

Kolmogorov-Loveland stochasticity and Kolmogorov complexity

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Abstract. Merkle et al. [13] showed that all Kolmogorov-Loveland stochastic infinite binary sequences have constructive Hausdorff dimension 1. In this paper, we go even further, showing that from an infinite sequence of dimension less than $\mathcal{H}(\frac{1}{2} + \delta)$ (\mathcal{H} being the Shannon entropy function) one can extract by an effective selection rule a biased subsequence with bias at least δ . We also prove an analogous result for finite strings.

1 Introduction

In 1919 R. von Mises gave the first definition of algorithmic randomness, which was inspired by the law of large numbers. According to his definition, an infinite binary sequence α of zeroes and ones is said to be “random” (instead of “random sequence”, von Mises used the term *collective*) if it is not biased, i.e. the frequency of zeroes goes to $\frac{1}{2}$, and if every sequence we can extract from α by an “admissible” selection rule is not biased. The second condition is important. Indeed, the infinite sequence 010101010.... is not biased; however, and this is why no one would call it “random”, the selection rule consisting in selecting the bits of even positions will select the subsequence 000000...., which this time is biased. R. von Mises never made completely precise what he meant by admissible selection rule. When computability theory emerged two decades later, Church proposed a formal definition: he defined an admissible selection rule to be a (total) computable process which, having read the first n bits of an infinite binary sequence α , decides if it wants to select the next bit or not, and then reads it (of course, it is crucial that the decision to select the bit or not is made before reading the bit). The sequence of selected bits is the selected subsequence w.r.t. to the selection rule. Later, Kolmogorov and Loveland proposed a more permissive definition of an admissible selection rule: they argued that in Church’s definition, the bits are read in order, which is too restrictive. Hence, they defined an admissible selection rule to be a (partial) computable process which, having read any n bits of an infinite binary sequence α , picks a bit that has not been read yet, decides whether it should be selected or not, and then reads it. Nowadays, the sequences α which are collectives w.r.t. to this last definition are called Kolmogorov-Loveland stochastic (KL-stochastic for short).

It turns out that even with this improvement, KL-stochasticity is too weak a notion of randomness. A method developed by van Lambalgen [17], which relies on randomness w.r.t. non-uniform probability measures, was used by A. Shen [16] to show that there exists a KL-stochastic sequence all of whose prefixes contain more zeroes than ones (this event has probability 0 for the uniform measure). In 1966, P. Martin-Löf introduced a notion of randomness which is now called Martin-Löf randomness and considered by many as the most satisfactory notion of algorithmic randomness. Its original definition involved effective measure theory, but after the work of Levin, Chaitin and Schnorr, we know that Martin-Löf randomness can be characterized in terms of Kolmogorov complexity (we assume that the reader is familiar with this notion; if not, see [8]): an infinite binary sequence α is Martin-Löf random if $K(\alpha_0\dots\alpha_n) \geq n + O(1)$ (K being the prefix Kolmogorov complexity).

Now that we have this good notion of randomness, it is worth looking back at KL-stochasticity in the light of Kolmogorov complexity. For example:

Question 1. Do the initial segments of a KL-stochastic sequence have to be of high Kolmogorov complexity?

Question 2. Conversely, given a string α with some randomness deficiency (i.e. in the case where $K(\alpha_0\dots\alpha_n) \leq n - f(n)$ for some unbounded function f), can we quantify the maximal bias we can get by selecting a subsequence from α ?

Concerning Question 1, the following two central theorems give a good picture of the situation:

Theorem 1 (Muchnik et al. [14]) *Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a computable function. If $f(n) = o(n)$, there exists a KL-stochastic sequence α such that*

$$K(\alpha_0\dots\alpha_n) \leq n - f(n) + O(1)$$

Theorem 2 (Merkle et al. [13]) *Let α be an infinite binary sequence. If*

$$\liminf_{n \rightarrow +\infty} \frac{K(\alpha_0\dots\alpha_n)}{n} < 1$$

then α is not KL-stochastic.

(as we will see later, the quantity $\liminf \frac{K(\alpha_0\dots\alpha_n)}{n}$ is called the constructive Hausdorff dimension of α).

Question 2 has been addressed in the case of finite binary sequences by Asarin [2], Durand and Vereshchagin in [5], who gave lower and upper bounds for the maximal bias one can extract from a sequence with a given randomness deficiency. However, not much is known in the case of infinite binary sequences. For example, Theorem 2 says nothing about the relation between the lim inf term and the maximal bias one can obtain by selecting a subsequence.

Both papers [13] and [5] use the same main three techniques, which are already present in [14] (where they were used to prove a weaker version of Theorem 2):

1. *Splitting technique.* Any sequence (finite or infinite) which has a linear randomness deficiency can be split into a finite number of subsequences such that at least two of the subsequences have a linear randomness deficiency relatively to the other ones.

2. *Competing strategies.* Given two finite sequences u and v with known randomness deficiencies (say, respectively $K(u) = |u| - d_1$ and $K(v) = |v| - d_2$), one can construct (the construction depending only on d_1 and d_2 , not on (u, v)) two strategies (the concept of strategy is formalized below) S_1 and S_2 such that: S_1 reads v and bets on u , S_2 reads u and bets on v , and either S_1 multiply its initial capital by a least 2^{d_1} or S_2 multiply its initial capital by at least 2^{d_2} . Hence, a good way to predict the bits of a string w with some random deficiency is to use the above technique 1 to split w into pieces such that two of them have some randomness deficiency and apply technique 2.

3. *Converting a strategy into a selection rule.* If a betting strategy wins on a sequence (finite or infinite) an amount of money which is exponential in the number of bets, one can construct from this strategy a selection rule which selects a biased subsequence.

In this paper, we address Question 2 for both the finite and infinite cases. In section 2, we introduce some game-theoretic notions, and in particular the notion of (selective) betting strategy. We prove a refinement of the conversion of a strategy into a selection rule (Theorem 6), which will be of crucial use in the sequel.

In section 3, we start with an account of effective Hausdorff dimension. This is a well-known approach of algorithmic randomness, which was first introduced by Lutz [10], where he defines constructive Hausdorff dimension. We will also define computable Hausdorff dimension, which was introduced by Downey et al. [4]. We then present our main result —Theorem 11— which is a quantitative version of the above Theorem 2, i.e. it relates explicitly the constructive Hausdorff dimension of a sequence to the maximal bias one can obtain by selecting a subsequence. More precisely, we provide a lower bound on this maximal bias, which we will prove to be optimal. We also give an interpretation of this result purely in terms of effective Hausdorff dimension.

Finally, in section 4, we will prove an analogous result in the framework of finite binary sequences, answering a question of Durand and Vereshchagin.

Before we move on to our discussion, we present the basic definitions and notation we will need in the sequel. We denote by 2^* the set of finite binary sequences, and by 2^ω the set of infinite ones. For every element $\alpha = \alpha_0\alpha_1\alpha_2\dots$ of 2^ω , and every $n, m \in \mathbb{N}$, we denote by $\alpha_{[n,m]}$ the string $\alpha_n\alpha_{n+1}\dots\alpha_m$. For all $u \in 2^*$, we denote by $u2^\omega$ the set of infinite sequences of which u is a prefix. We denote by $|u|$ the length of u .

We denote by $\#0(\alpha, n)$ and $\#1(\alpha, n)$ respectively the number of 0's and 1's among $\alpha_0 \dots \alpha_{n-1}$. We set

$$Bias(\alpha) = \limsup_{n \rightarrow +\infty} \left| \frac{\#0(\alpha, n)}{n} - \frac{1}{2} \right|$$

We denote by $\mathcal{H}(p)$ the entropy of the Bernoulli random variable with parameter p . Recall that for $p \in [0, 1]$, $\mathcal{H}(p) = -p \log p - (1-p) \log(1-p)$, and that $x \mapsto \mathcal{H}(\frac{1}{2} + x)$ is a decreasing bijection from $[0, \frac{1}{2}]$ to $[0, 1]$.

