Distance Labeling in Hyperbolic Graphs

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Abstract. A graph G is δ-hyperbolic if for any four vertices u, v, x, y of G the two larger of the three distance sums $d_G(u,v) + d_G(x,y), d_G(u,x) + d_G(v,y), d_G(u,y) + d_G(v,x)$ differ by at most δ, and the smallest $\delta \geqslant 0$ for which G is δ-hyperbolic is called the hyperbolicity of G.
In this paper, we construct a distance labeling scheme for bounded hyperbolicity graphs, that is a vertex labeling such that the distance between any two vertices of G can be estimated from their labels, without any other source of information. More precisely, our scheme assigns labels of $O(\log^2 n)$ bits for bounded hyperbolicity graphs with n vertices such that distances can be approximated within an additive error of $O(\log n)$. The label length is optimal for every additive error up to n^{ε}. We also show a lower bound of $\Omega(\log\log n)$ on the approximation factor, namely every s-multiplicative approximate distance labeling scheme on bounded hyperbolicity graphs with polylogarithmic labels requires $s = \Omega(\log\log n)$.

Keywords. Distance queries, distance labeling scheme, hyperbolic graphs.

1 Introduction

It is well-known that a metric space (V, d) embeds into a tree metric if and only if the 4-point condition holds, that is, for any 4 points u, v, x, y of V the two larger of the sums $d(u,v) + d(x,y), d(u,x) + d(v,y), d(u,y) + d(v,x)$ are equals [1]. More generally, if the two larger sums differ by at most δ, then the metric space is said to be δ-*hyperbolic*. Introduced by Gromov [21, 20], δ-hyperbolic spaces arise naturally in the area of geometric group theory. In a certain extend hyperbolicity measures the deviation from tree-likeness. And thus, it appears in a natural way as a generalization of the study of trees in metric graph theory [4, 5, 13], classification theory [8], phylogenetic analysis [29], and Gauber dynamics also known as Gibbs samplers [25].

A graph $G = (V, E)$ is δ-*hyperbolic* if (V, d_G) is a δ-hyperbolic metric space, where d_G is the shortest-path metric of G, associating to each pair of vertices the length of a shortest path connecting them. The *hyperbolicity* of G is the smallest $\delta \geqslant 0$ for which G is δ-hyperbolic. The graphs considered in this paper are unweighted, simple, and connected.

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0-hyperbolic graphs are precisely the block graphs [5, 12, 24], i.e., graphs in which every 2-connected subgraph is a clique, and chordal graphs, i.e., the graphs containing no induced cycles of length larger than three, are 2-hyperbolic [7]. It is not difficult to see from the definition that graphs of diameter D are $(2\lfloor D/2 \rfloor)$-hyperbolic. 1-hyperbolic graphs have been partially characterized in [26], and recently a full characterization has been given in terms of a convexity condition and forbidden isometric subgraphs [3].

This paper deals with the problem of the distance computation and distributed abilities of δ-hyperbolic graphs. Commonly, when we make a query concerning a set of nodes in a graph (adjacency, distance, connectivity, etc.), we need to make a global access to the structure. In our approach, the compromise is to store the maximum of information in a label associated with a vertex to have directly what we need with a local access. Motivation of localized data-structures in distributed computing is survey and widely discussed in [18].

We are especially interested in the distance labeling problem, introduced by Peleg in [30]. The problem consists in labeling the vertices of a graph in order to compute or estimate the distance between any two of its vertices x and y using only the information stored in the labels of x and y, without any other source of information. The main parameters taken into account when designing a solution is the maximum label length (in bits) assigned by the labeling. More formally, an (s, r)-*approximate distance labeling scheme* on a given graph family $\mathfrak{F}$ is a pair $\langle L, f \rangle$, L is called the *labeling* function and f the *distance decoder*, such that, for every $G \in \mathfrak{F}$ and for all $x, y \in V(G)$: $L(x, G) \in \{0, 1\}^*$, and $d_G(x, y) \leqslant f(L(x, G), L(y, G)) \leqslant s \cdot d_G(x, y) + r$. If $s = 1$ and $r = 0$, then we shortly deal with a distance labeling scheme (or DLS). Also, an $(s, 0)$-approximate DLS is called s-multiplicative, and a $(1, r)$-approximate DLS is called r-additive.

