

# Completeness of Kozen's Axiomatisation of the Propositional $\mu$ -Calculus

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## Abstract

Propositional  $\mu$ -calculus is an extension of the propositional modal logic with the least fixpoint operator. In the paper introducing the logic Kozen posed a question about completeness of the axiomatisation which is a small extension of the axiomatisation of the modal system K. It is shown that this axiomatisation is complete.

## 1 Introduction and summary

We consider  $\mu$ -calculus as defined by Kozen [4]. This is the logic obtained from modal logic by adding the least fixpoint operator:  $\mu X.\alpha(X)$ . The intended models of the logic are Kripke structures. Kozen's axiomatisation consists of the axiomatisation of the modal system K together with one axiom and one rule characterising the least fixpoint operator:

$$\alpha(\mu X.\alpha(X)) \Rightarrow \mu X.\alpha(X) \qquad \frac{\alpha(\varphi) \Rightarrow \varphi}{\mu X.\alpha(X) \Rightarrow \varphi}$$

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The completeness theorem considered here is sometimes called weak completeness because it deals with validity relation; it says that every valid formula is provable. Strong completeness refers to an axiomatisation of semantic consequence relation. It is not possible to have finitary strongly complete axiomatisation for the  $\mu$ -calculus because the compactness theorem fails for the logic. In the following completeness means weak completeness and provability means provability in the Kozen's system unless explicitly stated otherwise.

In [4] Kozen showed that the axiom system proves negations of all unsatisfiable formulas of a special kind called *aconjunctive formulas*. In [9] another finitary axiomatisation was proposed and proved to be complete for the whole  $\mu$ -calculus. This solved one part of the problem posed in [4] but the question of the completeness of the original axiomatisation remained still open. We give an affirmative answer to this question.

There are other reasons, apart from curiosity, to investigate the problem of the completeness of Kozen's system. Axiomatisation proposed in [9] makes essential use of the small model theorem for the  $\mu$ -calculus; this makes it impossible to use it for extensions of the logic not enjoying the finite model theorem. The other reason is that Kozen's system is very natural, one may say as natural as the notion of Kripke structures. Hence it is good to know that the class of Kripke structures is a complete subclass of a quasi-variety defined by Kozen's system.

Let us review some methods used in previous approaches to the completeness problem. First step is a tableau method of model construction of Streett and Emerson [8]. For a given formula one constructs a *tableau*, if the formula is satisfiable then one can construct a model from a part of this tableau. It was shown in [6] that if the initial formula is unsatisfiable, and one cannot find a model in the tableau, then one can construct for the formula another tableau-like structure called *refutation*. In [9] a stronger axiomatisation was proposed and it was shown that:

- (a) If there is a refutation for  $\varphi$  then  $\neg\varphi$  is provable in the stronger system

This proof does not work for, weaker, Kozen's axiomatisation and it does not look like any simple modification of the argument can help here.

It is also possible to look at the Kozen's proof for aconjunctive formulas from the point of view of refutations. One can introduce a notion of *thin refutation*, which is a refutation where reductions of conjunctions are restricted. A slight extension of Kozen's arguments gives us:

- (b) If there is a thin refutation for  $\varphi$  then  $\neg\varphi$  is provable

Thin refutations suggest the notion of *weakly aconjunctive formulas*. These formulas have the property that every refutation for such a formula is thin. As the name indicates, all aconjunctive formulas are weakly aconjunctive. Below we will use both fact (b) and the notion of weakly aconjunctive formulas.

Let us now give an outline of the proof presented here. As we noted above it seems very hard to improve the statement (b) directly, by trying to enlarge the class of refutations for which it holds. On the other hand by fact (b) in order to show completeness it is enough to prove:

- (c) For every formula  $\varphi$  there is a semantically equivalent aconjunctive formula  $\hat{\varphi}$  such that  $\varphi \Rightarrow \hat{\varphi}$  is provable

This cannot work because it is not true that every formula is equivalent to an aconjunctive formula. This obstacle can be avoided if we allow weakly aconjunctive formulas but still these formulas are not particularly easy to work with. It would certainly save us some work if we first tried to find some class of formulas with better properties.

We will prove a statement like (c) but instead of aconjunctive formulas we will use *disjunctive formulas* [2]. These formulas have several useful properties. First, tableaux for disjunctive formulas have very simple structure. Next, the proof of the fact that negation of every unsatisfiable disjunctive formula is provable is much easier than for weakly aconjunctive formulas (see Theorem 18). The third important property is that for every formula there is a semantically equivalent disjunctive formula. This last statement can be even strengthened as we will describe in the next paragraph.

The properties stated above suggest that instead of proving (c) we should try to prove:

- (d) For every formula  $\varphi$  there is a semantically equivalent disjunctive formula  $\hat{\varphi}$  such that  $\varphi \Rightarrow \hat{\varphi}$  is provable

The tool we will use to construct a proof of  $\varphi \Rightarrow \hat{\varphi}$  is *tableau equivalence*. As we have mentioned above, models for a formula can be constructed from a tableau for the formula. We will say that two tableaux are equivalent if they are essentially the same from the perspective of the model construction procedure. This induces equivalence on formulas which is stronger than semantical equivalence because there exist semantically equivalent formulas which do not have equivalent tableaux. Now it was shown in [2] that for every formula there is a disjunctive formula with an equivalent tableau. The use of tableau equivalence is important because it allows us to replace semantical equivalence with an equivalence which is much finer and syntactically defined.

Another important observation is that we can prove (d) in case  $\varphi$  is a weakly aconjunctive formula. This follows from:

- (e) If  $\alpha$  is a weakly aconjunctive formula,  $\delta$  is a disjunctive formula and the two formulas have equivalent tableaux then  $\alpha \Rightarrow \delta$  is provable

Observe that already with this statement we substantially increase the class of formulas which are known to be provable. We now know that some formulas of the form  $\neg(\varphi \wedge \neg\widehat{\varphi})$  are provable, where  $\neg\widehat{\varphi}$  may not be a weakly aconjunctive formula.

Let us try to use (e) to prove (d) by induction on the structure of  $\varphi$ . This way we will see what we can do and where the problems are.

Suppose  $\varphi = \nu X.\alpha(X)$ . By induction assumption we have a disjunctive formula  $\widehat{\alpha}(X)$  and a proof of  $\alpha(X) \Rightarrow \widehat{\alpha}(X)$ . Hence  $\nu X.\alpha(X) \Rightarrow \nu X.\widehat{\alpha}(X)$  is provable. Because  $\widehat{\alpha}(X)$  is a disjunctive formula,  $\nu X.\widehat{\alpha}(X)$  is a weakly aconjunctive formula although it may not be a disjunctive formula. Let  $\widehat{\varphi}$  be a disjunctive formula with a tableau equivalent to a tableau for  $\nu X.\widehat{\alpha}(X)$ . By (e) we have a proof of  $\nu X.\widehat{\alpha}(X) \Rightarrow \widehat{\varphi}$ . So  $\varphi \Rightarrow \widehat{\varphi}$  is provable.

The problems come only in one case when  $\varphi = \mu X.\alpha(X)$ . This is because  $\mu X.\widehat{\alpha}(X)$  may not be a weakly aconjunctive formula. Fortunately, by the fix-point rule, to prove  $\mu X.\widehat{\alpha}(X) \Rightarrow \widehat{\varphi}$  it is enough to prove  $\widehat{\alpha}(\widehat{\varphi}) \Rightarrow \widehat{\varphi}$ . Formula  $\widehat{\alpha}(\widehat{\varphi})$  is weakly aconjunctive but this time we meet another problem. There may be no tableau for  $\widehat{\alpha}(\widehat{\varphi})$  which is equivalent to a tableau for  $\widehat{\varphi}$ . This should not come as a big surprise as the notion of tableau equivalence is very restrictive; it would be rather surprising if it worked all the way. We remedy this by introducing a weaker relation between tableaux which we call *tableau consequence*. We prove that there is a tableau for  $\widehat{\alpha}(\widehat{\varphi})$  of which a tableau for  $\widehat{\varphi}$  is the consequence. On the other hand the notion of tableau consequence is still strong enough to show a statement similar to (e):

- (f) If  $\alpha$  is a weakly aconjunctive formula,  $\delta$  is a disjunctive formula and a tableau for  $\delta$  is a consequence of a tableau for  $\alpha$  then  $\alpha \Rightarrow \delta$  is provable

This way we obtain a proof of  $\widehat{\alpha}(\widehat{\varphi}) \Rightarrow \widehat{\varphi}$  hence also a proof of  $\varphi \Rightarrow \widehat{\varphi}$ .

The plan of the paper is as follows. We start by defining the  $\mu$ -calculus and some auxiliary notions like: positive guarded formulas, binding function or  $(a \rightarrow \Psi)$  construct. In the next section we recall the results from [2] which we will need here. The notions of tableau equivalence and disjunctive formula are introduced there. Next, we present Kozen's axiomatisation of the logic and

show some simple properties of it. The following section deals with weakly aconjunctive formulas. The last section gives the inductive proof of (d).

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## 2 Preliminary definitions

Let  $Prop = \{p, q, \dots\}$  be a set of propositional letters,  $Var = \{X, Y, \dots\}$  a set of variables and  $Act = \{a, b, \dots\}$  a set of actions. Formulas of the  $\mu$ -calculus over these three sets are defined by the following grammar:

$$F := \top \mid \perp \mid Var \mid Prop \mid \neg F \mid F \vee F \mid F \wedge F \mid \\ \langle Act \rangle F \mid [Act]F \mid \mu Var.F \mid \nu Var.F$$

Additionally we require that in formulas of the form  $\mu X.\alpha(X)$  and  $\nu X.\alpha(X)$ , variable  $X$  occurs in  $\alpha(X)$  only *positively*, i.e., under even number of negations.

We will use  $\sigma$  to denote  $\mu$  or  $\nu$ . Formulas will be denoted by lowercase Greek letters. Uppercase Greek letters will denote finite sets of formulas. We write  $\alpha \Rightarrow \beta$  for  $\neg\alpha \vee \beta$ . For a finite set of formulas  $\Gamma$  we denote by  $\bigwedge \Gamma$  the conjunction of formulas in  $\Gamma$ . Similarly  $\bigvee \Gamma$  denotes the disjunction of formulas in  $\Gamma$ . As usual the conjunction of the empty set is true and the disjunction of the empty set is false. Propositional constants, variables and their negations will be called *literals*.

Formulas are interpreted in Kripke structures  $\mathcal{M} = \langle S, R, \rho \rangle$ , where:  $S$  is a nonempty set of states,  $R : Act \rightarrow \mathcal{P}(S \times S)$  is a function assigning a binary relation on  $S$  to each action in  $Act$  and  $\rho : Prop \rightarrow \mathcal{P}(S)$  is a function assigning a set of states to each propositional letter in  $Prop$ .

The meaning of a formula in a model is a set of states where it is true. For a given model  $\mathcal{M}$  and a valuation  $V : Var \rightarrow \mathcal{P}(S)$ , the meaning of a formula  $\alpha$ , denoted  $\|\alpha\|_V^{\mathcal{M}}$ , is defined by induction on the structure of  $\alpha$  by the following

clauses (we will omit superscript  $\mathcal{M}$  when it causes no ambiguity):

$$\begin{aligned}
\| \top \|_V &= S & \| \perp \|_V &= \emptyset \\
\| X \|_V &= V(X) \\
\| p \|_V &= \rho(p) \\
\| \neg \alpha \|_V &= S - \| \alpha \|_V \\
\| \alpha \wedge \beta \|_V &= \| \alpha \|_V \cap \| \beta \|_V \\
\| \alpha \vee \beta \|_V &= \| \alpha \|_V \cup \| \beta \|_V \\
\| \langle a \rangle \alpha \|_V &= \{s : \exists t. (s, t) \in R(a) \wedge t \in \| \alpha \|_V\} \\
\| [a] \alpha \|_V &= \{s : \forall t. (s, t) \in R(a) \Rightarrow t \in \| \alpha \|_V\} \\
\| \mu X. \alpha(X) \|_V &= \bigcap \{T \subseteq S : \| \alpha \|_{V[T/X]} \subseteq T\} \\
\| \nu X. \alpha(X) \|_V &= \bigcup \{T \subseteq S : T \subseteq \| \alpha \|_{V[T/X]}\}
\end{aligned}$$

Sometimes we will write  $\mathcal{M}, s, V \models \alpha$  instead of  $s \in \| \alpha \|_V^{\mathcal{M}}$ .

**Definition 1 (Positive, guarded formulas)** We call a formula *positive* iff all negations in the formula appear only before propositional constants and free variables.

Variable  $X$  in  $\mu X. \alpha(X)$  is called *guarded* iff every occurrence of  $X$  in  $\alpha$  is in the scope of some modality operator  $\langle a \rangle$  or  $[a]$ . We say that a *formula is guarded* iff every bound variable in the formula is guarded.

**Proposition 2 (Kozen)** Every formula is equivalent to a positive guarded formula.

**Proof**

Let  $\varphi$  be a formula, we first show how to obtain an equivalent guarded formula. The proof proceeds by induction on the structure of the formula with the only nontrivial cases being fixpoint constructors. We present here the case for the least fixpoint. The case for the greatest fixpoint is similar.

Assume that  $\varphi = \mu X. \alpha(X)$  and  $\alpha(X)$  is a guarded formula. Suppose  $X$  is unguarded in some subformula of  $\alpha(X)$  of the form  $\sigma Y. \beta(Y, X)$ . By the assumption, the variable  $Y$  is guarded in  $\sigma Y. \beta(Y, X)$ . We can use the equivalence  $\sigma Y. \beta(Y, X) = \beta(\sigma Y. \beta(Y, X), X)$  to obtain a formula with all unguarded occurrences of  $X$  outside the fixpoint operator. This way we obtain a formula equivalent to  $\alpha(X)$  with all unguarded occurrences of  $X$  not in the scope of a fixpoint operator.

Now using the laws of classical propositional logic we can transform this formula to a conjunctive normal form (considering fixpoint formulas and formulas of the form  $\langle a \rangle \gamma$  and  $[a] \gamma$  as propositional constants). This way we obtain a formula

$$(1) \quad (X \vee \alpha_1(X)) \wedge \dots \wedge (X \vee \alpha_i(X)) \wedge \beta(X)$$

where all occurrences of  $X$  in  $\alpha_1(X), \dots, \alpha_i(X), \beta(X)$  are guarded. Observe that some of  $\alpha_j(X)$  may be just  $\perp$  and  $\beta(X)$  may be  $\top$ . The variable  $X$  occurs only positively in (1) because it did so in our original formula. Formula (1) is equivalent to

$$(X \vee (\alpha_1(X) \vee \dots \vee \alpha_i(X))) \wedge \beta(X)$$

We will show that  $\mu X.(X \vee \bar{\alpha}(X)) \wedge \beta(X)$  is equivalent to  $\mu X.\bar{\alpha}(X) \wedge \beta(X)$ . It is obvious that

$$(\mu X.\bar{\alpha}(X) \wedge \beta(X)) \Rightarrow (\mu X.(X \vee \bar{\alpha}(X)) \wedge \beta(X))$$

Let  $\gamma$  stand for  $\mu X.\bar{\alpha}(X) \wedge \beta(X)$ . To prove another implication it is enough to observe that  $\gamma$  is a pre-fixpoint of  $\mu X.(X \vee \bar{\alpha}(X)) \wedge \beta(X)$  as the following calculation shows:

$$\begin{aligned} (\gamma \vee \bar{\alpha}(\gamma)) \wedge \beta(\gamma) &\Rightarrow \\ ((\bar{\alpha}(\gamma) \wedge \beta(\gamma)) \vee \bar{\alpha}(\gamma)) \wedge \beta(\gamma) &\Rightarrow \\ \bar{\alpha}(\gamma) \wedge \beta(\gamma) &\Rightarrow \gamma \end{aligned}$$

If  $\varphi$  is a guarded formula then we use dualities of the  $\mu$ -calculus:

$$\begin{aligned} \neg(\alpha \vee \beta) &= \neg\alpha \wedge \neg\beta & \neg(\alpha \wedge \beta) &= \neg\alpha \vee \neg\beta \\ \neg\langle a \rangle \alpha &= [a]\neg\alpha & \neg[a]\alpha &= \langle a \rangle \neg\alpha \\ \neg\mu X.\alpha(X) &= \nu X.\neg\alpha(\neg X) & \neg\nu X.\alpha(X) &= \mu X.\neg\alpha(\neg X) \end{aligned}$$

to produce an equivalent positive formula. It is easy to see that it will be still a guarded formula.  $\square$

Next we introduce some tools which allow us to deal with occurrences of subformulas of a given formula. These tools are very similar to those used in [4] or [7]. We would like to have a different name (which will be a variable) for every fixpoint subformula of a given formula. We will also introduce a notion of a binding function which will associate subformulas to names.