If Z is a subset of \mathbb{N} , and α, β are two elements of 2^ω , we call Z -join of α and β , and denote by $\alpha \oplus_Z \beta$, the element of 2^ω we get by merging α and β , placing the bits of β in positions i 's such that $i \in Z$. Formally,

$$(\alpha \oplus_Z \beta)_i = \begin{cases} \alpha_{|\bar{Z} \cap \{0..i-1\}|} & \text{if } i \notin Z \\ \beta_{|Z \cap \{0..i-1\}|} & \text{if } i \in Z \end{cases}$$

If $Z = 2\mathbb{N} + 1$, we have $\alpha \oplus_Z \beta = \alpha_0 \beta_0 \alpha_1 \beta_1 \dots$, and we abbreviate $\alpha \oplus_Z \beta$ by $\alpha \oplus \beta$.

If a set $A \subseteq \mathbb{N}$ is recursively enumerable, we denote by $A[\tau]$ the finite set containing the elements of A that appear during the first τ steps of a fixed enumeration of A .

2 Selection rules vs Strategies

2.1 Selection rules

We formalize the notion of selection rule we discussed above. A selection rule is a (partial) function $\sigma : 2^* \rightarrow \mathbb{N} \times \{\mathbf{selects}, \mathbf{scans}\}$.

We run a selection rule σ on a sequence α as follows. Let s_0 and h_0 be empty words, and p_0 be the empty set. We define s_n, h_n and p_n by induction. Informally, s_n represents the selected bits after n moves, h_n represents the history, i.e. the bits that have been read during the first n moves, and p_n the positions in α of these bits. At n -th move (by convention, there is a 0-th move):

- If $\sigma(h_n) = (k, \mathbf{selects})$ and $k \notin p_n$, set $s_{n+1} = s_n \alpha_k$, $h_{n+1} = h_n \alpha_k$ and $p_{n+1} = p_n \cup \{k\}$.

- If $\sigma(h_n) = (k, \mathbf{selects})$ and $k \in p_n$, or $\sigma(h_n) = (k, \mathbf{scans})$, set $s_{n+1} = s_n$, $h_{n+1} = h_n \alpha_k$, and $p_{n+1} = p_n \cup \{k\}$.

(if for some n , $\sigma(h_n)$ is not defined, the selection process is immediately stopped). If infinitely many selections are performed, i.e. if the set $\{s_0, s_1, s_2, \dots\}$ is infinite, the s_i 's are prefixes of an infinite sequence β . In this case, we say that β is the subsequence of α selected by σ , which we write $\beta = \sigma[\alpha]$.

We say that α is Kolmogorov-Loveland stochastic if for every $\beta \in 2^\omega$ that can be selected from α by a computable selection rule, β satisfies the law of large numbers (i.e. $\lim_{n \rightarrow \infty} \frac{\#0(\beta, n)}{n} = \frac{1}{2}$).

As we want to quantify the bias one can extract from a sequence by a computable selection rule, we will focus our attention on the quantity:

$$\delta_{max}^{sel}(\alpha) = \sup \{ Bias(\sigma[\alpha]) : \sigma \text{ computable selection rule} \}$$

Remark 3 *We made the choice to define selection rules by partial functions (and hence, by computable selection rule we mean partial computable selection rule). It turns out that, by an argument of W. Merkle [12], defining them to be total functions would not change the notion of KL-stochasticity nor would affect the quantity δ_{max}^{sel} .*

2.2 Strategies

In [14], Muchnik et al., trying to improve on the notion of KL-stochasticity, suggested to adopt a game-theoretic point of view. We follow their approach.

Let us consider the following game, where Player plays against a sequence $\alpha \in 2^\omega$. The goal for Player is to make money while trying to guess the bits of α . Initially, all the bits are hidden. At each move, Player selects a bit that is not yet revealed. He can either scan it, or bet on its value some (rational) fraction ρ of his current capital. If his guess is correct, his stake is doubled (i.e. his capital is multiplied by $(1 + \rho)$). If not, his stake is lost (i.e. his capital is multiplied by $(1 - \rho)$).

Formally, a selective strategy is a (partial) function $S : 2^* \rightarrow (\mathbb{N} \times \{\mathbf{scans}\}) \cup (\mathbb{N} \times \{0, 1\} \times (\mathbb{Q} \cap [0, 1]))$. If the range of S is contained in $\mathbb{N} \times \{0, 1\} \times (\mathbb{Q} \cap [0, 1])$ (i.e. S never scans), it is said to be a strategy.

We run a (selective) strategy S on a sequence α as follows: let h_0 be the empty word, and p_0 the empty set. Set $W_0 = 1$ (initial capital), and $N_0 = 0$ (number of bets). At n -th move:

- If $S(h_n) = (k, \mathbf{b}, \rho)$ and $k \notin p_n$, set $h_{n+1} = h_n \alpha_k$, $p_{n+1} = p_n \cup \{k\}$, $N_{n+1} = N_n + 1$. Also set $W_{n+1} = (1 + \rho)W_n$ if $\alpha_k = \mathbf{b}$ and $W_{n+1} = (1 - \rho)W_n$ otherwise.
- If $S(h_n) = (k, \mathbf{b}, \rho)$ and $k \in p_n$, or $S(h_n) = (k, \mathbf{scans})$, set $h_{n+1} = h_n \alpha_k$, $p_{n+1} = p_n \cup \{k\}$, $N_{n+1} = N_n$. Also set $W_{n+1} = W_n$.

We denote by V_m Player's capital after the m -th bet, that is: $V_m = W_n$ with $n = \min\{i : N_i = m\}$. We denote by $V_m(\alpha, S)$ Player's capital after the m -th bet, when playing against α according to the selective strategy S (note that this could be undefined). We finally call a triple (k, \mathbf{b}, ρ) a bet.

Muchnik et al. defined an infinite sequence α to be unpredictable (we now say Kolmogorov-Loveland random) if there exists no computable strategy S such that $\limsup V_n(\alpha, S) = +\infty$ (here again, by Merkle's argument, it does not matter whether we take the strategies to be partial computable or total computable). While Kolmogorov-Loveland randomness is a priori weaker than Martin-Löf randomness, the two notions have been shown to be close ([14], [13]), and their equality remains a fundamental open question. We will not discuss

Kolmogorov-Loveland randomness here, but we will use extensively the notion of selective strategy.

Remark 4 *One may think at first that the notions of selective strategy and strategy are equivalent, since scanning a bit is the same as betting 0 on it. This is true if we just want to define Kolmogorov-Loveland randomness (and indeed Muchnik et al. did not make our distinction between strategy and selective strategy). However, this would not be suitable for our purposes, as we will be interested in strategies which succeed exponentially fast in the number of bets, and non-necessairly in the number of moves.*

It is a well-known fact that if a sequence α is biased, there exists a computable strategy S which makes money exponentially when betting on its bits. More precisely:

Proposition 5 *Let $\alpha \in 2^\omega$, and $\delta = \text{Bias}(\alpha)$. There exists a strategy S , computable with oracle δ , such that for all $t > \mathcal{H}(\frac{1}{2} + \delta)$:*

$$\limsup_{n \rightarrow +\infty} \frac{V_n(\alpha, S)}{2^{(1-t)n}} = +\infty$$

Proof. Without loss of generality, suppose that $\limsup \frac{\#0(\alpha, n)}{n} = \frac{1}{2} + \delta$. Using the oracle δ , let us compute a sequence $\{\delta_n\}_{n \in \mathbb{N}}$ of rational numbers, converging to δ . Let S be the strategy which at the n -th move bets $(n, 0, 2\delta_n)$. We then have, for all n :

$$V_n(\alpha, S) = \prod_{\substack{0 \leq i \leq n-1 \\ \alpha_i=0}} (1 + 2\delta_i) \prod_{\substack{0 \leq i \leq n-1 \\ \alpha_i=1}} (1 - 2\delta_i)$$

