

Algebraic Necessary Condition for Tractability of Valued CSP

joint work with Marcin Kozik

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Highlights' 15, Prague
18th September 2015

MAX-CUT

$$G = (V, E)$$

- a set of variables: V
- a set of their possible values: $\{0, 1\}$
- minimise: $\sum_{(x,y) \in E} \varrho_{XOR}(x, y)$

$$\varrho_{XOR}(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

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objective function

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weighted relation

2-coloring

$$G = (V, E)$$

- a set of variables: V
- a set of their possible values: $\{0, 1\}$
- minimise: $\sum_{(x,y) \in E} \varrho_{\neq}(x, y)$

$$\varrho_{\neq}(x, y) = \begin{cases} 0 & \text{if } x \neq y \\ \infty & \text{otherwise} \end{cases}$$

VCSP Instance

An instance of the VCSP:

- a finite set of *variables*: $V = \{x_1, \dots, x_n\}$
- a finite set of their possible *values*: D
- an *objective function*:

$$\varrho_1(x_{1,1}, \dots, x_{1,m_1}) + \dots + \varrho_k(x_{k,1}, \dots, x_{k,m_k}).$$

Goal: find an assignment that minimises the function

Complexity

Γ - a fixed set of weighted relations called a *language*

$\text{VCSP}(\Gamma)$ - the objective function is a sum of functions from Γ

- $\text{VCSP}(\{\varrho_{XOR}\})$
- $\text{VCSP}(\{\varrho_{\neq}\})$

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- $\text{VCSP}(\{\varrho_{\neq}\})$ - PTime (2-coloring)

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- $\text{VCSP}(\{\varrho_{XOR}\})$ - NP-hard (MAX-CUT)
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Main goal: classify the complexity of problems $\text{VCSP}(\Gamma)$

Valued Constraint Satisfaction Problems

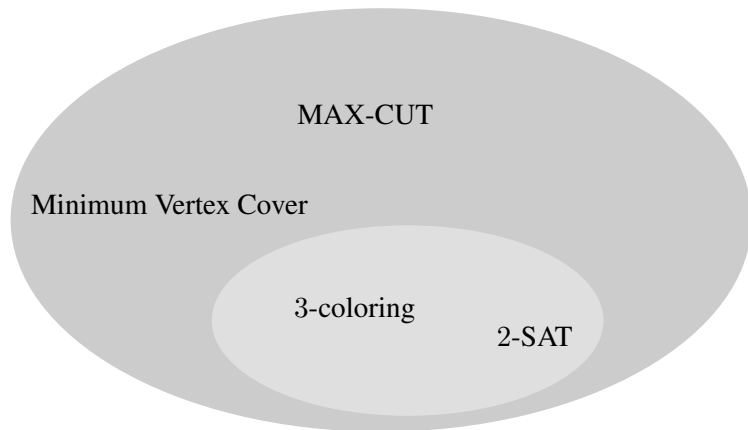
MAX-CUT

Minimum Vertex Cover

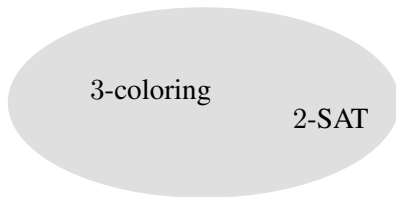
3-coloring

2-SAT

Valued Constraint Satisfaction Problems

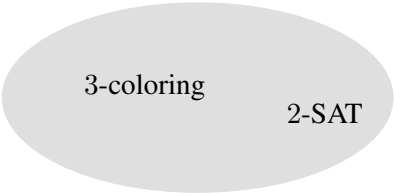


Constraint Satisfaction Problems



Constraint Satisfaction Problems

every weighted relation: $\rho: D^n \rightarrow \{0, \infty\}$



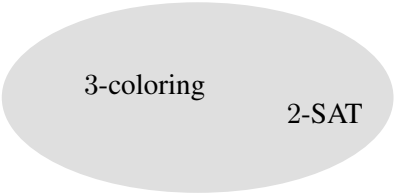
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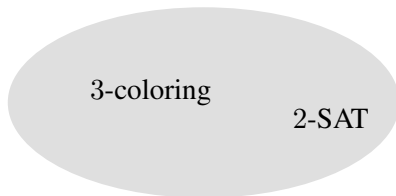
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Goal: Is there an assignment with cost 0?