**Definition 3 (Binding)** We call a formula *well named* iff every variable is

bound at most once in the formula and free variables are distinct from bound variables. For a variable  $X$  bound in a well named formula  $\alpha$  there exists the unique subterm of  $\alpha$  of the form  $\sigma X.\beta(X)$ , from now on called the *binding definition of  $X$  in  $\alpha$*  and denoted  $\mathcal{D}_\alpha(X)$ . We will omit subscript  $\alpha$  when it causes no ambiguity. We call  $X$  a  $\nu$ -variable when  $\sigma = \nu$ , otherwise we call  $X$  a  $\mu$ -variable.

The function  $\mathcal{D}_\alpha$  assigning to every bound variable its *binding definition* in  $\alpha$  will be called the *binding function* associated with  $\alpha$ .

**Remark:** Every formula is equivalent to a well-named one which can be obtained by some consistent renaming of bound variables. The substitution of a formula  $\beta$  for all free occurrences of a variable  $X$  in  $\alpha$ , denoted  $\alpha[\beta/X]$ , can be made modulo some consistent renaming of bound variables of  $\beta$ , so that the obtained formula  $\alpha[\beta/X]$  is still well-named.

**Definition 4 (Dependency order)** Given a formula  $\alpha$  we define the *dependency order*  $\leq_\alpha$  over the bound variables of  $\alpha$  as the least partial order relation such that if  $X$  occurs free in  $\mathcal{D}_\alpha(Y)$  then  $X \leq_\alpha Y$ . We will say that a bound variable  $Y$  depends on a bound variable  $X$  in  $\alpha$  when  $X \leq_\alpha Y$ .

**Example:** In case  $\alpha = \mu X.\langle a \rangle X \vee \nu Y.\langle b \rangle Y$ , variables  $X$  and  $Y$  are incomparable in  $\leq_\alpha$  ordering. On the other hand if  $\alpha$  is  $\mu X.\nu Y.\langle a \rangle X \vee \mu Z.\langle a \rangle (Z \vee Y)$  then  $X \leq_\alpha Z$ .  $\square$

**Definition 5** Let  $\alpha$  be a formula with an associated binding function  $\mathcal{D}_\alpha$ . For every subformula  $\beta$  of  $\alpha$  we define the *expansion* of  $\beta$  with respect to  $\mathcal{D}_\alpha$  as:

$$\llbracket \beta \rrbracket_{\mathcal{D}_\alpha} = \beta[\mathcal{D}_\alpha(X_n)/X_n] \cdots [\mathcal{D}_\alpha(X_1)/X_1]$$

where the sequence  $(X_1, X_2, \dots, X_n)$  is a linear ordering of all bound variables of  $\alpha$  compatible with the dependency partial order, i.e. if  $X_i \leq_\alpha X_j$  then  $i \leq j$ .

**Definition 6** In construction of our tableaux we will need to distinguish some occurrences of conjunction which should not be reduced by ordinary (*and*) rule. To do this we extend the syntax of the  $\mu$ -calculus by allowing the new construct of the form  $(a \rightarrow \Psi)$ , where  $a$  is an action and  $\Psi$  is a finite set of formulas. When semantics is concerned, we will consider such a formula as an abbreviation of a formula  $\bigwedge \{ \langle a \rangle \psi : \psi \in \Psi \} \wedge [a] \vee \Psi$ .

**Remark:** It is possible to express  $[a]$  and  $\langle a \rangle$  modalities with the construct introduced above. A formula  $[a]\psi$  is equivalent to  $(a \rightarrow \emptyset) \vee (a \rightarrow \{\psi\})$  and a formula  $\langle a \rangle \psi$  is equivalent to  $(a \rightarrow \{\psi, \top\})$ . All the notions from this section



like guarded formula, binding function etc. extend to formulas with this new construct.

**Definition 7 (Terminal formula)** Formula of the form  $(a \rightarrow \emptyset)$  will be called *terminal formula* because its meaning is that there are no  $a$ -transitions from a given state.

**Proviso:** If not otherwise stated all formulas are assumed to be well named, positive, guarded and use  $(a \rightarrow \Psi)$  construct instead of  $\langle a \rangle \psi$  and  $[a] \psi$  modalities. By observations stated above this is not a restriction if semantics is concerned. As we will mention later every formula is provably equivalent to a formula of this kind.

### 3 Tableau equivalence and disjunctive formulas

In this section we will recall results from [2] which we are going to use later on. We define the notions of tableau and tableau equivalence. It turns out that if two tableaux are equivalent then the formulas in the roots of the tableaux are semantically equivalent. In spite of the fact that the implication in the other direction does not hold, tableau equivalence turns out to be a very handy tool. Next we define a notion of disjunctive formula. Some of the properties of these formulas are discussed in [2]. Here we will recall only one result: for a given tableau which can be presented as a finite graph one can construct a disjunctive formula with an equivalent tableau.

**Definition 8 (Tableau rules)** For a formula  $\varphi$  and its binding function  $\mathcal{D}_\varphi$  we define the system of tableau rules  $\mathcal{S}^\varphi$  parameterised by  $\varphi$  or rather its binding function. The system is presented in Figure 1 (we use  $\{\alpha, \Gamma\}$  as a shorthand for  $\{\alpha\} \cup \Gamma$ ).

**Remark:** (1) We see applications of the rules as a process of reduction. Given a finite set of formulas  $\Gamma$  we want to derive, we look for the rule the conclusion of which matches our set. Then we apply the rule and obtain the assumptions of the instance of the rule in which  $\Gamma$  is the conclusion.

(2) There is no rule for reducing formulas of the form  $\langle a \rangle \varphi$  or  $[a] \varphi$  because we assume that this formulas are replaced by equivalent formulas using  $(a \rightarrow \Psi)$  notation.

$$\begin{array}{l}
\text{(and)} \quad \frac{\{\alpha, \beta, \Gamma\}}{\{\alpha \wedge \beta, \Gamma\}} \qquad \qquad \text{(or)} \quad \frac{\{\alpha, \Gamma\} \quad \{\beta, \Gamma\}}{\{\alpha \vee \beta, \Gamma\}} \\
\\
\text{(\mu)} \quad \frac{\{\alpha(X), \Gamma\}}{\{\mu X. \alpha(X), \Gamma\}} \qquad \qquad \text{(\nu)} \quad \frac{\{\alpha(X), \Gamma\}}{\{\nu X. \alpha(X), \Gamma\}} \\
\\
\text{(reg)} \quad \frac{\{\alpha(X), \Gamma\}}{\{X, \Gamma\}} \qquad \text{whenever } X \text{ is a bound variable of } \varphi \\
\qquad \qquad \qquad \text{and } \mathcal{D}_\varphi(X) = \sigma X. \alpha(X) \\
\\
\text{(mod)} \quad \frac{\{\psi\} \cup \{\bigvee \theta : (a \rightarrow \theta) \in \Gamma, \theta \neq \Psi\} \text{ for every } (a \rightarrow \Psi) \in \Gamma, \psi \in \Psi}{\Gamma}
\end{array}$$

Figure 1: The system  $\mathcal{S}^\varphi$

(3) The rule *(mod)* has as many assumptions as there are formulas in sets  $\Psi$ , s.t.,  $(a \rightarrow \Psi) \in \Gamma$ . For example

$$\frac{\{\varphi_1, \varphi_3\} \quad \{\varphi_2, \varphi_3\} \quad \{\varphi_1 \vee \varphi_2, \varphi_3\} \quad \{\psi_1\} \quad \{\psi_2\}}{\{(a \rightarrow \{\varphi_1, \varphi_2\}), (a \rightarrow \{\varphi_3\}), (b \rightarrow \{\psi_1, \psi_2\})\}}$$

is an instance of the rule. We will call a son labeled by an assumption obtained by *reducing* an action  $a$  an  $\langle a \rangle$ -son. In our example if a node  $n$  of a tableau is labeled by the conclusion of the rule then its son labeled by  $\{\varphi_1, \varphi_3\}$  is an  $\langle a \rangle$ -son of  $n$  and a son labeled by  $\{\psi_1\}$  is a  $\langle b \rangle$ -son of  $n$ .

**Definition 9 (Tableaux)** *Tableau* for a formula  $\varphi$  is a pair  $\langle T, L \rangle$ , where  $T$  is a tree and  $L$  is a labelling function such that

1. the root of  $T$  is labeled by  $\{\varphi\}$ ,
2. the sons of any internal node  $n$  are created and labeled according to the rules of the system  $\mathcal{S}_\varphi$ . Additionally we require that the rule *(mod)* is applied only when no other rule is applicable.

As our tableaux may be infinite we will be interested not only in the form of the leaves but also in the internal structure of tableaux. We are now going to distinguish some nodes of tableaux and define a notion of trace which captures the idea of a history of a regeneration of a formula.

**Definition 10 (Modal and choice nodes)** Leaves and nodes where reduction of modalities is performed, i.e., the rule (*mod*) is used, will be called *modal nodes*. The root of the tableau and sons of modal nodes will be called *choice nodes*.

If  $\varphi$  is a guarded formula then the sequence of all the choice nodes on the path of a tableau for  $\varphi$  induces a partition of the path into finite intervals beginning in choice nodes and ending in modal nodes. We will say that a modal node  $m$  is *near* a choice node  $n$  iff they are both in the same interval, i.e., in the tableau there is a path from  $n$  to  $m$  without an application of the rule (*mod*). Observe that in some cases a choice node may be also a modal node.

**Definition 11 (Trace)** Given a path  $\mathcal{P}$  of a tableau  $\mathcal{T} = \langle T, L \rangle$ , a *trace* on  $\mathcal{P}$  will be a function  $Tr$  assigning a formula to every node in some initial segment of  $\mathcal{P}$  (possibly to the whole  $\mathcal{P}$ ), satisfying the following conditions:

- If  $Tr(m)$  is defined then  $Tr(m) \in L(m)$ .
- Let  $m$  be a node with  $Tr(m)$  defined and let  $n \in \mathcal{P}$  be a son of  $m$ . If a rule applied in  $m$  does not reduce the formula  $Tr(m)$  then  $Tr(n) = Tr(m)$ . If  $Tr(m)$  is reduced in  $m$  then  $Tr(n)$  is one of the results of the reduction. This should be clear for all the rules except (*mod*). In case  $m$  is a modal node and  $n$  is labeled by  $\{\psi\} \cup \{\bigvee \theta : (a \rightarrow \theta) \in \Gamma, \theta \neq \Psi\}$  for some  $(a \rightarrow \Psi) \in L(m)$  and  $\psi \in \Psi$ , then  $Tr(n) = \psi$  if  $Tr(m) = (a \rightarrow \Psi)$  and  $Tr(n) = \bigvee \theta$  if  $Tr(m) = (a \rightarrow \theta)$  for some  $(a \rightarrow \theta) \in \Gamma, \theta \neq \Psi$ . Traces from all other formulas end in node  $m$ .

**Definition 12 ( $\mu$ -trace)** We say that there is a *regeneration* of a variable  $X$  on a trace  $Tr$  on some path of a tableau for  $\gamma$  iff for some node  $m$  and its son  $n$  on the path  $Tr(m) = X$  and  $Tr(n) = \alpha(X)$ , where  $\mathcal{D}_\gamma(X) = \sigma X.\alpha(X)$ .

We call a trace  *$\mu$ -trace* iff it is an infinite trace (defined for the whole path) on which the smallest with respect to  $\leq_\gamma$  ordering variable regenerated infinitely often is a  $\mu$ -variable. Similarly a trace will be called a  *$\nu$ -trace* iff it is an infinite trace where the smallest variable which regenerates infinitely often is a  $\nu$ -variable.

**Remark:** Every infinite trace is either a  $\mu$ -trace or a  $\nu$ -trace because all the rules except regenerations decrease the size of formulas and formulas are guarded hence every formula is eventually reduced.

We are now going to define what does it mean for two tableaux to be equivalent. It occurs that we can abstract from the order of application of non-modal rules, but the structure of a tree designated by modal nodes will be very important.

**Definition 13 (Tableau equivalence)** We say that two tableaux  $\mathcal{T}_1$  and  $\mathcal{T}_2$  are *equivalent* iff there is a bijection  $\mathcal{E}$  between choice and modal nodes of  $\mathcal{T}_1$  and  $\mathcal{T}_2$  such that:

1.  $\mathcal{E}$  maps the root of  $\mathcal{T}_1$  onto the root of  $\mathcal{T}_2$ , it maps choice nodes to choice nodes and modal nodes to modal nodes.
2. If  $n$  is a descendant of  $m$  then  $\mathcal{E}(n)$  is a descendant of  $\mathcal{E}(m)$ . Moreover if for some action  $a$ , node  $n$  is a  $\langle a \rangle$ -son of a modal node  $m$  then  $\mathcal{E}(n)$  is a  $\langle a \rangle$ -son of  $\mathcal{E}(m)$ .
3. For every modal node  $m$ , the sets of literals and terminal formulas (i.e. formulas of the form  $(a \rightarrow \emptyset)$ ) occuring in  $L(m)$  and in  $L(\mathcal{E}(m))$  are equal.
4. There is a  $\mu$ -trace on a path  $\mathcal{P}$  of  $\mathcal{T}_1$  iff there is a  $\mu$ -trace on a path of  $\mathcal{T}_2$  designated by the image of  $\mathcal{P}$  under  $\mathcal{E}$ .

**Theorem 14**

*If two formulas (satisfying our proviso) have equivalent tableaux then they are semantically equivalent.*

Next we define the notion of disjunctive formula.

**Definition 15 (Special conjunctions and disjunctive formulas)** A conjunction  $\alpha_1 \wedge \dots \wedge \alpha_n$  is called *special* iff every  $\alpha_i$  is either a literal or a formula of a form  $(a \rightarrow \Psi)$  and for every action  $a$  there is at most one conjunct of the form  $(a \rightarrow \Psi)$  among  $\alpha_1, \dots, \alpha_n$ .

The set of *disjunctive formulas*,  $\mathcal{F}_d$  is the smallest set defined by the following clauses:

1. every literal is a disjunctive formula,
2. if  $\alpha, \beta \in \mathcal{F}_d$  then  $\alpha \vee \beta \in \mathcal{F}_d$ ; if moreover  $X$  occurs only positively in  $\alpha$  and not in the context  $X \wedge \gamma$  for some  $\gamma$ , then  $\mu X.\alpha, \nu X.\alpha \in \mathcal{F}_d$ ,
3.  $(a \rightarrow \Psi) \in \mathcal{F}_d$  if  $\Psi \subseteq \mathcal{F}_d$ ,
4. special conjunction of disjunctive formulas is a disjunctive formula.

**Remark:** Modulo the order of application of (*and*) rules, disjunctive formulas have unique tableaux. Moreover on every infinite path there is one and only one infinite trace.

It turns out that every formula is equivalent to a disjunctive formula. This is unfortunately not a normal form because there may be many equivalent disjunctive formulas.

**Theorem 16**

For every formula  $\varphi$  and every regular tableau  $\mathcal{T}$  for  $\varphi$  (i.e. a tableau which can be presented as a finite graph) there is a disjunctive formula  $\widehat{\varphi}$  with the tableau equivalent to  $\mathcal{T}$ .