Hence,

$$\frac{\log V_n(\alpha, S)}{n} = \frac{1}{n} \sum_{\substack{0 \leq i \leq n-1 \\ \alpha_i=0}} \log(1 + 2\delta_i) + \frac{1}{n} \sum_{\substack{0 \leq i \leq n-1 \\ \alpha_i=1}} \log(1 - 2\delta_i)$$

It follows that:

$$\limsup_{n \rightarrow +\infty} \frac{\log V_n(\alpha, S)}{n} = \left(\frac{1}{2} + \delta\right) \log(1 + 2\delta) + \left(\frac{1}{2} - \delta\right) \log(1 - 2\delta) = 1 - \mathcal{H}\left(\frac{1}{2} + \delta\right)$$

□

Schnorr [15] proved conversely that if there exists a selective strategy S which, playing against α , makes money exponentially in the number of bets, then there exists a computable selection rule which selects from α a biased subsequence (although he did not quite use the same terminology as ours). However, Schnorr proved this in a purely qualitative way. We strengthen Schnorr's theorem by proving the converse of Proposition 5.

Theorem 6 Let $\alpha \in 2^\omega$. Suppose that there exists a real number s and a selective strategy S such that $\limsup \frac{V_n(\alpha, S)}{2^{\frac{(1-t)n}{2}}} = +\infty$ for all $t > s$. Then, there exists a selection rule σ , computable with oracle s , such that the bias $\delta = \text{Bias}(\sigma[\alpha])$ is large enough to satisfy $\mathcal{H}(\frac{1}{2} + \delta) \leq s$.

Proof. The basic idea of the proof is the following: by an argument of Ambos-Spies et al. [1], the above theorem would be easier to prove if S was only allowed to play moves of type (k, scans) or $(k, 0, q)$, where q is a fixed constant. Indeed, in this case, let σ be the computable selection rule which simulates S , scanning a bit if S scans it, and selecting a bit if S bets on it. We then have for all n :

$$V_n(\alpha, S) = (1+q)^{\#\mathbf{0}(\sigma[\alpha], n)} (1-q)^{\#\mathbf{1}(\sigma[\alpha], n)}$$

i.e.

$$\frac{\log V_n(\alpha, S)}{n} = \frac{\#\mathbf{0}(\sigma[\alpha], n)}{n} \log(1+q) + \frac{\#\mathbf{1}(\sigma[\alpha], n)}{n} \log(1-q)$$

Setting $\delta = \text{Bias}(\sigma[\alpha])$, it follows that

$$\limsup_{n \rightarrow +\infty} \frac{\log V_n(\alpha, S)}{n} \leq \left(\frac{1}{2} + \delta\right) \log(1+q) + \left(\frac{1}{2} - \delta\right) \log(1-q)$$

By definition of S :

$$\limsup_{n \rightarrow +\infty} \frac{\log V_n(\alpha, S)}{n} \geq 1-s$$

It follows that

$$1-s \leq \left(\frac{1}{2} + \delta\right) \log(1+q) + \left(\frac{1}{2} - \delta\right) \log(1-q)$$

The function $x \mapsto \left(\frac{1}{2} + \delta\right) \log(1+x) + \left(\frac{1}{2} - \delta\right) \log(1-x)$ taking its maximum for $x = 2\delta$, we then have

$$1-s \leq \left(\frac{1}{2} + \delta\right) \log(1+2\delta) + \left(\frac{1}{2} - \delta\right) \log(1-2\delta)$$

i.e.

$$s \geq \mathcal{H}\left(\frac{1}{2} + \delta\right)$$

Of course, our notion of strategy is not restricted as above. However, since the couples (value, stake) of the bets are contained in the compact set $\{0, 1\} \times [0, 1]$, we argue by a dichotomy technique that there must be some condensation point $(\mathbf{b}, \bar{\rho})$ in the neighbourhood of which bets are often successful.

Let us denote by $(k_i, \mathbf{b}_i, \rho_i)$ the i -th bet made by S while playing against α . We also denote by $\tilde{\rho}_i$ Player's reward for this bet (i.e. $\tilde{\rho}_i = \rho_i$ if $\alpha_{k_i} = \mathbf{b}_i$, and $\tilde{\rho}_i = -\rho_i$ otherwise). By assumption, we have for all $t > s$:

$$\limsup_{n \rightarrow +\infty} \prod_{i=0}^n 2^{t-1} (1 + \tilde{\rho}_i) = +\infty$$

It is then clear that at least one of the following holds:

$$\limsup_{n \rightarrow +\infty} \prod_{\substack{0 \leq i \leq n \\ b_i = 0}} 2^{t-1}(1 + \tilde{\rho}_i) = +\infty \text{ for all } t > s$$

or

$$\limsup_{n \rightarrow +\infty} \prod_{\substack{0 \leq i \leq n \\ b_i = 1}} 2^{t-1}(1 + \tilde{\rho}_i) = +\infty \text{ for all } t > s$$

Without loss of generality we assume the first holds. Hence, we can assume that S never predicts a bit of α to have value 1 (replacing such bets by scans). Now, let $\eta \in [0, \frac{1}{2}]$ be such that $\mathcal{H}(\frac{1}{2} + \eta) = s$. Using s as an oracle, let us compute a sequence $\{[s_m^-, s_m^+]\}_{m \in \mathbb{N}}$ of interval with rational bounds such that for all m , $s \in [s_m^-, s_m^+]$. Also, since \mathcal{H} is computable, η is computable with oracle s . Let then $\{\eta_m\}_{m \in \mathbb{N}}$ be a sequence of rational numbers, computable with oracle s , such that for all m , $\eta \in [\eta_m^-, \eta_m^+]$. We distinguish two cases:

Case 1: for all $m \in \mathbb{N}$,

$$\limsup_{n \rightarrow +\infty} \prod_{\substack{0 \leq i \leq n-1 \\ \rho_i \in [2\eta_m^-, 2\eta_m^+]}} 2^{t-1}(1 + \tilde{\rho}_i) = +\infty \text{ for all } t > s \quad (1)$$

In this case, let S' be the selective strategy which plays on the same bits as S , proceeding by stages. During the stage m , at the i -th move:

- if S plays $(k_i, 0, \rho_i)$, with $\rho_i \in [2\eta_m^-, 2\eta_m^+]$, S' plays the same thing.
- if S plays $(k_i, 0, \rho_i)$ with $\rho_i \notin [2\eta_m^-, 2\eta_m^+]$ or plays (k_i, scans) , S' plays (k_i, scans) .

S' stays in stage m until, for some n , $V_n(\alpha, S') \geq 2^{(1-s_m)n}$ (this will eventually happen by (1)). When this happens, S' enters stage $m+1$ and so on.

