Treewidth and duality for planar hypergraphs. ⋆

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Abstract. We prove a conjecture of Robertson and Seymour (Graph Minors. III. Planar tree-width): the treewidth of a planar graph is at most the treewidth of its dual plus one. To this purpose, we study hypermaps, which represent hypergraphs on a surface. We define a parallel composition on hypermaps which verifies five properties: it is commutative, associative, preserves planarity, commutes with the duality and with the parallel composition on hypergraphs. Finally, we introduce the notion of a equationel set of hypermaps and of a treewidth of a hypermap.

In their study of minors, Robertson and Seymour [8] introduce the notions of treewidth and tree-decomposition of a graph. These notions are essential in a algorithmical point of view: a lot of NP-problems become polynomial, when they are restricted to graphs of bounded tree-width (see [1]).

In his study of equational sets of graphs, Courcelle [4] (see also [2, 6]) introduces a parallel-composition of (hyper)graphs. This operation is commutative and associative, it follows that every graph can be described by an algebraic term. These two descriptions, the tree-decomposition and the algebraic term, are very close: an algebraic term can be viewed as a tree-decomposition defined on an ordered tree.

We prove a conjecture of Robertson and Seymour [8] that states:

the tree-width of any planar graph is at most the tree-width of its dual plus one.

To address such a conjecture, we shall manipulate not graphs but embeddings of them on the plane, or more generally on *orientable* surfaces. The encoding of such drawings is made by defining for each vertex v a circular order on the edges incident with v (see [3]); more precisely, we consider a *combinatorial map* that is a pair (σ, α) of permutations on the set of incidences: the first one defines the above mentioned circular order, the second one defining the edges. One of the interests of such encodings is the simplicity of the expression of the dual map, that is $(B, \sigma, \alpha^{-1} \circ \sigma)$.

In order to develop the algebraic approach of Courcelle [4], we define a parallel-composition (called in this paper splitting) of hypermaps that verifies five properties:

1. it is commutative.
2. it is weak-associative.
3. it commutes with the hypergraphs' parallel-composition.
4. it commutes with the duality.
5. it conserves planarity.

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And moreover, these properties yields a notion of equational set of hypermaps and a notion of *tree-width* of a hypermap. This notion of complexity verifies two natural properties. A hypermap and its dual have same complexity; a hypermap is more complex than every of its induced hypergraph.

Our main technical result is that every (loopfree) planar representation M of some graph of treewidth $\leq k$ may be decomposed into "basic" and planar hypermaps having at most $k+1$ vertices. As a consequence of the properties of the splitting mentioned above, this decomposition provides a decomposition of the dual hypermap M^* into planar hypermaps having at most 3 hyperedges and at most $(k+1)+1$ vertices. It follows the resolution of the Robertson and Seymour's conjecture mentioned above.

The paper is organized as follows. In section 1, we recall the notions of treewidth and splitting for hypergraphs. In section 2, we introduce the splitting of hypermaps. In section 3, we prove the conjecture of Robertson and Seymour mentioned above. In section 4, we introduce equational sets of hypermaps. In section 5, we introduce a notion of treewidth of hypermap.

Notation

We use := for "equal by definition" i.e, for introducing new notations. We denote by $\mathbb{N}$ the set of non-negative integers. We denote by $[i,j]$ the set of integers $\{i, i+1, \ldots, j\}$ and by $[n]$ the interval $[1,n]$. Let A be a set. The disjoint union of two sets A and B is denoted by $A \uplus B$. The cardinality of A is denoted by $\mathbf{card}(A)$, its powerset by $\mathcal{P}(A)$. The set of nonempty sequences of elements of A is denoted by A^+, and sequences are denoted by $(a_1, \ldots, a_n)$ with commas and parentheses.

The *domain* of some partial function $f : A \to B$ is the set $\{a \in A \mid f(a)$ is defined$\}$ and is denoted by $\mathbf{Dom}(f)$. The restriction of a partial function f to a subset A' of $\mathbf{Dom}(f)$ is denoted by $f \restriction A'$. If two partial functions $f : A \to B$ and $g : C \to D$ coincide on $\mathbf{Dom}(f) \cap \mathbf{Dom}(g)$, we denote by $f \cup g$ their common extension into a partial function $A \cup B \to C \cup D$. For all partial functions $f : A \to B$ and $g : B \to C$, we denote by $g \circ f$ the partial function $A \to C$ defined as usually. By a *mapping*, we mean a total function.

A *permutation* is a bijective mapping $\alpha : B \to B$. Let α be a permutation defined on a finite set B. For each $b \in B$, we denote by $[b]_\alpha$ the set $\{\alpha^i(b) \mid i \in \mathbb{N}\}$. The cardinality of $\{[b]_\alpha \mid b \in B\}$ is denoted $z(\alpha)$. For every subset $A \subseteq B$, we denote by $\alpha : A$ the permutation on A that associates with every $a \in A$ the element $\alpha^i(a)$ with $i = \min\{1 \leq j \mid \alpha^j(a) \in A\}$. Let $b_{1,0}, \ldots, b_{1,j_1}, \ldots, b_{l,0}, \ldots, b_{l,j_l}$ be some pairwise distinct elements and let B be their set. We denote by $[[b_{1,0}, \ldots, b_{1,j_1}], \ldots, [b_{l,0}, \ldots, b_{l,j_l}]]$ the permutation α on B that associates with every element of the form $b_{i,j}$ for some $i \in [l]$ and $j \in [0, j_l]$ the element $b_{i,(j+1) modulo(j_i)}$.

Let $\perp : \mathcal{B}^2 \to \mathcal{B}$ be some partially defined operation and $\mathcal{A}$ be some subset of $\mathcal{B}$. The $\perp$-*closure* of $\mathcal{A}$ is the minimal set w.r.t $\subseteq$ that contains $\mathcal{A}$ and that is closed under $\perp$.

1 Maps, hypermaps and hypergraphs

A graph is planar if it can be drawn in the plane or equivalently in the sphere. Such a drawing or embedding is made by selecting for each vertex a point in the sphere and for each edge (u, v) by drawing a line connecting the two points representing the vertices u and v in such a way that no lines intersect.

Now, we need a combinatorial way to describe such embeddings. Let be an orientation of this sphere. By looking at the neighbourhood of the point representing some vertex v, the orientation of the sphere determines a cyclic permutation of the edges incident with v, or more precisely in the case of loop, of the edge ends at v (see example below). In this paper, these edge ends, called sometimes in French “brins”, are called “darts”. Thus, every embedding in the oriented sphere determines a combinatorial hypermap, that is the set of darts D, the permutation σ on D union of all cyclic permutations mentioned above and a third permutation α on D relating the darts which are extremities of a same edge. In the case of graph, α has no invariant element (every edge has at least two extremities) and is involutive ($\alpha^2 = Id$: every edge edge has at most two extremities).

Example 1. In Figure 1, is represented at left side the graph having 5 vertices a, b, c, d, e and 5 edges having respectively as extremities $\{a, b\}$, $\{b, e\}$, $\{e, a\}$, $\{e, c\}$ and $\{e, d\}$. At center are depicted two embeddings of this graph in the sphere oriented anticlockwise (that is the meaning of the upper arrow). At right side, are represented the two combinatorial descriptions of these two embeddings. The upper (resp. lower) combinatorial map is $([10], \sigma_1, \alpha)$ (resp. $([10], \sigma_2, \alpha)$) with $\alpha = [[1, 2], \ldots, [9, 10]]$ and $\sigma_1 = [[1, 3, 5, 7], [2], [4], [6, 10], [8, 9]]$ (resp $\sigma_2 = [[1, 5, 3, 7], [2], [4], [6, 10], [8, 9]]$). □

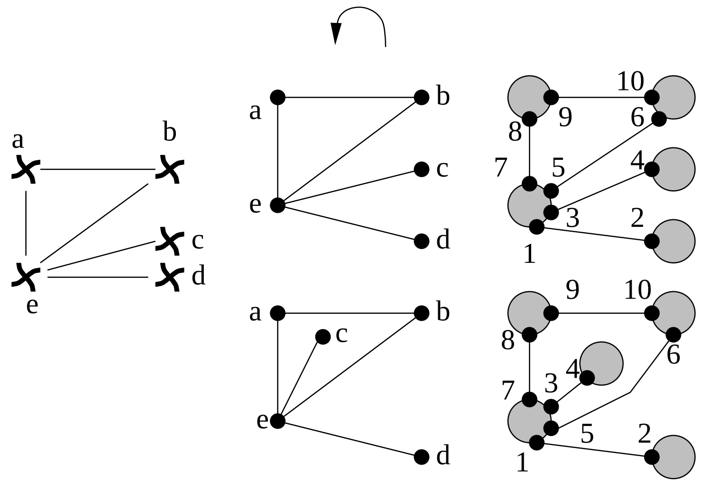

Fig. 1. Graphs, embeddings in the sphere and planar maps.

In fact, this combinatorial description does not concern only embeddings in the sphere but in any orientable surface. An orientable surface is obtained from the sphere by adding a finite number of handles. This number is called the “genus” of the surface. For instance, sphere has genus 0 and torus has genus 1. A classical result (see [3]): every combinatorial map describes an embedding on some orientable surface.

Now, a few words about dual graphs. In the presentation of embedding, we omit a point (for a formal definition, see [3]). The set of points which are not representative of vertices or that does not constitute the lines of the edges should be a disjoint union of set homeomorphic to a disk. These sets are called faces. For this reason, the embedded graph is sometimes supposed to be connected. Nevertheless, in this paper, graphs are connected or not. As a direct consequence, every embedding admits a dual embedding constructed as follow: for every face choose a point, for every line draw a line connecting the two points associated with the two faces bordering the line (or make a loop if the faces are identical). A remarkable property of this dual embedding is its combinatorial description which is simply $(D, \alpha^{-1}\sigma, \alpha^{-1})$ if (D, σ, α) is the combinatorial description of the former embedding.

Example 2. At the top of Figure 2, are represented an embedding $\mathcal{M}$ of the circuit with 4 vertices in the sphere oriented anticlockwise and its combinatorial description. That is the combinatorial map $M := ([-4,-1] \cup [1,4], \sigma, \alpha)$ with $\sigma := [[1,-4],[2,-1],[3,-2],[4,-3]]$ and $\alpha := [[1,-1],[2,-2],[3,-3],[4,-4]]$.

At the bottom, is drawn the dual of $\mathcal{M}$. In order to conserve the equality $(M^*)^* = M$ for map, we change the orientation of the surface on which the dual embedding is drawn (for more details see [3]). Thus, the lower surface is oriented clockwise. The vertices of the dual of $\mathcal{M}$ are the two faces of $\mathcal{M}$: the inner one and the outer one. Its combinatorial description is the the combinatorial map $([-4,-1] \cup [1,4], \sigma^*, \alpha^*)$ with $\sigma^* := [[1,2,3,4][-4,-3,-2,-1]]$ and $\alpha^* := [[1,-1],[2,-2],[3,-3],[4,-4]]$. Observe the equalities $\alpha^* = \alpha^{-1}$ and $\sigma^* = \alpha^{-1}\sigma$. $\square$

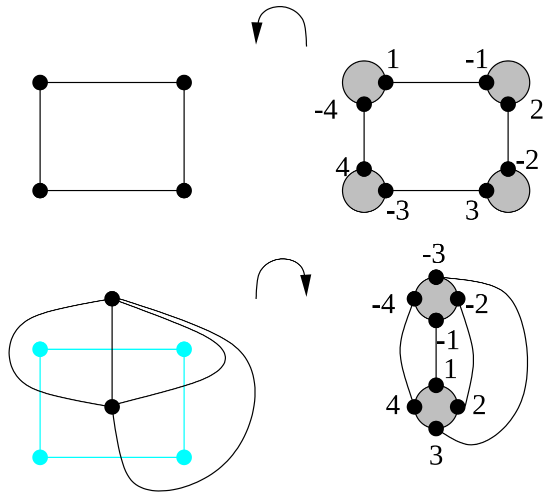

Fig. 2. Dual embeddings and dual maps.

Our aim is to study graphs of bounded tree-width. The treewidth, a measure of complexity for hypergraphs, admits different characterizations. One of them (given by Lemma 17) uses hypergraphs and some kind of concatenation of them: the splitting. By hypergraph, we mean a set of vertices plus a set of hyperedges plus a mapping that associates to every hyperedge its extremities. In consequence, we have to consider embeddings of hypergraphs and combinatorial descriptions of them (we shall call hypermaps). A standard way (see [3]) is to use combinatorial hypermaps, which are tuples of the form (D, σ, α) where σ and α are two permutations on D (here, α may have invariant element and is not necessarily involutive). In order to "contain" the induced hypergraph structure, this description is not sufficient. We have to add the set of hyperedges. Hence, a hypermap is a combinatorial hypermap

(D,σ,α) augmented with a set of hyperedges E and a mapping **dart** that associates with every hyperedge its extremities forming a unique cyclic permutation α.

In this section, we recall hypergraphs, hypermaps, treewidth of hypergraphs. We conclude by defining the splitting $\square$ of hypergraphs that provides a characterization of the treewidth: the hypergraphs of treewidth $\leq k$ are the closure under $\square$ of the hypergraphs having at most $k+1$ vertices.

Definition 3 (Graphs and hypergraphs). A *hypergraph* G is a tuple (V,E,f), denoted by $(\mathbf{V}_G,\mathbf{E}_G,\mathbf{vert}_G)$, where V is the set of *vertices*, E is the set of *hyperedges* and f is a mapping that associates with every hyperedge e an eventually empty subset of V, the set of *extremities* of e. A hyperedge and each of its extremities are said to be *incident*. A vertex is *isolated* if it is incident to no hyperedge. An *edge* is a hyperedge having at most two extremities. A *graph* is a hypergraph having only edges. $\square$

Definition 4 (Maps and hypermaps). A *combinatorial hypermap* is a tuple (D,σ,α) where σ and α are two permutations on D, called the set of *darts*. A *hypermap* M is a tuple (D,E,f,σ,α), denoted by $(\mathbf{D}_M,\mathbf{E}_M,\mathbf{dart}_M,\sigma_M,\alpha_M)$, where:

1. (D,σ,α) is a combinatorial hypermap.
2. f is a mapping $E\rightarrow\mathcal{P}(D)$ where $\{f(e)\mid e\in E: f(e)\neq\emptyset\}$ is a partition of D. A dart of some set of the form $f(e)$ with $e\in E$ is said an *extremity* of e.
3. every element $e\in E$, called an *hyperedge* of M, is *representative*, that means $f(e)=[d]_\alpha$ for every extremity d of e.

A *map* is a hypermap M where α_M is involutive with no invariant element. We denote by β_M the permutation $\alpha^{-1}\circ\sigma$. A *vertex* (resp. *face*) of M is a set of the form $[d]_{\sigma_M}$ (resp. $[d]_{\beta_M}$) for some dart $d\in D$; it is *incident* with some hyperedge e of M if it is not disjoint with $f(e)$. The set of vertices (resp. faces) of M is denoted by $\mathbf{V}_M$ (resp. $\mathbf{V}^*_M$). $\square$

Now, let us present our drawings conventions.

Example 5. In Figure 3, are represented two times the hypermap $M := (D,E,f,\sigma,\alpha)$ with $D:=[6]$, $E:=\{a,b,c\}$, $f:=\{(a,\{1\}),(b,\{3,4\}),(c,\{2,5,6\})\}$ $\sigma:=[[1,2,3],[4],[5],[6]]$ and $\alpha:=[[1],[3,4],[2,5,6]]$ and its induced hypergraph. The general and complete conventions used to drawing the left figure are the following. Every dart is represented with a black and small disk. Every hyperedge is represented with a white box. The permutation σ is represented by drawing grey disks. The sense of rotation is specified with an arrow. The permutation α is represented by using line. The sense of rotation is specified with an arrow too. The mapping f is represented with dotted lines linking white boxes and small disks.

In order to simplify, we adopt the following rules (used to draw the right figure). If no sense of rotation is specified for σ (resp. α), we take the anticlockwise's one (resp. clockwise's one). Every edge with a unique dart is represented by a single line connected to the dart. For instance, the white box and the lines of edge a are omitted. Every edge with exactly two dart-extremities (and mutually associated by α) is represented by a single line connecting these two darts. For instance, the white box and the lines of edge b are omitted.

For hypergraphs, we represent vertices with crosses and hyperedges by using same rules as for hypermaps, as it is illustrated by Figure 3. $\square$

Definition 6 (Induced hypergraphs). The *induced* hypergraph of some hypermap M is the hypergraph $(\mathbf{V}_M,\mathbf{E}_M,f)$ where f associates with every hyperedge e the set of all vertices of $\mathbf{V}_M$ incident in M with e. This hypergraph is denoted by $\underline{hgraph}(M)$. The hypermap M is said a *representation* of G. $\square$

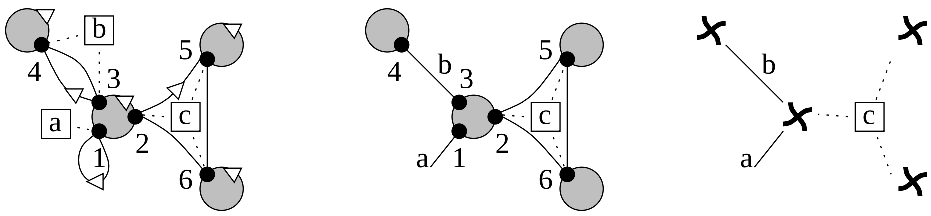

Fig. 3. Drawing conventions.