Related works for distance labeling. The main results on the field are that general graphs support an (exact) distance labeling scheme with labels of $O(n)$ bits [19], and that trees [2, 30], bounded tree-width graphs [19], distance-hereditary graphs [16], bounded clique-width graphs [10], some non-positively curved plane graphs [9], all support distance labeling schemes with $O(\log^2 n)$ bit labels. Since 0-hyperbolic graphs are block graphs, which are distance-hereditary, it follows that this class supports a $O(\log^2 n)$ bit label DLS.

The $O(n)$ bit upper bound is tight for general graphs (simply by counting the number of n-vertex graphs), and a lower bound of $\Omega(\log^2 n)$ bit on the label length is known for trees [19], implying that all the results mentioned above (including 0-hyperbolic graphs) are tight as well since all of them contains trees. Recently, [17, 6] showed an optimal bound of $O(\log n)$ bits for interval graphs,

permutation graphs, and their generalizations (circular-arc graphs and cicurlar permutation graphs).

Other results concern approximated distance labeling schemes. For arbitrary graphs, the best scheme in date is due to Thorup and Zwick [34]. They propose a $(2k-1)$-multiplicative DLS, for each integral parameter $k \geqslant 1$, with labels of $O(n^{1/k}\log^2 n)$ bits. Moreover, $\Omega(n^{1/k})$ bit labels are required in the worst-case for every s-multiplicative DLS with $s < 2k+1$. In fact, this result relies to a 1963 girth conjecture of Erdös [14] proved for $k = 1, 2, 3$ and 5. However, for all the other values of k, the results of [27] implies that the $\Omega(n^{1/k})$ lower bound is true for $s < 4k/3 + 2$.

In [15], it is proved that trees (and bounded tree-width graphs as well) enjoy a $(1 + 1/\log n)$-multiplicative DLS with labels of $O(\log n \cdot \log\log n)$ bits, and this is tight in terms of label length and approximation. They also design some $O(1)$-additive DLS with $O(\log^2 n)$ bit labels for several families of graphs including: the graphs with bounded longest induced cycle, and, more generally, the graphs of bounded tree-length, i.e., that admit a Robertson-Seymour tree-decomposition in bags of bounded diameter (see [11]). Interestingly, it is easy to show that every exact DLS for these families of graphs needs labels of $\Omega(n)$ bits in the worst-case (e.g., considering some chordal graphs, namely the split graphs [15]).

Recently, the graphs with doubling dimension α have been considered, i.e., the graphs for which, for every r, each ball of radius $2r$ can be covered by at most 2^α balls of radius r. It generalizes Euclidian metrics and bounded growth graphs, and includes many realistic networks. After several successive improvements [22, 32, 28], the best scheme in date, due to Slivkins [31], is a $(1+\varepsilon)$-multiplicative DLS with $O(\varepsilon^{-O(\alpha)}\log n \cdot \log\log n)$ bit labels. This is optimal for bounded α by combining the results of [28] and the lower bound of [15] for trees. Finally, in [33], it is shown that planar graphs enjoy a $(1+\varepsilon)$-multiplicative DLS with labels of $O(\varepsilon^{-1}\log^3 n)$ bits (see also [23]).

Our results. From the above list of results, it is clear that 0-hyperbolic graphs enjoy an (exact) DLS with $O(\log^2 n)$ bit labels, and that moreover every DLS for 2-hyperbolic graphs requires some labels of $\Omega(n)$ bits. Again, some chordal graphs, that are all 2-hyperbolic, require $\Omega(n)$ bit labels [15].

Our first contribution is a lower bound on an s-multiplicative DLS for bounded hyperbolicity graphs. We construct a family of bounded hyperbolic graphs for which, for every integer $k \geqslant 1$, every s-multiplicative DLS with $s < 2\log k + O(1)$ requires some labels of $\Omega(n/\log k)^{1/k}$ bits. In particular, for $k = \Theta(\log n/\log\log n)$, it implies that any s-multiplicative DLS using labels of any poly-logarithmic length requires $s = \Omega(\log\log n)$.

