

# Comment résumer le plan<sup>†</sup>

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Cet article concerne les graphes de recouvrement d'un ensemble fini de points du plan Euclidien. Un graphe de recouvrement  $H$  est de facteur d'étirement  $t$  pour un ensemble de points  $S$  si, entre deux points quelconques de  $S$ , le coût d'un plus court chemin dans  $H$  est au plus  $t$  fois leur distance Euclidienne. Les graphes de recouvrement d'étirement  $t$  (ci-après nommés  $t$ -spanneurs) sont à la base de nombreux algorithmes de routage et de navigation dans le plan. Le graphe (ou triangulation) de Delaunay, le graphe de Gabriel, le graphe de Yao ou le Theta-graphe sont des exemples bien connus de  $t$ -spanneurs. L'étirement  $t$  et le degré maximum des spanneurs sont des paramètres important à minimiser pour l'optimisation des ressources. En même temps le caractère planaire des constructions se révèle essentiel dans les algorithmes de navigation.

Nous présentons une série de résultats dans ce domaine, en particulier:

- Nous montrons que le graphe  $\Theta_6$  (le Theta-graphe où  $k = 6$  cônes d'angle  $\Theta_k = 2\pi/k$  par sommet sont utilisées) est l'union de deux spanneurs planaires d'étirement deux. En particulier, nous établissons que l'étirement maximum du graphe  $\Theta_6$  est deux, ce qui est optimal. Des bornes supérieures sur l'étirement du graphe  $\Theta_k$  n'étaient connues que lorsque  $k > 6$ . Pour  $k = 7$ , la meilleure borne connue est d'environ 7.56 et pour  $k = 6$  il était ouvert de savoir si le graphe était un  $t$ -spanneur pour une valeur constante de  $t$ .
- Nous montrons que le graphe  $\Theta_6$  contient comme sous-graphe couvrant un 3-spanneur planaire de degré maximum au plus 9.
- Finalement, en utilisant une variante du résultat précédant, nous montrons que le plan Euclidien possède un 6-spanneur planaire de degré maximum au plus 6.

La dernière construction, non décrite ici par manque de place, améliore une longue série de résultats sur le problème largement ouvert de déterminer la plus petite valeur  $\delta$  telle que tout ensemble du plan possède un spanneur planaire d'étirement constant et de degré maximum  $\delta$ . Le meilleur résultat en date montrait que  $3 \leq \delta \leq 14$ .

**Mots-clefs :** Theta-graph, spanner, Delaunay triangulation

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## 1 Introduction

A *geometric graph* is a weighted graph whose vertex set is a set of points of  $\mathbb{R}^d$ , and whose edge set consists of line segments joining two vertices. The weight of any edge is the Euclidean distance ( $L_2$ -norm) between its endpoints. The *Euclidean complete graph* is the complete geometric graph, in which all pairs of distinct vertices are connected by an edge.

Although geometric graphs are in theory specific weighted graphs, they naturally model many practical problems and in various fields of Computer Science, from Networking to Computational Geometry. Delaunay triangulations, Yao graphs, theta-graphs,  $\beta$ -skeleton graphs, Nearest-Neighborhood graphs, Gabriel graphs are just some of them [GO97]. A companion concept of the geometric graphs is the *graph spanner*. A  $t$ -spanner of a graph  $G$  is a spanning subgraph  $H$  such that for each pair  $u, v$  of vertices the distance in

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$H$  between  $u$  and  $v$  is at most  $t$  times the distance in  $G$  between  $u$  and  $v$ . The value  $t$  is called *stretch factor* of the spanner.

Spanners have been independently introduced in Computational Geometry by Chew [Che89] for the complete Euclidean graph, and in the fields of Networking and Distributed Computing by Peleg and Ullman [PU89] for arbitrary graphs. Literature in connection with spanners is vast and applications are numerous. We refer to Peleg’s book [Pel00], and Narasimhan and Smid’s book [NS07] for comprehensive introduction to the topic.