Definition 7 (Dual). The *dual* of some hypermap M is the hypermap $(\mathbf{D}_M, \mathbf{E}_M, \mathbf{dart}_M, \beta_M, \alpha_M^{-1})$ denoted by M^*. □

Basic property of the dual operation: it is involutive.

Lemma 8. *Every hypermap M verifies $(M^*)^* = M$.*

Proof. A direct consequence of the equality $\alpha_{M^*}^{-1} \circ \beta_{M^*} = \sigma_M$. □

Definition 9. Let $G = (V, E, f)$ and $H = (W, F, g)$ be two hypergraphs. G is a *subhypergraph* of H if $V \subseteq W$, $E \subseteq F$ and $f(e) = g(e)$ for every $e \in E$; that is denoted by $G \subseteq H$. If every $e \in E \cap F$ verifies $f(e) = g(e)$, we denote by $G \cup H$ the hypergraph $(V \cup W, E \cup F, f \cup g)$ and by $G \cap H$ the hypergraph $(V \cap W, E \cap F, f \upharpoonright (E \cap F))$.

A *path* of G is a nonempty sequence $(o_1, \ldots, o_m) \in (V \cup E)^+$ such that for every $i \in [m-1]$ o_i is incident with o_{i-1}. The path p is *elementary* if for all $1 \leq i < j \leq m$ with $\{o_i, o_j\} \subseteq E$ we have $o_i \neq o_j$. G is *connected* if it is nonempty and if every pair of elements of $V \cup E$ is the pair of extremities of some path of G. A *connected component* of G is a maximal subhypergraph of G (w.r.t $\subseteq$) which is connected.

A *forest* is a graph such that for every vertices s and t, there is at most one elementary path from s to t. A *tree* is a connected forest. Let T be a tree. A *leaf* of T is a node incident with at most one edge of T. A *node* of T is a vertex. □

This definition of a tree-decomposition is from [9].

Definition 10 (Treewidth). A *tree-decomposition* is a pair (T, g) where T is a tree and where g associates with every node t of T a hypergraph $g(t)$ such that:

- $\mathbf{E}_{g(s)} \cap \mathbf{E}_{g(t)} = \emptyset$, for all distinct nodes s, t of T.
- for all nodes s, u of T, every node t of the path of T from s to u verifies: $g(s) \cap g(u) \subseteq g(t)$.

The *width* of a tree-decomposition (T, g) is the maximum of $\mathbf{card}(\mathbf{V}_{g(t)}) - 1$ taken over all nodes t of T. The *treewidth* of a hypergraph G, denoted by $\underline{twd}(G)$, is the minimum width of all tree-decompositions of G, i.e tree-decomposition (T, g) such that $G = \bigcup_{s \in \mathbf{V}_T} g(s)$. □

The operation below defined concerns concrete hypergraphs, hence, it is formally not the "parallel-composition" defined in [4] that concerns "abstract" hypergraphs i.e, isomorphic classes of graphs. Nevertheless, the mechanism of "concatenation" of two hypergraphs is similar. This definition is illustrated by Example 12.

Definition 11 (splitting). Let H and K be two hypergraphs such that $\mathbf{E}_H \cap \mathbf{E}_K = \{e\}$ and such that $\mathbf{vert}_H(e) = \mathbf{V}_H \cap \mathbf{V}_K = \mathbf{vert}_K(e)$. We denote by $H \square K$ the hypergraph $(\mathbf{V}_H \cup \mathbf{V}_K, (\mathbf{E}_H \cup \mathbf{E}_K) \backslash e, f)$ where f is the restriction of $\mathbf{vert}_H \cup \mathbf{vert}_K$ to $(\mathbf{E}_H \cup \mathbf{E}_K) \backslash e$. □

Example 12. In Figure 4 are drawn three hypergraphs G, H, K that verify $G = H \square K$. The unique hyperedge a of $\mathbf{E}_H \cap \mathbf{E}_K$ is drawn with a white box and its extremities (vertices 1, 2, 3 and 4) with four crosses. □

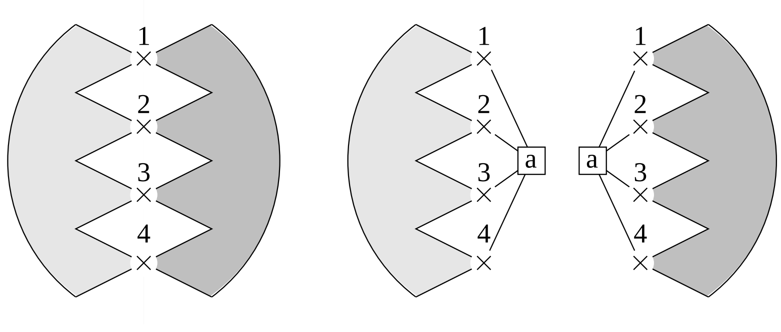

Fig. 4. splitting of hypergraphs.

The splitting verifies the three following basic and immediate properties:

Lemma 13 (Commutativity). *Every hypergraph of the form $H \square K$ for some hypergraphs H and K is equal to $K \square H$.*

Lemma 14 (Weak associativity). *Every hypergraph of the form $(H \square K) \square L$ for some hypergraphs H, K and L such that $\mathbf{E}_H \cap \mathbf{E}_K \cap \mathbf{E}_L = \emptyset$ is either equal to:*

- *$H \square (K \square L)$.*
- *or to $(H \square L) \square K$.*

Lemma 15. *The $\square$-closure of some set of hypergraphs $\mathcal{A}$ which is closed under isomorphism is the set of all hypergraphs of the form $((G_1 \square G_2) \ldots) \square G_n$ with $n \geq 1$ and $G_1, \ldots, G_n \in \mathcal{A}$.*

Thanks the splitting, we can decompose every hypergraph into more simple ones, which contain only 3 hyperedges.

Lemma 16. *The set of all hypergraphs is the $\square$-closure of the set of hypergraphs having at most 3 hyperedges.*

Proof. By definition, for every hypergraph of the form $H \square K$ for some hypergraphs H and K, the set $\mathbf{E}_{H \square K}$ is the disjoint union $\{e\} \cup (\mathbf{E}_N \backslash e) \cup (\mathbf{E}_P \backslash e)$ with $\{e\} = \mathbf{E}_N \cap \mathbf{E}_P$. It follows that every hypergraph having at least $m \geq 4$ hyperedges is of the form $H \square K$ where H and K denote two hypergraphs having respectively $1 + 2$ and $1 + m - 2$ hyperedges. That suffices to conclude. □

There are different ways to characterize the treewidth of some hypergraph, one of them is by using the splitting as it is explained below.

Lemma 17. *Let k be some integer. The set of all hypergraphs of treewidth at most k is the $\square$-closure of the set of all hypergraphs having at most $k + 1$ vertices.*

Proof. Let k be some integer. Let $\mathcal{T}$ be the set of all hypergraphs of treewidth at most k, let $\mathcal{A}$ be the set of all hypergraphs having at most $k + 1$ vertices and let $\mathcal{C}$ be the $\square$-closure of $\mathcal{A}$. Let us prove: $\mathcal{T} = \mathcal{C}$.

1. Obviously, we have: $\mathcal{A} \subseteq \mathcal{T}$. Suppose there is some integer $N \geq 1$ such that $\mathcal{T}$ contains every hypergraph of the form $((G_1 \Box G_2)\ldots)\Box G_N$ with $G_1, \ldots, G_N \in \mathcal{A}$. Let G be a graph of the form $((G_1 \Box G_2)\ldots)\Box G_{N+1}$ for some $G_1, \ldots, G_{N+1} \in \mathcal{A}$. By recurrence, the hypergraph $H := ((G_1 \Box G_2)\ldots)\Box G_N$ belongs to $\mathcal{T}$, admits some tree-decomposition (T, g) of width at most k. Let e be the unique hyperedge in H and in G_{N+1} and t be the node of T defined by $e \in g(t)$. The pair (U, h) where U is obtained from T by attaching to t some new node u and where h associates with u the hypergraph $G_{N+1} \backslash e$ and with every node $t \neq u$ of U the hypergraph $g(t) \backslash e$ is a tree-decomposition of G of width at most k. Hence, $\mathcal{C} \subseteq \mathcal{T}$.
2. For every integer N, we denote by $\mathcal{T}_N$ the set of all hypergraphs admitting a tree-decomposition of width $\leq k$ having at most N nodes. Clearly, every hypergraph of $\mathcal{T}_1$ has at most $k+1$ vertices and belongs to $\mathcal{A} \subseteq \mathcal{C}$. Suppose there is some integer $N \geq 1$ such that $\mathcal{T}_N \subseteq \mathcal{C}$. Let $G \in \mathcal{T}_{N+1} - \mathcal{T}_N$ and (T, g) be some tree-decomposition of G of with $\leq k$ and having $N+1$ node. Let u be some leaf of T and t the unique node of T adjacent with t. Let e be a hyperedge not in G. Let us consider the hypergraphs $H := g(\mathbf{V}_T \backslash u)$ and $I := g(u)$. Let J (resp. K) the hypergraph obtained from H (resp. L) by adding some new hyperedge (not in G) with extremities $\mathbf{V}_H \cap \mathbf{V}_J$. Clearly, we have $G = J \Box K$ and $J \in \mathcal{T}_N$ and $K \in \mathcal{T}_1$. By recurrence, J and K belong to $\mathcal{C}$. Thus, $G \in \mathcal{C}$. That suffices to conclude. □

A refinement of the precedent fact shows us how to produce every treewidth bounded set of hypergraphs with a finite set of hypergraphs (finite up to isomorphism):

Lemma 18. *Let k be some integer. The set of all hypergraphs of treewidth at most k is the $\Box$-closure of the set of all hypergraphs having at most $k+1$ vertices and having at most 3 hyperedges.*

Proof. Let k be some integer. As an immediate consequence of Definition 11 and of Lemma 16, the set of hypergraphs having at most $k+1$ vertices is the $\Box$-closure of the set having at most $k+1$ vertices and having at most 3 hyperedges. Lemmas 15 and 17 suffice to conclude. □

2 A splitting-operation for E-hypermaps

In the last section, we showed how to split a hypergraph in two new ones and proved the commutativity and the weak-associativity of such an operation. In this section, we generalize this splitting to hypermaps. To be more precise, this generalization fails. The result of such a splitting are not hypermaps but "extended-hypermaps", called simply E-hypermaps: the darts extremities of every hyperedge does not form necessarily a cyclic permutation α but a union of cyclic permutations (see Definition 19).

The proof of the well-foundness of the splitting of E-hypermaps is not immediate (we need some preliminaries!) but its properties are! In a few lines, we will prove that splitting of E-hypermaps:

1. is commutative.
2. is weak-associative.
3. commutes with the duality.
4. commutes with the splitting of hypergraphs.

Now, let us introduce E-hypermaps, which are a natural extension of hypermaps:

Definition 19 (E-hypermap, duality, induced hypergraph). A *E-hypermap* M is a tuple $(D, E, f, \sigma, \alpha)$ verifying the conditions (1.) and (2.) of Definition 4:

1. (D, σ, α) is a combinatorial hypermap.
2. f is a mapping $E \to \mathcal{P}(D)$ where $\{f(e) \mid e \in E : f(e) \neq \emptyset\}$ is a partition of D.

and a third condition that replaces condition (3.) of Definition 4:

3'. for every $e \in E$, $f(e)$ is closed under α.

All other definitions and notations of Definition 4 are extended to E-hypermaps. Similarly, we extend:

- Definition 6 to E-hypermaps by using same definitions and same notations, except that a E-hypermap is said a *E-representation* of a hypergraph G (instead of "a representation of G").
- Definition 7 to E-hypermaps by using same definitions and same notations. □

For same reasons as for hypermaps (see Lemma 8), the dual operation is involutive. The proof is left to the reader.

Corollary 20. *Every E-hypermap M verifies: $M = (M^*)^*$.*

We define the splitting of E-hypermaps, by defining first a more intuitive operation: its converse, which associates with every E-hypermap M and every set of hyperedges C of M, the subhypermap of M induced by C. This E-hypermap is obtained by replacing in M the set $\mathbf{E}_M - C$ by some hyperedge c having as extremities all the darts that "border" $\mathbf{dart}_M(C)$, that are in $\sigma_M^{-1}(\mathbf{dart}_M(C)) - \mathbf{dart}_M(C)$; this E-hypermap N is denoted by $M \uparrow_c C$. By defining $\sigma_N := (\sigma - M : \mathbf{D}_M)$ and $\beta_N = (\beta_M : \mathbf{D}_N)$, it follows that every vertex (resp. face) of N is a vertex (resp. face) of M ; in consequence, splitting of E-hypermaps commutes with splitting of hypergraphs.

In a similar way, we define the "context" of $M \uparrow_c C$ in M, that is $M \downarrow_c C := M \uparrow_c (\mathbf{E}_M - C)$. All these notions are formalized below and illustrated by Example 23.

Definition 21. Let $M = (D, E, g, \sigma, \alpha)$ be a E-hypermap, let $C \subseteq E$ be a set and let $c \notin C$ be an element. We denote by $M \uparrow_c C$ the tuple $(A, C \cup \{c\}, \sigma : A, (\sigma : A) \circ (\beta_M : A)^{-1}, f)$ where $A := g(C) \cup \sigma^{-1}(g(C))$ and f is the mapping defined by $f(c) = A - g(C)$ and $f(b) = g(b)$ for each $b \in C$.

For every $e \notin E - C$, we denote by $M \downarrow_e C$ the tuple $M \uparrow_e (E - C)$. An hyperedge d of M is *replaceable* if $\sigma^{-1}(g(d)) \cap g(d) = \emptyset$ or, equivalently, if $M \downarrow_d \{d\} = M$. □

By the following fact, we prove that $M \uparrow_c C$ is a hypermap having same permutation α on the extremities of the remained hyperedges.

Fact 22. *Let M be a E-hypermap, let $C \subseteq \mathbf{E}_M$ and let $c \notin C$ be an element. Then, $N := M \uparrow_c C$ is an E-hypermap such that $\alpha_N(a) = \alpha_M(a)$ for every $a \in \mathbf{dart}_M(C)$.*

Proof. Let M be some E-hypermap, let $C \subseteq \mathbf{E}_M$ and let $c \notin C$. Let $N := M \uparrow_c C$ and $A := \mathbf{dart}_M(C)$. By definition, $\{c\} = \mathbf{E}_N - C$; thus, to conclude it suffices to prove $\alpha_N(a) = \alpha_M(a)$ for every $a \in A$. Let $a \in A$. By construction, $b = \beta_M^{-1}(a)$ is equal to $\sigma_M^{-1}(\alpha_M(a))$, belongs to $A \cup \sigma_M^{-1}(A)$ and verifies $\beta_M(b) = a = \beta_N(b)$. The dart $c = \sigma_M(b)$ is equal to $\alpha_M(a)$, belongs to A, hence, is equal to $\sigma_N(b)$. Then $\alpha_N(a)$, by definition $\sigma_N(\beta_N^{-1}(a))$, is equal to $\sigma_M(\beta_M^{-1}(a)) = \alpha_M(a)$. □

Example 23. In Figure 5, are represented from left to right a map M and the two E-hypermaps $N := M \uparrow_c C$ and $P := M \downarrow_c C = M \uparrow_c (\mathbf{E}_M - C)$. The set of edges C of M are drawn with no thick lines, their dart-extremities are the white ones. The new hyperedge c of N (resp. P) is represented with a white box. As we shall see in Definition 27, the E-hypermap M is denoted $N \square P$.

Let detail how $N := M \uparrow_c C$ is obtained from M. All edges not in C are deleted. The permutation α on the extremities of this edges is preserved. The new hyperedge c is added. The dart-extremities of c are the darts not in $\mathbf{dart}_M(C)$ which are in $\sigma_M^{-1}(\mathbf{dart}_M(C))$ (that are the black darts having as σ_M-successor a "white" dart). The permutation σ_N is the permutation $\sigma_M : \mathbf{dart}_N$. The permutation σ_N is the permutation $\sigma_N(\beta_M : \mathbf{dart}_N)^{-1}$. □

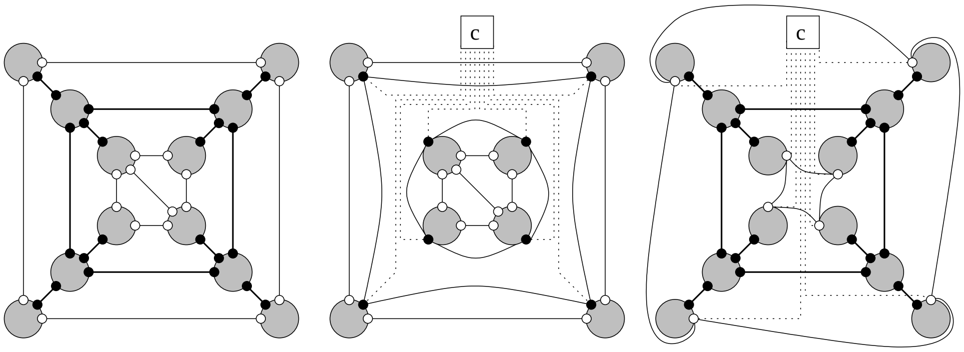

Fig. 5. the operation $M \uparrow_c C$ and splitting operation.

