Homotopy Equivalence and Isometry of Treelike Non Compact Hyperbolic 3-Manifolds

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1. Introduction

This paper investigates the problem of rigidity for non compact hyperbolic 3-manifolds with non finitely generated fundamental group. The study is restricted to *treelike* manifolds (cf Definition 1 below) which are shown to be characterized by their fundamental group up to isometry in the sense that two such manifolds are isometric if and only if they have isomorphic fundamental groups.

Sullivan showed in [Sul81] that, if the action of a discrete group of hyperbolic motions Γ on the sphere at infinity is conservative then the quotient manifold $\mathbb{H}^n/\Gamma$ is Mostov-rigid i.e. any pseudo-isometry between $\mathbb{H}^n/\Gamma$ and an other hyperbolic manifold is homotopic to an isometry. McMullen gave in [McM96] a sufficient condition for the conservativity of this action (see also [MT98]) : the action of Γ on the sphere at infinity is conservative if the injectivity radius of the quotient manifold is uniformly bounded. Our aim is to investigate constructions of pseudo-isometries for a class of non compact manifolds satisfying this criterion.

A non compact hyperbolic manifold is said to be treelike if it can be cutted into pieces along pairwise disjoint separating incompressible surfaces such that the following conditions are satisfied. The pieces have to be compact, prime and acylindrical; their number must be finite up to homeomorphism and finally, their sizes have to be uniformly bounded. This last condition will get a precise statement in Definition 1. These assumptions together with Theorem of Thurston about compacity of the Teichmüller spaces of acylindrical compact 3-manifolds [Thu86] allows us to control deformations of treelike manifolds. Constructions of pseudo-isometry make an extensive use of this control.

Section 3 is devoted to the construction of examples of treelike manifolds. It describes how to get such manifolds by gluing infinitely many hyperbolic compact 3-manifolds with totally geodesic boundaries.

The main theorem appears in section 4. It states that two treelike hyperbolic manifolds are homotopically equivalent if and only if they are isomorphic. The proof needs several steps. The first one consists in Lemma 6 : given an isomorphism between the fundamental groups of a couple of treelike hyperbolic manifolds, one can construct a homotopy equivalence inducing it which is k-lipschitz for some constant $k > 0$. This lemma together with a technical result obtained in second step allows to use a standard construction of pseudo-isometry in the third step. Conclusion comes in the fourth step with the use of Theorem of Sullivan and the criterion of McMullen.

2. Acknowledgments

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3. Treelike Decomposition

Definition 1 (treelike decomposition)
Let M be a complete hyperbolic 3-manifold without boundary. M is said to be *treelike* if

- There exists a geodesic triangulation of M such that lengths of edges vary in a compact interval not containing zero.
- This triangulation also satisfies the following condition :
  There exists a system of pairwise disjoint separating incompressible surfaces $\{S_u\}_u$ which cuts M into simple pieces in the sense that :
  There exists a finite number of bordered compact triangulated 3-manifolds $M_1, ..., M_n$ such that for all component C of $M \setminus \bigcup_u S_u$, there exists a simplicial homeomorphism $\varphi_C : \bar{C} \to M_{\iota(C)}$ where $\iota(C) \in [1, n]$.
- For all component C of $M \setminus \bigcup_u S_u$, $\bar{C}$ is acylindrical.

4. Example

This section is devoted to the construction of examples of treelike hyperbolic 3-manifolds.
Lemma 1 is due to D. Epstein and I. Rivin [ER].

LEMMA 1. – *Let N be a prime acylindrical compact 3-manifold whose boundary components are incompressible surfaces of genus greater than two, then N can be endowed with a complete hyperbolic structure with totally geodesic boundary.*

Proof. – Let $\bar{N}$ be the closed 3-manifold obtained by gluing two copies of N along their boundary, with respect to the identity.

$\bar{N}$ is a Haken manifold. Thus, by the uniformization theorem of Thurston, $\bar{N}$ can be endowed with a complete hyperbolic structure. Let $p : \mathbb{H}^3 \to \bar{N}$ define this structure.