On the positive side, we construct for δ-hyperbolic graphs an $\delta \log n$-additive DLS with labels of $O(\log^2 n)$ bits. The label length is optimal since every r-additive DLS for trees, and thus for δ-hyperbolic graphs for every $\delta \geqslant 0$, requires $\Omega(\log^2(n/r))$ bit labels [19]. In the full version, we show that any poly-log label DLS for bounded hyperbolic graphs requires $r = \Omega(\log n)$, proving the optimality of the approximation of our scheme.

Due to the lack of space, proofs appear in the full version.

2 Pyramidal Construction

Our lower bound combines several ingredients. First we show how to construct from any graph G a graph P, called the *pyramid* of G, such that: 1) G is a subgraph of P; 2) P has bounded hyperbolicity (i.e., bounded by some constant independent of G); and 3) $d_P(x,y) \geqslant 2\log d_G(x,y) - O(1)$ for all x,y in G. In particular the girth of P is at least the log of the girth of G.

Let G be a graph. Let D denote the diameter of G. Let us consider $\lceil \log D \rceil + 1$ copies of G denoted by $G_0, G_1, \ldots, G_{\lceil \log D \rceil}$. Let us consider a new graph constructed as follows: we start from the disjoint union of the G_i's. We add some new edge as follows: First, for any vertex v of G, let us denote v^i the copy of v in G_i. For any $i = 0, \ldots, \lceil \log D \rceil - 1$, let us add an edge between v^i and v^{i+1}. Such an edge shall be said to be vertical. Second, by induction, let us add a new edge between any two vertices of G_i, let say v_1^i and v_2^i, if their copies in G_{i-1}, v_1^{i-1} and v_2^{i-1}, are at distance 2. Such an edge but also any orginal edge of some G_i's shall be said to be transversal. The graph that we obtain is denoted by $P(G)$, it is called the *pyramid* graph of G, G_0 is called the *base* of $P(G)$.

Lemma 1. *There exists a constant K such that for any G, the hyperbolicity of $P(G)$ is at most K.*

Geodesics of $P(G)$. Here we consider the shape of geodesics of $P(G)$ in order to prove that $d_{P(G)}(x,y) \geqslant 2\log d_G(x,y) - O(1)$ for all x,y in G. The successive steps of this study are presented here along the following propositions. For any vertex v of $P(G)$, let us call the *height* of v the unique i such that $v \in G_i$, it shall be denoted by $h(v)$. For any two vertices v_1 and v_2 of the same height h, we denote $d_{G_h}(v_1, v_2)$ the distance between v_1 and v_2 in the subgraph of $P(G)$ generated by the vertices of G_h. We denote by D_{G_h} the maximum of d_{G_h}. If p is a geodesic, i.e., a shortest path, then $\ell(p)$ denotes its length.

Proposition 1. *Let v_1 and v_2 be two vertices of G. Then $d_{G_h}(v_1^h, v_2^h) = \lceil d_G(v_1, v_2)/2^h \rceil$. In particular $D_{G_h} \leqslant \lceil D/2^h \rceil$.*

Proposition 2. *Let p be a geodesic of $P(G)$ which only uses transversal edges. Then $\ell(p) \leqslant 5$.*

Let us consider a path $p = v_0 v_1 \dots v_t$ of length t. Let us consider the sequence of respective heights : $h_0 h_1 \dots h_t$. We say that p is *increasing* (resp. *decreasing*) if the sequence of heights is increasing (resp. decreasing).

Proposition 3. *Let $p = v_0 v_1 \dots v_t$ be a geodesic of $P(G)$. Let us assume that $h(v_0) \geqslant h(v_t)$. Then there exists a vertex v_i of p such that $v_0 \dots v_i$ is increasing and $v_i \dots v_t$ is decreasing.*

We consider a special kind of geodesic that we call *straight* geodesic. These are those having the following shape: first, it starts by using a sequence of vertical edges; second, it carries on by a sequence of transversal edges; and finally it uses a sequence of vertical edges.