The next lemma establishes that the final result does not depend on the order in which we replace successively a sequence of sets of hyperedges. That is the reason why the splitting is commutative and weak-associative.

Lemma 24. *Every E-hypermap of the form $(M \downarrow_b B) \downarrow_c C$ is equal to:*

- *$M \downarrow_c (B \cup C)$ if $b \in C$.*
- *$(M \downarrow_c C) \downarrow_b B$ if $b \notin C$ and $c \notin B$.*

Proof. Let M, N, P be three E-hypermaps such that $N = M \downarrow_b B$ and $P = N \downarrow_c C$ for some sets B and C and some elements b, c.

- Suppose $b \in C$. Let us establish $P = Q$ with $Q := M \downarrow_c (B \cup C)$.
  We have $\mathbf{E}_Q = \mathbf{E}_P$ because $\mathbf{E}_Q = (\mathbf{E}_M - (B \cup C)) \uplus \{c\} = (((\mathbf{E}_M - B) \uplus \{b\}) - C) \uplus \{c\} = \mathbf{E}_P$.
  Clearly, every $e \in \mathbf{E}_Q \backslash c$ verifies $\mathbf{dart}_Q(e) = \mathbf{dart}_M(e) = \mathbf{dart}_N(e) = \mathbf{dart}_P(e)$.
  Let $A := \mathbf{dart}_M(\mathbf{E}_Q \backslash \{c\})$. By Definition 21, every $a \in A$ verifies $\sigma_Q^{-1}(a) = \sigma_M^{-1}(a) = \sigma_N^{-1}(a) = \sigma_P^{-1}(a)$. Thus, $\mathbf{D}_Q$, by definition equal to $\sigma_M^{-1}(A) \cup A$, is equal to $\sigma_N^{-1}(A) - A = \mathbf{D}_P$. Then, $\mathbf{D}_Q = \mathbf{D}_P$.
  The set $\mathbf{dart}_Q(c)$, by definition $\mathbf{D}_Q - A$, is equal to $\mathbf{D}_R - A$, by definition equal to $\mathbf{dart}_P(c)$. Then, $\mathbf{dart}_Q = \mathbf{dart}_P$.
  As a consequence of the equality $\mathbf{D}_Q = \mathbf{D}_P$ and the inclusions $\mathbf{D}_P \subseteq \mathbf{D}_N \subseteq \mathbf{D}_M$, we have: $\sigma_Q = \sigma_M : \mathbf{D}_Q = (\sigma_M : \mathbf{D}_N) : \mathbf{D}_P = \sigma_P$; then, $\sigma_Q = \sigma_P$.
  With similar arguments, we prove: $\beta_Q = \beta_P$. That suffices to conclude.

– Suppose $b \notin C$ and $c \notin B$.
Observe that for every E-hypermap L, every subset $E \subseteq \mathbf{E}_L$ and every element e, we have $L \downarrow_e E = L' \downarrow_e \{e\}$ with $L' = (\mathbf{D}_L, (\mathbf{E}_L - E) \uplus \{e\}, f, \sigma_L, \alpha_L)$ where f associates with every $a \in (\mathbf{E}_L - E) \uplus \{e\}$ the set $\mathbf{dart}_L(a)$ if $a \neq e$ and $\mathbf{dart}_L(E)$ if $a = e$. Then, we can suppose, without loss of generality, that $B = \{b\}$ and $C = \{c\}$. Then, to conclude, let us establish $P = R$ with $Q := M \downarrow_c \{c\}$ and $R := Q \downarrow_b \{b\}$.
Clearly, $\mathbf{E}_M = \mathbf{E}_N = \mathbf{E}_P = \mathbf{E}_Q = \mathbf{E}_R$.
For every E-hypermap L and every hyperedge e of L, we denote by $\eta(L, e)$ the set $\{a \in \mathbf{dart}_L(e) \mid \sigma_L(a) \notin \mathbf{dart}_L(e)\}$. Clearly, every E-hypermap L verifies:
1. $\eta(L, e) = \mathbf{dart}_{L\downarrow_e\{e\}}(e)$ for every hyperedge e of L.
2. $\eta(L, e) = \eta(L \downarrow_f \{f\}, e)$ for all hyperedges e and f of L.

By definition, the restrictions of $\mathbf{dart}_M$, $\mathbf{dart}_N$,$\mathbf{dart}_P$,$\mathbf{dart}_Q$, $\mathbf{dart}_R$ to $\mathbf{E}_M - \{b, c\}$ are equals. By Point 2, the sets $\eta(M, c)$, $\eta(N, c)$, $\eta(P, c)$, $\eta(Q, c)$, $\eta(R, c)$ are equals. By Point 1, we have $\mathbf{dart}_R(c) = \mathbf{dart}_P(c)$. For similar reasons, we have $\mathbf{dart}_R(b) = \mathbf{dart}_P(b)$. Hence, we have $\mathbf{D}_R = \mathbf{D}_P$ and $\mathbf{dart}_R = \mathbf{dart}_P$.
As a consequence of the inclusions $\mathbf{D}_R \subseteq \mathbf{D}_Q \subseteq \mathbf{D}_M$ and $\mathbf{D}_P \subseteq \mathbf{D}_N \subseteq \mathbf{D}_M$, we have: $\sigma_R = (\sigma_M : \mathbf{D}_Q) : \mathbf{D}_R = \sigma_M : \mathbf{D}_R = (\sigma_M : \mathbf{D}_N) : \mathbf{D}_P = \sigma_P$.
With similar arguments, we establish $\beta_R = \beta_P$. That suffices to conclude. □

Now, present the condition that must verify every couple of E-hypermaps in order to be split. Such pairs are called "separations".

Definition 25 (Separation). A *separation* is a pair of E-hypermaps (N, P) such that $\mathbf{E}_N \cap \mathbf{E}_P$ is a singleton $\{e\}$ verifying:

1. for every $c \neq e$, $N \uparrow_c \{e\}$ is obtained from $P \uparrow_c \{e\}$ by renaming simultaneously e by c and c by e.
2. $\mathbf{D}_N \cap \mathbf{D}_P$ is the disjoint union $\mathbf{dart}_N(e) \cup \mathbf{dart}_P(e)$.

In order to simplify, are called separation (N, P, e) and (N, P, e, I), where $I := N \uparrow_\$ \{e\}$ for some fixed symbol \$. Observe that every separation (N, P) determines a unique tuple (N, P, e) (resp. (N, P, e, I)). □

Next result will be required to define splitting, it connects Definitions 21, 25 and 27.

Lemma 26. *For all E-hypermaps N and P and every e, the two following assertions are equivalent:*

1. *(N, P) is a separation and $\{e\} = \mathbf{E}_N \cap \mathbf{E}_P$.*
2. *there is a pair (M, C) such that $M \uparrow_e C = N$ and $M \downarrow_e C = P$. Furthermore, this pair is unique.*

Proof. Let N and P be two E-hypermaps and let e be some element. Before to prove that assertions (1.) and (2.) are equivalent, let us establish the uniqueness of the pair (M, C):

– two pairs (M, C) and (M', C') such that $M \uparrow_e C = N = M' \uparrow_e C'$ and $M \downarrow_e C = P = M' \downarrow_e C'$ are equal.
Let (M, C) and (M', C') be two such pairs. We have:
  - $C = \mathbf{E}_N - \{e\} = C'$.
  - $\mathbf{E}_M = (\mathbf{E}_M \cap C) \cup (\mathbf{E}_M - C) = (\mathbf{E}_N - \{e\}) \cup (\mathbf{E}_P - \{e\}) = \mathbf{E}_{M'}$.
  - $\mathbf{D}_M = \mathbf{D}_N \cup \mathbf{D}_P = \mathbf{D}_{M'}$.
  - $\mathbf{dart}_M$ is the union of the restrictions $\mathbf{dart}_N \restriction (\mathbf{E}_N \backslash e)$ and $\mathbf{dart}_P \restriction (\mathbf{E}_P \backslash e)$, hence, is equal to $\mathbf{dart}_{M'}$.

- Let d be a dart of $\mathbf{D}_M$. If $d \in \mathbf{dart}_M(C)$, we have: $\sigma_M^{-1}(d) = \sigma_N^{-1}(d) = \sigma_{M'}^{-1}(d)$; otherwise, d belongs to $\mathbf{dart}_M(\mathbf{E}_M - C)$ and verifies $\sigma_M^{-1}(d) = \sigma_P^{-1}(d) = \sigma_{M'}^{-1}(d)$. That implies $\sigma_M^{-1} = \sigma_{M'}^{-1}$ and $\sigma_M = \sigma_{M'}$.
- By an identical proof with the previous one, it comes $\beta_M = \beta_{M'}$, hence, $\alpha_M = \alpha_{M'}$.

Now let us prove that:

- if (2.) holds, (1.) holds.
  Suppose there is a E-hypermap M and a set $C \subseteq \mathbf{E}_M$ such that $M \uparrow_e C = N$ and $M \downarrow_e C = P$. Clearly, $\mathbf{E}_N \cap \mathbf{E}_P$ is the singleton $\{e\}$. By Definition 21 (resp. Lemma 24), we prove Condition 2. (resp. 1.) of Definition 25.
- if (1.) holds, (2.) holds.
  Let (N, P, e) be a separation. Let us prove there is (M, C) such that $(N, P) = (M \uparrow_e C, M \downarrow_e C)$. Let $A := \mathbf{D}_N - \mathbf{dart}_N(e)$ and $B := \mathbf{D}_P - \mathbf{dart}_P(e)$. By definition, $\mathbf{D}_N \cap \mathbf{D}_P = \mathbf{dart}_N(e) \uplus \mathbf{dart}_P(e)$, it follows that $\mathbf{D}_N \cup \mathbf{D}_P = A \uplus B$.
  Let $M := (A \cup B, E, g, \sigma, \alpha)$ where:
  - E is the difference $(\mathbf{E}_N \cup \mathbf{E}_P) \backslash e$.
  - g is the restriction of $\mathbf{dart}_N \cup \mathbf{dart}_P$ to E.
  - σ^{-1} (resp. β^{-1}) associates with every $a \in A$ the dart $\sigma_N^{-1}(a) \in A \cup B$ (resp. $\beta_N^{-1}(a)$) and with every $a \in B$ the dart $\sigma_P^{-1}(a) \in A \cup B$ (resp. $\beta_P^{-1}(a)$).
  - $\alpha := \sigma\beta^{-1}$.

  For every $a \in A$, $\alpha(a) = \sigma(\beta_N^{-1}(a)) = \sigma(\sigma_N^{-1}(\alpha_N(a)) = \sigma(\sigma^{-1}(\alpha_N(a)) = \alpha_N(a)$ (because $\alpha_N(a) \in A$). By a same argument, it comes $\alpha(a) = \alpha_P(a)$ for every $a \in B$. Thus, α is the union of the restrictions $\alpha_N \upharpoonright A$ and $\alpha_P \upharpoonright B$. Then, M is a E-hypermap. To conclude, it suffices to establish $(N, P) = (R, S)$ where $R := M \uparrow_e (E \cap \mathbf{E}_N)$ and $S := M \downarrow_e (E \cap \mathbf{E}_N)$.
  First, let us observe that:
  1. $\sigma_R^{-1}(a) = \sigma^{-1}(a) = \sigma_N^{-1}(a)$ and $\beta_R^{-1}(a) = \beta^{-1}(a) = \beta_N^{-1}(a)$ for every $a \in A$.
     Let $a \in A$ be a dart. The equality $\sigma^{-1}(a) = \sigma_N^{-1}(a)$ is trivial. Moreover, $\sigma^{-1}(a)$ belongs to $\mathbf{D}_R$, which contains $\sigma^{-1}(A) = \sigma^{-1}(\mathbf{dart}(E \cap \mathbf{E}_N))$. Thus, $\sigma_R^{-1}(a) = \sigma^{-1}(a)$. In s similar way, we prove: $\beta_R^{-1}(a) = \beta^{-1}(a) = \beta_N^{-1}(a)$.
  2. $\sigma_S^{-1}(a) = \sigma^{-1}(a) = \sigma_P^{-1}(a)$ and $\beta_S^{-1}(a) = \beta^{-1}(a) = \beta_P^{-1}(a)$ for every $a \in B$.
     The proof is similar to the above one.

  The equalities $N = R$ and $P = S$ is the consequence of the following facts:
  - $\mathbf{E}_R = \mathbf{E}_N$.
    Because $\mathbf{E}_R = (E \cap \mathbf{E}_N) \cup \{e\} = \mathbf{E}_N$.
  - $\mathbf{D}_R = \mathbf{D}_N$.
    By definition: $\mathbf{D}_R = \sigma^{-1}(A) \cup A$ and $\mathbf{D}_N = \sigma_N^{-1}(A) \cup A$. Point (1.) suffices to conclude.
  - $\mathbf{dart}_R = \mathbf{dart}_N$.
    By Point (1.): $\mathbf{dart}_R(e) = \sigma^{-1}(A) - A = \mathbf{dart}_N(e)$, hence, $\mathbf{dart}_R = \mathbf{dart}_N$.
  - $(\mathbf{E}_P, \mathbf{D}_P, \mathbf{dart}_P) = (\mathbf{E}_S, \mathbf{D}_S, \mathbf{dart}_S)$.
    The proof is similar to the above ones.
  - $\sigma_R = \sigma_N$.
    We establish that by proving $\sigma_R^{-1} = \sigma_N^{-1}$. Let a be a dart of R. If $a \in A$, Point (1.) implies $\sigma_R^{-1}(a) = \sigma_N^{-1}(a)$. Suppose $a \notin A$. Let x be a distinct element with e. From Lemma 24, $R \uparrow_x \{e\}$ and $S \uparrow_x \{e\}$ are equal modulo some renaming of their hyperedges. Thus, $\sigma_R^{-1}(a) = \sigma_{R\uparrow_x\{e\}}^{-1}(a) = \sigma_{S\uparrow_x\{e\}}^{-1}(a)$.
    Thus, a belongs to B and to $\sigma_R^{-1}(A) = \sigma^{-1}(A)$ (because a is a dart of R). Then, there is a sequence $(a_0, \ldots, a_m) \in B \times B^* \times A^*$ such that $a = a_0$ and $a_{i+1} = \sigma_S^{-1}(a_i) = \sigma_P^{-1}(a_i) = \sigma^{-1}(a_i)$ for every $i \in [0, m-1]$. It comes $a_m = \sigma_{S\uparrow_x\{e\}}^{-1}(a) = \sigma_{M\uparrow_x C}^{-1}(a) = \sigma_{P\uparrow_x\{e\}}^{-1}(a)$. Thus, $\sigma_R^{-1}(a) = \sigma_{P\uparrow_x\{e\}}^{-1}(a)$.
    From Definition 25, modulo some renaming of hyperedges, the E-hypermaps $P \uparrow_x \{e\}$ and $N \uparrow_x \{e\}$ are equal. Hence $\sigma_R^{-1}(a) = \sigma_{N\uparrow_x\{e\}}^{-1}(a)$.

$a \in B$ implies $\sigma_N^{-1}(a) \in A$. Thus, $\sigma_{N\uparrow_x\{e\}}^{-1}(a) = \sigma_N^{-1}(a)$, and $\sigma_R^{-1}(a) = \sigma_N^{-1}(a)$.

- $\beta_R = \beta_N$, $\sigma_S = \sigma_P$, $\beta_S = \beta_P$.
  The proof is similar with the above one. □

Now, we can define the splitting of E-hypermaps.

Definition 27 (splitting). For every separation (N, P), we denote by $N \Box P$ the unique E-hypermap M such that $M \uparrow_d (\mathbf{E}_M \cap \mathbf{E}_N) = N$ and $M \downarrow_d (\mathbf{E}_M \cap \mathbf{E}_N) = P$ with $\{d\} = \mathbf{E}_N \cap \mathbf{E}_P$. The pair (N, P) is called a *separation of* M. □

Theorem 28 (Commutativity). *Every separation* (N, P) *verifies:*

$$N \Box P = P \Box N$$

Proof. The equality $N \Box P = P \Box N$ is a direct consequence of Lemma 26 and the trivial equality $M \uparrow_c C = M \downarrow_c (\mathbf{E}_M - C)$ verified for every E-hypermap M, every set $C \subseteq \mathbf{E}_M$ and every element $c \notin \mathbf{E}_M$. □

Theorem 29 (Weak associativity). *Every E-hypermap of the form* $(N \Box P) \Box Q$ *for some E-hypermaps* N*,* P *et* Q *with* $\mathbf{E}_N \cap \mathbf{E}_P \cap \mathbf{E}_Q$ *empty is either equal:*

- *to* $N \Box (P \Box Q)$
- *or to* $(N \Box Q) \Box P$.

