks whether in $(g_1, g_2, \dots) \in (Q \to \mathcal{P}(Q))^\omega$ all the traces are winning for player 1.

• $\overline{\mathcal{A}}$ is a “product” of $\mathcal{A}''$ and $\mathcal{C}$

Slide 33

## Summarising

• Automata on tree-like structures are very similar to automata on trees:

$$\mathcal{A} = \langle Q, \Sigma, q \in Q, \delta : Q \times \Sigma \to \mathcal{P}(Q \times Q), \Omega : Q \to \mathbb{N} \rangle$$
$$\mathcal{A} = \langle Q, \Sigma, \iota \in I(|Q|), \delta : Q \times \Sigma \to \mathrm{BF}(|Q|), \Omega : Q \to \mathbb{N} \rangle$$

• The proof of the closure properties is almost the same with some complications because there may be infinitely many sons of a node.

Slide 34

## Muchnik's thm

• Fix a signature $Sig$.

**Thm:** For every sentence $\varphi$ over $Sig^+$ one can effectively find a sentence $\widehat{\varphi}$ over $Sig$ so that for every structure $M$:

$$M^+ \vDash \varphi \quad \text{iff} \quad M \vDash \widehat{\varphi}$$

**Cor:** If MSOL over $M$ is decidable then MSOL over $M^+$ is decidable

Slide 35

## Examples

• Rabin's tree Thm:

$MSOL(\{l,r\})$ decidable $\Rightarrow$ $MSOL(\{l,r\}^+)$ decidable $\Rightarrow$ $MSOL(\{l,r\}^*)$ decidable.

• MSOL over trees of arbitrary degree:

$MSOL(Set)$ decidable $\Rightarrow$ $MSOL(Set^+)$ decidable $\Rightarrow$ $MSOL(Trees)$ decidable.

• $MSOL(\langle \mathbb{N}, S, P\rangle)$ for $P$ being $\{n! : n \in \mathbb{N}\}$ or $\{n^2 : n \in \mathbb{N}\}$.

Slide 36

## The main lemma

**Lemma:** For every automaton $\mathcal{A}$ over one letter alphabet there is a sentence $\widehat{\varphi}$ over $Sig$, s.t., for every structure $M$:

$$M^+ \text{ is acepted by } \mathcal{A} \quad \text{iff} \quad M \vDash \widehat{\varphi}$$

• Take an automaton

$$\mathcal{A} = \langle Q = \{q_1, \ldots, q_n\}, \iota \in I(n), \delta : Q \to \mathrm{BF}(n), \Omega : Q \to \mathbb{N}\rangle$$

• Assume that $\Omega(q_i) = i$ and that $n$ is even.

Slide 37

## Proof

- As $\mathcal{A}$ is over one letter alphabet the acceptance game can be simplified.
- $V_0 = Q \times D_M$,
- $V_1 = (Q \to D_M)$,
- $m \to (q, d)$ iff $m(d) = q$,
- $(q, d) \to m$ iff $M, d, m \vDash \delta(q)$,
- $\Omega_G(q, d) = \Omega(q)$; for $v \in V_1$ we let $\Omega_G(v) = n$.
- The game can be represented by a relational structure:

$$G(M) = \langle V_0 \cup V_1, V_0^G, E^G, P_1^G, \ldots, P_n^G \rangle$$

Slide 38

## Proof

- The winning set for player 0 is described by a formula:

$$\mathcal{W}_0 = \| \, \mathsf{LFP}\, Z_1 \ldots \mathsf{GFP}\, Z_n . F_0(Z_1, \ldots, Z_n) \, \|^{G(M)}$$

$$F_0(Z_1, \ldots, Z_n) = \big\{x : V_0(x) \wedge \exists y. \big(E(x, y) \\ \wedge (\forall z.\ E(y, z) \Rightarrow \mathit{Next}(z, Z_1, \ldots, Z_n))\big)\big\}$$

$$\mathit{Next}(z, Z_1, \ldots, Z_n) = \bigwedge_{i \in \{1, \ldots, n\}} P_i(z) \Rightarrow Z_i(z)$$

- $W = \{x : V_0(x) \wedge \exists y. \big(E(x, y) \wedge \forall z.\ E(y, z) \Rightarrow z \in W\big)\}$

**Slide 39**

• Suppose, instead of $G(M)$ we are given

$$Q \times M = \langle Q \times D_M, Eq^{QM}, P_1^{QM}, \ldots, P_n^{QM}, r_1^{QM}, \ldots \rangle$$

where:

$+\ Eq^{QM}((q_1, d_1), (q_2, d_2))$ iff $d_1 = d_2$,

$+\ P_i^{QM}(q, d)$ iff $q = q_i$,

$+\ r_i^{QM}((q_1, d_1), \ldots, (q_k, d_k))$ iff $r_i^M(d_1, \ldots, d_k)$.

• We can adjust the formula to describe the winning positions in $Q \times M$.

• $Q \times M$ is definable from $M$ by means of FOL formulas.

For every formula $\varphi'$ there is a formula $\widehat{\varphi}$ with:

$$Q \times M \vDash \varphi' \quad \text{iff} \quad M \vDash \widehat{\varphi}$$

**Slide 40**

## $\boxed{\textbf{MSOL on } Set^+}$

• Let $Set$ be the class of structures over the empty signature.