By construction, the selective strategy S' also satisfies $\limsup_n \frac{V_n(\alpha, S')}{2^{(1-t)n}} = +\infty$ for all $t > s$. Let then σ be the selection rule which simulates S' , and selects a bit if S' bets on it, and scans a bit if S' scans it. Denoting by ρ'_i the stake of the i -th bet S' makes, we have:

$$V_n(\alpha, S') = \prod_{\substack{0 \leq i \leq n-1 \\ \sigma[\alpha]_i = 0}} (1 + \rho'_i) \prod_{\substack{0 \leq i \leq n-1 \\ \sigma[\alpha]_i = 1}} (1 - \rho'_i)$$

hence,

$$\frac{\log V_n(\alpha, S')}{n} = \frac{1}{n} \sum_{\substack{0 \leq i \leq n-1 \\ \sigma[\alpha]_i = 0}} \log(1 + \rho'_i) + \frac{1}{n} \sum_{\substack{0 \leq i \leq n-1 \\ \sigma[\alpha]_i = 1}} \log(1 - \rho'_i)$$

Set $\delta = \text{Bias}(\sigma[\alpha])$. Since $\rho'_i \rightarrow 2\eta$ and $\left| \{i \leq n-1 : \sigma[\alpha]_i = 0\} \right| \leq (\frac{1}{2} + \delta)n + o(n)$, we get:

$$\limsup_{n \rightarrow +\infty} \frac{\log V_n(\alpha, S')}{n} \leq (\frac{1}{2} + \delta) \log(1 + 2\eta) + (\frac{1}{2} - \delta) \log(1 - 2\eta)$$

It follows by (1) that

$$1 - s \leq (\frac{1}{2} + \delta) \log(1 + 2\eta) + (\frac{1}{2} - \delta) \log(1 - 2\eta)$$

But since $1 - s = (\frac{1}{2} + \eta) \log(1 + 2\eta) + (\frac{1}{2} - \eta) \log(1 - 2\eta)$, it follows that $\delta \geq \eta$, as wanted.

Case 2: If Case 1 does not hold, there exists m_0 such that

$$\limsup_{n \rightarrow +\infty} \prod_{\substack{0 \leq i \leq n \\ \rho_i \notin [2\eta_{m_0}^-, 2\eta_{m_0}^+]}} 2^{t-1}(1 + \tilde{\rho}_i) = +\infty \text{ for all } t > s$$

Thus, at least one of the following holds:

$$\limsup_{n \rightarrow +\infty} \prod_{\substack{0 \leq i \leq n \\ \rho_i \in [0, 2\eta_{m_0}^-)}} 2^{t-1}(1 + \tilde{\rho}_i) = +\infty \text{ for all } t > s$$

or

$$\limsup_{n \rightarrow +\infty} \prod_{\substack{0 \leq i \leq n \\ \rho_i \in (2\eta_{m_0}^+, 1]}} 2^{t-1}(1 + \tilde{\rho}_i) = +\infty \text{ for all } t > s$$

Choosing one that holds, we split the betting interval, $[0, 2\eta_{m_0}^-)$ or $(2\eta_{m_0}^+, 1]$, into two, and repeat the argument infinitely. By compactness, we get the existence of some $\bar{\rho}$ such that there exists arbitrarily small intervals $[q_1, q_2)$ (or $(q_1, q_2]$) if in the above we chose $(2\eta_{m_0}^+, 1]$, which does not change the proof) with $q_1, q_2 \in \mathbb{Q}^+$ containing $\bar{\rho}$ and such that:

$$\limsup_n \prod_{\substack{0 \leq i \leq n \\ \rho_i \in [q_1, q_2)}} 2^{t-1}(1 + \tilde{\rho}_i) = +\infty \text{ for all } t > s \quad (2)$$

Note that since $\bar{\rho} \in [0, 2\eta_{m_0}^-] \cup [2\eta_{m_0}^+, 1]$, we have $\bar{\rho} \neq 2\eta$. Moreover, the function $x \mapsto (\frac{1}{2} + \eta) \log(1 + x) + (\frac{1}{2} - \eta) \log(1 - x)$ takes its only maximum for $x = 2\eta$, it follows that:

$$(\frac{1}{2} + \eta) \log(1 + \bar{\rho}) + (\frac{1}{2} - \eta) \log(1 - \bar{\rho}) < (\frac{1}{2} + \eta) \log(1 + 2\eta) + (\frac{1}{2} - \eta) \log(1 - 2\eta)$$

Hence, taking them close enough to $\bar{\rho}$, we can find $q_1, q_2 \in \mathbb{Q}^+$ satisfying (2) and such that:

$$\left(\frac{1}{2} + \eta\right) \log(1 + q_2) + \left(\frac{1}{2} - \eta\right) \log(1 - q_1) < \left(\frac{1}{2} + \eta\right) \log(1 + 2\eta) + \left(\frac{1}{2} - \eta\right) \log(1 - 2\eta)$$

Let then S' be the selective strategy which runs S and whenever S plays $(k_i, 0, \rho_i)$:

- If $\rho_i \in [q_1, q_2)$, S' plays the same thing.
- Otherwise, S' plays (k_i, scans) .

Here again, we define σ to be the selection rule which runs S' and selects a bit if S' bets on it, and scans it if S' scans it.

For all n , we have

$$V_n(\alpha, S') \geq (1 - q_1)^{\#0(\sigma[\alpha], n)} (1 + q_2)^{\#1(\sigma[\alpha], n)}$$

hence

$$\frac{\log V_n(\alpha, S')}{n} \geq \frac{\#0(\sigma[\alpha], n)}{n} \log(1 + q_2) + \frac{\#1(\sigma[\alpha], n)}{n} \log(1 - q_1)$$

Setting $\delta = \text{Bias}(\sigma[\alpha])$, we have:

$$1 - s \leq \limsup_n \frac{\log V_n(\alpha, S')}{n} \leq \left(\frac{1}{2} + \delta\right) \log(1 + q_2) + \left(\frac{1}{2} - \delta\right) \log(1 - q_1)$$

By definition of q_1, q_2 this implies that $\delta \geq \eta$, as wanted. \square

3 Effective Hausdorff dimension and stochasticity

In this section, we investigate the relationship between stochasticity and effective Hausdorff dimension. Hausdorff dimension was defined by Hausdorff in [7] as a measure of dimension in an arbitrary metric space. In the sequel, we will restrict ourselves to the Hausdorff dimension of subsets of 2^ω . For a detailed survey of classical Hausdorff dimension, see [6].

Let X be a subset of 2^ω , and $s \geq 0$. X is said to be an s -nullset if there exists a sequence $(C_n)_{n \in \mathbb{N}}$ of subsets of 2^* such that for all n :

$$X \subseteq \bigcup_{u \in C_n} u2^\omega \quad \text{and} \quad \sum_{u \in C_n} 2^{-s|u|} \leq 2^{-n} \quad (3)$$

The classical Hausdorff dimension of X is defined by:

$$\dim_H(X) = \inf\{s : X \text{ is a } s\text{-nullset}\}$$

We now look at Hausdorff dimension from a computability viewpoint, effectivizing the above definition as Lutz [10] (for constructive dimension) and Downey et al. [4] (for computable dimension) did:

Definition 7 A subset X of 2^ω is a constructive s -nullset if there exists a computable sequence $(C_n)_{n \in \mathbb{N}}$ of computably enumerable subsets of 2^* satisfying (3).

A subset X of 2^ω is a computable s -nullset if there exists a computable sequence $(C_n)_{n \in \mathbb{N}}$ of computably enumerable subsets of 2^* satisfying (3) and such that the real numbers $\sum_{u \in C_n} 2^{-s|u|}$ are uniformly computable.

The constructive Hausdorff dimension $\dim_1(X)$ and the computable Hausdorff dimension $\dim_{comp}(X)$ are then defined by:

$$\dim_1(X) = \inf\{s : X \text{ is a constructive } s\text{-nullset}\}$$

$$\dim_{comp}(X) = \inf\{s : X \text{ is a computable } s\text{-nullset}\}$$

Merkle et al. used the term *Schnorr dimension* instead of computable dimension, but we believe the latter is more appropriate.

