

Euler Tour Lock-in Problem in the Rotor-Router Model

I choose pointers and you choose port numbers

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November 27th, 2009

Definitions

Anonymous graphs / networks

- No (used) node labeling
- **Local port numbering** at node v from 1 to $\deg(v)$

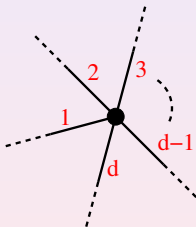
Mobile agent / robot / message / anything

Follows the rotor-router mechanism

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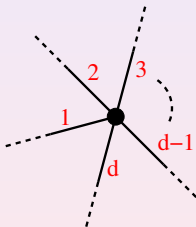
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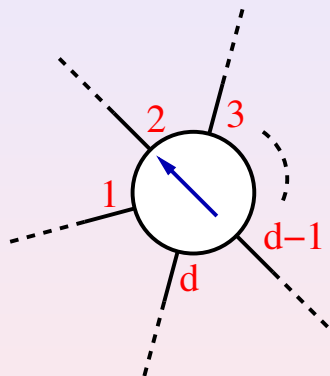
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Rotor-router mechanism

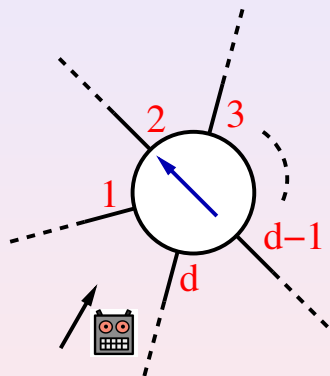


Very simple mechanism:

- Each node has a **pointer** ↗
- The agent follows the pointer and “increments” it with respect to the cyclic ordering \odot (induced by the port numbering)

Other names: Propp machine, Next-Port, Edge Ant Walk

Rotor-router mechanism

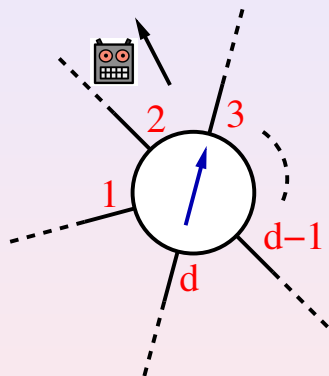


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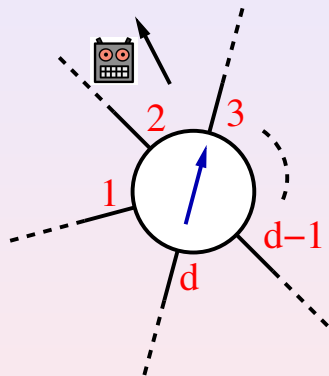


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Context

Known results

- The agent eventually **traverses each edge**
- The traversal stabilizes into an Euler tour: each edge is traversed once in each direction within a period
- Lock-in time: $\Theta(m \cdot D)$ (m : # edges, D : diameter)

Motivations / Applications

- Graph exploration (by a software agent / robot)
- Mutual exclusion
- Stabilisation of distributed processes
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- S. Bhatt, S. Even, D. Greenberg, and R. Tayar, *Journal of Graph Algorithms and Applications*, 2002.
- J.N. Cooper and J. Spencer, *Combinatorics, Probability and Computing*, 2006.
- B. Doerr and T. Friedrich, *Combinatorics, Probability and Computing*, 2009.
- A.S. Fraenkel, *Mathematics Magazine*, 1970.
- L. Gasieniec and T. Radzik, *Proc. WG*, 2008.
- V.B. Priezzhev, D. Dhar, A. Dhar, and S. Krishnamurthy, *Physics Review Letters*, 1996.
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Our results

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How does the **lock-in time** depend on the **initial configuration** of the ports \circlearrowleft and pointers \nearrow ?

