

Algebraic Properties of Valued Constraint Satisfaction Problem

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MAX-CUT

$$G = (V, E)$$

- a set of variables: V
- a set of their possible values: $\{0, 1\}$
- minimise: $\sum_{(x,y) \in E} \varrho_{XOR}(x, y)$

$$\varrho_{XOR}(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

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objective function

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weighted relation

2-coloring

$$G = (V, E)$$

- a set of variables: V
- a set of their possible values: $\{0, 1\}$
- minimise: $\sum_{(x,y) \in E} \varrho_{\neq}(x, y)$

$$\varrho_{\neq}(x, y) = \begin{cases} 0 & \text{if } x \neq y \\ \infty & \text{otherwise} \end{cases}$$

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- a set of variables: V
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- minimise: $\sum_{(x,y) \in E} \varrho_{\neq}(x, y)$

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$$\varrho_{\neq}(x, y) = \begin{cases} 0 & \text{if } x \neq y \\ \infty & \text{otherwise} \end{cases}$$

weighted relation

VCSP Instance

An instance of the VCSP:

- a finite set of **variables**: $V = \{x_1, \dots, x_n\}$
- a finite set of their possible **values**: D
- an **objective function**:

$$\varrho_1(x_{1,1}, \dots, x_{1,m_1}) + \dots + \varrho_k(x_{k,1}, \dots, x_{k,m_k}).$$

Goal: find an assignment that minimises the function

Complexity

Γ - a fixed set of weighted relations

$\text{VCSP}(\Gamma)$ - the objective function is a sum of functions from Γ

- $\text{VCSP}(\{\varrho_{XOR}\})$
- $\text{VCSP}(\{\varrho_{\neq}\})$

Complexity

Γ - a fixed set of weighted relations

$\text{VCSP}(\Gamma)$ - the objective function is a sum of functions from Γ

- $\text{VCSP}(\{\varrho_{XOR}\})$ - NP-hard (MAX-CUT)
- $\text{VCSP}(\{\varrho_{\neq}\})$ - PTime (2-coloring)

Main goal: classify the complexity of problems $\text{VCSP}(\Gamma)$

Valued Constraint Satisfaction Problems

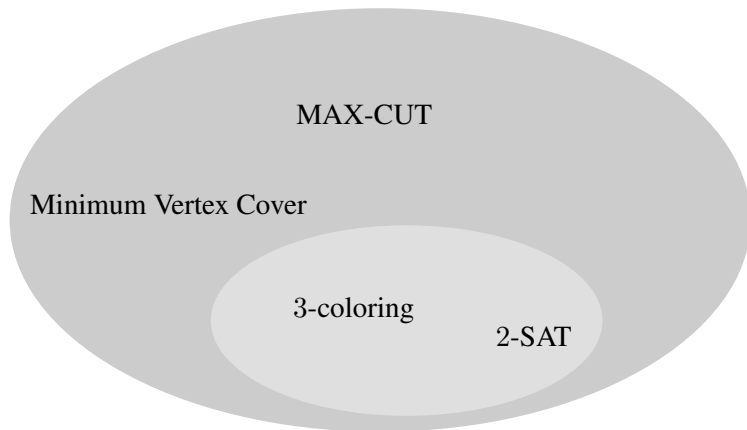
MAX-CUT

Minimum Vertex Cover

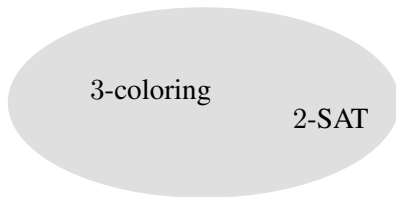
3-coloring

2-SAT

Valued Constraint Satisfaction Problems

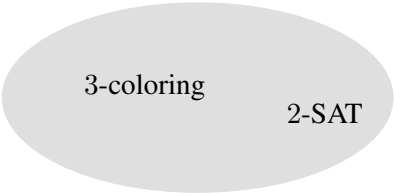


Constraint Satisfaction Problems



Constraint Satisfaction Problems

every weighted relation: $\rho: D^n \rightarrow \{0, \infty\}$



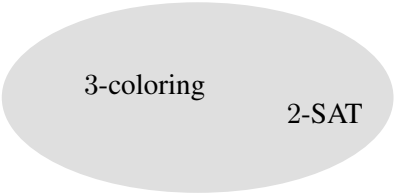
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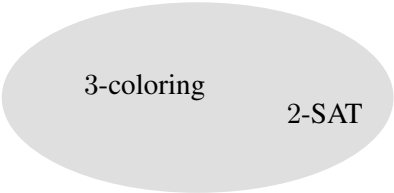
2-SAT

Constraint Satisfaction Problems

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Goal: Is there an assignment with cost 0?