Remark that, by the above definitions, for all $X \subseteq 2^\omega$:

$$0 \leq \dim_H(X) \leq \dim_1(X) \leq \dim_{comp}(X) \leq 1$$

For a single sequence $\alpha \in 2^\omega$, we abbreviate $\dim_1(\{\alpha\})$ by $\dim_1(\alpha)$ and $\dim_{comp}(\{\alpha\})$ by $\dim_{comp}(\alpha)$. The effective dimension of a singleton is not a trivial notion: although every singleton has classical Hausdorff dimension 0, the effectivity requirement can make a singleton have positive constructive (or computable) dimension. In particular, Mayordomo proved an elegant characterization of constructive Hausdorff dimension in terms of Kolmogorov complexity (there had been some earlier results in this direction, see the discussion in [3]):

Theorem 8 (Mayordomo [11]) For all $\alpha \in 2^\omega$:

$$\dim_1(\alpha) = \liminf_{n \rightarrow +\infty} \frac{K(\alpha_{[0,n]})}{n}$$

Hausdorff dimension and its effective versions have a game-theoretic characterization. It involves the notion of martingale. A (normed) martingale is a total function $d : 2^* \rightarrow [0, +\infty)$ such that $d(\emptyset) = 1$ (here \emptyset is the empty word) and for all $u \in 2^*$, $d(u) = \frac{d(u0) + d(u1)}{2}$.

A martingale is said to be s -successful on a sequence α if

$$\limsup_{n \rightarrow +\infty} \frac{d(\alpha_{[0,n]})}{2^{(1-s)n}} = +\infty$$

We have the following result, whose first part is due to Lutz [9] and second part to Downey et al. [4].

Theorem 9 For all $X \subseteq 2^\omega$:

$$\dim_H(X) = \inf\{s : \exists d \text{ martingale which } s\text{-succeeds on every } \alpha \in X\}$$

$$\dim_{comp}(X) = \inf\{s : \exists d \text{ computable martingale which } s\text{-succeeds on every } \alpha \in X\}$$

Constructive dimension can also be characterized by game-theoretic concepts (see [10]), but we will not need such a characterization.

The first thing one should remark is that a martingale can be interpreted as the capital of a strategy which bets on every bit (in order). Indeed, if d is a martingale, define S_d by

$$S_d(u) = \begin{cases} (|u|, 0, \frac{d(u0)}{d(u)} - 1) & \text{if } d(u0) \geq d(u1) \\ (|u|, 1, \frac{d(u1)}{d(u)} - 1) & \text{if } d(u0) < d(u1) \end{cases}$$

We then have, for all $\alpha \in 2^\omega$ and all n :

$$V_n(\alpha, S_d) = d(\alpha_{[0, n-1]})$$

Obviously, d is computable if and only if S_d is. Hence, given a computable martingale and a computable selection rule, one can canonically construct a computable selective strategy corresponding to their composition. This remark, together with Proposition 5 and Theorem 6, yields a characterization of KL-stochasticity in terms of computable dimension:

Proposition 10 A sequence α is KL-stochastic iff for every sequence β selected from α by a computable selection rule, $\dim_{comp}(\beta) = 1$

We now turn our attention to the relation between constructive Hausdorff dimension and KL-stochasticity. We shall prove the main theorem of this section:

Theorem 11 For all $\alpha \in 2^\omega$, $\mathcal{H}(\frac{1}{2} + \delta_{max}^{sel}(\alpha)) \leq \dim_1(\alpha)$

The proof follows the three steps we mentioned in the introduction. First, we use a splitting argument:

Lemma 12 Let $\alpha \in 2^\omega$ and s be such that $\dim_1(\alpha) \leq s$. There exists a recursive co-infinite $Z \subseteq \mathbb{N}$ such that, writing $\alpha = (\beta \oplus \beta') \oplus_Z \gamma$, we have:

$$\dim_1^{(\gamma)}(\beta) \leq s \text{ and } \dim_1^{(\gamma)}(\beta') \leq s$$

($\dim_1^{(\gamma)}$ is the dimension relative to the oracle γ).

Proof. In order to prove this lemma, we need the following result:

Lemma 13 (Merkle et al. [13]) For all $\beta', \beta'' \in 2^\omega$, we have:

$$\dim_1(\beta' \oplus \beta'') \geq \frac{1}{2} \dim_1(\beta') + \frac{1}{2} \dim_1^{(\beta')}(\beta'')$$

and symmetrically:

$$\dim_1(\beta' \oplus \beta'') \geq \frac{1}{2} \dim_1(\beta'') + \frac{1}{2} \dim_1^{(\beta'')}(\beta')$$

To obtain Lemma 12 from Lemma 13, let $\alpha \in 2^\omega$ and s with $\dim_1(\alpha) \leq s$. Let us set $t = \inf\{\dim_1^{(\gamma)}(\beta) : \exists Z \text{ recursive s.t. } \alpha = \beta \oplus_Z \gamma\}$. We distinguish two cases:

Case 1: $t = s$. We then write $\alpha = \alpha' \oplus \alpha''$. By Lemma 13, we have

$$\begin{aligned} s = \dim_1(\alpha) &\geq \frac{1}{2} \dim_1(\alpha') + \frac{1}{2} \dim_1^{(\alpha')}(\alpha'') \\ s = \dim_1(\alpha) &\geq \frac{1}{2} \dim_1(\alpha'') + \frac{1}{2} \dim_1^{(\alpha'')}(\alpha') \end{aligned}$$

But by definition of t : $\dim_1^{(\alpha'')}(\alpha') \geq t = s$, and $\dim_1^{(\alpha')}(\alpha'') \geq t = s$. Thus, $\dim_1(\alpha') \leq s$ and $\dim_1(\alpha'') \leq s$. We then get the desired result writing $\alpha = (\alpha' \oplus \alpha'') \oplus_\emptyset \mathbf{0}^\omega$.

Case 2: $t < s$. In this case, let $\beta, \gamma \in 2^\omega$ and Z recursive be such that $\alpha = \beta \oplus_Z \gamma$ and $\dim_1^{(\gamma)}(\beta) \leq \frac{s+t}{2}$. Then write $\beta = \beta' \oplus \beta''$, and let X_1, X_2, Y_1, Y_2 be the recursive subsets of \mathbb{N} such that $\alpha = \beta' \oplus_{X_1} (\beta'' \oplus_{X_2} \gamma)$ and $\alpha = \beta'' \oplus_{Y_1} (\beta' \oplus_{Y_2} \gamma)$. Relativizing Lemma 13 to the oracle γ , we get:

$$\begin{aligned} \frac{s+t}{2} &\geq \dim_1^{(\gamma)}(\beta) \geq \frac{1}{2} \dim_1^{(\gamma)}(\beta') + \frac{1}{2} \dim_1^{(\beta' \oplus_{Y_2} \gamma)}(\beta'') \\ \frac{s+t}{2} &\geq \dim_1^{(\gamma)}(\beta) \geq \frac{1}{2} \dim_1^{(\gamma)}(\beta'') + \frac{1}{2} \dim_1^{(\beta'' \oplus_{X_2} \gamma)}(\beta') \end{aligned}$$

By definition of t , $\dim_1^{(\beta' \oplus_{Y_2} \gamma)}(\beta'') \geq t$ and $\dim_1^{(\beta'' \oplus_{X_2} \gamma)}(\beta') \geq t$. Hence, $\dim_1^{(\gamma)}(\beta') \leq s$ and $\dim_1^{(\gamma)}(\beta'') \leq s$, as wanted. \square

The above splitting result allows us to apply the competing strategies technique, to get:

Lemma 14 Let $\alpha, \beta, \gamma \in 2^\omega$ and s such that $\dim_1^{(\gamma)}(\alpha) \leq s$ and $\dim_1^{(\gamma)}(\beta) \leq s$. There exists a selective strategy S , computable with oracle (s, γ) , such that for all $t > s$, $\limsup \frac{V_n(\alpha \oplus \beta, S)}{2^{(1-t)n}} = +\infty$.

Proof. We in fact prove the following proposition, which, relativized to oracle γ , implies Lemma 14.