Proof. Let M, N, P, Q be four E-hypermaps such that $M = (N \Box P) \Box Q$. The unique hyperedge e of $\mathbf{E}_{N \Box P} \cap \mathbf{E}_Q$ belongs to $\mathbf{E}_{N \Box P} = (\mathbf{E}_N \cup \mathbf{E}_P) - (\mathbf{E}_N \cap \mathbf{E}_P)$, hence, belongs either to N or to P. Thus, $N \Box Q$ or $P \Box Q$ is not defined.

Suppose that $e \in \mathbf{E}_P$ (by Theorem 28, the case $e \in \mathbf{E}_N$ is symmetrical) and let us prove that $N \Box (P \Box Q)$ is defined and is equal to M. Let d be the unique hyperedge of $\mathbf{E}_N \cap \mathbf{E}_P$. Let $C := \mathbf{E}_N \backslash d$, $D := \mathbf{E}_P - \{d, e\}$, $E := \mathbf{E}_Q \backslash e$. By Lemmas 24 and 26, we have $N = (M \uparrow_e (C \cup \{d\} \cup D)) \uparrow_d C = M \uparrow_d C$, $P = (M \uparrow_e (C \cup \{d\} \cup D)) \downarrow_d C = (M \downarrow_d C) \uparrow_e (\{d\} \cup D)$ and $Q = M \downarrow_e (C \cup \{d\} \cup D) = (M \downarrow_d C) \downarrow_e (\{d\} \cup D)$. By Lemma 26, $N \Box (P \Box Q)$ is defined and is equal to M. □

For the same reasons as for hypergraphs, we have:

Lemma 30. *The* $\Box$*-closure of some set of E-hypermaps* $\mathcal{A}$ *which is closed under isomorphism is the set of all E-hypermaps of the form* $((M_1 \Box M_2) \ldots) \Box M_n$ *with* $n \geq 1$ *and* $M_1, \ldots, M_n \in \mathcal{A}$.

Proof. Immediate consequence of the two precedent theorems. □

Other nice property of the splitting:

Theorem 31 (Duality). *Every separation* (N, P) *verifies:*

$$(N \Box P)^* = N^* \Box P^*$$

Proof. Let (N, P) be a separation of some E-hypermap M. By Lemma 26, there are $C \subseteq \mathbf{E}_M$ and $e \notin \mathbf{E}_M$ with $N = M \uparrow_e C$ and $P = M \downarrow_e C$. Let $Q := M^* \uparrow_e C$. Definitions 19 and 21 imply $\mathbf{E}_Q = C \cup \{e\} = \mathbf{E}_N$, $\mathbf{D}_Q = \mathbf{dart}_{M^*}(C) \cup \sigma_M^{-1}(\alpha_M(\mathbf{dart}_{M^*}(C))) = \mathbf{dart}_M(C) \cup \sigma_M^{-1}(\mathbf{dart}_M(C)) = \mathbf{D}_N$. We have $\sigma_Q = \sigma_{M^*} : \mathbf{D}_Q = \beta_M : \mathbf{D}_N = \beta_N$ and $\beta_Q = \beta_{M^*} : \mathbf{D}_Q = \sigma_M : \mathbf{D}_N = \sigma_N$. It follows $Q = N^*$. A similar proof implies: $M^* \downarrow_e C = P^*$. Lemma 26 implies: $M^* = (M^* \uparrow_e C) \Box (M^* \downarrow_e C) = N^* \Box P^*$. □

Now, let us prove that splitting commutes with induced hypergraph operation. Unfortunately, the definition of an induced hypergraph (Definition 19) and of the splitting (Definition 11) does not permit formally to establish next result (because two "same" vertices in the two sides of a separation are two sets of darts not disjoint but distinct). A good formalism to express Theorem 32 is to consider "abstract" hypergraphs i.e., isomorphic class of hypergraphs. That will be done in Section 4. Another way to express such result on concrete hypergraphs is to suppose there is a universe of darts given with an equivalent relation $\sim$ such that every vertex of any map is contained in some equivalent class and such that every equivalent class does not contain two vertices of a same map (observe that this new condition would not permit to define dual hypermap as it is done in Definition 19). Under this choice, the hypergraph induced operation $\underline{hgraph}$ associates with every hypermap M a hypergraph whose each vertex is not a vertex of the initial E-hypermap but the equivalent class $v_\sim$ containing some vertex v of M.

Theorem 32. *Every separation (N, P) verifies:*

$$\underline{hgraph}(N \Box P) = \underline{hgraph}(N) \Box \underline{hgraph}(P)$$

Proof. Let (N, P, e) be a separation. Let $G := \underline{hgraph}(N \Box P)$, $H := \underline{hgraph}(N)$, $K := \underline{hgraph}(P)$ and let $H \backslash e$ (resp. $K \backslash e$) be the hypergraph obtained from H (resp. K) by deleting e. By Definitions 21 and 27, we have $\mathbf{E}_H \cap \mathbf{E}_K = \{e\}$, $\mathbf{V}_H \cup \mathbf{V}_K = \mathbf{V}_G$, $\mathbf{V}_H \cap \mathbf{V}_K = \mathbf{vert}_H(e) = \mathbf{vert}_K(e)$ and $\mathbf{vert}_G$ is the union of the restrictions $\mathbf{vert}_H \restriction (\mathbf{E}_H \backslash e)$ and $\mathbf{vert}_K \restriction (\mathbf{E}_K \backslash e)$ (because $\mathbf{dart}_{N \Box P} = (\mathbf{dart}_N \restriction (\mathbf{E}_N \backslash e)) \cup (\mathbf{vert}_P \restriction (\mathbf{E}_P \backslash d)))$. Then, $G = H \Box K$. $\Box$

Example 33. In Figure 6, are represented from left to right a graph G and two hypergraphs H and K, which are respectively induced by the hypermaps M, N and P of Example 23. They verify $G = N \Box P$ with $\{c\} = \mathbf{E}_H \cap \mathbf{E}_K$. $\Box$

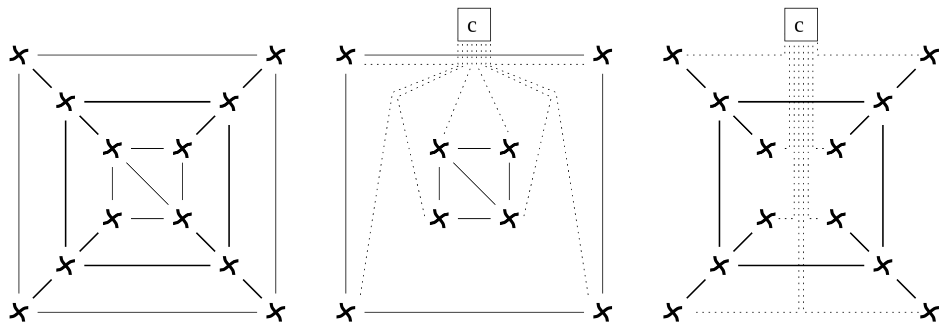

Fig. 6. The splitting operations of hypergraphs and of hypermaps commute.

3 A proof of a conjecture of Robertson and Seymour

In this section, we prove a generalization of the conjecture of Robertson and Seymour mentioned in the introduction. Our technique consists to split every hypermap in a number of hypermaps of "small" size. A problem appears: our splitting operation is too powerful. See for instance Example 23, where a planar hypermap was split in two E-hypermaps that seemed not to be planar and not to be simple!.

Thus, we restrict splitting into a "connected splitting" denoted by $\boxtimes$: a connected hypermap is split in a connected way, if the two hyperedges' sets of the splitting partition are connected. This operation keeps all good properties (commutativity, weak-associativity, commutation with duality and with the splitting of hypergraphs) and preserves planarity (an extension of this result is given by Theorem 51). It follows, that we can produce the set of all planar representations of hypergraphs of treewidth $\leq k$, that is the $\boxtimes$-closure of a set of "basic" and planar hypermaps having at most $k+1$ vertices (Theorem 57).

Definition 34 (Connected separation and splitting). A set of darts C of some combinatorial hypermap (σ, α) is *connected* if every partition $\{A, B\}$ of C is such that $A \cap (\alpha(B) \cup \sigma(B))$ is nonempty. This definition is extended in a natural way to hypermaps and E-hypermaps. A nonempty set of hyperedges D is *connected* in some hypermap M if $\mathbf{dart}_M(D)$ is connected in M (let us observe that this condition is equivalent with: "every partition $\{A, B\}$ of D is such that $\mathbf{dart}_M(A) \cap \sigma_M^{-1}(\mathbf{dart}_M(B))$ is nonempty"). A E-hypermap is *connected* if its set of hyperedges is connected.

A hyperedge e of M is *c-replaceable in M* if it is replaceable and if $\mathbf{E}_M$ and $\mathbf{E}_M \backslash e$ are connected in M. A separation (N, P, e) is *connected* if e is c-replaceable in N and in P. The separations (N, P) and (N, P, e, I) (see Definition 25) are said connected too. For such a separation, $N \square P$ is denoted by $N \boxtimes P$. $\square$

As a direct consequence of such a definition, a natural lemma. Its proof is left to the reader.

Lemma 35. *A E-hypermap is connected if and if only if its induced hypergraph is connected.*

In order to study the connected-splitting $\boxtimes$, three basic facts.

Fact 36. *Let M be a E-hypermap of the form $N \square P$. Every set $D \subseteq \mathbf{E}_N \cap \mathbf{E}_M$ is connected in M if and only it is connected in N.*

Proof. Let M be a E-hypermap of the form $N \square P$. Let D be a subset of $\mathbf{E}_N \cap \mathbf{E}_M$. As a consequence of Definition 27, every subset $A \subseteq D$ verify $\mathbf{dart}_N(A) = \mathbf{dart}_M(A)$ and $\sigma_N^{-1}(\mathbf{dart}_N(A)) = \sigma_M^{-1}(\mathbf{dart}_M(A))$. Thus, if D is connected in M, for every partition $\{A, B\}$ of D the intersection $\mathbf{dart}_N(A) \cap \sigma_N^{-1}(\mathbf{dart}_N(B))$ is equal to $\mathbf{dart}_M(A) \cap \sigma_M^{-1}(\mathbf{dart}_M(B))$, which is nonempty by the connectivity of M. Hence, D is connected in N. For the same reasons, the converse holds. $\square$

Fact 37. *A separation (N, P, e) is connected if $N \square P$ is connected, if $\mathbf{E}_N \backslash e$ is connected in N and if $\mathbf{E}_P \backslash e$ is connected in P.*

Proof. Let (N, P, e) be a separation of some connected E-hypermap M with $\mathbf{E}_N \backslash e$ connected in N and $\mathbf{E}_P \backslash e$ is connected in P.

Let $\{A, B\}$ be a partition of $\mathbf{E}_N$. Without loss of generality, we can suppose $e \in B$. By hypothesis, A and $\mathbf{E}_P \backslash e$ are connected, hence, are nonempty. Thus $\{A, B'\}$ with $B' := (B \cup \mathbf{E}_P) \backslash e$ is a partition of $\mathbf{E}_M$. The connectivity of M implies $\mathbf{dart}_M(A) \cap \sigma_M^{-1}(\mathbf{dart}_M(B'))$ nonempty. As a consequence of Definition 27, it comes $\mathbf{D}_N \cap \sigma_M^{-1}(\mathbf{dart}_M(B')) = (\mathbf{D}_N - \mathbf{dart}_N(e)) \cap \sigma_N^{-1}(\mathbf{dart}_N(B))$. Thus, the trivial inclusion $\mathbf{dart}_N(A) = \mathbf{dart}_M(A) \subseteq \mathbf{D}_N - \mathbf{dart}_N(e)$ implies $\mathbf{dart}_N(A) \cap \sigma_N^{-1}(\mathbf{dart}_N(B))$ nonempty. Thus, N is connected. $\square$

Fact 38. *Let (N, P, e) be a connected separation of some E-hypermap M. Let D be a subset of $\mathbf{E}_N$. The two following assertions are equivalent:*

– *D is connected in M.*
– *the set $(D \cup \mathbf{E}_P)\backslash e$ (resp. D) is connected if $e \in D$ (resp. $e \notin D$).*

Proof. Let (N, P, e) be a connected separation of some E-hypermap M. Let D be a subset of $\mathbf{E}_N$. If $e \notin D$, by Fact 36 D is connected in N if and only if it is connected in M. Furthermore, we suppose $e \in D$. Let $E := (D \cup \mathbf{E}_P)\backslash e$. Let us prove that D is connected in N if and only if E is connected in M. Let $\{D_1, \ldots, D_m\}$ be the partition of $D\backslash e$ into maximal subsets to be connected in N. By Fact 36, $\{D_1, \ldots, D_m\}$ is the partition of $E - \mathbf{E}_P$ into maximal subsets to be connected in M. Thus, if D is connected in N, for every $i \in [m]$ the set $\mathbf{dart}_N(e) \cap \sigma_N^{-1}(\mathbf{dart}_N(D_i))$ is nonempty, for every $i \in [m]$ the set $\mathbf{dart}_M(\mathbf{E}_P\backslash e) \cap \sigma_M^{-1}(\mathbf{dart}_M(D_i))$ is nonempty, thus E, union of the connected subsets $D_1 \cup \ldots \cup D_m \cup (\mathbf{E}_P\backslash e)$ is connected in M. By using same arguments, we prove the converse. □

In a few lines, let us establish the following natural but crucial 4 properties:

Corollary 39 (Commutativity). *Every connected separation (N, P) verifies:*

$$N \boxtimes P = P \boxtimes N$$

Proof. A direct consequence of Theorem 28 and Definition 34 □

Corollary 40 (Weak associativity). *Every E-hypermap of the form $(N \boxtimes P) \boxtimes Q$ for some E-hypermaps N, P and Q with $\mathbf{E}_N \cap \mathbf{E}_P \cap \mathbf{E}_Q$ empty is either equal:*

– *to $N \boxtimes (P \boxtimes Q)$*
– *or to $(N \boxtimes Q) \boxtimes P$.*

Proof. Let M be a E-hypermap of the form $(N \boxtimes P) \boxtimes Q$ for some E-hypermaps N, P and Q with $\mathbf{E}_N \cap \mathbf{E}_P \cap \mathbf{E}_Q$ empty. Let e be the unique hyperedge of $(\mathbf{E}_N \cup \mathbf{E}_P) \cap \mathbf{E}_Q$. By Theorem 29 and by Corollary 39, the two cases $e \in \mathbf{E}_N$ and $e \in \mathbf{E}_P$ are symmetrical. Then, without loss of generality we can suppose $e \in \mathbf{E}_N$. Let d be the unique hyperedge of $\mathbf{E}_N \cap \mathbf{E}_P$. From Theorem 29, $M = (N \square Q) \square P$. By hypothesis, $\mathbf{E}_Q$ and $\mathbf{E}_Q\backslash e$ are connected in Q. By hypothesis, $\mathbf{E}_{N \boxtimes P}\backslash e$ is connected in $N \boxtimes P$. By Fact 38, $\mathbf{E}_N\backslash e$ is connected in N. By hypothesis, $\mathbf{E}_N$ is connected in N. Thus $N \boxtimes Q$ is defined. By hypothesis, $\mathbf{E}_N\backslash d$ is connected in N. By Fact 38, $\mathbf{E}_{N \boxtimes Q}\backslash d$ is connected in $N \boxtimes Q$. Thus $(N \boxtimes Q) \boxtimes P$ is defined, is trivially equal to $(N \square Q) \square P$, hence to M. □

Corollary 41 (Duality). *Every connected separation (N, P) verifies:*

$$(N \boxtimes P)^* = N^* \boxtimes P^*$$

Proof. Let (N, P) be a connected separation of some E-hypermap M. Trivially, it comes: $M = N \square P$. By Theorem 31, we have: $M^* = N^* \square P^*$. For every set of hyperedges D of some E-hypermap Q, the equality $\sigma_{Q^*}^{-1}(\mathbf{dart}_{Q^*}(D)) = \sigma_Q^{-1}(\mathbf{dart}_Q(D))$ implies that D is connected in Q if and only if D is connected in Q^*. Thus, it follows first, that, every E-hypermap is connected if and only if its dual is connected and, secondly, that every hyperedge is c-replaceable in some E-hypermap if and only it is c-replaceable in its dual. It follows that $M^* = N^* \boxtimes P^*$. □

Corollary 42. *The $\boxtimes$-closure of some set of E-hypermaps $\mathcal{A}$ which is closed under isomorphism is the set of all E-hypermaps of the form $((M_1 \boxtimes M_2) \ldots) \boxtimes M_n$ with $n \geq 1$ and $M_1, \ldots, M_n \in \mathcal{A}$.*

Proof. Immediate consequence of the Corollaries 39 and 40. □

The next three lemmas as the analogous to Lemmas 16, 17 and 18 on hypergraphs.