**Proof**

We give an outline of the proof. A *tree with back edges* is a tree with added edges leading from some of the leaves to their ancestors. First we prove:

**Lemma 16.1** It is possible to construct a finite tree with back edges  $\mathcal{T}_l$ , satisfying the following conditions:

1.  $\mathcal{T}_l$  unwinds to  $\mathcal{T}$ .
2. Every node to which a back edge points can be assigned color magenta or navy in such a way that for every infinite path of the unwinding of  $\mathcal{T}_l$  we have: there is a  $\mu$ -trace on the path iff the closest to the root node of  $\mathcal{T}_l$  through which the path goes infinitely often is colored magenta.

Having such a tree one constructs from it a disjunctive formula  $\widehat{\varphi}$  which has a tableau equivalent to  $\mathcal{T}$ . The construction starts in the leaves of the tree and proceeds to the root. To every edge leading to a leaf  $n$  we assign a formula  $\widehat{n}$  which is a conjunction of all the literals in the label of  $n$  plus the formula  $\mu X_n.\top$  (with subscript  $n$  in  $X_n$  being the node). The later formula is used to distinguish the label of the node. To every back edge leading to a node  $n$  we assign the variable  $X_n$ . For every internal (*or*) node we take a disjunction of two formulas assigned to the edges going from it. For (*mod*) node we construct appropriate special conjunction. When coming to a node to which some back edge points we know that all the back edges pointing to this node are assigned the same variable  $X_n$ . The color of the node is used to decide which fixpoint operator should be used to close this variable. We take for  $\widehat{\varphi}$  the formula assigned to the root of  $\mathcal{T}_l$ .

It can be shown that  $\mathcal{T}$  is equivalent to a tableau  $\widehat{\mathcal{T}} = \langle \widehat{T}, \widehat{L} \rangle$  for  $\widehat{\varphi}$ . Let  $\mathcal{E} : \mathcal{T} \rightarrow \widehat{\mathcal{T}}$  be this equivalence. The addition of  $\mu X_n.\top$  components in the construction allows us to have the additional property:

**Observation 16.2** For a modal or choice node  $\widehat{m}$  of  $\widehat{\mathcal{T}}$  and a set of modal or choice nodes  $\widehat{N}$  of  $\widehat{\mathcal{T}}$ , if  $\widehat{L}(\widehat{m}) \subseteq \bigcup_{\widehat{n} \in \widehat{N}} \widehat{L}(\widehat{n})$  then  $L(\mathcal{E}^{-1}(\widehat{m})) \subseteq \bigcup_{\widehat{n} \in \widehat{N}} L(\mathcal{E}^{-1}(\widehat{n}))$ . In particular if  $\widehat{L}(\widehat{m}) = \widehat{L}(\widehat{n})$  then  $L(\mathcal{E}^{-1}(\widehat{m})) = L(\mathcal{E}^{-1}(\widehat{n}))$ .

□

## 4 The system

Here we would like to present an axiomatisation of the  $\mu$ -calculus proposed by Kozen [4] and show some simple properties of the system.

We adopt the original formulation of Kozen. The basic judgement of the system has the form  $\alpha = \beta$  with the intended meaning that the two formulas are semantically equivalent. Judgement  $\alpha \leq \beta$  is an abbreviation for  $\alpha \wedge \beta = \alpha$ . A formula  $\alpha$  is *provable* if  $\alpha = \top$  is provable.

The axiomatisation consists of the axioms and rules of equational logic (including substitution of equals by equals, i.e., cut rule) and the following axioms and rules:

$$\begin{aligned}
 (K1) \quad & \text{axioms for Boolean algebra} \\
 (K2) \quad & \langle a \rangle \varphi \vee \langle a \rangle \psi = \langle a \rangle (\varphi \vee \psi) \\
 (K3) \quad & \langle a \rangle \varphi \wedge [a] \psi \leq \langle a \rangle (\varphi \wedge \psi) \\
 (K4) \quad & \langle a \rangle \perp = \perp \\
 (K5) \quad & \alpha(\mu X. \alpha(X)) \leq \mu X. \alpha(X) \\
 (K6) \quad & \frac{\alpha(\varphi) \leq \varphi}{\mu X. \alpha(X) \leq \varphi}
 \end{aligned}$$

Because we have put  $\nu$  and box directly into the language we have to define them by equivalences:

$$\begin{aligned}
 [a] \alpha &= \neg \langle a \rangle \neg \alpha \\
 \nu X. \alpha(X) &= \neg \mu X. \neg \alpha(\neg X)
 \end{aligned}$$

It was proved in [4] that the following rules are admissible:

$$\begin{aligned}
 (\langle \rangle) \quad & \frac{\psi \wedge \bigwedge \{ \bigvee \theta : (a \rightarrow \theta) \in \Gamma, \theta \neq \Psi \} \leq \perp}{(a \rightarrow \Psi) \wedge \bigwedge \Gamma \leq \perp} \quad \text{for some } \psi \in \Psi \\
 (\text{fix}) \quad & \frac{\gamma \wedge \alpha(\mu X. \neg \gamma \wedge \alpha(X)) \leq \perp}{\gamma \wedge \mu X. \alpha(x) \leq \perp} \\
 (\text{mon}) \quad & \frac{\alpha \leq \beta}{\varphi(\alpha) \leq \varphi(\beta)} \quad X \text{ occurs only positively in } \varphi(X)
 \end{aligned}$$

According to our proviso we restrict ourselves to well-named, positive and guarded formulas. We must show that it is a harmless restriction if provability is concerned.

**Fact 17** Every formula is provably equivalent to a formula satisfying the proviso on page 9

**Proof:** Just observe that all the steps used in transforming a formula to a positive guarded form as described in Proposition 2 use provable equivalences.  $\square$

One of the nice properties of disjunctive formulas is that they are easy if provability is concerned.

**Theorem 18**

*For every unsatisfiable disjunctive formula  $\alpha$  the formula  $\neg\alpha$  is provable.*

**Proof**

In [2] it was shown:

- (2) A disjunctive formula  $\alpha$  is satisfiable iff  $\beta$  obtained from  $\alpha$  by replacing all  $\mu$ -variables by  $\perp$  and all  $\nu$ -variables by  $\top$  is satisfiable.

We prove the theorem by induction on the structure of  $\alpha$ .

Suppose  $\alpha$  is a special conjunction  $\alpha_1 \wedge \dots \wedge \alpha_n$ , we have two cases. If  $\alpha_i = \perp$  or  $\alpha_i = \neg\alpha_j$  for some  $i, j \in \{1, \dots, n\}$  then  $\neg\alpha$  is easily provable. Otherwise one of the conjuncts must be of the form  $(a \rightarrow \Psi)$  and one of the formulas from  $\Psi$  must be unsatisfiable. From induction assumption using rule ( $\diamond$ ) we obtain the proof of  $\neg\alpha$ .

If  $\alpha = \gamma \vee \delta$  then by induction assumption we have proofs of  $\neg\gamma$  and  $\neg\delta$  so we can use propositional calculus laws.

If  $\alpha = \mu X.\gamma(X)$  then because this formula is unsatisfiable so is  $\gamma(\perp)$ . By induction assumption there is a proof of  $\neg\gamma(\perp)$  and we can use the derivable rule:

$$\frac{\neg\gamma(\perp) = \top}{\neg\mu X.\gamma(X) = \top}$$

If  $\alpha = \nu X.\gamma(X)$  then we consider  $\gamma(\top)$ . It is of course a disjunctive formula. By (2)  $\gamma(\top)$  is satisfiable iff  $\nu X.\gamma(X)$  is satisfiable. As the later formula is not satisfiable we have by induction assumption the proof of  $\neg\gamma(\top)$  and we can use the derivable rule:

$$\frac{\neg\gamma(\top) = \top}{\neg\nu X.\gamma(X) = \top}$$

$\square$

## 5 Provability for weakly aconjunctive formulas

In this section we will consider a class of formulas for which the provability is easier than in the general case (although not as easy as for disjunctive formulas). We recall the notion of *aconjunctive formulas* [4] and propose its slight generalisation called *weakly aconjunctive formulas*. Our goal in this section is to obtain a generalisation of the main result from [4] which states that the negation of every unsatisfiable aconjunctive formula is provable. To do this we introduce the notion of *thin refutation* which isolates the cases for which the original proof still goes through. It turns out that every refutation of a weakly aconjunctive formula is thin.

**Definition 19 (Weakly aconjunctive formulas)** Let  $\varphi$  be a formula,  $\mathcal{D}_\varphi$  be its binding function and  $\leq_\varphi$  the dependency ordering (see Definitions 3 and 4).

- We say that a variable  $X$  is *active* in  $\psi$ , a subformula of  $\varphi$ , iff there is a variable  $Y$  appearing in  $\psi$  and  $X \leq_\varphi Y$ .
- Let  $X$  be a variable with its binding definition  $\mathcal{D}_\varphi(X) = \mu X.\gamma(X)$ . The variable  $X$  is called *aconjunctive* iff for all subformulas of  $\gamma$  of the form  $\alpha \wedge \beta$  it is not the case that  $X$  is active in  $\alpha$  as well as in  $\beta$ .
- A variable  $X$  as above is called *weakly aconjunctive* iff for all subformulas of  $\gamma$  of the form  $\alpha \wedge \beta$  if  $X$  is active both in  $\alpha$  and  $\beta$  then  $\alpha \wedge \beta$  is a special conjunction as defined in Definition 15.
- Formula  $\varphi$  is called (*weakly*) *aconjunctive* iff all  $\mu$ -variables in  $\varphi$  are (weakly) aconjunctive.

In the following we will be interested only in weakly aconjunctive formulas. The definition of aconjunctive formulas was recalled just to give a comparison of the two notions.

From the next observation follows that all formulas appearing in a tableau for a weakly aconjunctive formula are weakly aconjunctive.

**Fact 20** Every formula appearing in a tableau for  $\varphi$  is a subformula of  $\varphi$ .

The next proposition states some closure properties of the class of weakly aconjunctive formulas. Observe that weakly aconjunctive formulas are not closed under negation nor under the least fixpoint operation.

**Proposition 21 (Composition)** If  $\gamma(X)$  and  $\delta$  are weakly aconjunctive formulas then  $\gamma[\delta/X]$ ,  $\nu X.\gamma(X)$  and  $\delta \wedge \gamma(X)$  are also weakly aconjunctive formulas.



**Proof**

As we consider only well named formulas, when conjunction is formed we make sure that the bound variables in  $\delta$  and  $\gamma(X)$  are different. With this observation it should be easy to see that  $\nu X.\gamma(X)$  and  $\delta \wedge \gamma(X)$  are weakly aconjunctive.

Also while performing substitution  $\gamma[\delta/X]$  we keep bound variables of  $\delta$  distinct from the bound variables of  $\gamma$ . Let  $\alpha = \gamma[\delta/X]$  and let  $Y$  be a  $\mu$ -variable of  $\alpha$ . This variable is bound either in  $\gamma$  or in  $\delta$ . If it is a bound variable from  $\gamma$  then because no bound variable of  $\gamma$  is free in  $\delta$  we have that for every  $Y \leq_\alpha Z$ , variable  $Z$  is a bound variable of  $\gamma$ . Hence  $Y$  is weakly aconjunctive in  $\alpha$  iff it was weakly aconjunctive in  $\gamma$ . For the similar reason every  $\mu$ -variable of  $\delta$  is weakly aconjunctive in  $\alpha$ .  $\square$

**Definition 22 (Refutations)** *Refutation* for a formula  $\varphi$  is defined as tableau, but this time we modify system  $\mathcal{S}^\varphi$  (presented in Figure 1) by adding explicit weakening rule and instead of (*mod*) rule we take ( $\langle \rangle$ ) rule:

$$(\langle \rangle) \quad \frac{\{\psi\} \cup \{\bigvee \theta : (a \rightarrow \theta) \in \Gamma, \theta \neq \Psi\}}{\Gamma} \quad (a \rightarrow \Psi) \in \Gamma, \psi \in \Psi$$

This rule is similar to (*mod*) but has only one assumption. Additionally we require that every leaf of the refutation must be labeled by a set containing  $\perp$  or some literal and its negation and on every infinite path there should be a  $\mu$ -trace.

We call a refutation *thin* iff whenever a formula of the form  $\alpha \wedge \beta$  is reduced in some node of the refutation and some variable is active in  $\alpha$  as well as in  $\beta$  then either  $\alpha \wedge \beta$  is a special conjunction or one of the conjuncts is immediately discarded by the use of weakening rule.

The following is an easy consequence of Fact 20.

**Fact 23** Every refutation for a weakly aconjunctive formula is a thin refutation.

It was shown in [6] that every unsatisfiable formula has a refutation. From this perspective the next theorem essentially says that one can prove the negation of an unsatisfiable weakly aconjunctive formula. The theorem is stated more generally because in Lemma 31 we deal with thin proof tableaux for possibly not weakly aconjunctive formulas.

**Theorem 24**

*If a formula has a thin refutation then its negation is provable.*

**Proof**

The proof is a reformulation of Kozen’s arguments from [4]. Let  $\mathcal{R}$  be a thin refutation for  $\varphi$ . We can assume that in  $\mathcal{R}$  we reduce special conjunctions only when no other formula can be reduced by rules other than (*mod*). This restriction does not change the shape of the tableau. Let  $\mathcal{D}$  be the binding function associated with  $\varphi$  and let  $\leq_\varphi$  be the dependency ordering on the bound variables of  $\varphi$ . It will be convenient here to use some arbitrary linearisation of  $\leq_\varphi$ . We will write  $<_\varphi$  for strictly less relation determined by this linearisation.

We will assign to every node  $m$  of  $\mathcal{R}$  a formula which will contain some information about the path up to  $m$ . The information we are interested in is what variables and in what nodes were regenerated. To see what we mean consider a node  $m$  labeled  $\{X, \Gamma\}$  and its son  $n$  labeled  $\{\alpha(X), \Gamma\}$ . The formula assigned to  $m$  will have the form  $\neg(\gamma \wedge \mu X. \beta(X))$ . Now to remember the context in which  $X$  was regenerated we can use the rule (*fix*) and assign to  $n$  the formula  $\neg(\gamma \wedge \beta(\mu X. \neg\gamma \wedge \beta(X)))$ . If it ever happens that in some descendant  $o$  of  $n$  we regenerate  $X$  in the same context then we can use this recorded information in a sense that the formula assigned to  $o$  will be of the form  $\neg(\gamma \wedge \mu X. \neg\gamma \wedge \beta(X))$ ; hence it will be a provable formula. Summarising we want two properties from our assignment of formulas:

1. If the formula assigned to a node is unprovable then the formula assigned to one of the sons is unprovable.
2. At some nodes use remembering so that on every path there is a variable regenerated in the context which is already recorded.

If  $\neg\varphi$  is not provable then by the first property we can find an infinite path of  $\mathcal{R}$ , every node of which has associated an unprovable formula. By the second property we obtain the contradiction because a formula associated with the node where some variable was regenerated in the same context for the second time is provable.

Unfortunately the second condition is quite difficult to obtain. If we just used remembering trick in every possible node it could happen that we could get infinitely many different contexts. We have to be very careful about what information we remember and what we should “forget”. This is why the assignment of formulas to nodes is rather involved and is split into two steps. First, starting from the root of  $\mathcal{R}$  we assign a token list to every node; then we use this list assignment to define formulas.

We assume that we have a countable set of tokens. Tokens can be removed from the list and we can add tokens to the right end of the list. Removed tokens are never used again. Each token has its own counter. We also assign a pair (formula, bound variable of  $\varphi$ ) to every token on the list.