**Motivations.** In recent years, bounded degree plane spanners have been used as the building block of wireless network communication topologies. Emerging wireless distributed system technologies such as wireless ad-hoc and sensor networks are often modeled as proximity graphs in the Euclidean plane. Spanners of proximity graphs represent topologies that can be used for efficient unicasting, multicasting, *and/or* broadcasting. For these applications, spanners are typically required to be planar and have bounded degree: the planarity requirement is for efficient routing (given a source  $s$  and a target  $t$ , by following boundaries of polygons traversed by the edge  $(s, t)$ ), while the bounded degree requirement is due to the physical limitations of wireless devices (Bluetooth scatternets, for example, can be modeled as spanners of the geometric graph where master nodes must have at most 7 slave nodes [LSW04]).

**Theta-graphs.** Theta-graphs [Cla87, Kei88] are very popular geometric graphs that appear in the context of navigating graphs. Adjacency is defined as follows: the space around each point  $p$  is decomposed into  $k \geq 2$  regular cones, each with apex  $p$ , and a point  $q \neq p$  of a given cone  $C$  is linked to  $p$  if, from  $p$ ,  $q$  is the “nearest” point in  $C$ : the nearest neighbor of  $p$  is the point whose orthogonal projection onto the bisector of  $C$  minimizes the  $L_2$ -distance.

$\Theta_k$ -graphs are known to be efficient spanners: in [RS91], the stretch is shown to be at most  $1/(1 - 2\sin(\pi/k))$  for every  $k > 6$ . Very little is known for  $k \leq 6$ . For  $k = 7$ , and according to the current upper bound, we observe that the stretch of these graphs is larger than 7.562, and the stretch drops under 2 only from  $k \geq 13$ .

Our main result relies on a specific subgraph of the  $\Theta_k$ -graph, namely the *half- $\Theta_k$ -graph* and denoted by  $\frac{1}{2}\Theta_k$ , taking only half the edges, those belonging to non consecutive cones in the counter-clockwise order (see Section 2 for more a formal definition). For even  $k$ , every  $\Theta_k$ -graph is the union of two spanning half- $\Theta_k$ -graphs.

## 2 Our contribution

Given points in the two-dimensional Euclidean plane, the complete Euclidean graph  $\mathcal{E}$  is the complete weighted graph embedded in the plane whose nodes are identified with the points. In the following, given a graph  $G$ ,  $V(G)$  and  $E(G)$  stand for the set of nodes and edges of  $G$ . For every pair of nodes  $u$  and  $w$ , we identify with edge  $uw$  the segment  $[uw]$  and associate an edge length equal to the Euclidean distance  $|uw|$ . We say that a subgraph  $H$  of a graph  $G$  is a  *$t$ -spanner* of  $G$  if for any pair of vertices  $u, v$  of  $G$ , the distance between  $u$  and  $v$  in  $H$  is at most  $t$  times the distance between  $u$  and  $v$  in  $G$ ; the constant  $t$  is referred to as the *stretch factor* of  $H$  (with respect to  $G$ ). We will say that  $H$  is a spanner if it is a  $t$ -spanner of  $\mathcal{E}$  for some constant  $t$ .

A *cone*  $C$  is the region in the plane between two rays that emanate from the same point. Let us consider the rays obtained by a rotation of the positive  $x$ -axis by angles of  $i\pi/3$  with  $i = 0, 1, \dots, 5$ . Each pair of successive rays defines a cone whose apex is the origin. Let  $\mathcal{C}_6 = (\overline{C}_2, C_1, \overline{C}_3, C_2, \overline{C}_1, C_3)$  be the sequence of cones obtained, in counter-clockwise order, starting from the positive  $x$ -axis. The cones  $C_1, C_2, C_3$  are said to be *positive* and the cones  $\overline{C}_1, \overline{C}_2, \overline{C}_3$  are said to be *negative*. We assume a cyclic structure on the labels so that  $i + 1$  and  $i - 1$  are always defined. For a positive cone  $C_i$ , the clockwise next cone is the negative cone  $\overline{C}_{i+1}$  and the counter-clockwise next cone is the negative cone  $\overline{C}_{i-1}$ .