Lemma 43. *Let k be an integer. The set of all E-representations of connected graphs of treewidth $\leq k$ is the $\boxtimes$-closure of the set of all connected E-hypermaps having at most $k+1$ vertices.*

Proof. Let k be an integer. Let $\mathcal{A}$ be the set of all connected E-hypermaps having at most $k+1$ vertices. Let $\mathcal{B}$ be the $\boxtimes$-closure of $\mathcal{A}$. Let $\mathcal{C}$ be the set of all E-representations of connected graphs of treewidth $\leq k$. Let us prove $\mathcal{B} = \mathcal{C}$.

Let us prove: $\mathcal{B} \subseteq \mathcal{C}$. Let $M \in \mathcal{B}$ be a E-hypermap and G its induced hypergraph. By Fact 38, M is connected. By Lemma 35, G is connected. By Theorem 32, G belongs to the $\square$-closure of the set of all hypergraphs having at most $k+1$ vertices. By Lemma 17, its treewidth is at most k. Thus, $\mathcal{B} \subseteq \mathcal{C}$.

Let us prove: $\mathcal{C} \subseteq \mathcal{B}$. First, some definitions. The hypergraph obtained from a hypergraph (V, E, f) by *deleting* some subset $X \subseteq (V \cup E)$ is the hypergraph $(V - X, E - X, g)$ where g associates with every $e \in E - X$ the set $f(e) - X$. Clearly, this operation does not increase the treewidth.

For every graph $G = (V, E, f)$, for every subset $D \subseteq E$ and for every hyperedge e not in G, we denote by $G \uparrow_e D$ the hypergraph obtained from G by deleting $(E - D) \cup (V - f(D))$ and by adding the hyperedge e incident with all the vertices in $f(D) \cap f(E - D)$. The expression $G \downarrow_e D$ denotes the hypergraph $G \uparrow_e (E - D)$. Clearly, if G has no isolated vertex, we have: $G = (G \uparrow_e D) \square (G \downarrow_e D)$.

For every E-hypermap M, we denote by $|M|$ the integer $\mathbf{card}(\mathbf{V}_M)$. Suppose there is K such that $\mathcal{B}$ contains every E-representation M of some connected hypergraph of treewidth at most k such that $|M| \leq K$. Let M be a E-representation of some connected hypergraph G of treewidth $\leq k$ with $|M| = K + 1$.

Let (T, g) be a tree-decomposition of G of width $\leq k$ such that $n := \mathbf{card}(\mathbf{V}_T)$ is minimal. If $n = 1$, G and M have at most $k + 1$ vertices: trivially, $M \in \mathcal{B}$. Suppose $n > 1$. Let u be a leaf of T and t be the unique node incident with u in T. If we suppose $\mathbf{V}_{g(u)} \subseteq \mathbf{V}_{g(t)}$, the pair $(T \backslash u, h)$ where g associates with t the union $g(t) \cup g(u)$ and with every node s of $T - \{t, u\}$ the hypergraph $g(s)$, is a tree-decomposition of G of width $\leq k$ and having $n - 1$ nodes; contradiction. Thus, there is a vertex x in $\mathbf{V}_{g(u)} - \mathbf{V}_{g(t)}$. For same reasons, there is a vertex y in $\mathbf{V}_{g(t)} - \mathbf{V}_{g(u)}$. Let e be a hyperedge not in G. The hypergraphs $I := G \uparrow_e \mathbf{E}_{g(u)}$ and $J := G \downarrow_e \mathbf{E}_{g(u)}$ verify: $G = I \square J$, $\mathbf{card}(\mathbf{V}_I) \leq k + 1$, $\underline{twd}(J) \leq k$, $x \notin \mathbf{V}_J$ and $y \notin \mathbf{V}_I$.

Let E be a maximal subset of $\mathbf{E}_J \backslash e$ to be connected in M and that contains some hyperedge incident with y. By construction, $\mathbf{E}_I \backslash e$ contains no hyperedge incident with y. Thus, E contains all hyperedges incident with y. By construction, E contains no hyperedge incident with x. Let $F_1, \ldots, F_m$ be the maximal subsets of $\mathbf{E}_M - E$ to be connected in M where F_1 contains some hyperedge incident with x. For same reasons as above, F_1 contains all hyperedges incident in M with x. Let $F := E \cup F_2 \cup \ldots \cup F_m$. The E-hypermap M is connected, thus for every $i \in [m]$ the union $E \cup F_i$ is connected. It follows that $\{F, F_1\}$ is a partition of $\mathbf{E}_M$ into connected subsets in M of $\mathbf{E}_M$. Let $f_1, \ldots, f_m$ be m distinct hyperedges not in M and distinct with e. The E-hypermaps $N := M \downarrow_{f_1} F$ and $P := M \uparrow_{f_1} F$ are connected, verify $M = N \boxtimes P$. Thanks to $y \notin \mathbf{V}_N$ and $x \notin \mathbf{V}_P$, we have $|N| \leq K$ and $|P| \leq K$.

Let us prove $\{N, P\} \subseteq \mathcal{B}$ by establishing that the treewidth of their induced hypergraphs $I' := \underline{hgraph}(N)$ and $J' := \underline{hgraph}(P)$ is at most k. For every $i \in [m]$, the hypergraph $G_i := G \uparrow_{f_i} F_i$ is obtained from I by deleting all hyperedges not in $\{e\} \cup D_i$, by deleting the vertices not incident in M with some hyperedge of D_i and by renaming f_i the hyperedge e. Thus, for every $i \in [m]$, we have: $\underline{twd}(G_i) \leq \underline{twd}(I) \leq k$. The equality $G_1 = I' = \underline{hgraph}(N)$ implies: $\underline{twd}(\underline{hgraph}(N)) \leq k$. Let f be hyperedge not in M and distinct with all introduced above. The hypergraph

$G \uparrow_f E$ is obtained from J by deleting all hyperedges not in $\{e\} \cup E$, by deleting all vertices not incident in G (or in M) with some hyperedge in E and by renaming f the hyperedge e. It follows that: $\underline{twd}(G \uparrow_f E) \le \underline{twd}(J) \le k$. Let J_0 be the hypergraph obtained from $G \uparrow_f E$ by deleting f and by adding the m hyperedges $f_1, \ldots, f_m$ in such a way that for every $i \in [m]$, $\mathbf{vert}_{J_0}(f_i) = \mathbf{vert}_J(f) \cap \mathbf{V}_{G_i}$. It follows that for every $i \in [m]$, $\underline{twd}(J_0) \le \underline{twd}(J) \le k$. Moreover, we have: $\underline{hgraph}(P) = J' = (J_0 \square J_2) \ldots) \square J_m$. Lemma 17 implies $\underline{twd}(\underline{hgraph}(P)) \le k$.

Thus, N and P are two E-representations of two connected hypergraphs of treewidth $\le k$ and are of size $|N| \le K$ and $|P| \le K$. By induction hypothesis, N and P belong to $\mathcal{B}$. The equality $M = N \boxtimes P$ and Corollary 42 imply $M \in \mathcal{B}$. $\square$

Before to establish analogous lemmas with Lemmas 16 and 18, we have to define "basic" E-hypermaps. For hypergraphs, it is simple: a hypergraph is "basic" if it contains at most three hyperedges. For E-hypermaps, it is a little bit more complicated: for instance, the number of hyperedges is not bounded (see last figure of Example 45).

Definition 44 (Basic E-hypermaps). A connected E-hypermap M is *basic* if one of the two following conditions holds:

1. every hyperedge of M is c-replaceable, their number is at most 3.
2. there is a unique not c-replaceable hyperedge e such that every other hyperedge is c-replaceable and such that every pair in $\mathbf{E}_M \backslash e$ is not connected. $\square$

Example 45. Figure 7 represents 4 planar E-hypermaps N, P, Q and R. The first one contains 3 representative and c-replaceable (hyper)edges (the permutation α_N is represented by dotted lines). N contains 5 vertices (grey disks) and 6 faces. P contains a no c-replaceable edge (with two extremities) and two c-replaceable edges with one extremity. Q is the dual of N. R contains 4 hyperedges and 3 faces. The hyperedge represented by a box is the unique one not to be c-replaceable. $\square$

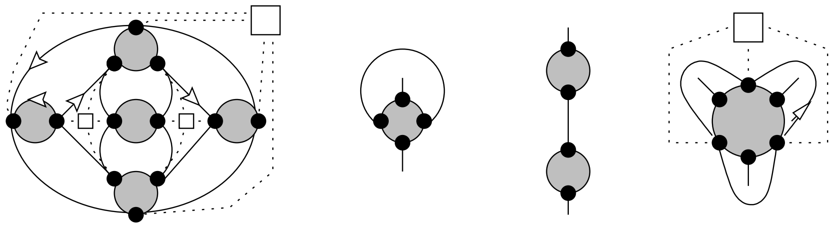

Fig. 7. Four basic E-hypermaps.

Lemma 46. *The set of all connected E-hypermaps is the $\boxtimes$-closure of the set of all basic E-hypermaps.*

Proof. Let $\mathcal{A}$ the set of all basic E-hypermaps and $\mathcal{B}$ the $\boxtimes$-closure of $\mathcal{A}$. By Fact 38, every E-hypermap of $\mathcal{B}$ is connected. Thus to conclude, it suffices to establish that every connected E-hypermap belongs to $\mathcal{B}$.

During this proof, the *size* of a E-hypermap M is $|M| := \mathbf{card}(\mathbf{E}_M) + \mathbf{card}(\mathbf{D}_M)$. Suppose there is an integer K such that $\mathcal{B}$ contains every connected E-hypermap of size $\le K$. Let M be a connected E-hypermap of size $K+1$. Different cases appear:

1. $\mathbf{card}(\mathbf{E}_M) \le 1$.
   M belongs obviously to $\mathcal{A} \subseteq \mathcal{B}$.

2. $\mathbf{card}(\mathbf{E}_M) \geq 2$, there is one hyperedge e in M not c-replaceable.
   Let $\{E_1, \ldots, E_m\}$ be the partition of the nonempty set $\mathbf{E}_M \backslash e$ into maximal subsets of $\mathbf{E}_M \backslash e$ to be connected in M. Let $f_1, \ldots, f_m$ be m distinct hyperedges not in M. Let $N := ((M \downarrow_{f_1} F_1) \ldots) \downarrow_{f_m} F_M$ and for every $i \in [m]$ let $P_i := M \uparrow_{f_i} F_i$. Clearly, we have: $M = ((N \square P_1) \ldots) \square P_m$. Let $\mathcal{L}$ the set that contains (N, f_1), every pair of the form (P_i, f_i) with $i \in [m]$ or of the form $(((N \square P_1) \ldots) \square P_i, f_{i+1})$ with $i \in [m-1]$. By hypothesis, M is connected and for every $i \in [m]$, E_i is connected in M. Fact 37 implies that for every $(L, l) \in \mathcal{L}$, l is c-replaceable in L. It comes: $M = ((N \boxtimes P_1) \ldots) \boxtimes P_m$.
   By Fact 38, for every $i \in [m]$, the set $\mathbf{E}_N \backslash f_i$ is connected in N. Thus, for every $i \in [m]$, f_i is c-replaceable in N (by construction, f_i is replaceable in N). For every $i \in [m]$, $\sigma_N^{-1}(\mathbf{dart}_N(f_i))$ is contained in $\sigma_M^{-1}(\mathbf{dart}_M(E_i)) - \mathbf{dart}_M(E_i)$ that is contained in $\mathbf{dart}_M(e) = \mathbf{dart}_N(e)$, because E_i is a maximal subset of $\mathbf{E}_N \backslash e$ to be connected in M. Thus, no pair of $\mathbf{E}_N \backslash e$ is connected.
   For same reasons, if $m \geq 2$, $\mathbf{E}_N \backslash e$ is not connected, thus, e is not c-replaceable in N: N belongs to $\mathcal{A} \subseteq \mathcal{B}$. If $m = 1$, by hypothesis, $\mathbf{E}_M \backslash e$ is connected in M; thus, e is not replaceable in M (it would be replaceable, it would be c-replaceable: contradiction): thus, e is not replaceable in N and is not c-replaceable in N. Hence, in the two cases, $N \in \mathcal{A} \subseteq \mathcal{B}$.
   Let $i \in [m]$. If e is not replaceable in M (or in N), $\mathbf{D}_{P_i}$ does not contain the nonempty subset $\mathbf{dart}_M(e) \cap \sigma_M^{-1}(\mathbf{dart}_M(e))$. If not, we have $m \geq 2$ and $\mathbf{D}_{P_i}$ does not contain the nonempty subset $\mathbf{dart}_M(E_1) \cup \mathbf{dart}_M(E_2)$. In the two cases, we have $|P_i| \leq K$. By induction hypothesis, $P_1, \ldots, P_m$ belong to $\mathcal{B}$.
   Thus, $M = ((N \boxtimes P_1) \ldots) \boxtimes P_m$ belongs to $\mathcal{B}$.
3. $\mathbf{card}(\mathbf{E}_M) \geq 2$, all hyperedges of M are c-replaceable.
   If $\mathbf{card}(\mathbf{E}_M) \leq 3$, M belongs trivially to $\mathcal{A}$. Suppose now $\mathbf{card}(\mathbf{E}_M) \geq 4$. Clearly, there are two hyperedges a and b such that $\{a, b\}$ is connected in M. Let $(E_1, \ldots, E_m)$ be the partition of $\mathbf{E}_M - \{a, b\}$ into maximal subsets of $\mathbf{E}_M - \{a, b\}$ to be connected in M. By hypothesis, a and b are c-replaceable. Thus, $\mathbf{E}_M \backslash a$ and $\mathbf{E}_M \backslash b$ are connected in M. Thus, for every $i \in [m]$, the unions $E_i \cup \{b\}$ and $E_i \cup \{a\}$ are connected in M. It follows that $(A, B, C) := (E_1, \{a\}, (\mathbf{E}_M \backslash a) - E_1)$ is a partition of $\mathbf{E}_M$ into connected sets in M such that $A \cup B$, $A \cup C$ and $B \cup C$ are connected in M.
   Let d, e two hyperedges not in M. Let $N := (M \downarrow_d A) \downarrow_e C$, $P := M \uparrow_d A$ and $Q := M \downarrow_e C$. It comes: $M = (N \square P) \square Q$. By Fact 37 and 38, N, P and Q are connected and verify $M = (N \boxtimes P) \boxtimes Q$. By construction, N (resp. P, Q) has at most $\mathbf{card}(\mathbf{E}_M) - 1$ hyperedges, hence its size is at most K. By induction hypothesis, N, P and Q belong to $\mathcal{B}$. It comes: $M = (N \boxtimes P) \boxtimes Q \in \mathcal{B}$. □

Lemma 47. *Let k be an integer. The set of all connected E-representations of connected graphs of treewidth $\leq k$ is the $\boxtimes$-closure of the set of all basic E-hypermaps having at most $k+1$ vertices.*

Proof. Let k be an integer. From Lemma 46, the set of all connected E-hypermaps having at most $k+1$ vertices is the $\boxtimes$-closure of the set of all basic E-hypermaps having at most $k+1$ vertices. Corollary 42 and Lemma 43 suffice to conclude. □

Now, let us recall Euler's formula that gives the genus of the oriented surface induced by some connected hypermap. We extend this formula for E-hypermaps: every hyperedge that is not representative should be viewed as a the representation of an open surface of genus $c-1$ where c is the number of cyclic permutations related with this hyperedge.

Definition 48 (Genus). The *genus* of some combinatorial hypermap $C = (D, \sigma, \alpha)$ is the integer $genus(C) := 1 + \frac{1}{2}(\mathbf{card}(D) - (z(\alpha) + z(\sigma) + z(\alpha^{-1}\sigma)))$.

The *genus* of a E-hypermap M, denoted by $genus(M)$, is the integer $z(\alpha_M) - \mathbf{card}(\mathbf{E}_M) + genus(\mathbf{D}_M, \sigma_M, \alpha_M)$. A connected E-hypermap is *planar* if its genus is null (note that a planar E-hypermap is necessarily a hypermap). A hypergraph is *planar* if it admits a planar representation. $\square$

Observe that from the above definition, a hypergraph (V, E, f) is planar (resp. of genus $\leq k$) if and only if its 2-partitioned graph $(V \cup E, \{(v, e) \in V \times E \mid v \in f(e)\}$ is planar (resp. of genus $\leq k$).

Definitions of oriented surface and genus of them are out of the scope of this article. For more details, the reader is invited to see for instance [3] that contains in particular the following and basic result:

Theorem 49. *The genus of every connected combinatorial hypermap is a natural number equal to the genus of smallest genus of the oriented surfaces in which the hypermap is embeddable.*

Lemma 50. *The genus of every connected E-hypermap is a natural number.*

Proof. Let M be a E-hypermap. Let n be the number of the connected components of the combinatorial hypermap $(\mathbf{D}_M, \sigma_M, \alpha_M)$. Theorem 49 implies $1 - n \leq genus(\mathbf{D}_M, \sigma_M, \alpha_M)$. Without difficulty, we establish that the connectivity of M implies $n - 1 \leq z(\alpha_M) - \mathbf{card}(\mathbf{E}_M)$. That suffices to conclude. $\square$

Now let us show why $\boxtimes$ preserves planarity. It is a direct consequence of two facts. The first, given by an equation, relates the genus of some split hypermap with the sum of the genus of its components. The second claims that the connected split-decomposition does not increase the genus.

