

FO-Definable Constraint Satisfaction Problems

joint work with Bartek Klin, Eryk Kopczyński
Sławek Lasota and Szymon Toruńczyk

Joanna Ochremiak

UPC Barcelona / Simons Institute

{Symmetry, Logic, Computation}, Berkeley
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Atoms

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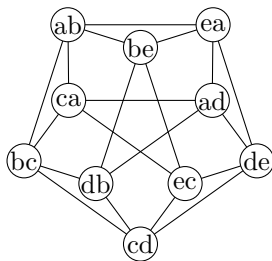
$(\mathbb{N}, =)$ (Mikołaj's talk)

Graph Colorability

G - an **infinite**, undirected graph:

- vertices: pairs of distinct atoms: ab , ad , ...
- edges: $ab - bc$, where a and c are distinct

Subgraph of G :



Question: Is the infinite graph G three-colorable?

Constraint Satisfaction Problem

$\mathbb{T} = (T; R_1, R_2, \dots, R_n)$ - a fixed finite *template*

Problem: CSP(\mathbb{T})

Input: a finite structure \mathbb{I} (over the same signature)

Decide: Is there a homomorphism from \mathbb{I} to \mathbb{T} ?

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Examples:

- $\mathbb{T} = (\{0, 1, 2\}; \neq)$ - three-colorability
- $\mathbb{T} = (\{0, 1\}; R_1, R_0)$ - linear equations mod 2
 $R_1 = \{(x, y) \in \{0, 1\}^2 \mid x + y = 1 \pmod{2}\}$
 $R_0 = \{(x, y, z) \in \{0, 1\}^3 \mid x + y + z = 0 \pmod{2}\}$

Definable Structures

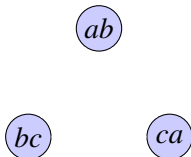
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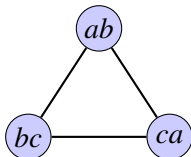
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Domain of the graph G :

$$\{(a, b) \mid a, b \in \mathbb{A}, a \neq b\}.$$

Edge relation:

$$\{((a, b), (b, c)) \mid a, b, c \in \mathbb{A}, a \neq b \wedge a \neq c \wedge b \neq c\}.$$



Definable Structures

Equivalently:

- structures FO-interpretable in $(\mathbb{N}, =)$ (model theory)

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Definable CSP

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Problem: $\text{CSP}_{\text{def}}(\mathbb{T})$

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Systems of Linear Equations mod 2

$$\mathbb{T} = (\{0, 1\}; R_1, R_0)$$

E - an **infinite** system of linear equations over \mathbb{Z}_2

- variables are ordered pairs of distinct atoms: ab, ad, \dots
- equations:

$$ab + ba = 1, \text{ where } a \text{ and } b \text{ are distinct}$$

$$ab + bc + ca = 0, \text{ where } a, b \text{ and } c \text{ are distinct}$$

Question: Does the system E have a solution?

$\text{CSP}_{def}(\mathbb{T})$ for Finite \mathbb{T}

Theorem [Klin, Kopczyński, O., Toruńczyk]. Given a definable structure \mathbb{I} and a finite structure \mathbb{T} , it is decidable whether there exists a homomorphism from \mathbb{I} to \mathbb{T} .

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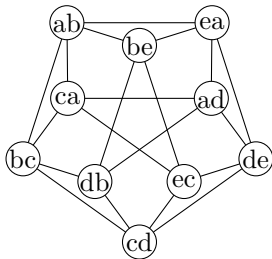
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Lemma. For a definable structure \mathbb{I} and a finite structure \mathbb{T} , tfae:

- there exists a homomorphism from \mathbb{I} to \mathbb{T} ,
- there exists a homomorphism from \mathbb{I} to \mathbb{T} which is constant on orbits under the action of monotone permutations of atoms.

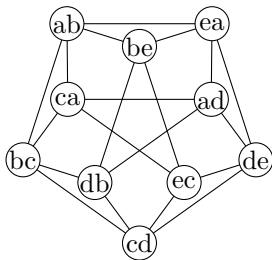
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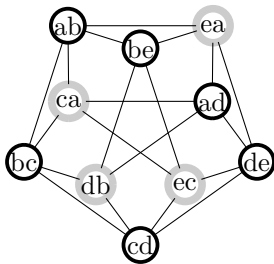


Fix a linear order on atoms (\mathbb{A}, \leq) isomorphic to (\mathbb{Q}, \leq) .

On atoms a, b, c, d, e it can be for example: $a < b < c < d < e$.

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- $(\mathbb{A}, =)$ - one orbit of variables
- (\mathbb{A}, \leq) - two orbits of variables: $(\cdot < \cdot)$ and $(\cdot > \cdot)$

Complexity of $\text{CSP}_{def}(\mathbb{T})$

Theorem [Klin, Kopczyński, O., Toruńczyk]. If $\text{CSP}(\mathbb{T})$ is C -complete then $\text{CSP}_{def}(\mathbb{T})$ is $\text{Exp}(C)$ -complete.

C	Exp(C)
P	EXPTime
NP	NEXPTime
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3-colorability of finite graphs – NP-complete



3-colorability of definable graphs – NEXPTIME-complete

Definable CSP

$\mathbb{T} = (T; R_1, R_2, \dots, R_n)$ - a fixed finite template

Problem: $\text{CSP}_{\text{def}}(\mathbb{T})$

Input: a **definable** structure \mathbb{I} (over the same signature)

Decide: Is there a homomorphism from \mathbb{I} to \mathbb{T} ?

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Theorem [Klin, Lasota, O., Toruńczyk]. Given definable structures \mathbb{I} and \mathbb{T} , it is decidable whether there exists a homomorphism from \mathbb{I} to \mathbb{T} .

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Lemma. For any definable structure \mathbb{T} , the problem $\text{CSP}_{def}(\mathbb{T})$ is polynomial-time reducible to $\text{CSP}_{def}(\mathbb{T}')$, where \mathbb{T}' is a finite structure computable from \mathbb{T} . (generalization of [Bodirsky, Mottet; LICS' 16])

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Corollary. For any definable structure \mathbb{T} , the problem $\text{CSP}_{def}(\mathbb{T})$ is in NEXPTIME.

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Definable CSP over a Definable Signature

$\mathbb{T} = (T; \{R_x\}_{x \in X_1}, \dots, \{R_x\}_{x \in X_n})$ - a fixed **definable** template

Problem: $\text{CSP}_{\text{def}}(\mathbb{T})$

Input: a **definable** structure \mathbb{I} (over the same signature)

Decide: Is there a homomorphism from \mathbb{I} to \mathbb{T} ?

$\text{CSP}_{def}(\mathbb{T})$ over a Definable Signature

Theorem [Klin, Lasota, O., Toruńczyk]. There exists a definable structure \mathbb{T} (over a definable signature) such that the problem $\text{CSP}_{def}(\mathbb{T})$ is undecidable.

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Problem: $\text{CSP}(\mathbb{T})$

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- **Isomorphism problem:** given definable structures \mathbb{A} and \mathbb{B} , decide whether they are isomorphic.

Thank you