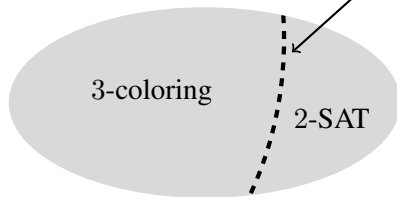
3-coloring

2-SAT

Constraint Satisfaction Problems

[Bulatov, Krokhin, Jeavons]

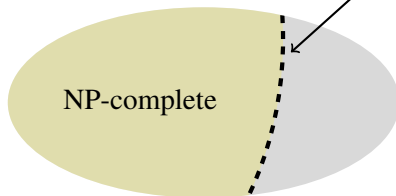
algebraic condition



Constraint Satisfaction Problems

[Bulatov, Krokhin, Jeavons]

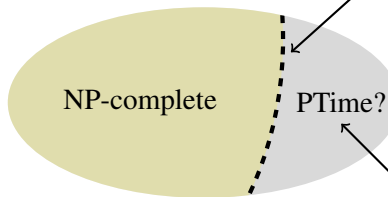
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Constraint Satisfaction Problems

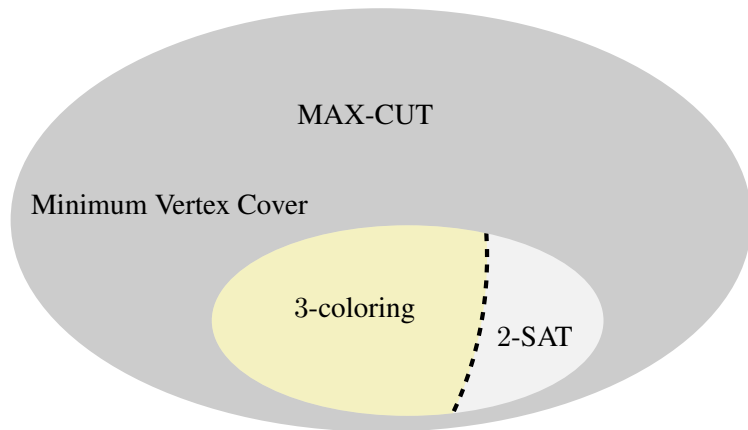
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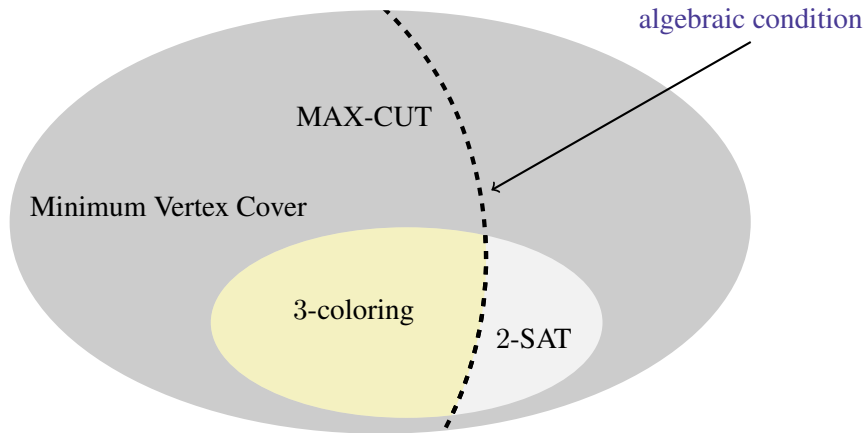