Theorem 51. *Every separation (N, P, e, I) verifies:*

$$genus(N \square P) + genus(I) = genus(N) + genus(P)$$

Moreover, if (N, P, e, I) is connected, we have:

$$0 \leq genus(I) \leq genus(N) \leq genus(N \boxtimes P)$$

Proof. Let (N, P, e, I) be a separation of some E-hypermap M. Recall that for every E-hypermap L, we have: $\mathbf{card}(V_L) = z(\sigma_L)$ and $\mathbf{card}(V_L^*) = z(\beta_L)$. As a consequence of Definition 27, we have $\mathbf{card}(\mathbf{E}_M) + \mathbf{card}(\mathbf{E}_I) = \mathbf{card}(\mathbf{E}_N) + \mathbf{card}(\mathbf{E}_P)$, $\mathbf{card}(\mathbf{D}_M) + \mathbf{card}(\mathbf{D}_I) = \mathbf{card}(\mathbf{D}_N) + \mathbf{card}(\mathbf{D}_P)$, $z(\sigma_M) + z(\sigma_I) = z(\sigma_N) + z(\sigma_P)$, $z(\beta_M) + z(\beta_I) = z(\beta_N) + z(\beta_P)$, $z(\alpha_M) + z(\alpha_I) = z(\alpha_N) + z(\alpha_P)$. All these equalities suffice to establish: $genus(M) + genus(I) = genus(N) + genus(P)$.

To treat the connected case, we establish a preliminary result (Point 9). For every E-hypermap M and for every dart a of M such that a and $\sigma_M(a)$ are extremities of the same hyperedge, we denote by $M \backslash a$ the tuple $(D, \mathbf{E}_M, \mathbf{dart}, \sigma_M : D, (\sigma_M : D)(\beta_M : D)^{-1})$ where $D := \mathbf{D}_M \backslash a$ and where $\mathbf{dart}(e) = D \cap \mathbf{dart}_M(e)$ for every hyperedge e. Let N be a tuple of the form $M \backslash a$ for some dart a of M. We have:

1. $\forall b \notin \{a, \alpha_M^{-1}(a), \alpha_M^{-1}(\sigma_M(a))\}$, $\alpha_N(b) = \alpha_M(b)$.
   Let $b \notin \{a, \alpha_M^{-1}(a), \alpha_M^{-1}(\sigma_M(a))\}$. Clearly, we have: $\beta_N^{-1}(b) = \beta_M^{-1}(b)$. By hypothesis, $\sigma_M(\beta_M^{-1}(b)) = \alpha_M(b) \neq a$. Thus, $\sigma_N(\beta_N^{-1}(b)) = \sigma_M(\beta_M^{-1}(b))$, hence, $\alpha_N(b) = \alpha_M(b)$.
2. N is a E-hypermap.
   By hypothesis, all darts in $\{a, \alpha_M^{-1}(a), \alpha_M^{-1}(\sigma_M(a))\}$ are extremity of the same hyperedge e. By precedent point, every extremity b of a distinct hyperedge with e, verifies $\alpha_N(b) = \alpha_M(b)$. Thus every set of extremity of every hyperedge is closed under α_N. N is a E-hypermap.

3. $z(\alpha_N) - z(\alpha_M) \leq 1$.
   A clear consequence of Point 1.
4. $z(\alpha_N) - z(\alpha_M) \leq 0$ if $\sigma_M(a) \notin [a]_{\alpha_M}$.
   A clear consequence of Point 1.
5. $genus(N) \leq genus(M)$.
   Let $\Delta := \frac{1}{2}\cdot(-1+z(\alpha_N)-z(\alpha_M)+z(\sigma_M)-z(\sigma_N)+z(\beta_M)-z(\beta_N))$. Definition 48 implies $genus(N) = \Delta + genus(M)$. As a consequence of Lemma 50, Δ is an integer. Thus to conclude, it suffices to establish $\Delta \leq \frac{1}{2}$. Different cases appear:
   - $\sigma_M(a) = a = \beta_M(a)$.
     It comes $\alpha_N(a)$. Thus $\Delta = \frac{1}{2}\cdot(-1-1+1+1) = 0$.
   - $\sigma_M(a) \neq a$.
     It comes $z(\sigma_N) = z(\sigma_M)$. Thus, $\Delta \leq \frac{1}{2}\cdot(-1+1+0+1) \leq \frac{1}{2}$.
   - $a \neq \beta_M(a)$.
     Symmetrical case with the precedent one.
6. $genus(N) \leq genus(M) - 1$ if $a \neq \sigma_M(a)$, $a \neq \beta_M(a)$ and $\sigma_M(a) \notin [a]_{\alpha_M}$.
   Clearly, we have: $z(\sigma_N) = z(\sigma_M)$, $z(\beta_N) = z(\beta_M)$. By Point 4, $z(\sigma_N) \leq z(\sigma_M)$. By using same notations as above, it comes $\Delta \leq -\frac{1}{2}$, hence $\Delta \leq -1$.

Let N be a E-hypermap of the form $M \downarrow_d D$ with M a E-hypermap, $D \subseteq \mathbf{E}_M$ a connected set and $d \notin \mathbf{E}_M - D$ a hyperedge. In order to simplify, we suppose $d \in D$. We have:

7. $genus(N) \leq genus(M)$ if D is a singleton.
   Clearly, N is $((M\backslash a_1)\ldots)\backslash a_n$ where $\{a_1,\ldots,a_n\}$ is the set $\mathbf{dart}_M(D) - \sigma_M^{-1}(\mathbf{dart}(\mathbf{E}_M - D))$. Point 5 suffices to conclude.
8. $genus(N) \leq genus(M)$ if D is a connected pair of hyperedges.
   Let $D = \{d, e\}$ be a connected pair of hyperedges. Let $P := M \downarrow_e \{e\} \downarrow_d \{d\}$. It comes $N := P \downarrow_d \{d, e\}$ and by precedent point, $genus(P) \leq genus(M)$. Let $Q := (\mathbf{D}_P, \mathbf{E}_N\backslash e, \mathbf{dart}, \sigma_P, \alpha_P)$ where $\mathbf{dart}$ associates with every hyperedge $f \in \mathbf{E}_N - \{d, e\}$ the set $\mathbf{dart}_P(f)$ and with d the union $\mathbf{dart}_P(d) \cup \mathbf{dart}_P(e)$. Clearly, $N := Q \downarrow_d d$ and $genus(Q) = genus(P) + 1 \leq genus(M) + 1$.
   By Fact 38, $\{d, e\}$ is connected in P. By definition of a E-hypermap, $\mathbf{dart}_P(e)$ and $\mathbf{dart}_P(e)$ are disjoint. Thus there is a dart $a \in \mathbf{dart}_P(d)$ such that $\sigma_P(a) \in \mathbf{dart}_P(e)$. By construction, we have $\sigma_P(a) \neq a$ and $\beta_P(a) \neq a$. By Point 6, we have: $genus(Q\backslash a) \leq genus(Q) - 1 \leq M$. Let $\{a_1,\ldots,a_m\}$ be the set $\mathbf{dart}_Q(d) - \sigma_Q^{-1}(\mathbf{dart}_Q(\mathbf{E}_Q\backslash d))$ with $a = a_1$. Clearly, we have: $N = ((Q\backslash a_1)\ldots)\backslash a_m$. Point 5 implies $genus(N) \leq genus(Q\backslash a_1) \leq genus(M)$.
9. $genus(N) \leq genus(M)$ if D is connected.
   Clearly, D is of the form $\{d_1,\ldots,d_m\}$ with $d = d_1$ and where for every $i \in [m]$ the set $\{d_1,\ldots,d_i\}$ is connected in M. The equality $N = (M \downarrow_d \{d, d_2\}\ldots) \downarrow_d \{d, d_m\}$, Fact 38 and precedent point imply $genus(N) \leq genus(M)$.

Let (N, P, e, I) be a connected separation of some E-hypermap M. By hypothesis $\mathbf{E}_P\backslash e$ is connected in P, hence by Fact 38 is connected in M. The equality $N = M \downarrow_e \mathbf{E}_P\backslash e$ and Point 9 implies $genus(N) \leq genus(M)$. By hypothesis, $\mathbf{E}_N\backslash e$ is connected in N. The equality $I = M \uparrow_e e$, Point 9 and Lemma 50 imply $0 \leq genus(I) \leq genus(N)$. Symmetrically, it comes $0 \leq genus(I) \leq genus(P) \leq genus(M)$. □

Corollary 52. *For every connected separation (N, P), $N \boxtimes P$ is planar if and only if N and P are planar.*

Proof. A direct consequence of the precedent theorem. □

Example 53. Figure 8 represents a connected separation (N, P, e) of a planar map M. The hyperedge e is represented by a box. □

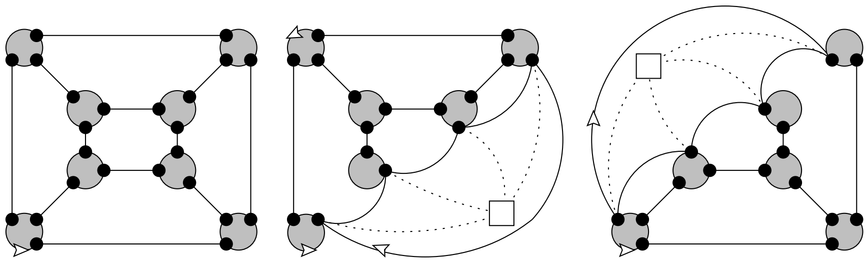

Fig. 8. Connected splitting preserves planarity.

A little but useful fact: it is required to prove Theorems 55, 71 and 77).

Fact 54. *Every basic E-hypermap M with all its hyperedges c-replaceable contains:*

1. *at least* $\mathbf{card}(\mathbf{V}_N) + \mathbf{card}(\mathbf{V}_N^*)$ *darts.*
2. *at most* $\mathbf{card}(\mathbf{V}_N) + 2 \cdot genus(M) + 1$ *distinct faces.*
3. *at most* $2 \cdot \mathbf{card}(\mathbf{V}_N) + 4 \cdot genus(M) + 2$ *distinct darts.*
4. *at most* $\mathbf{card}(\mathbf{V}_N) + \mathbf{card}(\mathbf{V}_N^*) + 2 \cdot genus(M) + 1$ *distinct darts.*

Proof. Let M be a basic E-hypermap with all its hyperedges c-replaceable with n vertices, m hyperedges, f faces and d darts. Let $e := z(\alpha_M)$. It comes:

a. $2 \cdot n \leq d$.
   By hypothesis, every hyperedge of M is replaceable. Thus, every dart a is distinct with $\sigma_M(a)$. Thus, every vertex contains at least two distinct darts
b. $2 \cdot f \leq d$.
   Similar argument as the above ones.
c. $e + d = n + f + 2 \cdot genus(M) - 2 + 2 \cdot m$.
   By definition of the genus of M.
d. $m \leq e$.
   Because the extremity of every hyperedge define proper isomorphic classes $[b]_{\alpha_M}$.
e. $m \leq 3$.
   By hypothesis.

   Then, to conclude, it suffices to observe:
1. $n + f \leq d$.
   By adding precedent inequalities as follow: $\frac{1}{2}((a.) + (b.))$.
2. $f \leq n + 2 \cdot genus(M) + 1$.
   By adding precedent inequalities as follow: $((b.) + (c.) + (d.) + (e.))$.
3. $d \leq 2 \cdot n + 4 \cdot genus(M) + 2$.
   By adding precedent inequalities as follow: $((b.) + 2 \cdot (c.) + 2 \cdot (d.) + 2 \cdot (e.))$.
4. $d \leq n + f + 2 \cdot genus(M) + 1$.
   By adding precedent inequalities as follow: $(b.) + (c.) + (d.)$. □

The main result of this section is to compare the treewidth of two dual hypergraphs embedded on any surface. A first corollary (see Corollary 56) is to prove Robertson and Seymour's conjecture that concerns the sphere.

For every E-hypermap M, we denote by $\delta(M)$ the maximum $\delta_M(d)$ taken over all hyperedges d of M, where $\delta_M(d)$ denotes the number of distinct vertices incident with d in M (if $\mathbf{E}_M = \emptyset$, we define $\delta(M) := 0$).

Theorem 55. *Let M be a E-hypermap, G its induced hypergraph and G^* the induced hypergraph of M^*. We have:*

$$\underline{twd}(G^*) \leq \max\{\underline{twd}(G) + 2 \cdot genus(M) + 1, \delta(M^*) - 1\}$$

Proof. First, let us consider the connected case. Let M be a connected E-hypermap, G its induced hypergraph and G^* the induced hypergraph of M^*. Let k be the treewidth of G. By Lemma 47, M is of the form $((M_1 \boxtimes M_2)\ldots)\boxtimes M_m$ for some $m \geq 1$ and some basic E-hypermaps $M_1,\ldots,M_m$ having at most $k+1$ vertices. By Corollary 41, $M^* = ((M_1^* \boxtimes M_2^*)\ldots)\boxtimes M_m^*$. By Lemma 47, to conclude the connected case it suffices to prove that every E-hypermap M_i^* (resp. M_i) with $i \in [m]$ has at most $\max\{k+2\cdot genus(M)+2, \delta(M^*)\}$ vertices (resp. faces). Let $L \in \{M_1,\ldots,M_m\}$. Two cases appear:

- L is basic with all its hyperedges c-replaceable.
  Theorem 51 implies $genus(L) \leq genus(M)$. By Fact 54, L has at most $k+1+2.genus(M)+1$ faces.
- L is basic and contains some hyperedge e not c-replaceable.
  By hypothesis, e is not c-replaceable. Thus, e belongs to M (resp. M^*). It follows that $\delta_{M^*}(e) \leq \delta(M^*)$.
  Let $f \subseteq \mathbf{D}_M$ be a face of M and d be a dart of f. If we suppose f not incident with e in L, d is some extremity of some hyperedge $f \in \mathbf{E}_L \backslash e$, $\beta_L^{-1}(d)$ does not belong to $\mathbf{dart}_L(f)$ because f is replaceable, does not belong to $\mathbf{dart}_L(\mathbf{E}_N \backslash e)$ because no pair of $\mathbf{E}_N \backslash e$ is connected (L is basic), hence, $\beta_L^{-1}(d)$ belongs to $\mathbf{dart}_L(e)$. Contradiction. Thus, every face of L is incident with e. Hence, L has at most $\delta(M^*)$ distinct faces.

Suppose now M not connected. For every pair of E-hypermaps N and P with disjoint domains ($(\mathbf{D}_N \cup \mathbf{E}_N) \cap (\mathbf{D}_P \cup \mathbf{E}_P) = \emptyset$), we denote by $N \cup P$ the E-hypermap $(\mathbf{D}_N \cup \mathbf{D}_P, \ldots, \alpha_N \cup \alpha_P)$. Clearly, such E-hypermaps N and P verify:

- $(N \cup P)^* = N^* \cup P^*$.
- $genus(N \cup P) + 1 = genus(N) + genus(P)$.
- $\underline{twd}(\underline{hgraph}(N \cup P)) = \max\{\underline{twd}(\underline{hgraph}(N)), \underline{twd}(\underline{hgraph}(P))\}$.
- $\delta(N \cup P) = \max\{\delta(N), \delta(P)\}$.

Clearly, M is of the form $((M_1 \cup M_2)\ldots)\cup M_m$ where $M_1,\ldots,M_m$ are connected. All above equalities and the inequality $\underline{twd}(\underline{hgraph}(M_i^*)) \leq \max\{\underline{twd}(\underline{hgraph}(M_i)) + 2\cdot genus(M_i)+1, \delta(M_i^*)-1\}$ for every $i \in [m]$ suffice to conclude. □

Corollary 56. *Let G and G^* be the induced graphs of two planar and mutually dual maps. We have:*

$$\underline{twd}(G) - 1 \leq \underline{twd}(G^*) \leq \underline{twd}(G) + 1$$

Proof. A consequence of Theorem 55 and the fact that every map M verifies $\delta(M^*) \leq 2$. □

To conclude this section, we show and prove in a few lines how to produce all planar representations of planar hypergraphs of treewidth at most k.

Theorem 57. *Let k be an integer. The set of all planar representations of connected hypergraphs of treewidth $\leq k$ is the $\boxtimes$-closure of the set of all planar and basic hypermaps having at most $k+1$ vertices.*

Proof. Let k be some integer. Let $\mathcal{B}$ be the set of all planar and basic hypermaps having at most $k+1$ vertices. Let M be a E-hypermap of the form $((M_1 \boxtimes M_2)\ldots)\boxtimes M_l$ for some $l \geq 0$ and some $M_1,\ldots,M_l \in \mathcal{B}$. By Lemma 47, M is connected, the treewidth of its induced hypergraph is at most k. By Theorem 51, $0 \leq genus(M) \leq \sum_{i\in[l]} genus(M_i) \leq 0$, hence, M is planar, connected and is a hypermap.