Proposition 4. *For any geodesic p, there exists a straight geodesic p' with same extremities. Moreover, p is totally included into a 5-neighbourhood of p' and conversely.*

Proposition 5. *Let x and y be two vertices of $P(G)$. Let p be such a straight geodesic between x and y. Let us assume that $h(x) \leqslant h(y)$. Let x' be the copy of x in $G_{h(y)}$. Let h be the minimal of the lengths of the vertical parts of p Then* $\log(d_{G_{h(y)}}(x', y)) - 3 \leqslant h \leqslant \log(d_{G_{h(y)}}(x', y)) - 1$

The following proposition compares distances of $P(G)$ with those of the G_i's.

Proposition 6. *Let x and y be two vertices of $P(G)$ with $h(x) \leqslant h(y)$. Let x' be the copy of x in $G_{h(y)}$. Then $h(y) - h(x) + 2\log(d_{G_{h(y)}}(x', y)) - 3 \leqslant d_{P(G)}(x, y) \leqslant h(y) - h(x) + 2\log(d_{G_{h(y)}}(x', y)) + 4$.*

In particular, for all $x, y \in G$, $2\log(d_G(x, y)) - 3 \leqslant d_{P(G)}(x, y) \leqslant 2\log(d_G(x, y)) + 4$.

Proposition 7. *If p and p' are two geodesics with same extremities, then p is totally included into a 11-neighbourhood of p' and conversely.*

Sketch of the Proof of Lemma 1 Let us be given with 3 vertices x, y and z of $P(G)$ (see Fig. 1). We consider 3 geodesics p_{xy}, p_{yz} and p_{xz} connecting respectively x and y, y and z, and x and z. By the criterion of Rips (cf. [20]), it suffices to show that there exists a constant K', independent of x, y and z, such that p_{xz} is included into the K'-neighbourhood of $p_{xy} \cup p_{yz}$. First, let us assume that p_{xy}, p_{yz} and p_{xz} are straight. We claim that in this case p_{xz} is included into a 5-neighbourhood of $p_{xy} \cup p_{yz}$. Let us consider the notations indicated in Figure 1. Let us look at vertices of p_{xz} case by case:

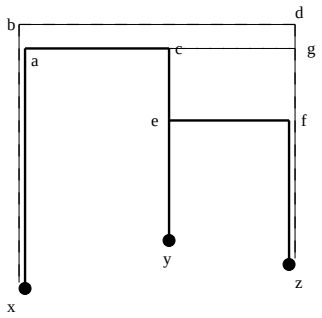

Fig. 1. Rips's Criterion.

- Vertices of p_{xz} which are located between x and a belong also to p_{xy}.
- Without loss of generality, let us suppose that p_{xy} is higher than p_{yz}. Vertices between a and b are at distance at most 3 from p_{xy}. Indeed, if a is higher than b, it is true seeing that the segment ab is totally included into p_{xy}. If b is higher than a, one can verify the previous claim by applying Proposition 5.
- By Proposition 2, vertices between b and d are at distance at most 5 from b, and therefore at most 8 from a.
- Vertices between d and g are within a distance at most 3 from d and therefore at most 11 from a.
- Vertices between g and f are at most at distance 5 from the segment ec, because of the length of ef which is at most 5.
- Finally, vertices between f and z belong to p_{yz}.

We conclude that p_{xz} is totally included into the 11-neighbourhood of $p_{xy} \cup p_{yz}$.

The general case where p_{xy}, p_{yz} and p_{xz} are not straight can be obtained from the above discussion by applying Proposition 4: we get that in general, p_{xz} is included into the 21-neighbourhood of $p_{xy} \cup p_{yz}$ $\square$

3 Distance Labeling Lower Bound

We consider the conjecture of Erdös according to which for any pair of integers $k \geqslant 1$ and $n \geqslant 1$, the maximal number of edges of a graph of girth $2k+2$ with n vertices is $\Omega(n^{1+1/k})$ (see [14]). It is true for $k = 1, 2, 3, 5$; it is also true if we consider graphs of girth $4k/3 + 3$ (see [27]). In the following, for any k and n we shall consider a graph $G_{n,k}$ of girth $4k/3 + 3$ with n vertices and with maximal number of edges equal to $\Omega(n^{1+1/k})$.