Proposition 15 Let α and β be in 2^ω and s be such that $\dim_1(\alpha) \leq s$ and $\dim_1(\beta) \leq s$. There exists a selective strategy S , computable with oracle s , such that for all $t > s$, $\limsup \frac{V_n(\alpha \oplus \beta, S)}{2^{(1-t)n}} = +\infty$.

To do so, we will need the following lemmas:

Lemma 16 (Muchnik et al. [14]) *Let $\{(u_0, q_0), \dots, (u_n, q_n)\}$ be a finite subset of $2^* \times \mathbb{Q}^{>0}$ such that: $\sum_{i \leq n} q_i \leq 1$. There exists a computable martingale d such that for all $i \leq n$, $d(u_i) \geq q_i 2^{|u_i|}$. As we have seen, we can associate to d a strategy S_d . We denote this strategy by $\text{Strat}(\{(u_0, q_0), \dots, (u_n, q_n)\})$*

Lemma 17 (Merkle et al. [13]) *Let $\alpha \in 2^\omega$ such that $\dim_1(\alpha) = s < 1$. Let $\{s_m\}_{m \in \mathbb{N}}$ be a decreasing sequence converging to s . For all $N, l, m \in \mathbb{N}$, there exists an N' such that $N' > N + l$ and $K(\alpha_{[N:N']}) \leq s_m(N' - N)$.*

Now, let α, β be in 2^ω , with $\dim_1(\alpha) \leq s$ and $\dim_1(\beta) \leq s$. Using s as an oracle, let us compute a decreasing sequence $\{s_m\}_{m \in \mathbb{N}}$ of rational numbers converging to s . Let also D be a computable integer-valued function such that $\lim \frac{D(n)}{n} = +\infty$. Let S_1 and S_2 be the selective strategies which, on $\alpha \oplus \beta$, work as follows:

We proceed by stages. At the beginning of stage m , S_1 and S_2 have both read the first p_m bits of α , and the first p_m bits of β (we set $p_0 = 0$, the sequence $\{p_m\}_{m \in \mathbb{N}}$ will be defined inductively during the game), and nothing else. At stage m , S_1 (resp. S_2) first enumerates $A_m = \{u : |u| > D(p_m) \text{ and } K(u) \leq s_m |u|\}$ and finds the smallest τ_m (resp. τ'_m) such that $\beta_{[p_m, +\infty)}$ (resp. $\alpha_{[p_m, +\infty)}$) has a prefix in $A_m[\tau_m]$ (resp. $A_m[\tau'_m]$), reading exactly as many bits of β (resp. α) as necessary. The existence of τ_m and τ'_m is asserted by Lemma 17. We set $L_m = \max\{|u| : u \in A_m[\tau_m]\}$ and $L'_m = \max\{|u| : u \in A_m[\tau'_m]\}$.

Then, S_1 (resp. S_2) saves half of its current capital, and uses the rest to apply $\text{Strat}(\{(u, 2^{-s_m|u|}) : u \in A_m[\tau_m]\})$ (resp. $\text{Strat}(\{(u, 2^{-s_m|u|}) : u \in A_m[\tau'_m]\})$) to $\alpha_{[p_m, +\infty)}$ (resp. $\beta_{[p_m, +\infty)}$) during L_m (resp. L'_m) moves. Notice that $\text{Strat}(\{(u, 2^{-s_m|u|}) : u \in A_m[\tau_m]\})$ is well-defined since:

$$\sum_{u \in A_m[\tau_m]} 2^{-s_m|u|} \leq \sum_{u \in A_m[\tau_m]} 2^{-K(u)} \leq \sum_{u \in 2^*} 2^{-K(u)} \leq 1$$

The first inequality is due to the definition of A_m , and the third one is a well known fact in Kolmogorov complexity (see for example [8]).

Finally, S_1 (resp. S_2) computes $p_{m+1} = p_m + 1 + \max(L_m, L'_m)$, reading exactly as many bits of α (resp. β) as necessary. Finally, both S_1 and S_2 read those of the first p_{m+1} bits of α and β they have not read yet. This finishes stage m .

It is clear that either $\tau'_m \leq \tau_m$ infinitely often, or $\tau_m \leq \tau'_m$ infinitely often. Without loss of generality, assume the first holds.

Notice that whenever $\tau'_m \leq \tau_m$, $\alpha_{[p_m, +\infty)}$ has a prefix $\alpha_{[p_m, p_m + N_m]}$ in $A_m[\tau_m]$. S_1 , by definition, applies $\text{Strat}(\{(u, 2^{-s|u|}) : u \in A_m[\tau_m]\})$ to $\alpha_{[p_m, +\infty)}$ during L_m moves, with $L_m \geq N_m$. After $p_m + N_m$ moves:

- The capital of S_1 is at least $2^{-(m+1)} 2^{(1-s_m)N_m}$. Indeed, S_1 lost at most half of its capital during each previous stage, hence the term $2^{-(m+1)}$. The term $2^{(1-s_m)N_m}$ is an application of Lemma 16: it represents the factor

by which the capital is multiplied when playing on $\alpha_{[p_m, N_m]}$ according to $\text{Strat}(\{(u, 2^{-s|u|}) : u \in A_m[\tau_m]\})$ during N_m moves.

- The number of bets made up until then is at most $p_m + N_m$ (since the number of bets cannot be greater than the number of moves!)

Thus, the ratio $\frac{\log(\text{capital})}{\text{number of bets}}$ is at that point greater than $\frac{(1-s_m)N_m - (m+1)}{p_m + N_m}$.

But by definition of A_m , $p_m = o(N_m)$ (and a fortiori $m = o(N_m)$). It follows that

$$\limsup_m \frac{(1-s_m)N_m - (m+1)}{p_m + N_m} \geq 1-s$$

And thus

$$\limsup_n \frac{\log V_n(\alpha \oplus \beta, S_1)}{n} \geq 1-s$$

which completes the proof. \square

Lemma 12 and Lemma 14 yield:

Proposition 18 *Let $\alpha \in 2^\omega$ and s such that $\dim_1(\alpha) \leq s$. There exists a selective strategy S , computable with oracle s , such that for all $t > s$,*

$$\limsup_{n \rightarrow +\infty} \frac{V_n(\alpha, S)}{2^{(1-t)n}} = +\infty$$

Proof. Let $\alpha \in 2^\omega$ such that $\dim_1(\alpha) \leq s$. Let β, β', γ be as in Lemma 12. By Lemma 14 there exists a strategy S , computable with oracle (s, γ) and playing on $\beta' \oplus \beta''$ such that $\limsup_n \frac{V_n(\beta' \oplus \beta'', S)}{n} = +\infty$. From S we can deduce a strategy S' , computable with oracle s , and playing on α such that $\limsup_n \frac{V_n(\alpha, S')}{n} = +\infty$. Indeed, let S' be defined as follows: it makes exactly the same moves as S i.e. scans and bets on the same bits of β' and β'' as S , except that whenever S uses oracle γ , S' scans the corresponding bits of γ in $\alpha = (\beta' \oplus \beta'') \oplus_Z \gamma$, to get the same information. Since no extra bet is made this way, we have for all n , $V_n(\beta' \oplus \beta'', S) = V_n(\alpha, S')$, hence the result. \square

Finally, converting the strategy S into a selection rule according to Theorem 6, the above proposition can be rephrased as follows:

Proposition 19 *Let $\alpha \in 2^\omega$ and s such that $\dim_1(\alpha) \leq s$. There exists a selection rule σ , computable with oracle s , such that, setting $\delta = \text{Bias}(\sigma[\alpha])$, we have $\mathcal{H}(\frac{1}{2} + \delta) \leq s$.*

To get Theorem 11 from Proposition 19, remark that in Proposition 19, if s is a rational number, σ is computable. Hence, let us take a decreasing sequence $\{s_m\}_m$ of rational numbers converging to s . For all m , by Proposition 19, there exists a computable strategy σ_m selecting a subsequence β with bias δ_m such that $\mathcal{H}(\frac{1}{2} + \delta_m) \leq s_m$. Setting $\delta = \sup_m \delta_m$, it follows that $\mathcal{H}(\frac{1}{2} + \delta) \leq s$, and

hence $\mathcal{H}(\frac{1}{2} + \delta_{max}^{sel}(\alpha)) \leq s$.