Let M be a planar representation of some connected hypergraph of treewidth $\leq k$. Trivially, M is connected. By Lemma 47, M is of the form $((M_1 \boxtimes M_2)\dots)\boxtimes M_l$ for some $l \geq 0$ and some basic E-hypermaps $M_1, \dots, M_l \in \mathcal{B}$ having at most $k+1$ vertices. By Theorem 51, for every $i \in [l]$ we have $0 \leq genus(M_i) \leq genus(M) \leq 0$, hence, $genus(M_i) = 0$. Thus M_i is planar. That suffices to conclude. □

The right side inequality of Theorem 55 is reached for planar hypermaps: the two dual and planar graphs, that are the complete graph with 3 vertices and the graph with two vertices and 3 edges, are of treewidth respectively 3 and 2. For general surfaces, it is open. In next example, we present two dual graphs according some no-planar embedding of treewidth respectively 0 and n.

Example 58. For every odd integer $n \geq 3$, let G_n be the grid-graph $n \times n$ and H_n be the connected graph with one vertex and with $2 \cdot (n-1) \cdot n$ edges. Let L_n be the planar grid-map $n \times n$ (that is unique up isomorphism) and let M_n be the non-planar map obtained from L_n by defining $\sigma_{M_n} := (\sigma_{L_n} : I) \cup (\sigma_{L_n}^{-1} : \mathbf{D}_{L_n} - I)$ where I is the set of darts which belong to vertices of "even" columns (see Figure 9). Clearly, M_n is a representation of G_n and M_n^* of H_n (M_n has a unique face). Hence, the two (planar) graphs G_n and H_n, of treewidth respectively n and 0, are "dual" w.r.t the non-planar embedding M_n of genus $\frac{1}{2} \cdot (n-1)^2$.

In Figure 9, are drawn the E-hypermaps M_3, at the left, and M_3^*, at the right (the permutation $\sigma_{M_3^*} := \beta_{M_3}$ is drawn by using one dotted line). □

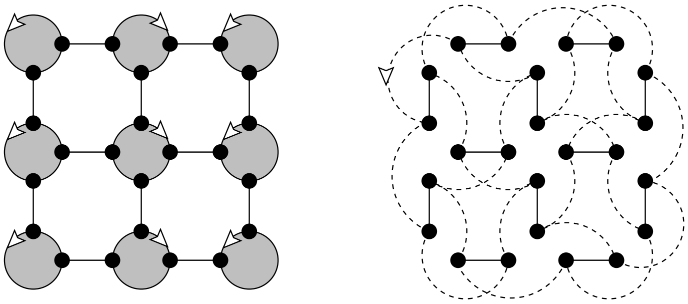

Fig. 9. Two dual graphs of treewidth n and 0.

4 Hypermap hyperedge-replacement grammars

In this section, from our splitting operation, we define an algebra of hypermaps. It follows a notion of hypermap hyperedge-replacement grammars (or equation system) which are a generalization of hypergraph hyperedge-replacement grammars defined by Courcelle [5].

Such formalism permits for instance to extend Theorem 57 by proving by Theorem 71 that the representations of genus g of graphs of treewidth $\leq k$ is equational.

Definition 59 (sort). A *sort* is a combinatorial hypermap $s = (\sigma, \alpha)$ with domain a set of the form $[-n] \cup [n]$ for some $n \geq 0$. Their set is denoted by $\mathcal{S}$. In order to simplify some definitions, we denote for every such pair s by s^+ the tuple $(\sigma : [n], \alpha : [n], \alpha^{-1}\sigma : [n])$ and by s^- the tuple $(\sigma : [-n], \alpha : [-n], \alpha^{-1}\sigma : [-n])$. □

Let U be a fixed ordered set of *labels*, each of them associated thanks **sort** with a sort in $\mathcal{S}$.

Definition 60 (abstract E-hypermap). A *concrete* E-hypermap M is a tuple $(D, E, f, \sigma, \alpha, d, l)$ denoted by $(\mathbf{D}_M, \mathbf{E}_M, \mathbf{dart}_M, \sigma_M, \alpha_M, \$_M, \mathbf{lab}_M)$ where:

- $(D, E, f, \sigma, \alpha)$ verifies the conditions of Definition 19, except that f associates with every hyperedge $e \in E$ not a set of darts but a sequence of distinct darts.
- d is a replaceable hyperedge (see Definition 21).
- l is a partial mapping $E \backslash d \to U$ that associates with certain replaceable hyperedge e its *label*. This label must verify: $\mathbf{sort}(l(e))^+ = \phi^{-1}(\sigma : A, \alpha : A, \alpha^{-1}\sigma : A)\phi$ where A is the set $f(e)$ and where ϕ is the bijection that associates with every integer $i \in [\mathbf{card}(A)]$ the dart $f(e, i)$.

With every concrete E-hypermap M, we associate its *sort*, denoted by $\mathbf{sort}(M)$, that is a sort $s \in \mathcal{S}$ verifying $s^- = \phi^{-1}(\sigma_M : A, \sigma_M : A, \beta_M : A)\phi$ where A is the set $\mathbf{dart}_M(\$_M)$ and where ϕ is the bijection that associates with every $i \in [-\mathbf{card}(A)]$ the dart $\mathbf{dart}_M(\$_M, -i)$. To be precise, a concrete E-hypermap is such a tuple plus its sort.

An *abstract* E-hypermap is an isomorphic class of concrete E-hypermaps (of same sort). In a natural way, we define for every abstract E-hypermap its *sort*. For every sort $s \in \mathcal{S}$, we denote by $\mathbf{M}_s$ the set of all abstract E-hypermaps of sort s and for every $s = (s_1, \ldots, s_m) \in \mathcal{S}^m$ we let $\mathbf{M}_s := \mathbf{M}_{s_1} \times \ldots \times \mathbf{M}_{s_m}$. □

Now, we have to define how to substitute some hyperedge in some E-hypermap with some E-hypermap. Such a substitution will be defined by deriving the splitting.

Definition 61 (hyperedge replacement). Let N and P be two concrete E-hypermaps. Let $e \neq \$_N$ be an hyperedge of N such that $\mathbf{sort}(\mathbf{lab}_N(e)) = \mathbf{sort}(P)$. Moreover, we suppose that N and P verifies the following properties (if not, we take two isomorphic copies of N and P defined in some deterministic way; for technical reasons, we suppose moreover that during these eventual isomorphic copies hyperedges of P may be renamed but not those of N):

- for every $i \in [n]$, $\mathbf{dart}_N(e, i) = i$.
- for every $i \in [n]$, $\mathbf{dart}_P(\$_P, i) = -i$.
- $\mathbf{sort}(\mathbf{lab}_N(e)) = (\sigma_N : A, \alpha_N : A)$ where $A := [n] \cup \sigma_N^{-1}([n])$.
- $\mathbf{sort}(P) = (\sigma_P : B, \alpha_P : B)$ where $B := [-n] \cup \sigma_P^{-1}([-n])$.
- $\mathbf{E}_N \cap \mathbf{E}_P = \{e\} = \{\$_P\}$.
- $\mathbf{D}_N \cap \mathbf{D}_P = [-n] \cup [n]$.
  where n is number of extremities of e in N.

The concrete E-hypermap obtained by *replacing in N the hyperedge e by P*, denoted by $N[e/P]$, is $(D, E, f, \sigma, \alpha, \$_N, g)$ where $(D, E, f, \sigma, \alpha)$ is $(\mathbf{D}_N, \mathbf{E}_N, \mathbf{dart}_N, \sigma_N, \alpha_N) \Box (\mathbf{D}_P, \mathbf{E}_P, \mathbf{dart}_P, \sigma_P, \alpha_P)$ and where g is the union of $\mathbf{lab}_P$ and the restriction of $\mathbf{lab}_N$ on $\mathbf{E}_N \backslash e$. We extend this definition in the following way: for every distinct hyperedges $e_1, \ldots, e_l$ of N and for every E-hypermaps $P_1, \ldots, P_l$ of sorts respectively $\mathbf{sort}(\mathbf{lab}_N(e_1)), \ldots, \mathbf{sort}(\mathbf{lab}_N(e_l))$, we define $N[e_1/P_1, \ldots, e_l/P_l]$ as the concrete E-hypermap $(\ldots(N[e_1/P_1])\ldots)[e_l/P_l]$.

Let u be a label of sort $\mathbf{sort}(u) = \mathbf{sort}(P)$ labelling at most one hyperedge f of N. The term $N[u/P]$ denotes the abstract E-hypermap N if no hyperedge of N is labelled u and $N[f/P]$ if not. If $N, P_1, \ldots, P_l$ denote $l+1$ concrete E-hypermaps and if $u_1, \ldots, u_l$ denote l labels such that for every $i \in [l]$ the sort of u_i is $\mathbf{sort}(P_i)$ and at most one hyperedge of N is labelled by u_i, we denote by $N[u_1/P_1, \ldots, u_l/P_l]$ the concrete E-hypermap defined recursively as $(N[u_1/P_1, \ldots, u_{n-1}/P_{n-1}])[e_l/P_l]$ if u_l is the label of some hyperedge e_l in N and $N[u_1/P_1, \ldots, u_{l-1}/P_{l-1}]$ if not. The two precedent definitions are extended in a natural way to abstract E-hypermaps. □

Definition 62 (A E-hypermap algebra). Let N be a *linear* abstract E-hypermap, i.e, such that $\mathbf{lab}_N$ is injective. The *arity* of N, denoted by $\mathbf{arity}(N)$, is the sequence $(\mathbf{sort}(u_1), \ldots, \mathbf{sort}(u_m))$ where $(u_1, \ldots, u_m)$ is the ordered set of labels of $\mathbf{lab}_M(\mathbf{E}_M)$. We denote by $\Box_N$ the mapping $\mathbf{M}_{\mathbf{arity}(N)} \to \mathbf{M}_{\mathbf{sort}(N)}$ that associates with every $(P_1, \ldots, P_m) \in \mathbf{M}_{\mathbf{arity}(N)}$ the abstract E-hypermap $N[u_1/P_1, \ldots, u_m/P_m]$ where $(u_1, \ldots, u_m)$ denotes the ordered set $\mathbf{lab}_N(\mathbf{E}_N)$.

Now, let us associate to this set of operations a $\mathcal{S}$*-signature*, i.e a set $\mathcal{F}$ given with two mappings the *arity mapping* $\mathbf{arity} : \mathcal{F} \to \mathcal{S}^*$ and the *sort mapping* $\mathbf{sort} : \mathcal{F} \to \mathcal{S}$. Furthermore, we suppose fixed such a $\mathcal{S}$-signature $\mathcal{F}$. For every symbol $f \in \mathcal{F}$, we denote by $f_{\mathbf{M}}$ the operation $\mathbf{M}_{\mathbf{arity}(f)} \to \mathbf{M}_{\mathbf{sort}(f)}$, which is supposed to be equal to $\Box_N$ for some abstract E-hypermap N. The $\mathcal{S}$-algebra $< (\mathbf{M}_s)_{s\in\mathcal{S}}, (f_{\mathbf{M}})_{f\in\mathcal{F}} >$ is denoted by $\mathbf{M}$. $\Box$

Now, we can define E-hypermaps' grammars, or in an equivalent way, equational sets of E-hypermaps (for more details, see [2, 5]).

Definition 63 (Equational sets). First, we define a new signature $\mathcal{F}_+$ that is the $\mathcal{S}$-signature $\mathcal{F}$ enlarged for every sort $s \in \mathcal{S}$ with a new symbol $+_s$ of type $s \times s \to s$ and with a new constant Ω_s of sort $\to s$. The power-set algebra of $\mathbf{M}$, denoted by $\mathcal{P}(\mathbf{M})$ is the pair $< (\mathcal{P}(\mathbf{M}_s))_{s\in\mathcal{S}}, (f_{\mathcal{P}(\mathbf{M})})_{f\in\mathcal{F}_+} >$ where:

- $A +_{s\mathcal{P}(\mathbf{M})} B := A \cup B$ for every $s \in \mathcal{F}$ and every $A, B \in \mathcal{P}(\mathbf{M}_s)$.
- $f_{\mathcal{P}(\mathbf{M})}(A_1, \ldots, A_k) := \{f_{\mathbf{M}}(a_1, \ldots, a_k) \mid a_1 \in A_1, \ldots, a_k \in A_k\}$ for $A_1 \subseteq \mathbf{M}_{s_1}, \ldots, A_k \subseteq \mathbf{M}_{s_k}$ where $\mathbf{arity}(f) = (s_1, \ldots, s_k)$.
- $\Omega_{s\mathcal{P}(\mathbf{M})} := \emptyset$.

A *polynomial system* over $\mathcal{F}$ is a sequence of equations $S =< v_1 = p_1, \ldots, v_n = p_n >$, where $V = \{v_1, \ldots, v_n\}$ is an $\mathcal{S}$-sorted set of variables called the set of *unknowns* of S. Each term p_i is a *polynomial*, i.e, a term of the form Ω_s or $t_1 +_s \ldots +_s t_m$ where the terms t_j are monomials of sort $s = \mathbf{sort}(v_i)$. A *monomial* is an algebraic term defined from the signature $V \cup \mathcal{F}$.

A mapping $S_{\mathcal{P}(\mathbf{M})}$ of $\mathcal{P}(\mathbf{M}_{\mathbf{sort}(v_1)}) \times \ldots \times \mathcal{P}(\mathbf{M}_{\mathbf{sort}(v_n)})$ into itself is associated with S and $\mathbf{M}$ as follows: for $A_1 \subseteq \mathbf{M}_{\mathbf{sort}(v_1)}, \ldots, A_n \subseteq \mathbf{M}_{\mathbf{sort}(v_n)}$, we let $S_{\mathcal{P}(\mathbf{M})}(A_1, \ldots, A_n) := (p_{1\mathcal{P}(\mathbf{M})}(A_1, \ldots, A_n), \ldots, p_{n\mathcal{P}(\mathbf{M})}(A_1, \ldots, A_n))$. A *solution* of S in $\mathcal{P}(\mathbf{M})$ is an n-tuple $(A_1, \ldots, A_n)$ such that $A_i \subseteq \mathbf{M}_{\mathbf{sort}(v_i)}$ for every $i \in [n]$ and $(A_1, \ldots, A_n) = S_{\mathcal{P}(\mathbf{M})}(A_1, \ldots, A_n)$, i.e, $A_i = p_{i\mathcal{P}(\mathbf{M})}(A_1, \ldots, A_n)$ for every $i \in [n]$.

Such a system has a least solution in $\mathcal{P}(\mathbf{M})$ denoted by $(\mathbf{L}((S, M), v_1), \ldots, \mathbf{L}((S, M), v_n))$. This n-tuple can be concretely described as follows :

- $\mathbf{L}((S, \mathbf{M}), v_i) = \bigcup\{A_i^j \mid j \geq 0\}$ where $A_i^0 = \emptyset$ for all $i \in [n]$.
- $\{A_1^{j+1}, \ldots, A_n^{j+1}\} = S_{\mathcal{P}(\mathbf{M})}(A_1^j, \ldots, A_n^j)$.

The components of the least solution in $\mathcal{P}(\mathbf{M})$ of a polynomial system are the $\mathbf{M}$-*equational* sets. $\Box$

Let us prove in a few lines, that the dual set of every equational set is equational too. In order to define a dual of an abstract E-hypermap, we use the same symbol $*$ to denote first the involution on $\mathcal{S}$ that associates with every sort $s = (\sigma, \alpha) \in \mathcal{S}$ its dual sort $s^* := (\alpha^{-1}\sigma, \alpha^{-1})$ and to denote secondly an involution on the set of labels U that commutes with $\mathbf{sort}$ ($\mathbf{sort}(u^*) = \mathbf{sort}(u)^*$ for every $u \in U$).

Definition 64. The *dual* of some concrete E-hypermap M is the concrete E-hypermap $(\mathbf{D}_M, \mathbf{E}_M, \mathbf{dart}_M, \beta_M, \alpha_M^{-1}, \$_M, *\circ\mathbf{lab}_M)$ of sort $\mathbf{sort}(M^*) := \mathbf{sort}(M)^*$. Such a definition is extended in a natural way to abstract E-hypermaps. $\Box$

Theorem 65. *Every abstract E-hypermap of the form $N[u/P]$ with $u \in U$ verifies: $(N[u/P])^* = N^*[u^*/P^*]$.*

Proof. As a direct consequence of Theorem 31 and of Definition 64. $\square$

In order to simplify, if an unary mapping like $*$ is defined on each element of some sequence $s = (s_1, \ldots, s_l)$, the term s^* denotes the sequence $(s_1^*, \ldots, s_1^*)$.