We consider subgraphs defined by subsets of edges: given a graph G, a subset E of edges of G defines a subgraph H whose vertices are the vertices of G and whose edges are the elements of E.

Proposition 8. *Let us fix $k \geqslant 1$ and $n \geqslant 1$, and let us consider a subgraph H of $G_{n,k}$. Let us consider $P(H)$ the pyramid graph of H, and a pair (x, y) made of two vertices of the base of $P(H)$ which are connected by an edge in $G_{n,k}$. Then either $d_{P(H)}(x, y) = 1$ or $d_{P(H)}(x, y) \geqslant 2\log(4k/3 + 2) - 3$.*

Theorem 1. *For $n \geqslant 1$ and $k \geqslant 1$, there exists a family $\mathfrak{F}_{n,k}$ of graphs of bounded hyperbolicity with $O(n \log k)$ vertices for which every (s, r)-approximated distance labeling scheme such that $s + r < 2\log(4/3k + 2) - 3$ requires labels of $\Omega(n^{1/k})$ bits.*

In particular, for $k = \Theta(\log n/ \log\log n)$, every s-multiplicative DLS on n-vertex bounded hyperbolic graphs with poly-log label length requires $s = \Omega(\log\log n)$.

Proof. Let us consider the family $\mathfrak{F}_{n,k}$ of the pyramid graphs of the connected subgraphs of $G_{n,k}$. By maximality of the number of edges, it is not difficult to see that $G_{n,k}$ has diameter $O(k)$. We restrict ourself to connected subgraphs of diameter $O(k)$ by fixing some shortest path spanning tree in $G_{n,k}$. Observe that pyramid graphs that we obtain have $O(n \log k)$ vertices. By Lemma 1, $\mathfrak{F}_{n,k}$ is of bounded hyperbolicity. Let us be given with an (s, r)-approximated distance labeling scheme $\langle L, f\rangle$ for $\mathfrak{F}_{n,k}$.

For each $H \in \mathfrak{F}_{n,k}$, let us denote by S_H the word $L(1, H)\#L(2, H)\# \ldots \#L(n, H)$ obtained by concatenation of the labels of all the vertices of its base. We suppose that the vertex set of $G_{n,k}$ is $\{1, 2, \ldots, n\}$. Besides, we use a special symbol $\#$ as delimiter.

Let us assume that $\max_{H\in\mathfrak{F}_{n,k},x\in V(H)}\{|L(x, H)|\} < c \cdot n^{1/k}$ for some constant $c > 0$. It follows that the number of words for $\mathfrak{F}_{n,k}$ is at most $2^{c\cdot n^{1+1/k}}$. Because $|\mathfrak{F}_{n,k}| = 2^{|E(G_{n,k})-(n-1)|} \geqslant 2^{c'\cdot n^{1+1/k}}$ for some suitable constant $c' > 0$. This implies, for $c < c'$ that there exists a pair H_1 and H_2 of distinct graphs of $\mathfrak{F}_{n,k}$ such that L does not distinguish H_1 and H_2, i.e., $S_{H_1} = S_{H_2}$. Let us choose a pair of vertices (x, y) of $G_{n,k}$ such that (x, y) is an edge of the base of H_1 but not of the base of H_2. If such a pair does not exist, we exchange H_1 and H_2. If we cannot find such a pair, this means that $H_1 = H_2$ which is a contradiction.

$S_{H_1} = S_{H_2}$ implies $L(x, H_1) = L(x, H_2)$ and $L(y, H_1) = L(y, H_2)$; and thus $f(L(x, H_1), L(y, H_1)) = f(L(x, H_2), L(y, H_2))$.

Besides, by definition of $\langle L, f\rangle$, we have $d_{H_1}(x, y) \leqslant f(L(x, H_1), L(y, H_1) \leqslant s \cdot d_{H_1}(x, y) + r$ and $d_{H_2}(x, y) \leqslant f(L(x, H_2), L(y, H_2) \leqslant s \cdot d_{H_2}(x, y) + r$.

All together we get $d_{H_2}(x,y) \leqslant s \cdot d_{H_1}(x,y) + r$. But $d_{H_1}(x,y) = 1$ by assumption, and $d_{H_2}(x,y) \geqslant 2\log(4k/3+2) - 3$ by Proposition 8. Finally we get $s + r \geqslant 2\log(4k/3+2) - 3$.