For each cone  $C \in \mathcal{C}_6$ , let  $\ell_C$  be the bisector ray of  $C$  (in Figure 1 (a), for example, the bisector rays of the positive cones are shown). For each cone  $C$  and each point  $u$ , we define  $C^u := \{x + u : x \in C\}$ , the translation of cone  $C$  from the origin to point  $u$ . We set  $\mathcal{C}_6^u := \{C + u : C \in \mathcal{C}_6\}$ , the set of all six cones at  $u$ . Observe that  $w \in C_i^u$  if and only if  $u \in \overline{C}_i^w$ .

Comment résumer le plan

Let  $v$  be a point in a cone  $C^u$ . The *projection distance* from  $u$  to  $v$ , denoted  $d_P(u, v)$ , is the Euclidean distance between  $u$  and the projection of  $v$  onto  $\ell_{C^u}$ . For any two points  $v$  and  $w$  in  $C^u$ ,  $v$  is *closer* to  $u$  than  $w$  if and only if  $d_P(u, v) < d_P(u, w)$ . We denote by  $\text{parent}_i(u)$  the closest point from  $u$  belonging to cone  $C_i^u$ .

We say that a given set of points  $S$  are in *general position* if no two points of  $S$  form a line parallel to one of the rays that define the cones of  $\mathcal{C}_6$ . For the sake of simplicity, in the rest of the paper we only consider sets of points that are in general position. This will imply that it is impossible that two points  $v$  and  $w$  have equal projective distance from another point  $u$ . Note that, in any case, ties can be broken arbitrarily when ordering points that have the same distance (for instance, using a counter-clockwise ordering around  $u$ ).

Our starting point is a geometric graph  $\frac{1}{2}\Theta_6$  which represents the first step of our construction.

**Step 1** Every node  $u$  of  $\mathcal{E}$  chooses  $\text{parent}_i(u)$  in each non-empty cone  $C_i^u$ . We denote by  $\frac{1}{2}\Theta_6$  the resulting subgraph.

While we consider  $\frac{1}{2}\Theta_6$  to be undirected, we will refer to an edge in  $\frac{1}{2}\Theta_6$  as *outgoing* with respect to  $u$  when chosen by  $u$  and *incoming* with respect to  $v = \text{parent}_i(u)$ , and we color it  $i$  if it belongs to  $C_i^u$ . Note that edge  $uv$  is in the negative cone  $\overline{C}_i^v$  of  $v$  (see Figure 1).

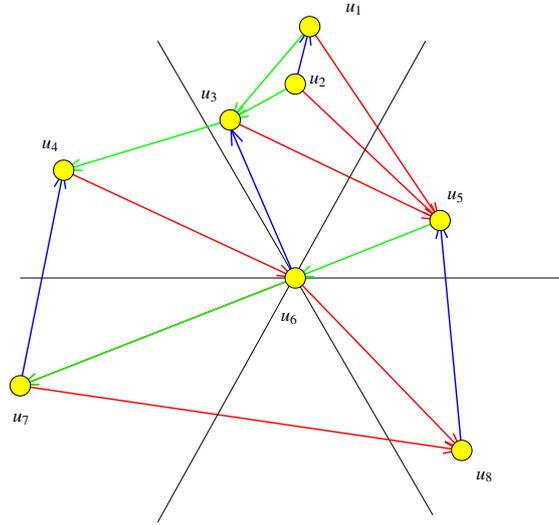


Fig. 1: An example of  $\frac{1}{2}\Theta_6$ .

**Theorem 1** The subgraph  $\frac{1}{2}\Theta_6$  of  $\mathcal{E}$ :

- is a plane graph such that every face (except the outerface) is a triangle,
- is a 2-spanner of  $\mathcal{E}$ , and
- has at most one (outgoing) edge in every positive cone of every node.