Constraint Satisfaction Problems

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objective function takes values 0 or ∞

essentially a relation

Goal: Is there an assignment with cost 0?

a satisfying assignment

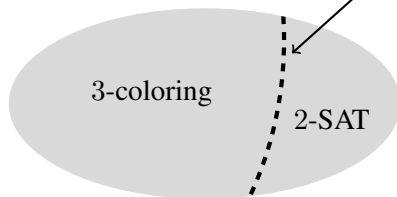
3-coloring

2-SAT

Constraint Satisfaction Problems

[Bulatov, Krokhin, Jeavons]

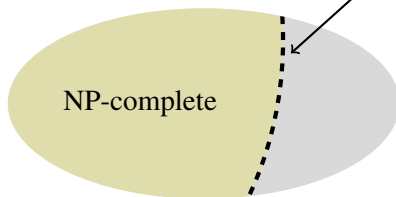
algebraic condition



Constraint Satisfaction Problems

[Bulatov, Krokhin, Jeavons]

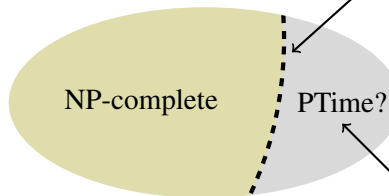
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Constraint Satisfaction Problems

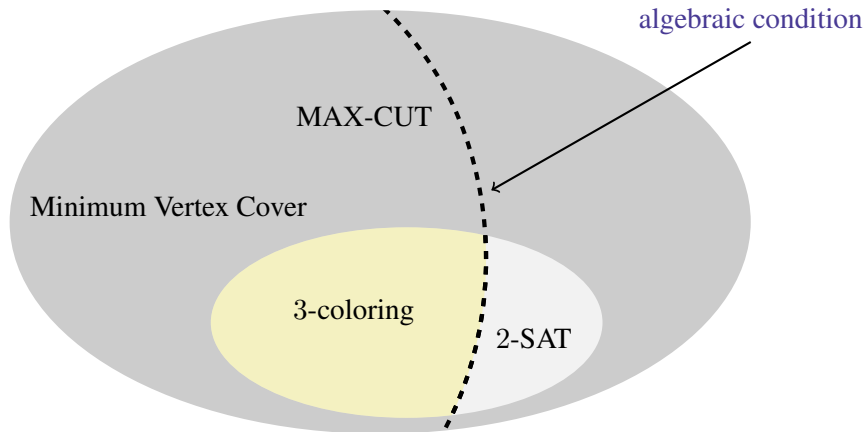
[Bulatov, Krokhin, Jeavons]