To the root of  $\mathcal{R}$  we assign an empty list of tokens. A list of tokens for an internal node  $n$  is constructed from the list for its father  $m$  according to the following rules:

1. Suppose  $\beta$  is obtained by reduction of  $\alpha$  and  $\beta$  appears already in  $L(m)$ , i.e., contraction is performed. Let  $X_\alpha$  be the smallest in  $<_\varphi$  ordering variable such that some token  $t_\alpha$  in the list for  $m$  is labeled  $(\alpha, X_\alpha)$ . Similarly find  $X_\beta$  and  $t_\beta$ . If  $X_\beta <_\varphi X_\alpha$  or  $X_\beta = X_\alpha$  and  $t_\beta$  appears to the left of  $t_\alpha$  on the list then remove all the tokens labeled  $(\alpha, Y)$  for  $Y$  active in  $\beta$  otherwise remove all the tokens having  $\beta$  in its label. The intuition is that when contraction is performed two formulas in the label become the same but they may not have the same history. We have to choose which history to remember discarding the other.
2. Suppose (*or*) rule is applied in  $m$  to  $\alpha \vee \beta$  and, say,  $\alpha$  is the result of reduction which appears in  $n$ . The token list for  $n$  is obtained by taking the token list for  $m$  and replacing every occurrence of  $\alpha \vee \beta$  in the labels with  $\alpha$ .
3. Suppose in  $m$  we apply the rule:

$$(\text{reg}) \quad \frac{\{\alpha(X), \Gamma\}}{\{X, \Gamma\}} \quad \mathcal{D}(X) = \sigma X.\alpha(X)$$

For every variable  $Y$  replace the label  $(X, Y)$  by  $(\alpha(X), Y)$ . Next we increase by one the counter of the token labeled by  $(\alpha(X), X)$  and reset to 0 the counters of all the tokens to the right of it.

4. Suppose in  $m$  we apply rule ( $\mu$ ) or ( $\nu$ ):

$$(\sigma) \quad \frac{\{X, \Gamma\}}{\{\sigma X.\alpha(X), \Gamma\}} \quad \mathcal{D}(X) = \sigma X.\alpha(X)$$

For every variable  $Y$  replace the label  $(\sigma X.\alpha(X), Y)$  by  $(\alpha(X), Y)$  After that if  $X$  is a  $\mu$ -variable we add a new token to the right end of the list labeled  $(\alpha(X), X)$  and set its counter to 0.

5. Suppose we apply the rule (*and*) to  $\alpha \wedge \beta$  which is not a special conjunction. Because our proof tableau is thin we know that either one of the conjuncts is deleted or every variable is active in at most one of the formulas  $\alpha$  or  $\beta$ . If only one of the formulas, say  $\alpha$  remains then we proceed exactly as in the case of (*or*) rule. If none of the formulas is deleted then for every

$Y$  we change the label of a token labeled  $(\alpha \wedge \beta, Y)$  to  $(\alpha, Y)$  if  $Y$  is active in  $\alpha$  otherwise we change it to  $(\beta, Y)$ .

6. Suppose we have zero or more applications of (*and*) rule to special conjunctions followed by an application of (*mod*) rule. Let  $n'$  be a node on  $\mathcal{P}$  labeled by the result of application of this (*mod*) rule and let  $a$  be an action reduced by this rule application. We will skip all the nodes between  $m$  and  $n'$  and assign a token list directly to  $n'$ . A special conjunction is of the form

$$\gamma \wedge (a_1 \rightarrow \Phi_1) \wedge (a_2 \rightarrow \Phi_2) \wedge \dots \wedge (a_i \rightarrow \Phi_i)$$

where  $\gamma$  is a conjunction of literals and there is at most one conjunct, say number  $j$ , with action  $a$ . In a token list for  $n'$  we replace the above formula with  $\bigvee \Phi_j$  or some  $\varphi_j \in \Phi_j$  depending on which formula appears in  $n'$ .

7. After the above steps we remove tokens which are either: (i) labeled with pairs  $(\alpha, Y)$  with  $Y$  not active in  $\alpha$  or (ii) labeled with formulas which do not appear in the label of the node.

**Observation 24.1** For every path  $\mathcal{P}$  of  $\mathcal{R}$  there is a counter which gets arbitrary big on  $\mathcal{P}$ .

**Proof**

Let us take a  $\mu$ -trace  $T$  on  $\mathcal{P}$  and let  $X$  be the smallest  $\mu$ -variable regenerated i.o. on the trace. After some time  $X$  is the smallest variable regenerated on the trace. Let  $n_0$  be a node of  $\mathcal{P}$  where  $X$  is regenerated on  $T$  and after which no variable smaller than  $X$  is regenerated on  $T$ . Let  $t_0$  be a token from the list for  $n_0$  labeled  $(X, Y_0)$  for the smallest possible (in  $<_\varphi$  ordering) variable  $Y_0$ . We call  $t_0$  a support of the trace in  $n_0$ . There is a support because  $(X, X)$  is a label of some token. Because of step 7 of the construction we have  $Y_0 \leq_\varphi X$

Suppose  $n_1$  is a node where  $t$  is deleted. As  $X$  is active in  $T(n_1)$ , it might have happened only because of step 1 of the construction. This means that the formula  $T(n_1)$  appeared in the label of the father of  $n_1$  and there is a token  $t_1$  which is either labeled by  $(T(n_1), Y_1)$  for some variable  $Y_1 <_\varphi Y_0$  or  $t_1$  is labeled  $(T(n_1), Y_0)$  but it is to the left of  $t_0$ . This means that the support for the trace changes.

It should be clear that a support can change only finitely many times. Let  $m$  be a node after this last change, where  $X$  is regenerated on the trace. Let  $t$  be a token labeled  $(X, X)$  on the list of  $m$ . From this point  $t$  is never deleted and its counter is increased every time  $X$  is regenerated.

If the counter of  $t$  is not unbounded then it must be the case that there is a token to the left of it in the list which counter is being increased i.o. The counter of the leftmost such token is unbounded.  $\square$

Next we assign a formula to every node of  $\mathcal{R}$ . To do this for every node  $n$  of  $\mathcal{R}$  and every formula  $\beta \in L(n)$  we define a binding function  $\mathcal{D}_{n,\beta}$  from the token list for  $n$ . This binding functions will be obtained from  $\mathcal{D}$  by modifications of one kind. For some  $\mu$ -variables instead of  $\mathcal{D}(X) = \mu X.\alpha(x)$  we will have  $\mathcal{D}_{n,\delta}(X) = \mu X.\neg\gamma_1 \wedge \dots \wedge \neg\gamma_k \wedge \alpha(X)$ , where formulas  $\gamma_1, \dots, \gamma_k$  are determined in the following way:

(\*) Consider ancestors of  $n$  up to the nearest node where a token now labeled  $(\delta, X)$  is added or its counter is reset to zero. Among these ancestors choose all  $n_1, \dots, n_k$  where the counter of the token was increased, then for  $i = 1, \dots, k$ :

$$\gamma_i = \bigwedge \{ \langle \delta \rangle_{\mathcal{D}_{n_i, \delta}} : \delta \in L(n_i), \delta \neq \alpha(X) \}$$

The formula assigned to a node  $n$  is:

$$\neg \bigwedge \{ \langle \beta \rangle_{\mathcal{D}_{n, \beta}} : \beta \in L(n) \}$$

**Observation 24.2** If for some node  $m$  formula  $\neg \bigwedge \{ \langle \beta \rangle_{\mathcal{D}_{m, \beta}} : \beta \in L(m) \}$  is unprovable then there is a son  $n$  of  $m$  such that  $\neg \bigwedge \{ \langle \beta \rangle_{\mathcal{D}_{n, \beta}} : \beta \in L(n) \}$  is unprovable. (In case special conjunctions are reduced in  $m$ , node  $n$  is not a son of  $m$  but a son of the modal node near  $m$ .)

**Proof:** The proof is by cases depending on the rule which was applied in  $m$ . We will consider only one case when the rule applied in  $m$  is a regeneration of a  $\mu$ -variable:

$$\text{(reg)} \quad \frac{\{ \alpha(X), \Gamma \}}{\{ X, \Gamma \}} \quad \mathcal{D}(X) = \mu X.\alpha(X)$$

Assume tokens with  $X$  in the labels were not deleted in the first step of the construction. Looking at the changes to the token list for the son  $n$  of  $m$  we can see that for every  $\beta \in \Gamma$  and every variable  $Y$ ,  $\mathcal{D}_{n,\beta}(Y)$  is either  $\mathcal{D}_{m,\beta}(Y)$  or  $\mathcal{D}(Y)$ . This implies that  $\langle \beta \rangle_{\mathcal{D}_{m,\beta}} \leq \langle \beta \rangle_{\mathcal{D}_{n,\beta}}$  is provable for all  $\beta \in \Gamma$ .

By definition  $\langle X \rangle_{\mathcal{D}_{n,\alpha(X)}}$  is of the form  $\mu X.\gamma \wedge \alpha(X)$  and let us denote  $\{ \langle \beta \rangle_{\mathcal{D}_{n,\beta}} : \beta \in L(n), \beta \neq \alpha(X) \}$  by  $\theta$ . We know that

$$\neg \left( \bigwedge \theta \wedge (\mu X.\gamma \wedge \alpha(X)) \right)$$

is unprovable. By rule (*fix*)

$$\neg(\bigwedge \theta \wedge \gamma \wedge \alpha(\mu X. \neg \bigwedge \theta \wedge \gamma \wedge \alpha(X)))$$

is unprovable, hence

$$\neg(\bigwedge \theta \wedge \alpha(\mu X. \neg \bigwedge \theta \wedge \gamma \wedge \alpha(X)))$$

is unprovable. But  $\alpha(\mu X. \neg \bigwedge \theta \wedge \gamma \wedge \alpha(X)) = \llbracket \alpha(X) \rrbracket_{\mathcal{D}_{n, \alpha(X)}}$ .  $\square$

For the root  $n_0$  of  $\mathcal{R}$  we have  $\mathcal{D}_{n_0, \varphi} = \mathcal{D}$ . Using the assumption that  $\neg\varphi$  is unprovable and the above observation we obtain an infinite path  $\mathcal{P}$  of  $\mathcal{R}$  such that for every node  $n$  of  $\mathcal{P}$  the formula  $\neg \bigwedge \{\llbracket \beta \rrbracket_{\mathcal{D}_{n, \beta}} : \beta \in L(n)\}$  is unprovable.

Let  $t$  be a token for which the counter can be arbitrary big on  $\mathcal{P}$ . Let  $X$  be a variable from the label of  $t$  and let  $\mathcal{D}(X) = \mu X. \alpha(X)$  be its original definition. Because the counter of  $t$  is unbounded there must be two nodes  $n_1, n_2$  such that: (i)  $L(n_1) = L(n_2)$ , (ii) the parts of the lists to the left of  $t$  are identical, (iii)  $t$  is labeled by  $(\alpha(X), X)$  and (iv) the counter of  $t$  was increased and it was not reset between  $n_1$  and  $n_2$ . Let us assume that  $n_2$  is a descendant of  $n_1$ . We will show that  $\neg \bigwedge \{\llbracket \gamma \rrbracket_{\mathcal{D}_{n_2, \gamma}} : \gamma \in L(n_2)\}$  is provable.

As binding functions are established by (\*) we have that

$$(3) \quad \mathcal{D}_{n_2, \delta} = \mathcal{D}_{n_1, \delta} \quad \text{for every formula } \delta \in L(n_1), \delta \neq \alpha(X)$$

This is because by (iii) and (iv) the counters of all the tokens to the right of  $t$  are 0 and all the counters to the left of  $t$  are the same in  $n_1$  and  $n_2$ . Of course the counter of  $t$  in  $n_1$  is strictly smaller than in  $n_2$ .

We have:

$$\mathcal{D}_{n_1, X}(X) = \mu X. \neg\gamma_1 \wedge \dots \wedge \neg\gamma_i \wedge \alpha(X)$$

$$\mathcal{D}_{n_2, X}(X) = \mu X. \neg\gamma_1 \wedge \dots \wedge \neg\gamma_j \wedge \alpha(X)$$

where  $j > i$  and formulas  $\gamma_1, \dots, \gamma_j$  are determined by the rule (\*). We know that  $\gamma_i$  is  $\bigwedge \{\llbracket \delta \rrbracket_{\mathcal{D}_{n_1, \delta}} : \delta \in L(n_1), \delta \neq \alpha(X)\}$  and by (3) it is the same as  $\bigwedge \{\llbracket \delta \rrbracket_{\mathcal{D}_{n_2, \delta}} : \delta \in L(n_2), \delta \neq \alpha(X)\}$ . Finally we have that  $\neg \bigwedge \{\llbracket \gamma \rrbracket_{\mathcal{D}_{n_2, \gamma}} : \gamma \in L(n_2)\}$  is of the form

$$\neg(\neg\gamma_i \wedge \beta(\mu X. \neg\gamma_i \wedge \beta(X))) \vee \neg \bigwedge \{\llbracket \delta \rrbracket_{\mathcal{D}_{(n_2, \delta)}} : \delta \neq \alpha(X), \delta \in L(n_2)\}$$

which is just an instance of the propositional tautology  $\neg(\alpha \wedge \beta) \vee \alpha$ . A contradiction with the choice of  $\mathcal{P}$ .  $\square$

## 6 Completeness

Our main goal is:

### Theorem 25 (Completeness)

*For every unsatisfiable formula  $\varphi$  formula  $\neg\varphi$  is provable.*

Having Theorem 18 to prove completeness it is enough to show that for every unsatisfiable formula  $\varphi$  there is a disjunctive unsatisfiable formula  $\widehat{\varphi}$  such that  $\varphi \leq \widehat{\varphi}$  is provable. Of course we could just take  $\widehat{\varphi}$  to be  $\perp$  but then the proof of this fact would be exactly as difficult as showing completeness. So in general we will look for more complicated formulas than  $\perp$ . Because we will prove this fact by induction on  $\varphi$  we clearly need to consider also satisfiable formulas. From this considerations it follows that we need:

### Theorem 26

*For every positive, guarded formula  $\varphi$  there is a semantically equivalent disjunctive formula  $\widehat{\varphi}$  such that  $\varphi \leq \widehat{\varphi}$  is provable. Moreover if a variable occurs only positively in  $\varphi$  then it occurs only positively in  $\widehat{\varphi}$ .*

Before proving this theorem let us show how to use it in the completeness proof.

#### Proof (Completeness)

Let  $\varphi$  be an unsatisfiable formula. By Proposition 17 we may assume that  $\varphi$  satisfies our proviso from page 9. From Theorem 26 it follows that there is a disjunctive formula  $\widehat{\varphi}$  equivalent to  $\varphi$  and  $\varphi \leq \widehat{\varphi}$  is provable. Hence it is enough to show that  $\neg\widehat{\varphi}$  is provable. But this follows from Theorem 18.  $\square$

The rest of this section is devoted to the proof of Theorem 26.

#### Proof (Theorem 26)

The proof is by induction on the structure of the formula  $\varphi$ .

**Case:  $\varphi$  is a literal** In this case  $\widehat{\varphi}$  is just  $\varphi$ .

**Case:  $\varphi = \alpha \vee \beta$**  By induction assumption there are disjunctive formulas  $\widehat{\alpha}, \widehat{\beta}$  equivalent to  $\alpha$  and  $\beta$  respectively. We let  $\widehat{\alpha \vee \beta}$  to be  $\widehat{\alpha} \vee \widehat{\beta}$ . Because  $\alpha \leq \widehat{\alpha}$  and  $\beta \leq \widehat{\beta}$  are provable,  $\alpha \vee \beta \leq (\widehat{\alpha} \vee \widehat{\beta})$  is also provable.

**Case:  $\varphi = (a \rightarrow \Phi)$**  This case is very similar to the one above.

**Case:**  $\varphi = \mu X.\alpha(X)$  The proof of this case will take a significant part of this section. Fortunately the tools developed here can be also used for the remaining cases.

By induction assumption there is a disjunctive formula  $\hat{\alpha}(X)$  equivalent to  $\alpha(X)$ . It is easy to see that  $\mu X.\alpha(X)$  is semantically equivalent to  $\mu X.\hat{\alpha}(X)$  and  $\mu X.\alpha(X) \leq \mu X.\hat{\alpha}(X)$  is provable. Unfortunately  $\mu X.\hat{\alpha}(X)$  may not be a disjunctive or even weakly aconjunctive formula. This is because  $X$  may occur in a context  $X \wedge \gamma$  for some  $\gamma$ . Therefore we have to construct  $\hat{\varphi}$  from scratch.