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\mathcal{P} =Player, \mathcal{A} =Adversary

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- Consider any choice of ports \odot and pointers \nearrow
- Run virtually the agent for $\Theta(m \cdot D)$ steps
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Useful properties (known results)

Definitions

- **Phase:** Interval between two traversals of the first edge
- G_i : Graph induced by the edges traversed in Phase i
- **Saturated node:** Node whose incident edges are all traversed in both directions during the current phase

(Known) Properties

- Each arc is traversed at most once during a phase
- $G_i \subseteq G_{i+1}$
- If $v \in G_i$, then v is saturated in G_{i+1}
- $\Rightarrow G_{i+1} \supseteq \text{neighborhood}(G_i)$



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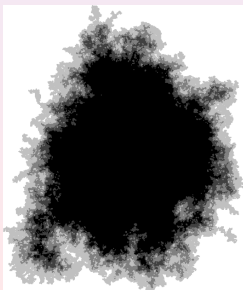
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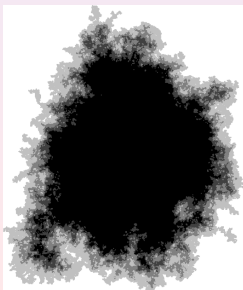
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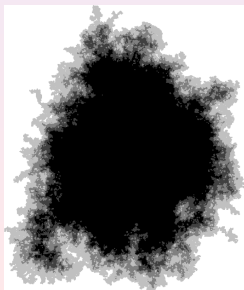
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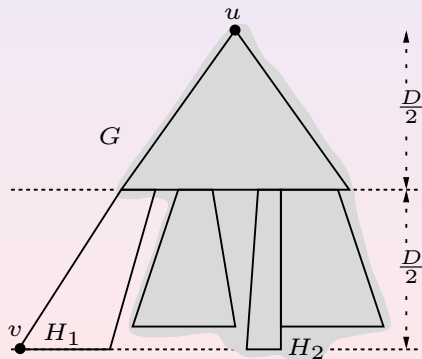
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- Let $[u, v]$ be a diameter
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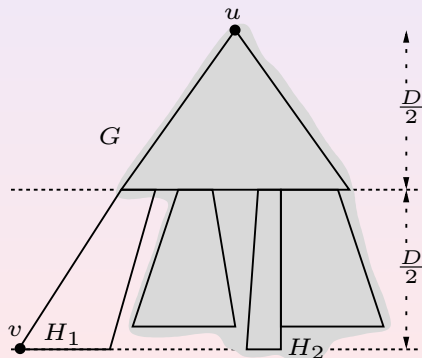


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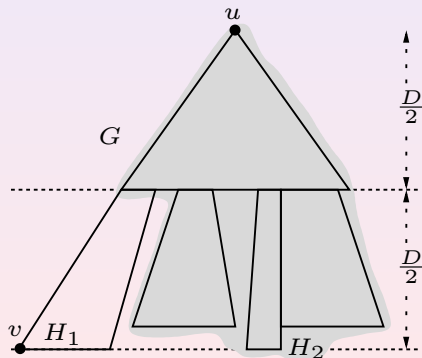
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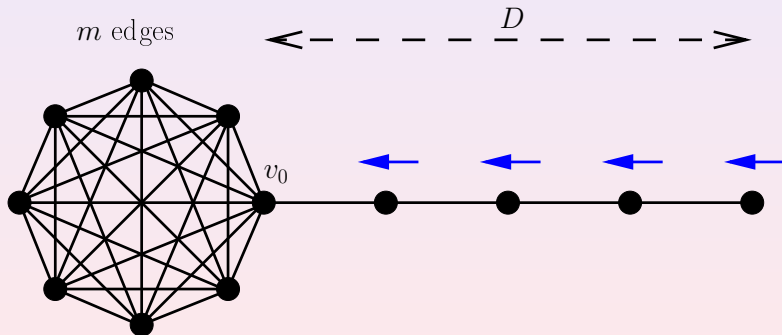