algebraic condition

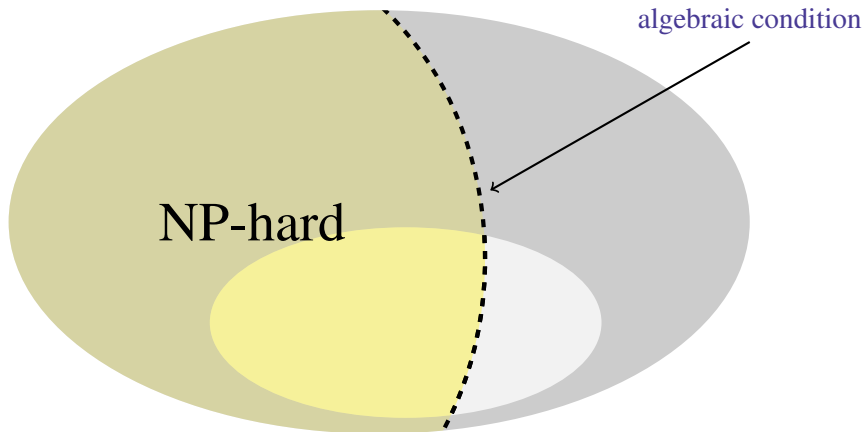


Algebraic
Dichotomy Conjecture

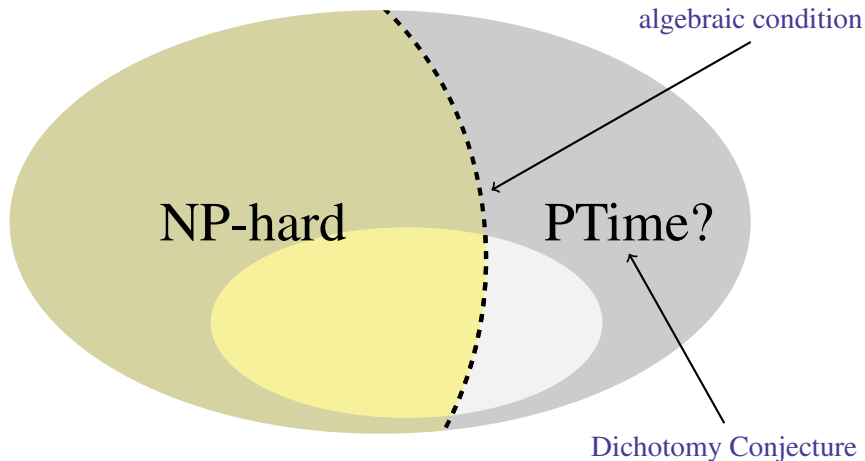
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Polymorphisms

$$\varrho: D^n \rightarrow \mathbb{Q} \cup \{\infty\}$$

$$f: D^k \rightarrow D$$

$$\mathbf{x}_1 = (x_1^1, \quad \dots \quad , x_1^n) \quad \text{such that } \varrho(\mathbf{x}_1) < \infty$$

$$\vdots$$

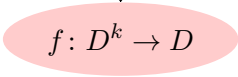
$$\mathbf{x}_k = (x_k^1, \quad \dots \quad , x_k^n) \quad \text{such that } \varrho(\mathbf{x}_k) < \infty$$

$$\varrho(f(x_1^1, \dots, x_k^1), \quad \dots \quad , f(x_1^n, \dots, x_k^n)) < \infty$$

Polymorphisms

polymorphism of ϱ

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Weighted polymorphisms

Polymorphisms characterize the complexity of CSP.

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weighted polymorphism - probability distribution on the set of polymorphisms

Theorem [Cohen, Cooper, Creed, Jeavons, Živný]

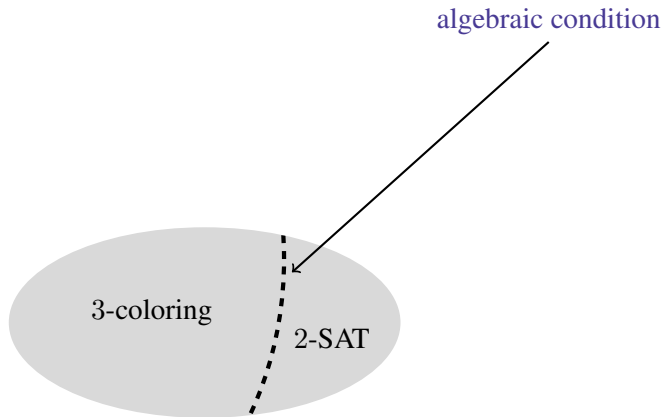
Weighted polymorphisms characterize the complexity of VCSP.