By Theorem 16 there is a disjunctive formula  $\hat{\varphi}$  which has the tableau equivalent to some tableau  $\mathcal{T}$  for  $\mu X.\hat{\alpha}(X)$ . By Theorem 14 the two formulas are equivalent. We are left to show that  $\mu X.\hat{\alpha}(X) \leq \hat{\varphi}$  is provable in Kozen's system. To do this it is enough to prove  $\hat{\alpha}(\hat{\varphi}) \leq \hat{\varphi}$  and then use the rule (K6). Now, it is possible to show that if  $\alpha$  and  $\delta$  have equivalent tableaux,  $\alpha$  is weakly aconjunctive and  $\delta$  is disjunctive then  $\alpha \leq \delta$  is provable. Unfortunately the notion of tableau equivalence is too strong for us because there may be no tableau for  $\hat{\alpha}(\hat{\varphi})$  equivalent to a tableau for  $\hat{\varphi}$ . We need some weaker notion of correspondence between tableaux but it should be strong enough to allow us to construct a proof of  $\hat{\alpha}(\hat{\varphi}) \leq \hat{\varphi}$ . Below we propose such a notion which we call tableau consequence. It is defined in terms of games.

**Definition 27 (Tableau consequence)** Given a pair of tableaux  $(\tilde{\mathcal{T}}, \mathcal{T})$ , where  $\tilde{\mathcal{T}} = \langle \tilde{T}, \tilde{L} \rangle$  and  $\mathcal{T} = \langle T, L \rangle$ , we define a two player game  $\mathcal{G}(\tilde{\mathcal{T}}, \mathcal{T})$  with the following rules.

1. The starting position is a pair of the roots of both tableaux.
2. Suppose a position of a play is  $(\tilde{n}, n)$ , both nodes being choice nodes of  $\tilde{\mathcal{T}}$  and  $\mathcal{T}$  respectively. Player *I* must choose a modal node  $\tilde{m}$  near  $\tilde{n}$  and player *II* must respond by choosing a modal node  $m$  near  $n$ . Node  $m$  must have the property that every literal and terminal formula from  $L(m)$  appears in  $\tilde{L}(\tilde{m})$ .
3. Suppose a position of a play is  $(\tilde{N}, N)$  with  $\tilde{N}, N$  being sets of choice nodes of  $\tilde{\mathcal{T}}$  and  $\mathcal{T}$  respectively. Player *I* must choose a modal node  $\tilde{m}$  near some  $\tilde{n} \in \tilde{N}$  and player *II* must respond with a modal node  $m$  near some  $n \in N$ , such that, every literal and terminal formula from  $L(m)$  appears in  $\tilde{L}(\tilde{m})$ .
4. Suppose a position consists of a pair of modal nodes  $(\tilde{m}, m)$  from  $\tilde{\mathcal{T}}$  and  $\mathcal{T}$  respectively. Player *I* chooses some action  $a$  and has two possibilities afterwards. He can choose a  $\langle a \rangle$ -son  $n$  of  $m$  and player *II* has to respond



with a  $\langle a \rangle$ -son  $\tilde{n}$  of  $\tilde{m}$ . Otherwise player  $I$  can choose all  $\langle a \rangle$ -sons of  $m$  and player  $II$  must respond with the set of all  $\langle a \rangle$ -sons of  $\tilde{m}$ .

The game may end in a finite number of steps because one of the players cannot make a move. In this case the opposite player wins. When the game has infinitely many steps we get as the result two infinite paths:  $\tilde{\mathcal{P}}$  from  $\tilde{\mathcal{T}}$  and  $\mathcal{P}$  from  $\mathcal{T}$ . Player  $I$  wins if there is no  $\mu$ -trace on  $\tilde{\mathcal{P}}$  but there is a  $\mu$ -trace on  $\mathcal{P}$ , otherwise player  $II$  is the winner.

**Definition 28 (Strategy)** A strategy  $\mathcal{S}$  for the second player in the game  $\mathcal{G}(\tilde{\mathcal{T}}, \mathcal{T})$  is a function assigning to a position consisting of two modal nodes  $(\tilde{m}, m)$  and a son  $n$  of  $m$  a son  $\mathcal{S}(\tilde{m}, n)$  of  $\tilde{m}$  of the same type as  $n$ . If  $(\tilde{n}, n)$  is a pair of choice nodes and  $\tilde{m}$  is a modal node near  $\tilde{n}$  then the strategy gives us a modal node  $\mathcal{S}(\tilde{m}, n)$  near  $n$ . If a position consists of two sets  $(\tilde{N}, N)$  then for every modal node  $\tilde{m}$  near some  $\tilde{n} \in \tilde{N}$  strategy  $\mathcal{S}$  gives a modal node  $\mathcal{S}(\tilde{m}, N)$  near some  $n \in N$ . A strategy is *winning* if it guarantees that player  $II$  wins no matter what the moves of player  $I$  are.

We will say that a wide tableau  $\mathcal{T}$  is a *consequence* of a wide tableau  $\tilde{\mathcal{T}}$  iff player  $II$  has a winning strategy in  $\mathcal{G}(\tilde{\mathcal{T}}, \mathcal{T})$ .

The definition of the game is based on the following intuition about tableaux. Tableau for a formula describes “operationally” semantics of a formula. In order to satisfy formulas in a choice node  $n$  we must provide a state which satisfies the label of one of the modal nodes near  $n$ . The sons of a modal node describe the transitions from a hypothetical state satisfying its label. Every  $\langle a \rangle$ -son describes an  $a$ -successor which is required. The set of all  $\langle a \rangle$ -sons puts a restriction on all possible  $a$ -successors of the node. In this way tableau of a formula describes all possible models of that formula.

The game is defined so that whenever player  $II$  has a winning strategy from a position  $(\tilde{n}, n)$  then every model of the label of  $\tilde{n}$ ,  $\tilde{L}(\tilde{n})$ , is also a model of the label of  $n$ ,  $L(n)$ . If  $\tilde{n}$  and  $n$  are both choice nodes then a model of  $\tilde{L}(\tilde{n})$  must satisfy the label of one of the modal nodes near  $\tilde{n}$ . Hence for every modal node near  $\tilde{n}$  we must show a modal node near  $n$  which label is implied by it. If  $\tilde{n}, n$  are modal nodes then every  $\langle a \rangle$ -son of  $n$  describes a state the existence of which is required in order to satisfy  $L(n)$ . We must show that existence of such a state is also required by  $\tilde{L}(\tilde{n})$ . The set of all the  $\langle a \rangle$ -sons represents general requirements on states reachable by action  $a$  imposed by  $L(n)$ . We must show that they are implied by the general requirements in  $\tilde{L}(\tilde{n})$ .

The following lemma shows that tableau consequence is indeed weaker than equivalence.

**Lemma 29** If two tableaux  $\mathcal{T}_1$  and  $\mathcal{T}_2$  are equivalent then  $\mathcal{T}_1$  is a consequence of  $\mathcal{T}_2$ .

**Proof:** Let  $\mathcal{E} : \mathcal{T}_1 \rightarrow \mathcal{T}_2$  be an equivalence function. Consider the game  $\mathcal{G}(\mathcal{T}_1, \mathcal{T}_2)$ . The strategy for player *II* is to keep to positions of the form  $(n, \mathcal{E}(n))$ . The initial position is of this form. The strategy is defined by the following rules:

- If a position of a play is a pair of choice or modal nodes  $(m, \mathcal{E}(m))$ , then player *I* chooses some node  $n$  and player *II* replies by choosing  $\mathcal{E}(n)$ .
- If a position of a play is  $(N, \mathcal{E}(N))$  with  $N$  being a set of choice nodes of  $\mathcal{T}_1$  and  $\mathcal{E}(N) = \{\mathcal{E}(n) : n \in N\}$  then player *I* chooses a modal node  $m$  near some  $n \in N$  and player *II* responds with the node  $\mathcal{E}(m)$ .

By the definition of the equivalence this strategy is winning.  $\square$

**Lemma 30** The tableau  $\mathcal{T}$  for  $\mu X.\hat{\alpha}(X)$  is a consequence of the tableau  $\tilde{\mathcal{T}}$  for  $\hat{\alpha}(\hat{\varphi})$ .

**Proof**

Let  $\mathcal{T} = \langle T, L \rangle$ ,  $\tilde{\mathcal{T}} = \langle \tilde{T}, \tilde{L} \rangle$  and let  $\hat{\mathcal{T}} = \langle \hat{T}, \hat{L} \rangle$  be the tableau for  $\hat{\varphi}$ . Recall that  $\hat{\mathcal{T}}$  was constructed from  $\mathcal{T}$  using Theorem 16. Hence we can assume that  $\hat{\mathcal{T}}$  satisfies the properties from Observation 16.2. As the tableaux for  $\mu X.\hat{\alpha}(X)$  and  $\hat{\alpha}(\mu X.\hat{\alpha}(X))$  differ just by one application of the fixpoint rule in the root we will denote by  $\mathcal{T}$  also the tableau for  $\hat{\alpha}(\mu X.\hat{\alpha}(X))$ .

By assumption  $\hat{\varphi}$  and  $\hat{\alpha}(X)$  are disjunctive formulas. We will use  $\beta[\mu X.\hat{\alpha}(X)/\hat{\varphi}]$  and  $\beta[\hat{\varphi}/\mu X.\hat{\alpha}(X)]$  to stand for the obvious replacement, it will be always the case that no free variable in  $\mu X.\hat{\alpha}(X)$  or  $\hat{\varphi}$  is bound by the context  $\beta$ . From Fact 20 we obtain:

**Observation 30.1** For every node  $\tilde{n}$  of  $\tilde{\mathcal{T}}$ , every formula in  $\tilde{L}(\tilde{n})$  is either a disjunctive formula or of the form  $\delta(\hat{\varphi})$  with  $\delta(X)$  being a disjunctive formula.

As the first step, for every node  $\tilde{m}$  of  $\tilde{\mathcal{T}}$  we will define two functions:

$$p_{\tilde{m}} : \tilde{L}(\tilde{m}) \rightarrow \mathbb{N} \cup \{\infty\} \quad \widehat{nd}_{\tilde{m}} : \text{Ran}(p_{\tilde{m}}) \cap \mathbb{N} \rightarrow \mathcal{P}(\hat{T})$$

The first function assigns a priority which is a natural number or  $\infty$  to every formula in  $\tilde{L}(\tilde{m})$ . The function  $\widehat{nd}_{\tilde{m}}$  assigns sets of nodes of  $\hat{T}$  to finite priorities in the range of  $p_{\tilde{m}}$ ;  $\text{Ran}(p_{\tilde{m}}) = \{q : p_{\tilde{m}}^{-1}(q) \neq \emptyset\}$ . Sometimes we will identify a singleton set  $\{m\}$  with the element  $m$ . For example we will write  $\widehat{L}(\widehat{nd}_{\tilde{m}}(q))$  when  $\widehat{nd}_{\tilde{m}}(q)$  is a singleton.

These two functions will satisfy the following condition which we call *II*.

- if  $\widehat{nd}_m(q)$  is a singleton then:

$$p_m^{-1}(q) \subseteq \widehat{L}(\widehat{nd}_m(q)) \subseteq \bigcup_{q' \leq q} p_m^{-1}(q')$$

- (II) • if  $\widehat{nd}_m(q)$  is not a singleton then:

$$p_m^{-1}(q) = \{\bigvee\{\psi : \{\psi\} = \widehat{L}(\widehat{m}), \widehat{m} \in \widehat{nd}_m(q)\}\}$$

- if  $\widehat{m}$  is a modal node then  $\widehat{nd}_m(q)$  is a singleton for all  $q \in \text{Ran}(p_m) \cap \mathbb{N}$

The idea behind these two functions comes from considering  $\widetilde{\mathcal{T}}$  to be some kind of composition of  $\mathcal{T}$  and many copies of  $\widehat{\mathcal{T}}$ . To see what we mean consider a part of a path of  $\widetilde{\mathcal{T}}$  which is represented in the middle of Figure 2. To the left of it we have put a corresponding path of  $\mathcal{T}$  and to the right we have represented a part of  $\widehat{\mathcal{T}}$ . Arrows represent traces.

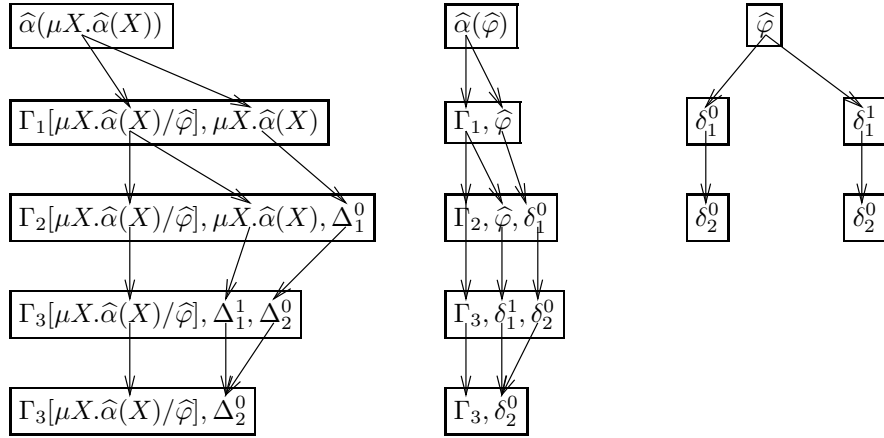


Figure 2: Decomposition of a path of  $\widetilde{\mathcal{T}}$

The label of a node of  $\widetilde{\mathcal{T}}$  can be divided into a set of formulas to which there is no trace going through  $\widehat{\varphi}$ , and the rest which have such a trace. Every formula  $\gamma$  of the first kind corresponds to a formula  $\gamma[\mu X. \widehat{\alpha}(X)/\widehat{\varphi}]$  of  $\mathcal{T}$ . These formulas will have priority  $\infty$ . In our figure they are represented by  $\Gamma$  with indices. For a formula  $\delta$  of the second type there is the earliest occurrence of  $\widehat{\varphi}$  from which

there is a trace to  $\delta$ . This occurrence determines the priority of the formula and the whole trace determines the node of  $\widehat{\mathcal{T}}$ . We use priorities when contraction occurs between two formulas. Consider for example a situation represented in the last two nodes of the part of the path in Figure 2. Formula  $\delta_1^1$  is reduced and becomes the same as  $\delta_2^0$ . Nevertheless the node associated with  $\delta_2^0$  may be different than the son of the node associated with  $\delta_1^1$  because the histories of reductions of these formulas could have been different. The priority tells us that the path from the son of  $\delta_1^1$  should not be followed but the path from the left occurrence of  $\delta_2^0$  should continue. We can say that the trace jumps from  $\delta_1^1$  to the left occurrence of  $\delta_2^0$ . Arranging formulas this way we have that every trace on a path of  $\widetilde{\mathcal{T}}$  is either some trace on the corresponding path of  $\mathcal{T}$  or is eventually (after finitely many jumps) a trace on a path of  $\widehat{\mathcal{T}}$  designated by  $\widehat{nd}$  function.

The two functions will be defined by simultaneous induction on the distance of the node from the root. For the root  $\widetilde{r}$  of  $\widetilde{\mathcal{T}}$  we let  $p_{\widetilde{r}}(\widehat{\alpha}(\widehat{\varphi})) = \infty$ . The induction step is handled by the following two observations.

**Observation 30.2** Suppose  $\widetilde{m} \in \widetilde{\mathcal{T}}$  is a modal node, both  $p_{\widetilde{m}}$  and  $nd_{\widetilde{m}}$  are defined and satisfy condition *I1*. For every son  $\widetilde{n}$  of  $\widetilde{m}$  we can define  $p_{\widetilde{n}}$  and  $nd_{\widetilde{n}}$  so that *I1* will be satisfied.