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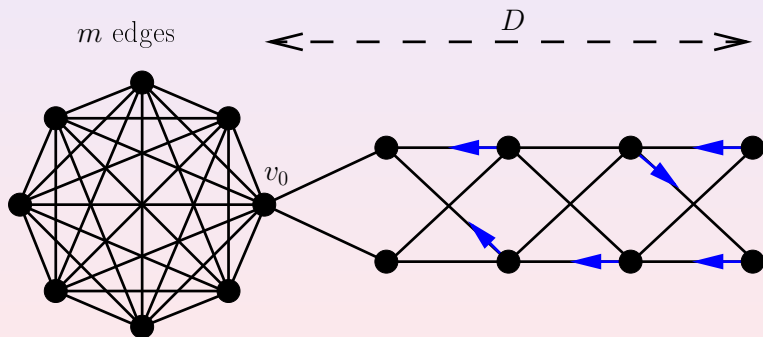
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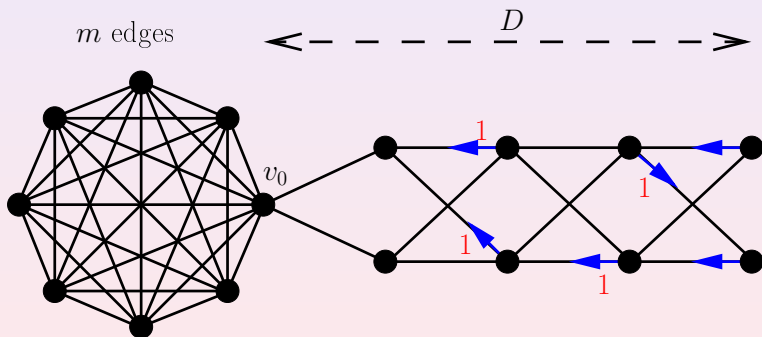
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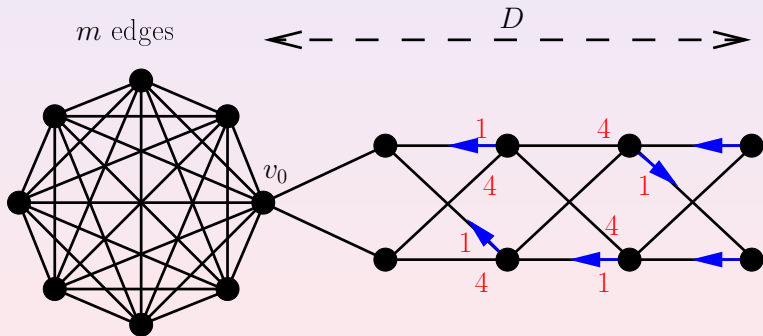
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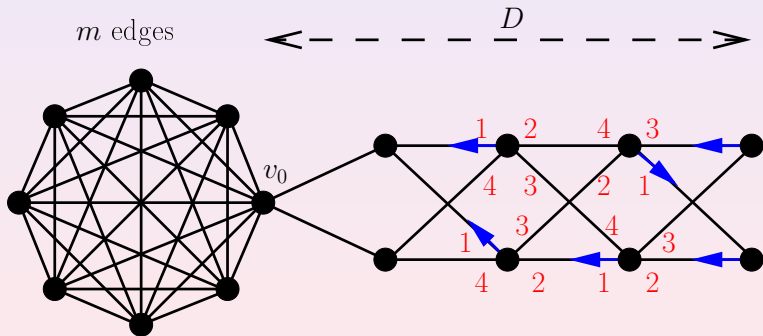
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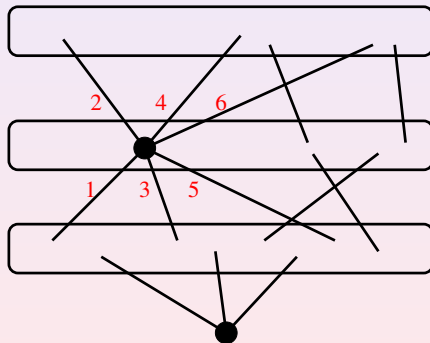
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