Cyclic polymorphism

cyclic polymorphism - for every $x_1, \dots, x_k \in D$

$$f(x_1, x_2, \dots, x_k) = f(x_2, \dots, x_k, x_1)$$

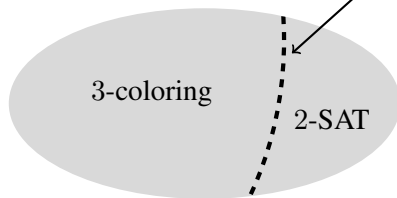
Constraint Satisfaction Problems



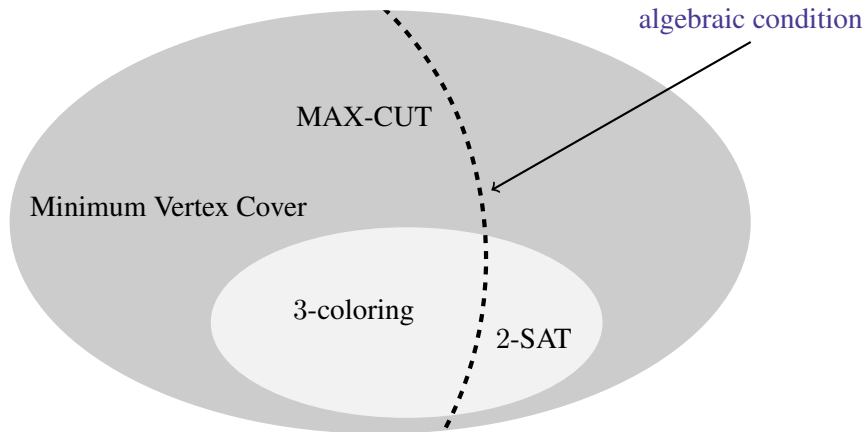
Constraint Satisfaction Problems

[Barto, Kozik]

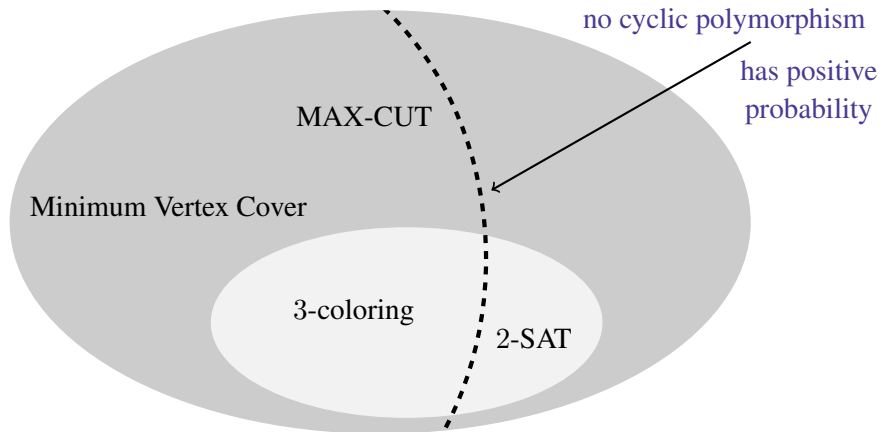
no cyclic polymorphism



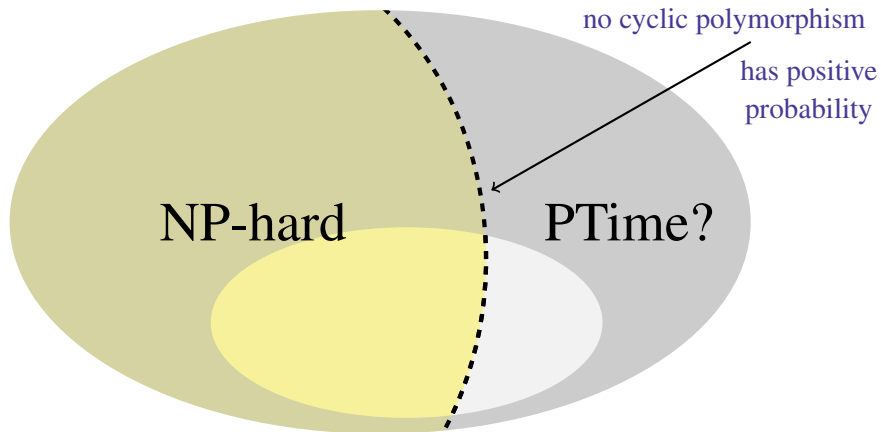
Valued Constraint Satisfaction Problems



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Thank you