Let us consider $\tau : \bar{N} \to \bar{N}$ the homeomorphism exchanging the two copies of N. Let B be a component of the boundary of the copies of N in $\bar{N}$. Let P be a plane embedded in $\mathbb{H}^3$ which is a lift of B. Let $\tilde{\tau} : \mathbb{H}^3 \to \mathbb{H}^3$ be the lifting of τ which fix P pointwise. Let us note that $\tilde{\tau}^2 = \mathrm{Id}_{\mathbb{H}^3}$. By theorem 11.6.1 of [Rac94], $\tilde{\tau}$ is a pseudo-isometry.

By Mostov's Theorem the extension of $\tilde{\tau}$ to the sphere at infinity denoted by $\tilde{\tau}^\infty : S^2_\infty \to S^2_\infty$ is a Moëbus transformation. Since $\tilde{\tau}$ is of order two, $\tilde{\tau}^\infty$ is a symmetry. So, the limit of P is a circle at infinity. It defines a hyperbolic plane P'. It remains to deform P in P' Γ-equivariantly in order to induce a deformation of B into a totally geodesic surface.

We claim that $p(P')$ is a compact surface. To see that, we first show that the action of H on P' is cocompact, where H is the stabilisator of P. To prove this last assertion, we show that there exists K such that for all $x \in P$, $d_{\mathbb{H}^3}(x, P') \leq K$ and for all $x' \in P'$, $d_{\mathbb{H}^3}(x', P) \leq K$.
Let us note that H also stabilizes P'. Let $x_0 \in P$. Let us pick $x'_0 \in P'$ be such that $d_{\mathbb{H}^3}(x_0, x'_0)$ is minimal. Since B is compact and $B = P/H$, there exists $R > 0$ such that for all $x \in P$, there exists $h \in H$ such that $d_{\mathbb{H}^3}(x_0, h(x)) \leq R$. Then,

$$\begin{aligned} d_{\mathbb{H}^3}(x, h^{-1}(x'_0)) &= d_{\mathbb{H}^3}(h(x), x'_0) \\ &\leq d_{\mathbb{H}^3}(h(x), x_0) + d_{\mathbb{H}^3}(x_0, x'_0) \\ &\leq R + d_{\mathbb{H}^3}(x_0, x'_0) \end{aligned}$$

Let $K = R + d_{\mathbb{H}^3}(x_0, x'_0)$. The first assertion is proved.
Let $x' \in P'$. Let γ be the geodesic of $\mathbb{H}^3$ intercepting P' perpendicularly in x'. $\gamma \cap P \neq \emptyset$, let $x \in \gamma \cap P$. Easily :

$$d_{\mathbb{H}^3}(x', P) \leq d_{\mathbb{H}^3}(x', x) = d(P, x) \leq K$$

Let us now prove that the action of H on P' is cocompact. To do that, we prove that there exists $K > 0$ such that $\forall x'_1, x'_2 \in P'$ there exists $h \in H$ such that $d_{\mathbb{H}^3}(h(x'_1), x'_2) \leq K$. Let $x'_1, x'_2 \in P'$. Let

$x_1, x_2 \in P$ be such that $d_{\mathbb{H}^3}(x_i, x_i')$ is minimal. Let $h \in H$ be such that $d_{\mathbb{H}^3}(h(x_1), x_2) \leq R$. Then we have :

$$\begin{aligned} d_{\mathbb{H}^3}(h(x_1'), x_2') &\leq d_{\mathbb{H}^3}(h(x_1'), h(x_1)) + d_{\mathbb{H}^3}(h(x_1), x_2) + d_{\mathbb{H}^3}(x_2, x_2') \\ &\leq d_{\mathbb{H}^3}(x_1', x_1) + R + K \\ &\leq 2K + R \end{aligned}$$