**Proof:** Let  $\{(a \rightarrow \theta_1), \dots, (a \rightarrow \theta_k)\}$  be all the formulas from  $\widetilde{L}(\widetilde{m})$  having the form  $(a \rightarrow \theta)$ .

Let  $\widetilde{n}$  be a  $\langle a \rangle$ -son of  $\widetilde{m}$  and to keep indexing simple say it is labeled by  $\{\varphi_1\} \cup \{\bigvee \theta_2, \dots, \bigvee \theta_k\}$  for some  $\varphi_1 \in \theta_1$ .

If  $\varphi_1 \neq \bigvee \theta_i$  for all  $i = 2, \dots, k$  then set  $p_{\widetilde{n}}(\bigvee \theta_i) = p_{\widetilde{m}}(a \rightarrow \theta_i)$  and let  $\widehat{nd}_{\widetilde{n}}(p_{\widetilde{n}}(\bigvee \theta_i))$  be the set of all  $\langle a \rangle$ -sons of  $nd_{\widetilde{m}}(p_{\widetilde{m}}(a \rightarrow \theta_i))$ . For  $\varphi_1$  let  $p_{\widetilde{n}}(\varphi_1) = p_{\widetilde{m}}(a \rightarrow \theta_1)$  and let  $\widehat{nd}_{\widetilde{n}}(p_{\widetilde{n}}(\varphi_1))$  be the son of  $\widehat{nd}_{\widetilde{m}}(p_{\widetilde{m}}(a \rightarrow \theta_1))$  which is labeled by  $\{\varphi_1\}$ . Of course we set  $\widehat{nd}_{\widetilde{n}}$  only whenever the priority is finite.

Suppose now that  $\varphi_1 = \bigvee \theta_i$  for some  $i = 2, \dots, k$ . We must decide whether to treat this formula as  $\varphi_1$  or as  $\bigvee \theta_i$ .

- If  $p_{\widetilde{m}}(a \rightarrow \theta_i) < p_{\widetilde{m}}(a \rightarrow \theta_1)$  then let  $p_{\widetilde{n}}(\bigvee \theta_i) = p_{\widetilde{m}}(a \rightarrow \theta_i)$  and let  $\widehat{nd}_{\widetilde{n}}(p_{\widetilde{n}}(\bigvee \theta_i))$  be the set of all the  $\langle a \rangle$ -sons of  $\widehat{nd}_{\widetilde{m}}(p_{\widetilde{m}}(a \rightarrow \{\theta_i\}))$ .
- If  $p_{\widetilde{m}}(a \rightarrow \theta_i) > p_{\widetilde{m}}(a \rightarrow \theta_1)$  then let  $p_{\widetilde{n}}(\varphi_1) = p_{\widetilde{m}}(a \rightarrow \theta_1)$  and let  $\widehat{nd}_{\widetilde{n}}(p_{\widetilde{n}}(\varphi_1))$  to be the son of  $\widehat{nd}_{\widetilde{m}}(p_{\widetilde{m}}(a \rightarrow \theta_1))$  which is labeled by  $\varphi_1$ .

With all  $\bigvee \theta_j$  for  $j = 2, \dots, k, j \neq i$  we proceed as before.  $\square$

**Observation 30.3** Suppose  $\tilde{m} \in \tilde{T}$  is not a modal node, both  $p_{\tilde{m}}$  and  $nd_{\tilde{m}}$  are defined and satisfy condition *I1*. For every son  $\tilde{n}$  of  $\tilde{m}$  we can define  $p_{\tilde{n}}$  and  $nd_{\tilde{n}}$  so that *I1* will be satisfied.

**Proof:** If  $\tilde{m}$  is not a modal node then only one formula, say  $\beta$  is reduced by the rule applied in  $\tilde{m}$ . Let  $q = p_{\tilde{m}}(\beta)$ . Let  $\tilde{n}$  be a son of  $\tilde{m}$  and let  $\gamma \in L(\tilde{n})$  be one of the formulas obtained by reducing  $\beta$ . We have several cases depending on the type of formula  $\beta$ .

- If  $\widehat{nd}_{\tilde{m}}(q)$  is not a singleton then  $\beta = \bigvee \{\psi : \{\psi\} = \widehat{L}(\tilde{m}), \tilde{m} \in \widehat{nd}_{\tilde{m}}(q)\}$ . In this case  $\gamma$  is a disjunct of  $\beta$ . Let  $p_{\tilde{n}}(\gamma) = q$  and let  $\widehat{nd}_{\tilde{n}}(q)$  be an appropriate subset of  $\widehat{nd}_{\tilde{m}}(q)$ . For every  $\delta \in \tilde{L}(\tilde{n}), \delta \neq \gamma$  let  $p_{\tilde{n}}(\delta) = p_{\tilde{m}}(\delta)$  and  $\widehat{nd}_{\tilde{n}}(p_{\tilde{n}}(\delta)) = \widehat{nd}_{\tilde{m}}(p_{\tilde{m}}(\delta))$ .
- If  $\beta = \mu X.\widehat{\alpha}(X)$  then we let  $p_{\tilde{n}}(\gamma)$  to be the smallest priority not in the range of  $p_{\tilde{m}}$  and set  $\widehat{nd}_{\tilde{n}}(p_{\tilde{n}}(\gamma))$  to the root of  $\tilde{T}$ . For every  $\delta \in \tilde{L}(\tilde{n}), \delta \neq \gamma$  we proceed as before.
- If not the previous cases,  $\gamma \in \tilde{L}(\tilde{m})$  and  $p_{\tilde{m}}(\gamma) > p_{\tilde{m}}(\beta)$  then for every  $\delta \in \tilde{L}(\tilde{n})$  let  $p_{\tilde{n}}(\delta) = p_{\tilde{m}}(\delta)$  and  $\widehat{nd}_{\tilde{n}}(p_{\tilde{n}}(\delta)) = \widehat{nd}_{\tilde{m}}(p_{\tilde{m}}(\delta))$ .
- If not the previous cases,  $\gamma \notin \tilde{L}(\tilde{m})$  or  $p_{\tilde{m}}(\gamma) \leq p_{\tilde{m}}(\beta)$  then  $p_{\tilde{n}}(\gamma) = p_{\tilde{m}}(\beta)$  and  $\widehat{nd}_{\tilde{n}}(p_{\tilde{n}}(\gamma))$  is the son of  $\widehat{nd}_{\tilde{m}}(p_{\tilde{m}}(\beta))$  containing  $\gamma$ . For all  $\delta \in \tilde{L}(\tilde{n}), \delta \neq \gamma$  we proceed as in the first case.

□

Next step is to define a winning strategy in the game  $\mathcal{G}(\tilde{T}, \mathcal{T})$ . We will write  $nd_{\tilde{m}}(q)$  for  $\mathcal{E}^{-1}(\widehat{nd}_{\tilde{m}}(q))$ . All positions  $(\tilde{m}, m)$  reachable in a game played according to the strategy will have the following property.

If  $\tilde{m}$  is a choice or modal node then (whenever defined)  $nd_{\tilde{m}}(q)$  is either a singleton or a set of all the  $\langle a \rangle$ -sons of some node. Let  $L_{\tilde{m}}(q)$  stand for  $L(n)$  if  $\{n\} = nd_{\tilde{m}}(q)$  and let  $L_{\tilde{m}}(q) = \{\bigvee \theta_1, \dots, \bigvee \theta_k\}$  if  $nd_{\tilde{m}}(q)$  is the set of all  $\langle a \rangle$ -sons of some node  $m$  and  $\{(a \rightarrow \theta_1), \dots, (a \rightarrow \theta_n)\} \subseteq L(m)$  is the set of all the formulas of the form  $(a \rightarrow \theta)$  in  $L(m)$ . With this definition we have the property:

$$(I2) \quad L(m) \subseteq \bigcup_{q \in \text{Ran}(p_{\tilde{m}}) \cap \mathbb{N}} L_{\tilde{m}}(q) \cup \{\psi[\mu X.\widehat{\alpha}(X)/\widehat{\varphi}] : \psi \in p_{\tilde{m}}^{-1}(\infty)\}$$

Condition *I2* allows us to define function  $p_{\tilde{m}, m} : L(m) \rightarrow \mathbb{N} \cup \{\infty\}$ . For every  $\gamma \in L(m)$  let  $p_{\tilde{m}, m}(\gamma)$  be the smallest priority  $q$  such that  $\gamma \in L_{\tilde{m}}(q)$ ; in case there is no such  $q$  let  $p_{\tilde{m}, m}(\gamma) = \infty$ .

The strategy is described in the next three observations.

**Observation 30.4** Assume the game is in a position  $(\tilde{m}, m)$  consisting of two modal nodes and the condition *I2* is satisfied. Suppose player *I* chooses a  $\langle a \rangle$ -son  $n$  of  $m$ . We can find a  $\langle a \rangle$ -son  $\tilde{n}$  of  $\tilde{m}$  so that: (i) condition *I2* will be satisfied, and (ii) if  $\beta \in L(n)$  is obtained from  $\alpha \in L(m)$  then  $p_{n,n}^{\sim}(\beta) \leq p_{m,m}^{\sim}(\alpha)$ .

**Proof:** Let  $n$  be a  $\langle a \rangle$ -son of  $m$ . It is labeled by

$$\{\xi\} \cup \{\bigvee \theta : (a \rightarrow \theta) \in L(m), \theta \neq \Xi\}$$

for some  $(a \rightarrow \Xi) \in L(m)$  and  $\xi \in \Xi$ . Let  $q = p_{m,m}^{\sim}(a \rightarrow \Xi)$ . If  $q = \infty$  we take a son of  $\tilde{m}$  labeled

$$\{\xi[\widehat{\varphi}/\mu X.\widehat{\alpha}(X)]\} \cup \{\bigvee \theta : (a \rightarrow \theta) \in \tilde{L}(\tilde{m}), \theta \neq \Xi[\widehat{\varphi}/\mu X.\widehat{\alpha}(X)]\}$$

The case when  $q \in \mathbb{N}$  is represented in Figure 3.

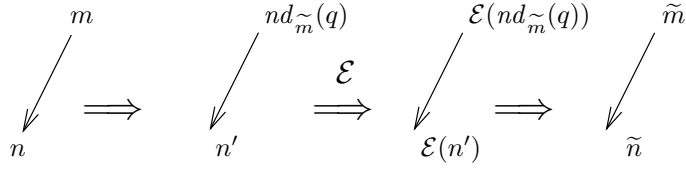


Figure 3: Finding a  $\langle a \rangle$ -son of  $\tilde{m}$  for a  $\langle a \rangle$ -son of  $m$ .

By *I2* there is a  $\langle a \rangle$ -son  $n'$  of  $nd_m^{\sim}(q)$  labeled

$$\{\xi\} \cup \{\bigvee \theta : (a \rightarrow \theta) \in L(nd_m^{\sim}(q)), \theta \neq \Xi\}$$

Using equivalence  $\mathcal{E}$  we get  $\langle a \rangle$ -son  $\mathcal{E}(n')$  of  $\widehat{nd}_m^{\sim}(q) = \mathcal{E}(nd_m^{\sim}(q))$ . It is labeled by some  $\{\psi\}$  for  $\psi \in \Psi$  and  $(a \rightarrow \Psi) \in \widehat{L}(\widehat{nd}_m^{\sim}(q))$ . By *I1* we can take as  $\tilde{n}$  a  $\langle a \rangle$ -son of  $\tilde{m}$  labeled:

$$\{\psi\} \cup \{\bigvee \theta : (a \rightarrow \theta) \in \tilde{L}(\tilde{m}), \theta \neq \Psi\}$$

It is quite straightforward to show that (i) and (ii) are satisfied if we use the fact that for every two choice nodes  $\widehat{n}_1, \widehat{n}_2$  of  $\widehat{\mathcal{T}}$ , whenever  $\widehat{L}(\widehat{n}_1) = \widehat{L}(\widehat{n}_2)$  then  $L(\mathcal{E}^{-1}(\widehat{n}_1)) = L(\mathcal{E}^{-1}(\widehat{n}_2))$ . We can assume this property by Observation 16.2.  $\square$

**Observation 30.5** Assume the game is in a position  $(\tilde{m}, m)$  consisting of two choice nodes and the condition *I2* is satisfied. Suppose player *I* chooses a modal node  $\tilde{n}$  near  $\tilde{m}$ . We can find a modal node  $n$  near  $m$  so that: (i) the condition *I2*

will be satisfied, and (ii) the traces from  $m$  to  $n$  will be preserved. This means that whenever there is a trace from  $\alpha \in L(m)$  to  $\beta \in L(n)$  and  $Y$  is the smallest in  $\leq_{\mu X.\hat{\alpha}(X)}$  ordering variable regenerated on the trace then either:

- $p_{m,m}^{\sim}(\alpha) > p_{n,n}^{\sim}(\beta)$  or
- $p_{m,m}^{\sim}(\alpha) = p_{n,n}^{\sim}(\beta) = q$  and when  $q \in \mathbb{N}$  there is a trace from  $\alpha \in L(nd_m^{\sim}(q))$  to  $\beta \in L(nd_n^{\sim}(q))$  or when  $q = \infty$  there is a trace from  $\alpha[\hat{\varphi}/\mu X.\hat{\alpha}(X)] \in L(\tilde{m})$  to  $\beta[\hat{\varphi}/\mu X.\hat{\alpha}(X)] \in L(\tilde{n})$ . In both cases  $Y$  is the smallest regenerated variable on the trace.

**Proof:** We will find desired  $n$  by constructing a path to it from  $m$ . In some sense  $n$  is determined by  $\tilde{n}$  and all  $nd_n^{\sim}(q)$  for  $q \in \mathbb{N}$ . For every  $q \in \text{Ran}(p_m^{\sim}) \cap \mathbb{N}$  let  $\theta_q = \{\psi : \{\psi\} = \hat{L}(\tilde{m}), \tilde{m} \in \widehat{nd}_m^{\sim}(q)\}$ . By *I1* we have:

$$\tilde{L}(\tilde{m}) = \{\bigvee \theta_q : q \in \text{Ran}(p_m^{\sim}) \cap \mathbb{N}\} \cup p_m^{\sim^{-1}}(\infty)$$

On the path to  $\tilde{n}$  there is a node  $\tilde{s}$  where exactly one disjunct is chosen from each  $\theta_q$ . Say its label is

$$\tilde{L}(\tilde{s}) = \{\psi_q : q \in \text{Ran}(p_m^{\sim}) \cap \mathbb{N}\} \cup p_m^{\sim^{-1}}(\infty)$$

for some  $\psi_q \in \theta_q$ , ( $q \in \text{Ran}(p_m^{\sim}) \cap \mathbb{N}$ ). For this node we can define  $\widehat{nd}_s^{\sim}(q)$  to be a node from  $\widehat{nd}_m^{\sim}(q)$  labeled  $\{\psi_q\}$ . As before we define  $nd_s^{\sim}(q) = \mathcal{E}^{-1}(\widehat{nd}_s^{\sim}(q))$ .