The bound we give for $\delta_{max}^{sel}(\alpha)$ in Theorem 11 is optimal. Indeed, let us generate a sequence α by choosing its bits at random and independently, in such a way that for all i , the probability of α_i to be 1 is $\frac{1}{2} + \delta$. Then, with probability 1:

- Every sequence β selected from α by a computable selection rule has bias exactly δ (see van Lambalgen [17], Shen [16])
- $\lim \frac{K(\alpha_{[0,n]})}{n} = \mathcal{H}(\frac{1}{2} + \delta)$ (see Zvonkin and Levin [18])

Hence, for all α satisfying these two conditions, we have $\delta_{max}^{sel}(\alpha) = \delta$ and $\dim_1(\alpha) = \mathcal{H}(\frac{1}{2} + \delta)$.

Note that although the bound of Theorem 11 is optimal, there are some cases where $\mathcal{H}(\frac{1}{2} + \delta_{max}^{sel}(\alpha))$ is much smaller than $\dim_1(\alpha)$: take a Martin-Löf random sequence α and consider $\beta = \alpha \oplus_Z 0^\omega$ with $Z = \{n^2 : n \in \mathbb{N}\}$. In this case, $\delta_{max}^{sel}(\beta) = \frac{1}{2}$ (one just needs to select the bits whose position is in Z) which means $\mathcal{H}(\frac{1}{2} + \delta_{max}^{sel}(\beta)) = 0$, whereas $\dim_1(\beta) = 1$.

The martingale characterization of computable Hausdorff dimension, together with Proposition 19, provides the following relation between the two notions of effective dimension:

Proposition 20 *Let $\alpha \in 2^\omega$. There exists a selection rule σ , computable with oracle $\dim_1(\alpha)$, such that $\dim_{comp}(\sigma[\alpha]) \leq \dim_1(\alpha)$.*

4 Kolmogorov-Loveland stochasticity for finite binary sequences

The study of Kolmogorov-Loveland stochasticity for finite sequences was initiated by E. Asarin [2]. The extension of Kolmogorov-Loveland stochasticity to finite sequences is more quantitative, i.e. contrary to infinite sequences, there is no clear separation between stochastic and non-stochastic. Rather, for each finite sequence u of length N , and each selection rule σ , there are three key-parameters:

- the Kolmogorov complexity of σ : $K(\sigma|N)$
- the size of the selected subsequence: $|\sigma[u]|$
- the bias of the selected subsequence: $Bias(\sigma[u]) = \left| \frac{\#0(\sigma[u], N')}{N'} - \frac{1}{2} \right|$ (where $N' = |\sigma[u]|$)

The smaller the first, and the bigger the two others, the less stochastic u is. Asarin [2], Durand and Vereshchagin [5] proved respectively an upper bound and a lower bound of the bias one can obtain by selecting a subsequence of a sequence with some randomness deficiency, these bounds depending on the

randomness deficiency, the Kolmogorov complexity of the selection rule and the size of the selected subsequence. Moreover, these bounds are very general as they require (almost) no restriction of their three parameters. We instead focus on a particular case, which we believe is very natural given the above discussion on infinite sequences: for a finite sequence u with randomness deficiency $(1-s)|u|$, what bias can we obtain if we require the Kolmogorov complexity of the selection rule to be $O(1)$, and the size of the selected subsequence to be $\Omega(|u|)$? This question was raised by Durand and Vereshchagin (open question 1 of [5]). The following two theorems provide an answer to this question and show that in the case of finite sequences too, the constant δ such that $\mathcal{H}(\frac{1}{2} + \delta) = s$ is a threshold for the extraction of biased subsequences.

Theorem 21 *For all $s \in [0, 1]$ and all δ such that $\mathcal{H}(\frac{1}{2} + \delta) > s$, there exist real constants c_1, c_2 such that for all large enough N and every finite sequence u of length N satisfying $K(u) \leq sN$, there exists a selection rule σ such that*

$$K(\sigma|N) \leq c_1, |\sigma[u]| \geq c_2N \text{ and Bias}(\sigma[u]) \geq \delta$$

Proof. The idea of the proof is very similar to the one of Proposition 19. However, it requires some extra care as we want the selected subsequence to be comparable in length to the initial sequence. Indeed, in the proof of Proposition 19, the selection rule we construct might possibly select a very sparse subsequence of the initial sequence, and hence the construction cannot be modified in a straightforward way to prove Theorem 21.

We start with a lemma, which is a slightly modified version of the lemma 7 of [5].

Lemma 22 *Let $s \in [0, 1]$ and $\varepsilon > 0$. For all large enough N , the following holds. Every finite sequence u of length N such that $K(u) \leq s|u|$ can be chopped into 2^k (with $k \leq \lceil 2s/\varepsilon \rceil$) blocks w_1, \dots, w_{2^k} of length $\frac{N}{2^k} + O(\log k)$, such that there exist two consecutive blocks w_i and w_{i+1} which satisfying*

$$K(w_i|v, v') \leq (s + \varepsilon)|w_i|$$

$$K(w_{i+1}|v, v') \leq (s + \varepsilon)|w_{i+1}|$$

where v, v' are such that $u = vw_iw_{i+1}v'$

Proof of Lemma 22: We prove this by induction on the quantity $\lceil 2s/\varepsilon \rceil$.

If $\lceil 2s/\varepsilon \rceil = 1$, then split u into two halves of equal (up to 1) length $u = u_1u_2$. We have, up to an additive constant $K(u_1) \leq K(u) + O(1) \leq s|u| \leq 2s|u_1|$. But since by the hypothesis $2s/\varepsilon \leq 1$ (i.e. $\varepsilon \geq 2s$) this yields $K(u_1) \leq \varepsilon|u_1| + O(1)$. Similarly, $K(u_2) \leq \varepsilon|u_2| + O(1)$, hence the result.

Induction step: we first recall the Levin-Kolmogorov identity. For all finite sequences u, v holds $K(uv) = K(u) + K(v|u) + O(\log |uv|)$. Let u be a finite sequence such that $K(u) \leq s|u|$. Here again, split u into two halves $u = u_1u_2$. If $K(u_1) \leq (s + \varepsilon)|u_1|$ and $K(u_2) \leq (s + \varepsilon)|u_2|$, we are done. If not, but the Kolmogorov-Levin identity, it follows that either $K(u_1|u_2) \leq (s - \varepsilon)|u_1|$

or $K(u_2|u_1) \leq (s - \varepsilon)|u_2| + O(\log |u_1|)$. Without loss of generality, suppose that the first holds. If $|u_1|$ is big enough, we have $K(u_1|u_2) \leq (s - \frac{\varepsilon}{2})|u_1|$. By the induction hypothesis, conditioned by u_2 , u_1 can be chopped into 2^k (with $k \leq \lceil 2(s - \frac{\varepsilon}{2})/\varepsilon \rceil = \lceil 2s/\varepsilon \rceil - 1$) blocks w_1, \dots, w_{2^k} of length $\frac{n}{2^k} + O(\log k)$ such that there exist two consecutive blocks w_i, w_{i+1} satisfying $K(w_i|v, v', u_2) \leq (s + \varepsilon)|w_i|$ and $K(w_{i+1}|v, v', u_2) \leq (s + \varepsilon)|w_{i+1}|$. This completes the induction.