Algebraic
Dichotomy Conjecture

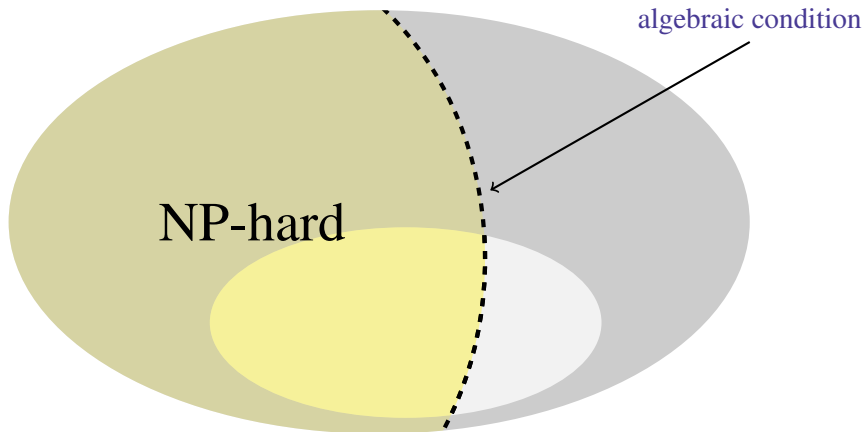
Valued Constraint Satisfaction Problems



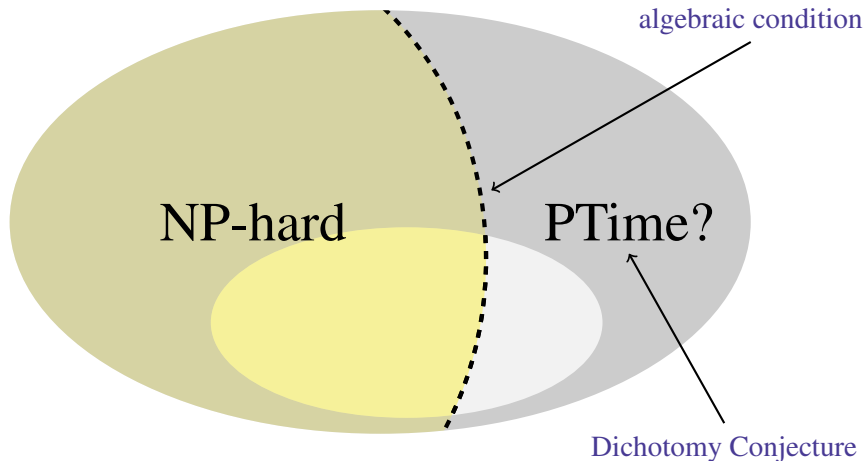
Valued Constraint Satisfaction Problems



Valued Constraint Satisfaction Problems



Valued Constraint Satisfaction Problems



Polymorphisms

$$R \subseteq D^n$$

$$f: D^k \rightarrow D$$

For any n -tuples $\mathbf{x}_1, \dots, \mathbf{x}_k \in R$, we have $f(\mathbf{x}_1, \dots, \mathbf{x}_k) \in R$.

$$\begin{aligned} \mathbf{x}_1 &= (x_1^1, \dots, x_1^n) \in R \\ &\quad \vdots \\ \mathbf{x}_k &= (x_k^1, \dots, x_k^n) \in R \end{aligned}$$

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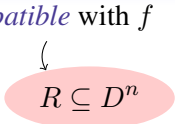
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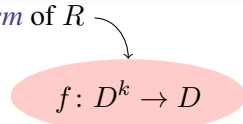
$$(\quad f(x_1^1, \dots, x_k^1), \quad \dots \quad , f(x_1^n, \dots, x_k^n) \quad) \in R$$

Polymorphisms

compatible with f


$$R \subseteq D^n$$

polymorphism of R


$$f: D^k \rightarrow D$$

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Polymorphisms

$$\varrho(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 1) \\ \infty & \text{if } (x, y) = (1, 0) \\ 1 & \text{otherwise} \end{cases} \quad D = \{0, 1\}$$

$\text{Feas}(\varrho) = \{(0, 1), (0, 0), (1, 1)\} \subseteq \{0, 1\}^2$ - binary relation

polymorphism of Γ - polymorphism of every $\varrho \in \Gamma$

$\text{Pol}(\Gamma)$ - algebraic object called a **clone**

Polymorphisms

Γ - CSP language (values 0 or ∞)

$\varrho: D^n \rightarrow \{0, \infty\}$ - weighted relation

Theorem [Bulatov, Krokhin, Jeavons]

$\text{Feas}(\varrho)$ compatible with $\text{Pol}(\Gamma)$

\Downarrow

$\text{CSP}(\Gamma \cup \{\varrho\})$ polynomial-time reducible to $\text{CSP}(\Gamma)$

Polymorphisms characterize the complexity of CSP.