• Let $Set^+ = \{M^+ : M \in Set\}$

• Let $\mathrm{BF}(n)$ be the set of disjunctions of formulas of the form:

$$\begin{aligned} &\exists y_1, \ldots, y_k.\ \mathrm{diff}(y_1, \ldots, y_k) \wedge Q_{i_1}(y_1) \wedge \cdots \wedge Q_{i_k}(y_k) \\ &\qquad\qquad \wedge \forall z.\ \mathrm{diff}(z, y_1, \ldots, y_k) \Rightarrow (Q_{j_1}(z) \vee \cdots \vee Q_{j_l}(z)) \end{aligned}$$

**Lemma:** The family $\{\mathrm{BF}(n)\}_{n \in \mathbb{N}}$ satisfies (over $Set$) the closure conditions C1–C5.

Slide 41

**Thm:** For every MSOL formula over the signature $\{son\}$ with free variables $Z_1, \ldots, Z_k$ there is an equivalent, over $Set^+$, automaton of the form:

$$\langle Q, \Sigma, \iota \in I(|Q|), \delta : Q \times \Sigma \to \mathrm{BF}(|Q|), \Omega : Q \to \mathbb{N} \rangle$$

where $\Sigma = \mathcal{P}(\{1, \ldots, k\})$ and $I(|Q|) = \mathrm{BF}(|Q|)$ are as on the previous slide.

Slide 42

## Automata on trees

- Let *Trees* be the class of trees.
- Consider automata of the form:

$$\mathcal{A} = \langle Q, \Sigma, q_1 \in Q, \delta : Q \times \Sigma \to \mathrm{BF}(|Q|), \Omega : Q \to \mathbb{N} \rangle$$

**Thm:** These automata are equivalent in expressive power to MSOL on *Trees*.

Slide 43

## FPL syntax

- $Var_1 = \{x, y, \dots\},\ Var_2 = \{X, Y, \dots\}$
- Predicates: $X$, $\{x : \alpha\}$, $\mathsf{LFP}\, X.F$, $\mathsf{GFP}\, X.F$ ($X$ positive in $F$)
- Formulas: $x = y$, $son(x, y)$, $\neg\alpha$, $\alpha \vee \beta$, $\exists x.\alpha$, $F(x)$
- Example:

$$\mathsf{LFP}\, X.\ \{z : P(z) \vee \exists y.\ son(z, y) \wedge X(y)\}$$

Slide 44

## FPL semantics

- Let $M$ be a tree. Let $Val_1 : Var_1 \to D_M,\ Val_2 : Var_2 \to \mathcal{P}(D_M)$ be valuations of variables
- The meaning of a predicate $F$ is a subset $\parallel F \parallel^M_{Val_1, Val_2} \subseteq D_M$.
- Consider the function:

$$\lambda S.\ \parallel F(X) \parallel^M_{Val_1, Val_2[S/X]} : \mathcal{P}(D_M) \to \mathcal{P}(D_M)$$

It is monotone if $X$ appears only positive in $F$

- $\parallel \mathsf{LFP}\, X.\ F(X) \parallel^M_{Val_1, Val_2}$ is the least fixpoint of this function.

Slide 45

## $\boxed{\text{MSOL=FPL over trees}}$

**Lemma:** For every automaton $\mathcal{A}$ over an alphabet $\Sigma = \{0,1\}^k$ there is, an equivalent over *Set*, FPL formula $\alpha$ with free variables $\{Z_1, \ldots, Z_k\}$.

**Thm:** For every MSOL formula $\varphi$ there is a FPL formula $\alpha$ s.t., $Trees \vDash \varphi \Leftrightarrow \alpha$.

Slide 46

## $\boxed{\text{Bisimulation invariance}}$

- Transition system

$$M = \langle S^M, sr^M \in S^M, son^M \subseteq S^M \times S^M, \{p^M\}_{p \in Prop} \rangle$$

- A MSOL sentence $\varphi$ defines the class of transition systems:

$$\{M : M \vDash \varphi\}$$

- A MSOL sentence is *bisimulation invariant* iff it defines bisimulation closed class of transition systems.

**Fact:** Not every MSOL sentence is bisimulation invariant.

**Fact:** Every $\mu$-calculus sentence is bisimulation invariant.

Slide 47

## $\omega$-expansion

$$M = \langle S^M, sr^M, son^M, \{p^M\}_{p \in Prop} \rangle$$

• $\omega$-indexed path of $M$:

$$u = s_0(a_1, s_1)(a_2, s_2) \cdots (a_n, s_n)$$

where $s_0, \ldots, s_n$ is a path from $sr^M$ and $a_1, \ldots, a_n \in \mathbb{N}$.

• $\omega$-expansion of $M$

+ $S^{\widehat{M}}$ is the set of $\omega$-indexed paths of $M$,

+ $sr^{\widehat{M}} = sr^M$,

+ $son^{\widehat{M}}(u, v)$ iff $v = u(a, s)$, for some $a$ and $s$,

+ $p^{\widehat{M}} = \{u(a, s) : s \in p^M, \quad u, a \text{ arbitrary}\} \cup \{sr^{\widehat{M}} : sr^M \in p^M\}$

Slide 48

## Two facts

**Fact:** A MSOL bisimulation invariant sentence is equivalent to a $\mu$-calculus sentence iff both sentences are satisfied in the same $\omega$-expansions.

**Fact:** For every automaton $\mathcal{A}$ on trees over the alphabet $\{1, \ldots, k\}$ there is a $\mu$-calculus sentence equivalent to $\mathcal{A}$ over $\omega$-expansions.

Slide 49

## $\boxed{\textbf{Bisimulation invariant MSOL} = \mu\textbf{-calculus}}$

**Thm:** [Janin & W.]

Every MSOL sentence defining bisimulation closed class of structures is equivalent to a $\mu$-calculus sentence.

Slide 50

## Some related results

**Thm:** [Janin & W.] Characterisation of the $\mu$-calculus in terms of automata.

**Thm:** [D'Agostino & Hollenberg] Uniform interpolation holds for the $\mu$-calculus.