Lemma 23 *Let $s \in [0, 1]$, and δ such that $\mathcal{H}(\frac{1}{2} + \delta) > s$. For all large enough N , and every finite sequence u of length N , the following holds. For every selective strategy S which, playing against u , multiplies its initial capital by at least $2^{(1-s)N}$, there exist a selection rule σ such that $K(\sigma|N) \leq K(S|N) + O(1)$, $|\sigma[u]| = \Omega(N)$, and $\text{Bias}(\sigma[u]) \geq \delta$.*

Proof of Lemma 23: Fix $\varepsilon > 0$. Let ρ_i be the stake of S during the i -th bet. As in the proof of Theorem 6, we can assume that S always predicts the value of a bit to be 0 (up to replacing the other bets by bets with stake 0). We can also assume that the strategy bets on every bit. We have

$$\prod_{i=1}^N (1 + \tilde{\rho}_i) \geq 2^{(1-s)N}$$

and thus

$$\prod_{i=1}^N (1 + \tilde{\rho}_i) 2^{(s-1+\varepsilon)N} \geq 2^{\varepsilon N}$$

Let us split the interval $[0, 1]$ into m subintervals $[a_k, b_k]$ such that for all k , $\log(1 + b_k) \leq \log(1 + a_k) + \varepsilon$. We then have:

$$\prod_{1 \leq k \leq m} \prod_{\substack{1 \leq i \leq N \\ \rho_i \in [a_k, b_k]}} (1 + \tilde{\rho}_i) 2^{(s-1+\varepsilon)N} \geq 2^{\varepsilon N}$$

Hence, there exists some j such that

$$\prod_{\substack{1 \leq i \leq N \\ \rho_i \in [a_j, b_j]}} (1 + \tilde{\rho}_i) 2^{(s-1+\varepsilon)N} \geq 2^{\varepsilon N/m}$$

Let us first estimate the size of $I_j = \{i : \rho_i \in [a_j, b_j]\}$. Since for all i , $\tilde{\rho}_i \leq 1$, by the last inequality above:

$$2^{\varepsilon N/m} \leq \prod_{\substack{1 \leq i \leq N \\ i \in I_j}} (1 + \tilde{\rho}_i) 2^{(s-1+\varepsilon)|I_j|} \leq 2^{(s+\varepsilon)|I_j|}$$

and hence

$$|I_j| \geq \frac{\varepsilon N}{(s + \varepsilon)m}$$

Now, let σ be the selection rule which selects a bit whenever S bets on it a fraction $\rho \in [a_j, b_j]$ of its capital. Notice that $K(\sigma|N) \leq K(S|N) + O(1)$ since σ can be specified using S , m and j . Let $n = |I_j| = |\sigma[u]|$ (notice that $n = \Omega(N)$), and $\delta = \text{Bias}(\sigma[u])$. Similarly to the proof of Theorem 6, we have:

$$2^{(1-s-\varepsilon)n} \leq (1+b_j)^{(\frac{1}{2}+\delta)n} (1-a_j)^{(\frac{1}{2}-\delta)n}$$

hence

$$\begin{aligned} 1-s-\varepsilon &\leq \left(\frac{1}{2}+\delta\right) \log(1+b_j) + \left(\frac{1}{2}-\delta\right) \log(1-a_j) \\ &\leq \left(\frac{1}{2}+\delta\right) \log(1+a_j) + \left(\frac{1}{2}-\delta\right) \log(1-a_j) + \varepsilon \\ &\leq 1 - \mathcal{H}\left(\frac{1}{2}+\delta\right) + \varepsilon \end{aligned}$$

(The second inequality is a consequence of the definition of the a_i 's and b_i 's, the third inequality is a consequence of the fact that the function $x \mapsto (\frac{1}{2}+\delta) \log(1+x) + (\frac{1}{2}-\delta) \log(1-x)$ reaches its maximum for $x = 2\delta$). Finally, the above yields $\mathcal{H}(\frac{1}{2}+\delta) \leq s+2\varepsilon$. Since ε can be taken arbitrarily small, we get the desired result.

The proof of Theorem 21 now goes as follow. Fix $\varepsilon > 0$. Let u be a finite sequence of length N such that $K(u) \leq s|u|$. By Lemma 22, one can chop u into at most $2^{\lceil 2s/\varepsilon \rceil}$ blocks of equal (up to a constant independant on N) length N' , with $N' = \Omega(N)$, such that there exist two consecutive blocks w_i and w_{i+1} such that, writing $u = vw_1w_2v'$, one has $K(w_1|v, v') \leq (s+\varepsilon)N'$ and $K(w_2|v, v') \leq (s+\varepsilon)N'$.

Then, as in the proof of Theorem 6, one can construct two strategies S_1, S_2 such that:

- S_1 scans v, v', w_{i+1} , and bets on w_i
- S_2 scans v, v', w_i , and bets on w_{i+1}
- either S_1 or S_2 multiplies its capital by $2^{N'}$

Moreover, the strategies are of complexity $O(1)$, as they are specified by the positions of w_i and w_{i+1} , which are of complexity $O(\lceil 2s/\varepsilon \rceil)$.

Then, apply Lemma 23 to the successful strategy to get a selection rule σ of complexity $O(1)$ which selects a subsequence of length $\Omega(N') = \Omega(N)$ which is biased with bias δ such that $\mathcal{H}(\frac{1}{2}+\delta) \leq s+2\varepsilon$. Once again, since ε can be chosen arbitrarily small, this completes the proof of Theorem 21. \square

Theorem 24 *There is no tuple (s, δ, c_1, c_2) , with $s \in [0, 1]$, $\mathcal{H}(\frac{1}{2}+\delta) < s$ and c_1, c_2 positive real constants such that for all large enough N and all finite sequence u of length N satisfying $K(u) \leq sN$, there exists a selection rule σ satisfying:*

$$K(\sigma|N) \leq c_1, \quad |\sigma[u]| \geq c_2N \quad \text{and} \quad \text{Bias}(\sigma[u]) \geq \delta$$

Proof. We here again use a probabilistic argument. Let (s, δ, c_1, c_2) be such a tuple. Recall that if a finite binary sequence of u length N is chosen at random using N independant Bernoulli random variables of parameter $\frac{1}{2} + \eta$, we have (see Zvonkin and Levin [18])

$$\mathbb{E}(K(u)) = \mathcal{H}\left(\frac{1}{2} + \eta\right) N + o(N)$$

Let $\eta > 0$ be such that $\mathcal{H}(\frac{1}{2} + \delta) < \mathcal{H}(\frac{1}{2} + \eta) < s$, and choose a finite binary sequence u_N of length N at random using N independant Bernoulli random variables of parameter η . By the above, we have

$$\lim_{N \rightarrow +\infty} \text{Prob}\left[K(u_N) \leq sN\right] = 1 \quad (4)$$

Moreover for each selection rule σ :

$$\lim_{N \rightarrow +\infty} \text{Prob}\left[|\sigma[u_N]| \geq c_2 N \text{ and } \text{Bias}(\sigma[u_N]) \geq \delta\right] = 0$$

Indeed, the Bernoulli variables being independant, the selected subsequence $\sigma[u_N]$ can itself be seen as a sequence obtained with independant Bernoulli random variables of parameter η . And since $\delta > \eta$, by the law of large numbers we get the above identity. Finally, since for all N there are at most 2^{c_1} selection rules such that $K(\sigma|N) \leq c_1$, it follows that

$$\lim_{N \rightarrow +\infty} \text{Prob}\left[\exists \sigma : K(\sigma|N) \leq c_1 \text{ and } |\sigma[u_N]| \geq c_2 N \text{ and } \text{Bias}(\sigma[u_N]) \geq \delta\right] = 0 \quad (5)$$

By (4) and (5), the theorem is proved. \square

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