Weighted polymorphisms

ϱ - weighted relation

ω - probability distribution on the set of m -ary operations

$\text{supp}(\omega)$ - operations of positive probability

For any n -tuples $\mathbf{x}_1, \dots, \mathbf{x}_m \in \text{Feas}(\varrho)$, we have

$$\sum_{g \in \text{supp}(\omega)} \omega(g) \varrho(g(\mathbf{x}_1, \dots, \mathbf{x}_m)) \leq \frac{1}{m} (\varrho(\mathbf{x}_1) + \dots + \varrho(\mathbf{x}_m)).$$

Weighted polymorphisms

improved by ω

ϱ - weighted relation

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weighted polymorphism of ϱ

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Weighted polymorphisms

Γ - VCSP language

$\varrho: D^n \rightarrow \overline{\mathbb{Q}}$ - weighted relation

Theorem [Cohen, Cooper, Creed, Jeavons, Živný]

ϱ improved by $\text{wPol}(\Gamma)$

\Downarrow

$\text{VCSP}(\Gamma \cup \{\varrho\})$ polynomial-time reducible to $\text{VCSP}(\Gamma)$

Weighted polymorphisms characterize the complexity of VCSP.

Support clone

$$\text{supp}(\Gamma) = \{g \mid g \in \text{supp}(\omega), \omega \in \text{wPol}(\Gamma)\}$$

↓

support clone of Γ

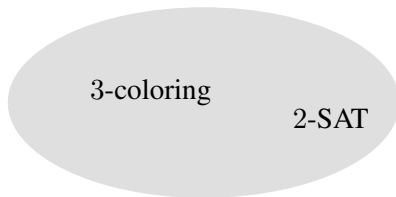
$$\text{supp}(\Gamma) \subseteq \text{Pol}(\Gamma)$$

Cyclic operations

cyclic operation - for every $x_1, \dots, x_k \in D$

$$f(x_1, x_2, \dots, x_k) = f(x_2, \dots, x_k, x_1)$$

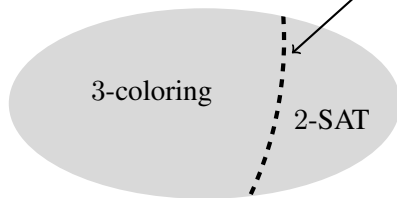
Constraint Satisfaction Problems



Constraint Satisfaction Problems

[Bulatov, Krokhin, Jeavons]

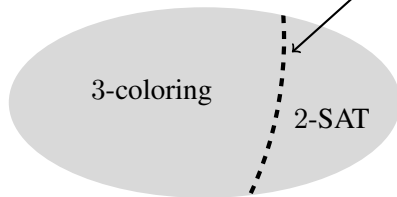
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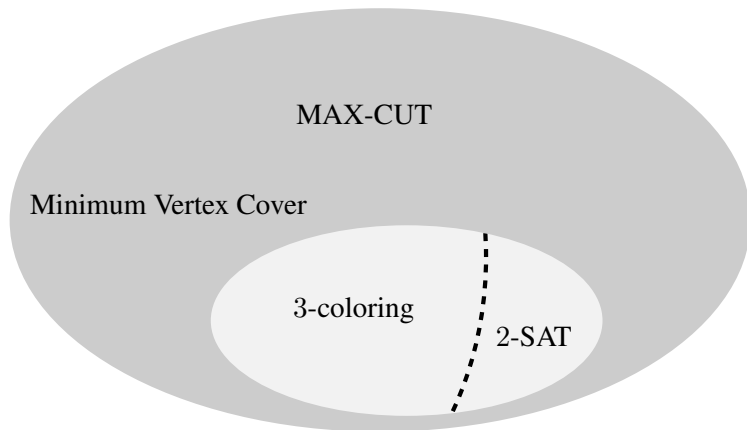
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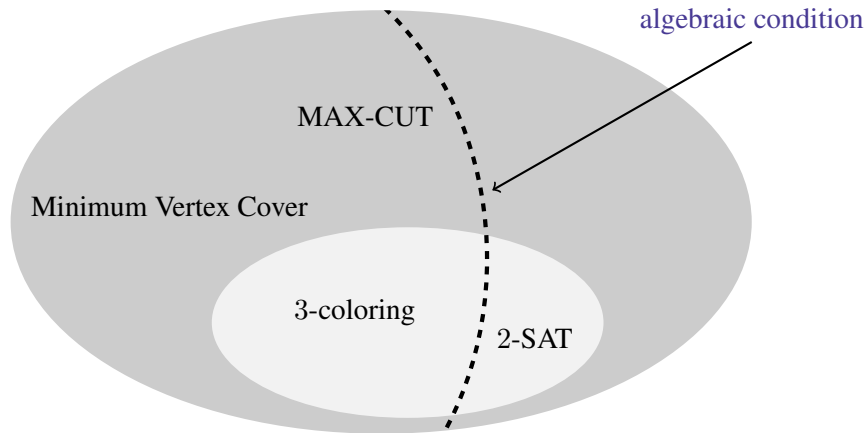
no cyclic operation
in $\text{Pol}(\Gamma)$