**Main ingredients of the proof.** Chew introduced in [Che89] the triangular distance-Delaunay graphs, *TD-Delaunay graphs* for short and denoted **TDDel**, whose convex distance function is defined from an equilateral triangle (instead of a cycle). He proved that TD-Delaunay graphs are plane 2-spanners. The stretch 2 is optimal because of some 3-gons. The properties of the graph  $\frac{1}{2}\Theta_6$  are proved by showing a general equivalence between the **TDDel** graph and  $\frac{1}{2}\Theta_6$ .

Note that the number of incoming edges at a particular node of  $\frac{1}{2}\Theta_6$  is not bounded.

In our construction of the subsequent subgraph  $\mathcal{H}$  of  $\frac{1}{2}\Theta_6$ , for every node  $u$  some neighbors of  $u$  will play an important role. Given  $i$ , let  $\text{children}_i(u)$  be the set of points  $v$  such that  $u = \text{parent}_i(v)$ . Note that  $\text{children}_i(u) \subseteq \overline{C}_i^u$ . In  $\text{children}_i(u)$ , three special points are named:

- $closest_i(u)$  is the closest point of  $children_i(u)$ ;
- $first_i(u)$  is the first point of  $children_i(u)$  in counter-clockwise order starting from  $x$  axis;
- $last_i(u)$  is the last point of  $children_i(u)$  in counter-clockwise order starting from  $x$  axis.

Note that some of these nodes can be undefined if the cone  $\bar{C}_i^u$  is empty.

Let  $(u, v)$  be an edge such that  $v = parent_i(u)$ . A node  $w$  is  $i$ -relevant with respect to (wrt)  $u$  if  $w \in \bar{C}_i^v = \bar{C}_i^{parent_i(u)}$ , and either  $w = first_{i-1}(u) \neq closest_{i-1}(u)$ , or  $w = last_{i+1}(u) \neq closest_{i+1}(u)$ .

**Step 2** Let  $\mathcal{H}$  be the graph obtained by choosing edges of  $\frac{1}{2}\Theta_6$  as follows: for each node  $u$  and each negative cone  $\bar{C}_i^u$ :

- add edge  $(u, closest_i(u))$  if  $closest_i(u)$  exists,
- add edge  $(u, first_i(u))$  if  $first_i(u)$  exists and is  $(i+1)$ -relevant wrt to  $u$  and
- add edge  $(u, last_i(u))$  if  $last_i(u)$  exists and is  $(i-1)$ -relevant wrt to  $u$ .

Note that  $\mathcal{H}$  is a subgraph of  $\frac{1}{2}\Theta_6$  that is easily seen to have maximum degree no greater than 12 (there are at most 3 incident edges per negative cone and 1 incident edge per positive cone). Surprisingly, we prove that:

**Theorem 2** *The graph  $\mathcal{H}$  has maximum degree 9 and is a 3-spanner of  $\frac{1}{2}\Theta_6$ , and thus a 6-spanner of  $\mathcal{E}$ .*

**Main ingredients of the proof.** From the construction of  $\mathcal{H}$ , it is obvious that the degree of every node is at most 12 (at most three in-neighbours per negative sector and one out-neighbour for positive sector). The main tool consists in charging the relevant edges to positive sectors and show that it is impossible that a positive sector is charged three times. For the stretch, we show, for every nodes  $v_1, \dots, v_l$  sharing the same out-neighbor  $u$  in  $\frac{1}{2}\Theta_6$ , the existence of a weighted path  $P$  from  $v_1$  to  $v_l$ . This path can be followed to reach  $u$  through the node  $closest_i(u)$  for given  $i$ . Then we prove that, in  $P \cup closest_i(u)$ ,  $u$  can be reached from any  $v_j$  with a stretch at most 3.

Due to space constraint, we omit to present the construction of the final spanner but we get:

**Theorem 3** *From  $\mathcal{H}$ , it is possible to build a 6-spanner of  $\mathcal{E}$  of maximal degree 6.*

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