Corollary 66. *A set of abstract E-hypermaps L is equational if and only if its dual set $L^* := \{M^* \mid M \in L\}$ is equational.*

Proof. Rather to add a dual "copy" to the $\mathcal{S}$-signature $\mathcal{F}$, we can suppose there is an involution $*$ on $\mathcal{F}$ such that for every $f \in \mathcal{F}$, we have $\mathbf{sort}(f^*) = \mathbf{sort}(f)^*$ and $\mathbf{arity}(f^*) = \mathbf{arity}(f)^*$. As a consequence of Definition 64, we can suppose for every $f \in \mathcal{F}$, $(f^*)_{\mathbf{M}} = \square_{M^*}$ for some abstract E-hypermap M satisfying $f_{\mathbf{M}} = \square_M$. Thus by Theorem 65, for every $f \in \mathcal{F}$, $f^*{}_{\mathbf{M}}$ is defined on $\mathbf{M}_{\mathbf{arity}(f)^*}$ and is such that for every $(N_1, \ldots, N_l) \in \mathbf{M}_{\mathbf{arity}(f)}$, we have $(f_{\mathbf{M}}(N_1, \ldots, N_l))^* = f^*{}_{\mathbf{M}}(N_1^*, \ldots, N_l^*)$.

Let L be an equational set, that is, by definition, a component of the least solution K of some polynomial system S. The system S is of the form $< v_1 = p_1, \ldots, v_l = p_l >$ for some variables' set $V = \{v_1, \ldots, v_l\}$ and some polynomials set $P = \{p_1, \ldots, p_l\}$. For every $v \in V$, let v^* be a variable of sort $\mathbf{sort}(v)^*$. For every $p \in P$, let p^* be the polynomial obtained from p by replacing every symbol of the form $+_s$ (resp. Ω_s) for some $s \in \mathcal{S}$ with $+_{s^*}$ (resp. Ω_{s^*}), by replacing every variable $v \in V$ with v^* and every symbol $f \in \mathcal{F}$ with f^*. It is easy to prove by recurrence that the polynomial system $S^* :=< v_1^* = p_1^*, \ldots, v_m^* = p_m^* >$ has as least solution the sequence K^*. Hence L^*, which is a component of K^*, is equational.

The equality $L = (L^*)^*$ suffices to establish the reciprocal. $\square$

Before to prove that every the hypergraphs induced by some equational set of of E-hypermaps is equational too, some definitions. In order to simplify, we suppose that U is given with a second sort mapping $\mathbf{sort}_g$ that associates with every symbol $u \in U$ the natural number $z(\sigma)$ where $(\sigma, \alpha) = \mathbf{sort}(u)$. This choice permits to label hyperedges of hypergraphs and hyperedges of E-hypermaps with same labels.

Definition 67 (Abstract hypergraphs). A *concrete hypergraph G* is a tuple (V, E, f, d, l) denoted by $(\mathbf{V}_G, \mathbf{E}_G, \mathbf{vert}_G, \$_G, \mathbf{lab}_G)$ and such that :

- (V, E, f) verifies the conditions of Definition 51 except that f associates not a set of vertices but a sequence of distinct vertices.
- d is a hyperedge of E.
- l is a partial mapping on $E \backslash e \to U$ such that for every labelled hyperedge, its number of extremities is the sort of its label.

Its sort, denoted by $\mathbf{sort}(G)$, is the length of the sequence $\mathbf{vert}_G(\$_G)$. An abstract hypergraph is an isomorphic class of concrete hypergraphs.

Let H be a concrete hypergraph, e be a hyperedge of H and K be a hypergraph of sort $\mathbf{sort}(K) = \mathbf{sort}(\mathbf{lab}_H(e))$. Rather to consider isomorphic copies of H and K, we may suppose that:

- $\mathbf{vert}_H(e, i) = \mathbf{vert}_K(\$_K, i)$ for every $i \in [n]$.
- $\mathbf{V}_H \cap \mathbf{V}_K = \{\mathbf{vert}_H(e, i) \mid i \in [n]\}$.
- $\mathbf{E}_H \cap \mathbf{E}_K = \{e\} = \{\$_K\}$.

  where n is sort $\mathbf{sort}(K) = \mathbf{sort}(\mathbf{lab}_H(e))$.

The hypergraph *obtained from H by replacing e by K* is the hypergraph $(V, E, f, \$_H, g)$ where $(V, E, f) = (\mathbf{V}_H, \mathbf{E}_H, \mathbf{vert}_H) \square (\mathbf{V}_K, \mathbf{E}_K, \mathbf{vert}_K)$ and where g is the union of $\mathbf{lab}_K$ and the restriction of $\mathbf{lab}_H$ on $\mathbf{E}_H \backslash e$. In a similar way as for E-hypermaps, we extend that definition for concrete (resp. abstract) hypergraphs and a label of U.

In a similar way as for E-hypermaps, all these definitions permit to define an algebra for hypergraphs and a notion of equational set of hypergraphs. For more details see [5]. $\square$

Definition 68. The *induced* hypergraph of some concrete E-hypermap M, denoted by $\underline{hgraph}(M)$, is the concrete hypergraph $(\mathbf{D}_M/_{\sigma_M}, \mathbf{E}_M, f, \mathbf{lab}_M, \$_M)$ where f associates with every hyperedge e the sequence $([d_1]_{\sigma_M}, \dots, [d_m]_{\sigma_M})$ where $(d_1, \dots, d_m)$ is the subsequence obtained from $\mathbf{dart}_M(e) = (e_1, \dots, e_n)$ by deleting every dart e_j such that there is $i < j$ with $e_j \in [e_i]_{\sigma_M}$. In a natural way, this definition is extended to abstract hypergraphs. □

Theorem 69. *Every abstract E-hypermap of the form $N[u/P]$ with $u \in U$ verifies: $\underline{hgraph}(N[u/P]) = \underline{hgraph}(N)[u/\underline{hgraph}(P)]$.*

Proof. As a direct consequence of Theorem 32 and of Definition 68. □

Corollary 70. *If a set of abstract E-hypermaps L is equational, then the set $\underline{hgraph}(L) := \{\underline{hgraph}(M) \mid M \in L\}$ is equational too.*

Proof. The proof is similar with that of Corollary 66. □

As we shall see in Example 78, the converse of above fact is not true.

Now, let us present some example of equational set of hypermaps:

Theorem 71. *Let k and g be two integers. The set of all representations of genus g of graphs of treewidth $\leq k$ is equational.*

Proof. In order to simplify the proof, we restrict the proof to connected graphs: the generalization to the general case is obvious. Let k and g be two integers. Our proof consists to construct a polynomial S such that its least solution contains every set of the form $L_{(s,h)}$ with $s \in \mathcal{S}$ and $h \in [0, g]$ where $L_{(s,h)}$ denotes the set of connected maps of sort s and of genus h having an induced hypergraph of treewidth $\leq k$.

Let $\mathcal{M}$ be the set of all abstract E-hypermaps M such that:

- M has at most $k+1$ vertices.
- the genus of M is at most g.
- M is basic.
- if M contains some not c-replaceable hyperedge e, e is an edge, is not labelled and is not $\$_M$.

By Fact 54 and by the fact that every basic E-hypermap that contains an edge that is not c-replaceable contains at most 4 darts, every E-hypermap in $\mathcal{M}$ contains at most $2 \cdot (k + 1 + 1 + 2 \cdot g)$ darts. Thus, the set of operations $\{\Box_M \mid M \in \mathcal{M}\}$ is finite. Let $\mathcal{F}'$ be a finite subset of $\mathcal{F}$ such that $\{f_{\mathbf{M}} \mid f \in \mathcal{F}'\} = \{\Box_M \mid M \in \mathcal{M}\}$.

For every sort $s = (\sigma, \alpha) \in \mathcal{S}$, we denote by $Egenus(s)$ the integer $z(\alpha) - 2 + genus((\sigma, \alpha))$ (here a sort should be viewed as a E-hypermap having exactly 2 hyperedges). Let $\mathcal{S}'$ be the set that contains every sort $s \in \mathcal{S}$ having at most $k+1$ vertices and of genus $Egenus(s) \leq g$. By Fact 54, every sort of $\mathcal{S}'$ has at most $k + 2 + 2 \cdot g$ darts. Thus, $\mathcal{S}'$ is finite.

Let V be a (finite) set of unknowns in bijection with $\mathcal{S}' \times [0, g]$; thus every unknown $v \in V$ has a sort denoted by $\mathbf{sort}(v)$ and a genus, denoted by $genus(v)$.

Let T be the set of monomials having a unique symbol $f \in \mathcal{F}'$ and having as unknowns some variables $v_1, \dots, v_l$ of V'; the genus of such monomial m, denoted by $genus(m)$, is the integer $genus(M) + \sum_{i \in [l]}(genus(v_i) - genus(s_i))$ where M is some abstract E-hypermap satisfying $f_{\mathbf{M}} = \Box_M$. The finitude of $\mathcal{F}'$ and V implies T finite. Let S be the polynomial system that contains every equation of the form $v = \sum_{m \in T_{(\mathbf{sort}(v), genus(v))}} m$ where for every $(s, h) \in \mathcal{S} \times [0, g]$, $T_{(s,h)}$ denotes the set of all monomials of T of sort s and of genus h. To conclude, let us prove that every component L_v with $v \in V$ of the least solution of S is $L_{(s,h)}$ with $s := \mathbf{sort}(v)$ and $h := genus(v)$.

1. As a consequence of Lemma 43, for every $v \in V$, every E-hypermap L_v, trivially of sort $\mathbf{sort}(v)$, has a treewidth $\leq k$. As a consequence of Theorem 51, for every symbol $f \in \mathcal{F}$ of type $(s_1, \ldots, s_m) \to s$ and for every E-hypermaps $P_1, \ldots, P_m$ of sorts $s_1, \ldots, s_m$, we have: $genus(f_{\mathbf{M}}(P_1, \ldots, P_m)) = genus(N) + \sum_{i \in [m]}(genus(P_i) - genus(s_i))$ for some E-hypermap satisfying $f_{\mathbf{M}} = \Box_N$. Thus for every monomial $m = f(v_1, \ldots, v_n) \in T$ of sort s and of genus h, if $(P_1, \ldots, P_n) \in \mathbf{M}_{\mathbf{arity}(f)}$ are n E-hypermaps of genus respectively $genus(v_1), \ldots, genus(v_n)$, the genus of the E-hypermap $f_{\mathbf{M}}(P_1, \ldots, P_n)$ is $genus(N) + \sum_{i \in [m]}(genus(v_i) - genus(s_i)) = genus(m) = h$. Thus, by recurrence, we prove for every $v \in V$ the inclusion: $L_v \subseteq L_{(\mathbf{sort}(v), genus(v))}$.
2. By using Lemma 47 and same arguments as above, we prove for every $v \in V$ the inclusion: $L_v \supseteq L_{(\mathbf{sort}(v), genus(v))}$. □

Note that the precedent theorem may be improved in two directions. First by considering representations of genus g of hypergraphs of treewidth $\leq k$, which are (the representations) of hyperedge's degree uniformly bounded by some integer. The proof is similar with the precedent one. Secondly, by considering representations of genus g of graphs (or of hypergraphs with same restrictions as above) of treewidth k. A proof of this fact requires new notions (for instance Monadic Second Order Logic) out of the scope of this article.

5 Treewidth of E-hypermaps

Let us conclude this paper by introducing an extension of treewidth to hypermaps. As for hypergraphs, this measure of complexity is a good tool to prove the nonequationality of some given set of hypermaps: every equational set is uniformly treewidth bounded.

To define the treewidth of a hypermap, it suffices in a similar way as for a tree-decomposition of a hypergraph to partition the hyperedges in the nodes of some tree and to place in each node the darts of the E-hypermaps, according to some "connectivity" requirement. The width of such a tree-decomposition is the maximal number of darts contained in some node.

Definition 72 (Treewidth of a E-hypermap). A *tree-decomposition* of some E-hypermap M is a pair (T, g) where T is a tree and where g is a mapping which associates with every node t of T a set of hyperedges of M such that $\mathbf{E}_M$ is the disjoint union $\bigcup_{t \in \mathbf{V}_T} g(t)$. Let h be the mapping that associates with every node t of T the union $\bigcup_{(s,u) \in P_t} B_s \cap B_u$ where for each node t of T:

- P_t contains every pair of extremities of every elementary path in T containing t.
- B_t denotes $\mathbf{dart}_M(g(t)) \cup \sigma_M^{-1}(\mathbf{dart}_M(g(t)))$.

The *width* of X is the maximum of $\mathbf{card}(h(t))$ taken over all nodes t of T. The *treewidth* of a E-hypermap M is the smallest width of all its tree-decomposition and is denoted by $\underline{twd}(M)$. □

Lemma 73. *Every E-hypermap and its dual have same treewidth.*

Proof. As a direct consequence of the precedent definition and the fact that a tree-decomposition of some E-hypermap M is a tree-decomposition of its dual of same width (in particular for every set of hyperedges E of M $\sigma_M^{-1}(\mathbf{dart}_M(E)) = \sigma_{M^*}^{-1}(\mathbf{dart}_{M^*}(E))$. □

Now, a very simple characterization of the treewidth of a hypermap analogous with that for hypergraphs stated in Lemma 17. The proof quite similar with the proof of Lemma 17 is left to the reader.

Lemma 74. *The treewidth of some E-hypermap M is the smallest integer k such that the $\Box$-closure of the set of all E-hypermaps having at most k darts contains M.*

Corollary 75. *Every equational set of E-hypermaps is uniformly treewidth bounded.*

Proof. By Lemma 74, every E-hypermap of the form $N \Box P$ verifies $\underline{twd}(N \Box P) \leq max\{\underline{twd}(N), \underline{twd}(P)\}$. It follows that for every $f \in \mathcal{F}$ and for every $(P_1, \ldots, P_l) \in \mathbf{M}_{\mathbf{arity}(f)}$, we have $\underline{twd}(f_{\mathbf{M}}(P_1, \ldots, P_l)) \leq max\{\underline{twd}(X) \mid X \in \{M, P_1, \ldots, P_l\}\}$ where M is some E-hypermap such that $f_{\mathbf{M}} = \Box_M$. It follows that the least solution of some polynomial system S is at most $max\{\underline{twd}(f) \mid f \in \mathcal{F}_S\}$ where $\mathcal{F}_S$ is the set of all symbols of $\mathcal{F}$ that appear in some monomial of S and where for every $f \in \mathcal{F}$ $\underline{twd}(f)$ denotes the treewidth of some abstract E-hypermap M satisfying $f_{\mathbf{M}} = \Box_M$ (we recall that this abstract E-hypermap is unique up some relabelling of its hyperedges). $\Box$

As a corollary of Lemma 74 and as a direct consequence of the fact that the number of abstract E-hypermaps having at most k darts is finite, we can reformulate Lemma 74 by using not a $\Box$-closure but a polynomial system. This polynomial system may be refined in order to produce not E-hypermaps but hypermaps or maps. The formal proof is left to the reader.

Corollary 76. *For every integer k, the set of all E-hypermaps (resp. hypermaps, maps) of treewidth at most k is equational.*

In this last fact, we compare the three measures of complexity which are the treewidth of a graph, the treewidth of a hypermap and the genus of a surface.

Theorem 77. *Let M be a map, G its induced graph and G^* the induced graph of its dual. We have:*

$$\underline{twd}(G) + \underline{twd}(G^*) + 1 \leq \underline{twd}(M) \leq \underline{twd}(G) + \underline{twd}(G^*) + 3 + 2 \cdot genus(M)$$

Proof. These inequalities are a direct consequence of Lemma 47, of Fact 54 and of the fact that every basic hypermap that contains a no c-replaceable edge (having necessarily two extremities) contains at most 4 darts. $\Box$

The precedent inequality is reached for the sphere. The planar embedding of the path with 4 vertices has as treewidth 4, its induced graph a treewidth equal to 1 and its dual 0. For other surfaces, it is open.

A natural question arises: is the set of all representations of hypergraphs of some equational set equational too? The answer is no. The reason is that this set is, in non trivial case, not uniformly bounded treewidth. Example 78 illustrates this fact.

Example 78. The set of all graphs having a unique vertex is equational but not the set of all its representations: this set is not uniformly treewidth-bounded (a simple proof is made by using Lemma 47, Fact 54 and the fact that every basic E-hypermaps issued from the decomposition of such hypermaps has at most 3 hyperedges). For instance, the treewidth of hypermaps associated with the embeddings of Figure 10 are respectively 2, 4, 4, 4, 5, 6 and 6. $\Box$

If we add some restrictions, the answer is positive. For instance, we established in [7], that for every equational set of hypergraphs L, we can construct an equational set of hypermaps that contains at least one planar representation of every planar hypergraph of L.

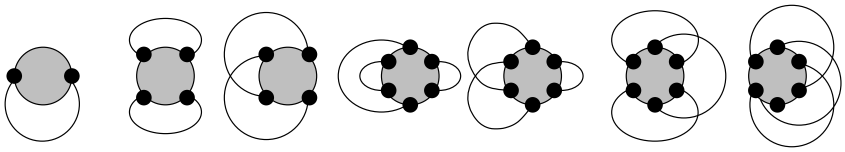

Fig. 10. Hypermaps of treewidth uniformly not bounded.

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