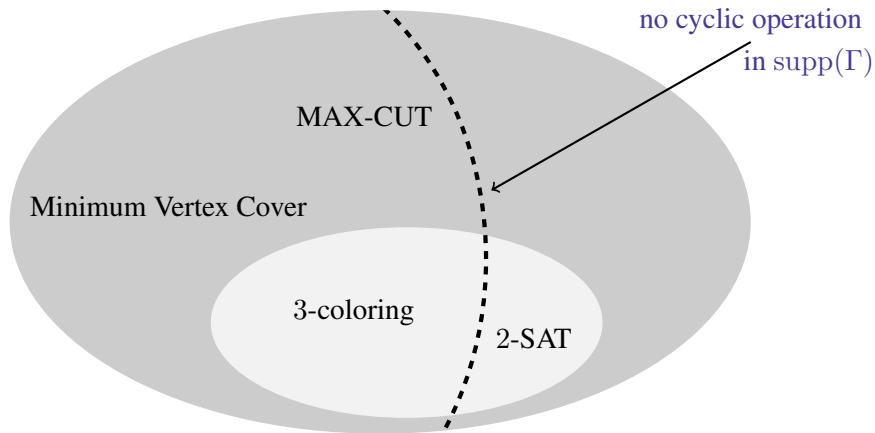
Valued Constraint Satisfaction Problems



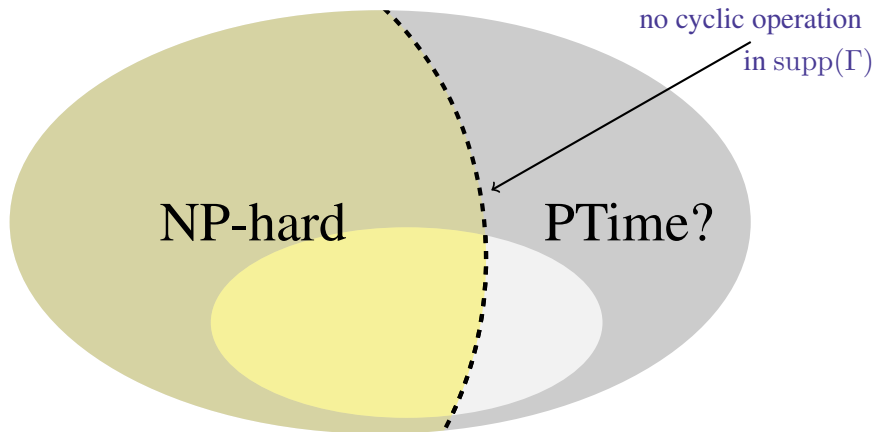
Valued Constraint Satisfaction Problems



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Valued Constraint Satisfaction Problems



Necessary condition for tractability

Theorem

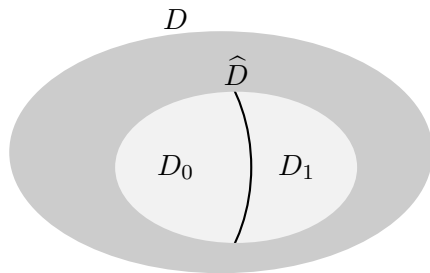
no cyclic operation in $\text{supp}(\Gamma)$



VCSP(Γ) is NP-hard

Necessary condition for tractability

$$S = \{(x_1, x_2, x_3) \in \widehat{D}^3 : \text{exactly one of } x_1, x_2, x_3 \text{ belongs to } D_1\}$$



Lemma 1 [Taylor and Barto, Kozik]

no cyclic operation in $\text{supp}(\Gamma) \Rightarrow S$ compatible with $\text{supp}(\Gamma)$

Necessary condition for tractability

$$R \subseteq D^n \quad \rightarrow \quad \varrho_R(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} \in R \\ \infty & \text{otherwise} \end{cases}$$