Switching to the other tableau by *I2* we have:

$$L(m) \subseteq \bigcup_{q \in \mathbb{N}} L_m^{\sim}(q) \cup \{\psi[\mu X.\hat{\alpha}(X)/\hat{\varphi}] : \psi \in p_m^{\sim^{-1}}(\infty)\}$$

If  $nd_m^{\sim}(q)$  is a singleton then  $\bigvee \theta_q = \psi_q$  and  $L(nd_m^{\sim}(q)) = L(nd_s^{\sim}(q))$ . Otherwise  $L_m^{\sim}(q) = \{\bigvee \Delta_1, \dots, \bigvee \Delta_k\}$  and  $L(nd_s^{\sim}(q)) = \{\bigvee \Delta_1, \dots, \delta_i, \dots, \bigvee \Delta_k\}$  for some  $i = 1, \dots, k$  and  $\delta_i \in \Delta_i$ . First apply (*or*) rules from  $m$  to obtain a node  $s$  such that:

$$(4) \quad L(s) \subseteq \bigcup_{q \in \mathbb{N}} L(nd_s^{\sim}(q)) \cup \{\psi[\mu X.\hat{\alpha}(X)/\hat{\varphi}] : \psi \in p_m^{\sim^{-1}}(\infty)\}$$

The obtained situation and the rest of the construction is represented in Figure 4. From  $s$  we will construct a path choosing one node at the time. For every considered node  $o$  we will define a priority function  $p_{s,o}^{\sim} : L(o) \rightarrow \mathbb{N} \cup \{\infty\}$ . We

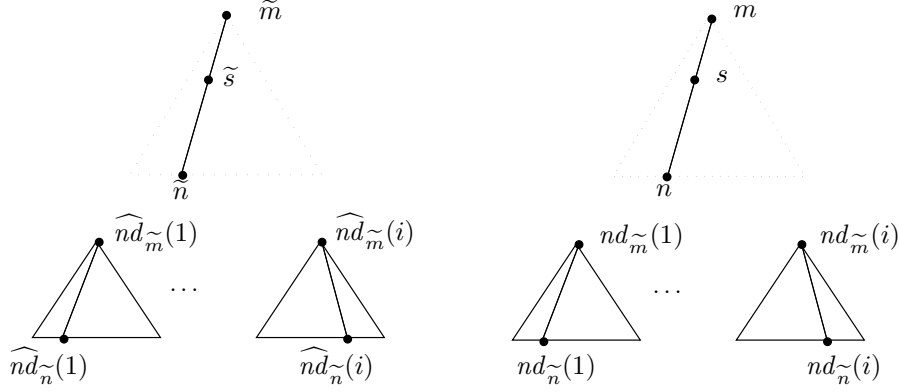


Figure 4: Finding a modal node  $n$  for the modal node  $\tilde{n}$

will assume that for every considered node  $o$  and every  $\psi \in L(o)$ :

- (I3) If  $p_{s,o}^{\sim}(\psi) = \infty$  then  $\psi[\widehat{\varphi}/\mu X.\widehat{\alpha}(X)]$  appears on the path from  $\tilde{m}$  to  $\tilde{n}$ , otherwise if  $p_{s,o}^{\sim}(\psi) = q \in \mathbb{N}$  then  $\psi$  appears on the path from  $nd_{\tilde{m}}^{\sim}(q)$  to  $nd_{\tilde{n}}^{\sim}(q)$

Function  $p_{s,s}^{\sim}$  is defined by (4) by letting  $p_{s,s}^{\sim}(\psi)$  to be the smallest  $q$  such that  $\varphi \in L(nd_{\tilde{n}}^{\sim}(q))$  or  $\infty$  if there is no such  $q$ . Actually it may happen that  $s$  does not satisfy I3 or rather I3 does not make sense because  $Ran(p_n^{\sim}) \neq Ran(p_m^{\sim})$ . Let us extend  $\widehat{nd}_{\tilde{m}}^{\sim}$  and  $\widehat{nd}_{\tilde{n}}^{\sim}$ . The only element which can appear in  $Ran(p_n^{\sim}) \setminus Ran(p_m^{\sim})$  is the smallest priority  $q$  which does not appear in  $Ran(p_m^{\sim})$ . We take care about this by extending the definition of  $\widehat{nd}_{\tilde{m}}^{\sim}$  and letting  $\widehat{nd}_{\tilde{m}}^{\sim}(q)$  to be the root of  $\widehat{T}$ . Let  $q \in Ran(p_m^{\sim}) \setminus Ran(p_n^{\sim})$  and let  $\Gamma \subseteq \widehat{L}(\tilde{n})$  be the set of all the formulas to which there is a trace from the unique formula of priority  $q$  in  $\widehat{L}(\tilde{m})$ .  $\Gamma$  is the label of some modal node  $\hat{n}$  near  $\widehat{nd}_{\tilde{m}}^{\sim}(q)$ . Let  $\widehat{nd}_{\tilde{n}}^{\sim}(q) = \hat{n}$ . As  $\Gamma \subseteq \bigcup \{\widehat{L}(\widehat{nd}_{\tilde{n}}^{\sim}(q')) : q' \in Ran(p_n^{\sim}), q' < q\}$  we know by Observation 16.2 that

$$(5) \quad nd_{\tilde{n}}^{\sim}(q) \subseteq \bigcup \{L(nd_{\tilde{n}}^{\sim}(q')) : q' \in Ran(p_n^{\sim}), q' < q\}$$

With this extensions I3 is satisfied for the node  $s$  and we may proceed with the construction of the path.

- If  $\psi$  is not a disjunction then there is only one son  $o'$  of  $o$ . Let  $\psi'$  be the result of reducing  $\psi$ . For every  $\beta \in L(o')$ ,  $\beta \neq \psi'$  we let  $p_{s,o'}^{\sim}(\beta) = p_{s,o}^{\sim}(\beta)$ . If  $\psi' \notin L(o)$  then let  $p_{s,o'}^{\sim}(\psi') = p_{s,o}^{\sim}(\psi)$  otherwise let  $p_{s,o'}^{\sim}(\psi') = \min\{p_{s,o}^{\sim}(\psi), p_{s,o}^{\sim}(\psi')\}$ . If  $\psi \neq \mu X.\widehat{\alpha}(X)$  then function  $p_{s,o'}$  satisfies the condition I3. If  $\psi = \mu X.\widehat{\alpha}(X)$  then  $\psi' = X$  and letting  $p_{s,o'}^{\sim}(X) = \infty$



would be unsound with respect to *I3*. We let  $p_{s,o'}^{\sim}(X)$  to be the smallest priority  $q$  not in  $Ran(p_n^{\sim})$ . This is sound as  $nd_n^{\sim}(q)$  is a modal node near the root of  $\mathcal{T}$ .

- If  $\psi = \alpha \vee \beta$  then  $o$  has two sons  $o_1, o_2$  and we have to choose one of them. If  $p_{s,o}^{\sim}(\psi) = \infty$  then  $\psi[\widehat{\varphi}/\mu X.\widehat{\alpha}(X)]$  is on the path from  $\widetilde{n}$  to  $\widetilde{m}$  otherwise  $\psi$  appears on the path from  $nd_m^{\sim}(p_{s,o}^{\sim}(\psi))$  to  $nd_n^{\sim}(p_{s,o}^{\sim}(\psi))$ . We choose a son of  $o$  with the same disjunct as the one appearing on the appropriate path. For the chosen  $o'$  we define  $p_{s,o'}^{\sim}$  as in the case of unary rule. It should be easy to check that for so defined  $o'$  and  $p_{s,o'}^{\sim}$  the condition *I3* holds.

Repeating this procedure we arrive at a modal node  $n$  near  $m$ . Let us check that condition *I2* holds. Suppose  $\psi \in L(n)$  and  $q = p_{s,n}^{\sim}(\psi)$ . Because  $n$  is a modal node,  $\psi$  can be reducible only by application of (*mod*) rule. By *I3* if  $q = \infty$  then  $\psi[\widehat{\varphi}/\mu X.\widehat{\alpha}(X)] \in \widetilde{L}(\widetilde{n})$  otherwise  $q \in \mathbb{N}$  and  $\psi \in L(nd_n^{\sim}(q))$ . In the later case either  $q \in Ran(p_n^{\sim})$  or by (5) we have  $q' \in Ran(p_n^{\sim})$ ,  $q' \leq q$  with  $\psi \in nd_n^{\sim}(q')$ .

Finally it is easy to see that the traces are preserved.  $\square$

**Observation 30.6** Suppose a position in the game is  $(\widetilde{N}, N)$  for  $\widetilde{N}$  being a set of all the  $\langle a \rangle$ -sons of some node  $\widetilde{m}$  and  $N$  being a set of all the  $\langle a \rangle$ -sons of some node  $m$ . Suppose also that *I2* holds for the pair  $(\widetilde{m}, m)$ . For every modal node  $\widetilde{o}$  near some  $\widetilde{n} \in \widetilde{N}$  we can find a modal node  $o$  near some  $n \in N$  so that: (i) condition *I2* holds for position  $(\widetilde{o}, o)$  and (ii) the traces from  $m$  to  $o$  are preserved.

**Proof:**

For every  $q \in Ran(p_n^{\sim}) \cap \mathbb{N}$  let  $\theta_q = \bigvee \{ \psi : \{ \psi \} = L(\widetilde{n}), \widetilde{n} \in \widehat{nd}_n^{\sim}(q) \}$ . By *II* we have:

$$\widetilde{L}(\widetilde{n}) = \{ \bigvee \theta_q : q \in Ran(p_n^{\sim}) \cap \mathbb{N} \} \cup p_n^{\sim^{-1}}(\infty)$$

On the way to  $\widetilde{o}$  we reach a node  $\widetilde{s}$  where exactly one formula  $\psi_q$  is chosen from each  $\theta_q$ .

$$\widetilde{L}(\widetilde{s}) = \{ \psi_q : q \in Ran(p_m^{\sim}) \cap \mathbb{N} \} \cup p_m^{\sim^{-1}}(\infty)$$

For this node we can define  $\widehat{nd}_s^{\sim}(q)$  to be a node from  $\widehat{nd}_m^{\sim}(q)$  labeled  $\{ \psi_q \}$ . As before we define  $nd_s^{\sim}(q) = \mathcal{E}^{-1}(\widehat{nd}_s^{\sim}(q))$ .

Now  $L(nd_s^{\sim}(q)) = \{ \delta_1 \} \cup \{ \bigvee \Delta_2, \dots, \bigvee \Delta_k \}$  and  $L(nd_n^{\sim}(q))$  is either the same set of formulas or it is  $\{ \bigvee \Delta_1, \dots, \bigvee \Delta_k \}$  for some  $\Delta_1 \ni \delta_1$ .

We can find a choice node  $n \in N$  and a descendant  $s$  of  $n$  such that:

$$L(s) \subseteq \bigcup_{q \in \mathbb{N}} L(nd_{\sim}^{-1}(q)) \cup \{\psi[\mu X.\widehat{\alpha}(X)/\widehat{\varphi}] : \psi \in p_m^{-1}(\infty)\}$$

From this point we can repeat the arguments from the previous observation.  $\square$

Finally we would like to show that the defined strategy is winning. Let us take some play of  $\mathcal{G}(\widetilde{\mathcal{T}}, \mathcal{T})$  where  $II$  plays according to the strategy. By the three observations above player  $II$  can always make a move so  $II$  cannot loose in a finite number of steps. Assume that the play was infinite. The result of the play are two paths  $\widetilde{\mathcal{P}} = \{\widetilde{n}_0, \widetilde{n}_1, \dots\}$  of  $\widetilde{\mathcal{T}}$  and  $\mathcal{P} = \{n_0, n_1, \dots\}$  of  $\mathcal{T}$ . If there is no  $\mu$ -trace on  $\mathcal{P}$  then  $II$  wins, so assume that there is a  $\mu$ -trace on  $\mathcal{P}$ .

By condition *I2* for every choice or modal node  $n_i$  of  $\mathcal{P}$  we can define priority  $p_{\widetilde{n}_i, n_i}$ . By trace preservation this priority cannot increase, hence after some index  $j$  it is constant, say equal  $q$ .

If  $q = \infty$  then  $j = 0$  and for every  $k \geq 0$  by *I2* we have  $T(n_k)[\widehat{\varphi}/\mu X.\widehat{\alpha}(X)] \in \widetilde{L}(\widetilde{n}_k)$ . By trace preservation we obtain a  $\mu$ -trace on  $\widetilde{\mathcal{P}}$  going thorough this formulas.

If  $q \in \mathbb{N}$  then for  $k \geq j$  by *I2* we have  $T(n_k) \in L(nd_{\sim}^{-1}(q))$  which by trace preservation gives us a  $\mu$ -trace on the path  $\mathcal{P}' = \{nd_{\sim}^{-1}(q), nd_{\sim}^{-1}(q), \dots\}$  of  $\mathcal{T}$ . By equivalence  $\mathcal{E}$  we have a  $\mu$ -trace on the path  $\widehat{\mathcal{P}} = \{\widehat{nd}_{\sim}^{-1}(q), \widehat{nd}_{\sim}^{-1}(q), \dots\}$  of  $\widehat{\mathcal{T}}$ . By condition *I1* we have a  $\mu$ -trace on  $\widetilde{\mathcal{P}}$  which means that player  $II$  wins.  $\square$

In the next lemma we show how to use the fact that a tableau for a formula  $\delta$  is a consequence of a tableau for a formula  $\alpha$  to prove  $\alpha \leq \delta$ .

**Lemma 31** Suppose that we have a weakly aconjunctive formula  $\alpha$  and a disjunctive formula  $\delta$ . If there is a tableau for  $\delta$  which is a consequence of a tableau for  $\alpha$  then  $\neg(\alpha \wedge \neg\delta)$  is provable.

**Proof**

Let  $\mathcal{T}_\alpha = \langle T_\alpha, L_\alpha \rangle$  and  $\mathcal{T}_\delta = \langle T_\delta, L_\delta \rangle$  be tableaux for  $\alpha$  and  $\delta$  respectively such that the second is a consequence of the first. Let  $\mathcal{S}$  be a winning strategy for player  $II$  in the game  $\mathcal{G}(\mathcal{T}_\alpha, \mathcal{T}_\delta)$ . We will construct a thin refutation  $\mathcal{R} = \langle T, L \rangle$  for  $\alpha \wedge \neg\delta$ .

To facilitate the construction we will define two correspondence functions  $\mathcal{C}_\alpha$  and  $\mathcal{C}_\delta$  which assign to every considered node of  $\mathcal{R}$  (that is not to all the nodes) a node of  $\mathcal{T}_\alpha$  and  $\mathcal{T}_\delta$  respectively. It will be always the case that:

$$C1 \quad L(n) = L_\alpha(\mathcal{C}_\alpha(n)) \cup \{\neg \wedge L_\delta(\mathcal{C}_\delta(n))\},$$

C2 strategy  $\mathcal{S}$  is defined for the position  $(\mathcal{C}_\alpha(n), \mathcal{C}_\delta(n))$ .

The root of  $\mathcal{R}$  will be of course labeled by  $\{\alpha \wedge \neg\delta\}$ , the next node, say  $m_0$ , by  $\{\alpha, \neg\delta\}$ . We let  $\mathcal{C}_\alpha(m_0)$  and  $\mathcal{C}_\delta(m_0)$  to be the roots of  $\mathcal{T}_\alpha$  and  $\mathcal{T}_\beta$  respectively. The next two observations show how to prolong  $\mathcal{R}$ .

**Observation 31.1** Suppose we have already constructed  $\mathcal{R}$  up to a node  $m$ ,  $\mathcal{C}_\alpha(m)$ ,  $\mathcal{C}_\delta(m)$  are choice nodes of appropriate tableaux and satisfy C1, C2. We can construct a finite part of  $\mathcal{R}$  and define for each leaf  $n$  of the constructed part  $\mathcal{C}_\alpha(n)$  and  $\mathcal{C}_\delta(n)$  so that: (i)  $\mathcal{C}_\delta(n) = \mathcal{S}(\mathcal{C}_\alpha(n), \mathcal{C}_\delta(m))$ , (ii) conditions C1 and C2 are satisfied and (iii) traces from  $m$  to  $n$  are *reflected*. This last property means that the traces from  $m$  to  $n$  are exactly the traces from  $\mathcal{C}_\alpha(m)$  to  $\mathcal{C}_\alpha(n)$  with the exception of the trace from  $\neg \wedge L_\delta(\mathcal{C}_\delta(m))$  to  $\neg \wedge L_\delta(\mathcal{C}_\delta(n))$  which corresponds to negated (unique) trace from  $\mathcal{C}_\delta(m)$  to  $\mathcal{C}_\delta(n)$ .

**Proof:** By assumption  $L(m) = L_\alpha(\mathcal{C}_\alpha(m)) \cup \{\neg\gamma\}$  as  $\mathcal{C}_\delta(m)$  is labeled by one formula because  $\delta$  is a disjunctive formula. The idea of the construction is represented in Figure 5.