By contraposition, we have thus proved that for any k and any n, $s + r < 2\log(4k/3+2) - 3$ implies that $\max_{H \in \mathfrak{F}_{n,k}, x \in V(H)}\{|L(x,H)|\} \geqslant c \cdot n^{1/k}$. □

4 Tree Approximation and Distance Labeling

This section is devoted to the proof of Theorem 2. It is based on the classical result about approximation of hyperbolic metric spaces by real trees (cf. e.g. [20, Thm. 12, p. 33]. We set up a combinatorial version of this result based on the same method of proof.

We use the characterization of hyperbolicity in terms of Gromov product. Let G be a connected finite graph. Let x, y and w be vertices of G. One defines the *Gromov product* of x and y regarding w to be $(x|y)_w = \frac{1}{2}(|x-w| + |y-w| - |x-y|)$ where $|u-v|$ denotes $d_G(u,v)$. Let G be a connected undirected finite graph. Then the hyperbolicity of G is equal to $2\max_{x,y,z,w \in G}\{\min\{(x|z)_w, (z|y)_w\} - (x|y)_w\}$ (see [20]).

Proposition 9. *Let X be a finite 0-hyperbolic metric space with integral distances; let D be the diameter of X. Then there exists a mapping $\sigma : X \to T$ where T is a tree of at most $2(|X|-1) \cdot D$ nodes such that for any pair (x,y) of elements of X, $d_T(x,y) = 2d_X(x,y)$.*

Let G be a connected undirected finite graph. Let us fix a vertex w_0 of G. In the following, $|x - w_0|$ shall be denoted by $|x|$ for any vertex x of G, it shall be called the length of x (regarding w_0). Following [20], let us define $(x|y)' = \max\{\min_{2 \leqslant j \leqslant \ell}\{(x_{j-1}|x_j)_{w_0}\}$ where $x_1, \ldots, x_\ell$ denotes any sequence of vertices. And from this, let $|x-y|' = |x| + |y| - 2(x|y)'$.

Lemma 2. *Let δ be the hyperbolicity of G. Then for any pair of vertices x and y of G, we have $|x-y| - \delta\log n \leqslant |x-y|' \leqslant |x-y|$*

Then we consider the equivalence relation defined by $x \equiv y$ if and only if $|x-y|' = 0$. And the metric space whose elements are $G/\equiv$ provided with the distance $d'([x]_\equiv, [y]_\equiv) = |x-y|'$. We have the following property:

Lemma 3. *$(G/\equiv, d')$ is a 0-hyperbolic metric space.*

Theorem 2. *The family of δ-hyperbolic graphs with n vertices have a $\delta\log n$-additive distance labeling scheme with $O(\log^2 n)$ bit labels.*

Proof. Let us be given with G a δ-hyperbolic graph with n vertices. We consider the mapping chain $G \xrightarrow{\pi} G/\equiv \xrightarrow{\sigma} T$ where $G \xrightarrow{\pi} G/\equiv$ is defined as above and $G/\equiv \xrightarrow{\sigma} T$ as in Proposition 9 (let us recall that $G/\equiv$ is 0-hyperbolic).

Since T is a tree, there exists an exact distance labeling scheme $\langle L_T, f_T\rangle$ using labels of length $0(\log^2 |T|)$ (cf. [19]). By Proposition 9, $|T| \leqslant 2(n-1)^2$ because $|G/\equiv| \leqslant n$. So, labels used by $\langle L_T, f_T\rangle$ are of length $O(\log^2 n)$. Besides we have $|x-y| - \delta \log n \leqslant \frac{1}{2} d_T(\sigma\circ\pi(x), \sigma\circ\pi(y)) \leqslant |x-y|$.

Finally, let us define $L(x,G) = L_T(\sigma \circ \pi(x))$ and $f(\ell_1,\ell_2) = \frac{1}{2} f_T(\ell_1,\ell_2) + \delta \log n$. Then $\langle L, f\rangle$ satisfies the conditions of the Theorem. □

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