Lemma 2 [Kozik, O. and Fulla, Živný]

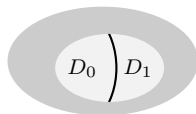
R compatible with $\text{supp}(\Gamma) \Rightarrow \varrho_R$ improved by $\text{wPol}(\Gamma)$

Necessary condition for tractability

no cyclic operation in $\text{supp}(\Gamma)$



S compatible with $\text{supp}(\Gamma)$



ϱ_S improved by $\text{wPol}(\Gamma)$ $\varrho_S(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} \in S \\ \infty & \text{otherwise} \end{cases}$



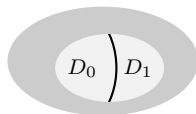
$\text{VCSP}(\{\varrho_S\})$ polynomial-time reducible to $\text{VCSP}(\Gamma)$

Necessary condition for tractability

no cyclic operation in $\text{supp}(\Gamma)$



S compatible with $\text{supp}(\Gamma)$



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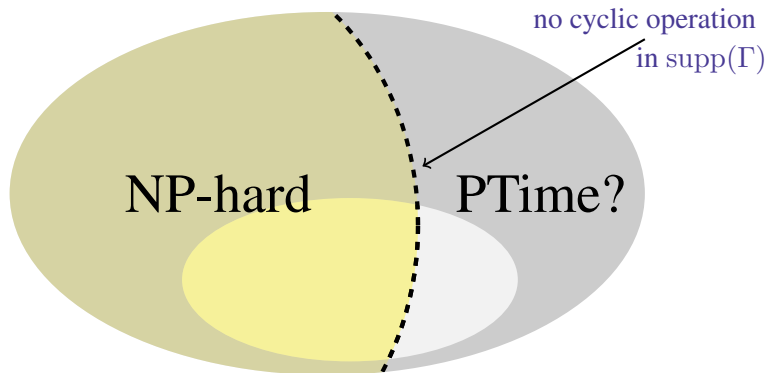


VCSP($\{\varrho_S\}$) polynomial-time reducible to VCSP(Γ)

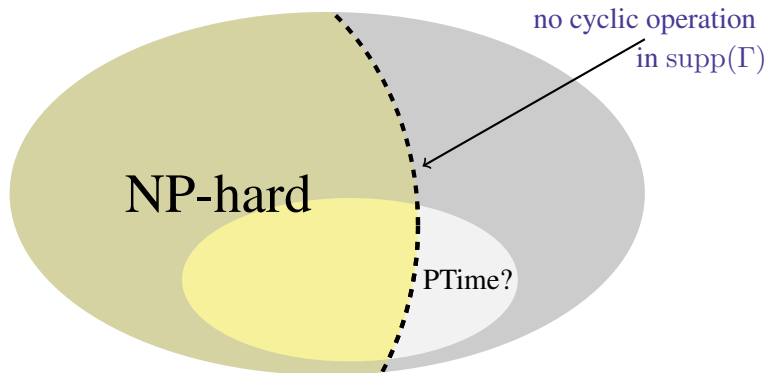
1IN3-SAT



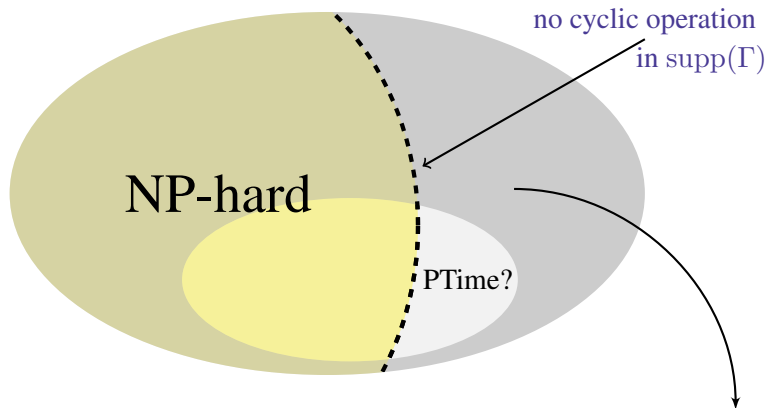
Dichotomy Conjecture



Dichotomy Conjecture



Dichotomy Conjecture



[Kolmogorov, Krokhin, Rolinek]

solving VCSP = solving corresponding CSP + applying BLP