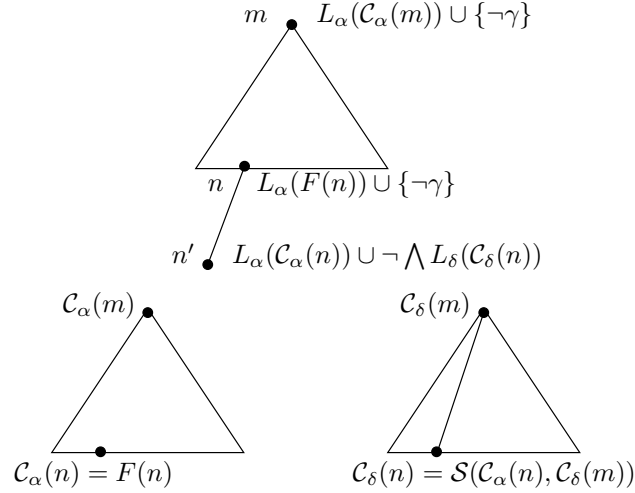


Figure 5: Construction of  $\mathcal{R}$

From  $m$  we apply as long as possible rules other than  $(\langle \rangle)$  and weakening to all formulas in  $L(m)$  except  $\neg\gamma$ . We apply them in the same order as they were applied from  $\mathcal{C}_\alpha(m)$ . This way we obtain a finite tree rooted in  $m$ . This

tree is isomorphic to the part of  $\mathcal{T}_\alpha$  between  $\mathcal{C}_\alpha(m)$  and nearest modal nodes. Denoting this isomorphism  $F$  we have the property that for every leaf  $n$  of this part  $L(n) = L_\alpha(F(n)) \cup \{\neg\gamma\}$ . Set  $\mathcal{C}_\alpha(n) = F(n)$ . Strategy  $\mathcal{S}$  gives us a node  $\mathcal{C}_\delta(n)$  which is a reply of player  $II$  to choosing  $\mathcal{C}_\alpha(n)$  by player  $I$ . From the definition of the game it follows that  $\mathcal{C}_\delta(n)$  is a modal node near  $\mathcal{C}_\delta(m)$ . Let us look at the path from  $\mathcal{C}_\delta(m)$  to  $\mathcal{C}_\delta(n)$  in  $\mathcal{T}_\delta$ . Because  $\delta$  is a disjunctive formula, on this path first only  $(\sigma)$ ,  $(cons)$  and  $(or)$  rules may be applied and then we have zero or more applications of  $(and)$  rule. Let us apply dual rules to  $\neg\gamma$  (dual to  $(\mu)$  is  $(\nu)$ ,  $(cons)$  is self-dual). When it comes to an application of  $(or)$  rule in  $\mathcal{T}_\delta$ , apply  $(and)$  rule followed by weakening to leave only the conjunct which appears on the path to  $\mathcal{C}_\delta(n)$ . This way we make sure that the resulting tableau will be thin.

After these reductions we get a node  $n'$  which is labeled by  $L(\mathcal{C}_\alpha(n)) \cup \{\neg \wedge L_\delta(\mathcal{C}_\delta(n))\}$ . Setting  $\mathcal{C}_\alpha(n') = \mathcal{C}_\alpha(n)$  and  $\mathcal{C}_\delta(n') = \mathcal{C}_\delta(n)$  establishes conditions  $C_1$  and  $C_2$ . Finally trace reflection follows directly from the construction.  $\square$

**Observation 31.2** Suppose we have constructed the tableau up to a node  $m$ . Assume that  $\mathcal{C}_\alpha(m)$  and  $\mathcal{C}_\delta(m)$  are modal nodes and  $C1, C2$  are satisfied. We can construct a finite part of the tableau and define  $\mathcal{C}_\alpha(n), \mathcal{C}_\delta(n)$  for every leaf  $n$  of this part in such a way that: (i) position  $(\mathcal{C}_\alpha(n), \mathcal{C}_\delta(n))$  will be reachable from  $(\mathcal{C}_\alpha(m), \mathcal{C}_\delta(m))$  when player  $II$  plays according to  $\mathcal{S}$  (ii) conditions  $C1, C2$  will be satisfied, (iii) traces will be reflected.

**Proof:** Let  $\gamma = \wedge L_\delta(\mathcal{C}_\delta(m)) = \wedge \{\gamma_1, \dots, \gamma_l\}$  and  $\Gamma = L_\alpha(\mathcal{C}_\alpha(m))$ . By  $C1$  we have  $L(m) = \{\neg\gamma\} \cup \Gamma$ . Node  $\mathcal{C}_\delta(m)$  is a modal node hence every  $\gamma_i$  is either a literal or a formula of the form  $(a \rightarrow \Phi)$ . When we negate  $\gamma$  we obtain a disjunction of negations of such formulas. Let us apply  $(or)$  rule to eliminate these disjunctions. This way we obtain new leaves  $m_1, \dots, m_l$ . For every  $i \in 1, \dots, l$  node  $m_i$  is labeled by  $\{\neg\gamma_i\} \cup \Gamma$ . We use  $\gamma_i$  to decide what rule to apply in  $m_i$ .

If  $\gamma_i$  is a literal or a terminal formula then we are done because  $\gamma_i$  appears in  $\Gamma$ . This follows directly from  $C2$  and the definition of the game.

If  $\gamma_i$  is of the form  $(a \rightarrow \Phi)$  with  $\Phi \neq \emptyset$  then negated it becomes

$$\bigvee \{[a]\neg\varphi : \varphi \in \Phi\} \vee \langle a \rangle \bigwedge \{\neg\varphi : \varphi \in \Phi\}$$

or rather a translation of this formula to  $(a \rightarrow \theta)$  notation. We apply disjunction rules as long as possible. This way we obtain a part of a tree. Each leaf  $u$  of this part is labeled by  $\Gamma$  and one of the disjuncts.

- Suppose this disjunct is  $(a \rightarrow \emptyset)$ . As  $\Phi \neq \emptyset$ , there is an  $\langle a \rangle$ -son of  $\mathcal{C}_\delta(m)$  so there is a  $\langle a \rangle$ -son of  $\mathcal{C}_\alpha(n)$ . Hence there is  $(a \rightarrow \theta) \in \Gamma$  with  $\theta \neq \emptyset$ . We obtain an axiom after one application of  $(\langle \rangle)$ .
- If it is  $(a \rightarrow \{\neg\varphi\})$  for some  $\varphi \in \Phi$  then let us consult the strategy in the case when player  $I$  chooses the  $\langle a \rangle$ -son  $u_\delta$  of  $\mathcal{C}_\delta(m)$  labeled by  $\{\varphi\}$ . Strategy  $\mathcal{S}$  gives us in this case a  $\langle a \rangle$ -son  $u_\alpha$  of  $\mathcal{C}_\alpha(m)$ . This son is labeled by  $\{\psi\} \cup \{\bigvee \theta : (a \rightarrow \theta) \in \Gamma, \theta \neq \Psi\}$  for some  $(a \rightarrow \Psi) \in \Gamma$  and  $\psi \in \Psi$ . We apply  $(\langle \rangle)$  rule to  $(a \rightarrow \Psi)$  in  $L(u)$  and obtain a son  $u'$  of  $u$  labeled  $\{\psi\} \cup \{\bigvee \theta : (a \rightarrow \theta) \in \Gamma, \theta \neq \Psi\} \cup \{\neg\varphi\}$ . We let  $\mathcal{C}_\alpha(u') = u_\alpha$  and  $\mathcal{C}_\delta(u') = u_\delta$ .
- If it is  $(a \rightarrow \{\bigwedge \{\neg\varphi : \varphi \in \Phi\}, \top\})$  then we apply  $(\langle \rangle)$  rule to this formula and obtain a son  $u'$  of  $u$  labeled

$$\{\bigwedge \{\neg\varphi : \varphi \in \Phi\}\} \cup \{\bigvee \theta : (a \rightarrow \theta) \in \Gamma\}$$

The construction from this point is presented in Figure 6.

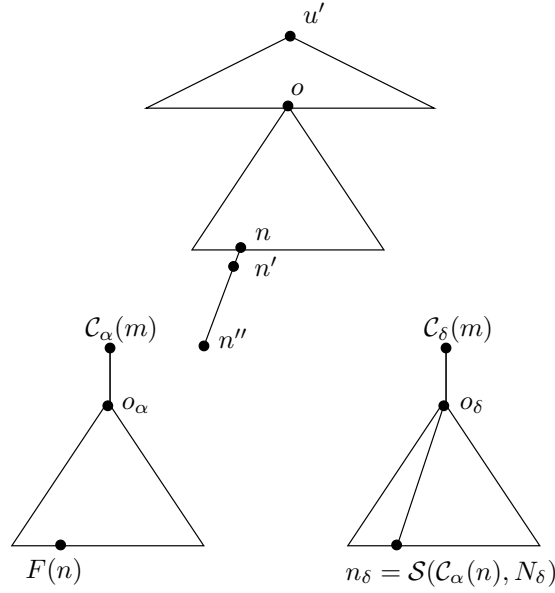


Figure 6: Constructing part of  $\mathcal{R}$

Let us choose one formula  $\bigvee \Psi \in \{\bigvee \theta : (a \rightarrow \theta) \in \Gamma\}$  and apply  $(or)$  rules to it. This way we obtain a part of  $\mathcal{R}$  each leaf of which is labeled by the

set:

$$\{\bigwedge\{\neg\varphi : \varphi \in \Phi\}\} \cup \{\psi\} \cup \{\bigvee\theta : (a \rightarrow \theta) \in \Gamma, \theta \neq \Psi\}$$

for some  $\psi \in \Psi$ . Let  $o$  be one of such leaves and let  $o_\alpha$  be a  $\langle a \rangle$ -son of  $\mathcal{C}_\alpha(m)$  labeled by  $\{\psi\} \cup \{\bigvee\theta : (a \rightarrow \theta) \in \Gamma, \theta \neq \Psi\}$ . As in the previous observation apply rules other than (*mod*) and weakening to this set to obtain a finite part of a tree isomorphic to the part of  $\mathcal{T}_\alpha$  between  $o_\alpha$  and the nearest modal nodes. Let  $F$  denote this isomorphism. Let  $n$  be a leaf of this part. We have:  $L(n) = L(F(n)) \cup \bigwedge\{\neg\varphi : \varphi \in \Phi\}$ . Now it is time to consult the strategy.

Let player *I* choose  $N_\delta$ , the set of all  $\langle a \rangle$ -sons of  $\mathcal{C}_\delta(n)$ . Player *II* responds with  $N_\alpha$  being the set of all  $\langle a \rangle$ -sons of  $\mathcal{C}_\alpha$ . For every modal node  $n_\alpha$  near  $o_\alpha \in N_\alpha$  the strategy  $\mathcal{S}$  gives us a modal node  $n_\delta = \mathcal{S}(n_\alpha, N_\delta)$ . Let  $n_\delta = \mathcal{S}(F(n), N_\delta)$  and let  $o_\delta$  be a modal node on the path to  $n_\delta$ . We have  $L_\delta(o_\delta) = \{\varphi\}$  for some  $\varphi \in \Phi$ . Apply (*and*) rule followed by contraction to obtain a node  $n'$  labeled  $\{\neg\varphi\} \cup L(n)$ . Then reduce  $\neg\varphi$  in  $n'$  as in the proof of the previous observation. We arrive at a node  $n''$  labeled  $\{\neg \bigwedge L_\delta(n_\delta)\} \cup L_\alpha(F(n))$ . Let  $\mathcal{C}_\alpha(n'') = F(n)$  and  $\mathcal{C}_\delta(n'') = n_\delta$ .

□

The above two observations describe  $\mathcal{R}$  completely. All the leaves are labeled by sets containing  $\perp$  or some literal and its negation. For every infinite path  $\mathcal{P}$  we have two possibilities. There may be a  $\mu$ -trace on a path of  $\mathcal{T}_\alpha$  designated by the image of  $\mathcal{P}$  under  $\mathcal{C}_\alpha$ . If it is so then by trace reflection there is also a  $\mu$ -trace on  $\mathcal{P}$ . If there is no  $\mu$ -trace on  $\mathcal{C}_\alpha(\mathcal{P})$  then there cannot be a  $\mu$ -trace on  $\mathcal{C}_\delta(\mathcal{P})$  because we were choosing our moves accordingly to the strategy  $\mathcal{S}$ . Hence there is a  $\nu$ -trace on  $\mathcal{C}_\delta(\mathcal{P})$  which negated in  $\mathcal{R}$  becomes a  $\mu$ -trace.

This shows that  $\mathcal{R}$  is a refutation.  $\mathcal{R}$  is also a thin refutation because  $\alpha$  is a weakly aconjunctive formula and whenever we reduce a conjunction coming from  $\neg\delta$  we leave only one of the conjuncts. Hence by Theorem 24  $\neg(\alpha \wedge \neg\delta)$  is provable.

□

Summarising the case of the proof of Theorem 26 for  $\varphi = \mu X.\alpha(X)$ . By induction assumption we have a disjunctive formula  $\widehat{\alpha}(X)$  equivalent to  $\alpha(X)$  and know that  $\alpha(X) \leq \widehat{\alpha}(X)$  is provable. By Theorem 24 we obtain a disjunctive formula  $\widehat{\varphi}$  which has a tableau  $\widehat{\mathcal{T}}$  equivalent to some tableau  $\mathcal{T}$  for  $\mu X.\widehat{\alpha}(X)$ . By Theorem 14, formula  $\widehat{\varphi}$  is equivalent to  $\varphi$ . By Lemma 30  $\mathcal{T}$  is a consequence of  $\widetilde{\mathcal{T}}$ . By Lemma 29  $\widehat{\mathcal{T}}$  is a consequence of  $\mathcal{T}$ . Hence, as consequence relation is transitive,  $\widehat{\mathcal{T}}$  is a consequence of  $\widetilde{\mathcal{T}}$ . Now by Proposition 21,  $\widehat{\alpha}(\widehat{\varphi})$  is a weakly

aconjunctive formula and  $\widehat{\varphi}$  is by definition a disjunctive formula. By Lemma 31  $\widehat{\alpha}(\widehat{\varphi}) \leq \widehat{\varphi}$  is provable. Then  $\mu X.\widehat{\alpha}(X) \leq \widehat{\varphi}$  is provable by rule (K6) and  $\varphi \leq \mu X.\widehat{\alpha}(X)$  is provable by induction assumption, hence  $\varphi \leq \widehat{\varphi}$  is provable.

**Case:**  $\varphi = \nu X.\alpha(X)$  By induction assumption we have equivalent disjunctive formula  $\widehat{\alpha}(X)$  and  $\alpha(X) \leq \widehat{\alpha}(X)$  is provable. By Theorem 24 we obtain a disjunctive formula  $\widehat{\varphi}$  which has a tableau  $\widehat{T}$  equivalent to some tableau  $\mathcal{T}$  for  $\nu X.\widehat{\alpha}(X)$ . Fortunately, by Proposition 21,  $\nu X.\widehat{\alpha}(X)$  is a weakly aconjunctive formula and by Lemma 29  $\widehat{T}$  is a consequence of  $\mathcal{T}$ . Hence we can use Lemma 31 to show that  $\nu X.\alpha(X) \leq \widehat{\varphi}$  is provable.

**Case:**  $\varphi = \alpha \wedge \beta$  By induction assumption there are disjunctive formulas  $\widehat{\alpha}, \widehat{\beta}$  equivalent to  $\alpha$  and  $\beta$  respectively and such that both  $\alpha \leq \widehat{\alpha}$  and  $\beta \leq \widehat{\beta}$  are provable. Hence  $\alpha \wedge \beta \leq \widehat{\alpha} \wedge \widehat{\beta}$  is provable. By Theorem 24 there is a disjunctive formula  $\widehat{\varphi}$  which has a tableau equivalent to some tableau for  $\widehat{\alpha} \wedge \widehat{\beta}$ . Because, by Proposition 21,  $\widehat{\alpha} \wedge \widehat{\beta}$  is a weakly aconjunctive formula, we can as in the case before use Lemma 31 to show that  $\alpha \wedge \beta \leq \widehat{\varphi}$  is provable. □

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