It remains to prove that each point of $p(P')$ has a neighborhood homeomorphic to a disk, with respect to the topology induced by $\bar{N}$.
Let $x' \in p(P')$ and $\tilde{x}' \in p^{-1}(x') \cap P'$. Let D be an open disk of P' containing $\tilde{x}'$. Let us suppose that for all elementary neighborhood V of x', there exists $z_V \in p(P') \cap V$ which is not in $p(D)$. Let us consider $\tilde{V}$, the component of $p^{-1}(V)$ containing $\tilde{x}'$. Let $\tilde{z}_V \in p^{-1}(z_V) \cap \tilde{V}$. For V small enough, $\tilde{V} \cap P' = \tilde{V} \cap D$ and thus $\tilde{z}_V \notin P'$. Therefore, there exists $\gamma_V \in \Gamma \backslash H$ such that $\tilde{z}_V \in \gamma_V(P')$. Let us consider the family $\mathcal{P}$ of all the hyperbolic planes obtained as image of P' by some γ_V.

- If the family $\mathcal{P}$ is finite, then there exists $\gamma \in \Gamma$ such that $\gamma = \gamma_V$ for V varying in a topological basis of elementary neighborhood of x'. In this case $\gamma(P')$ intercepts P'. Then, $\gamma(P)$ also intercepts P transversally, which is impossible since B is a surface.
- Let us assume that $\mathcal{P}$ is infinite. The above argument shows that two Γ-translated of P' are either equal or disjoint. Therefore, there is an infinity of pairwise disjoint plane of the form $\gamma_V(P')$. Since the action of H on P is cocompact, for V small enough, there exists $\gamma_V' \in \Gamma$ such that $\gamma_V'(\tilde{x}') \in \gamma_V(P')$ and $d_{\mathbb{H}^3}(\gamma_V'(\tilde{x}'), z_V) \leq R$. If we choose V small enough to be in a ball of radius $\epsilon > 0$ around x', then $\gamma_V'(\tilde{x}')$ stays in a bounded neighborhood of $\tilde{x}'$. By compacity, there exists a sequence $(V_k)_k$ of elementary neighborhoods such that $(\gamma_{V_k}'(\tilde{x}'))_k$ converges. But the action of Γ is properly discontinuous, this is a contradiction.

The claim is proved.

Therefore, even if it means deforming slightly B, there exists a geodesic triangulation of $\bar{N}$ such that B and $p(P')$ are simplicial. This triangulation is lifted in $\mathbb{H}^3$ by a Γ-equivariant triangulation for which P and P' are simplicial. This triangulation allows us to define, tetrahedron after tetrahedron, an isotopy of $\bar{N}$ deforming B in $p(P')$.
□

Now, given a prime acylindrical compact 3-manifold N with $m > 1$ incompressible boundary components of genus greater than 2, we can construct a treelike non compact hyperbolic manifold. By the above lemma, N can be endowed with a hyperbolic complete metric with totally geodesic boundary. For each boundary component B of N, let us consider a copy of N, denoted by N_B. Let us glue N_B on N along B regarding to the identity of B. We get a manifold with $m(m-1)$ boundary components. Then we do again the operation on it, and so on. The reader can check that this process leads to the construction of a treelike non compact hyperbolic manifold.

Let us note that the fundamental group of the manifold obtained in this way may be not finitely generated.

5. Main Theorem

THEOREM 1. – *Let M, M' be two treelike complete hyperbolic 3-manifolds. Then every isomorphism between $\pi_1(M)$ and $\pi_1(M')$ is induced by an isometry.*
The proof of Theorem 1 involves several steps. But before beginning, we have to fix some notations.

Let M be a treelike complete hyperbolic 3-manifold. Let $\{S_u\}_u$, M_i and φ_C be like in Definition 1. Let $\{B_{ij}\}_{1 \leq j \leq b_i}$ be the components of the boundary of M_i.
Let $S_u \subset \bar{C}$, then $\varphi_C|_{S_u}$ defines an isomorphism between the triangulations of S_u and of a boundary component of one of the M_i. But there is only a finite number of B_{ij}. So there is a finite number

of S_u up to isomorphism of triangulations. Let $\{T_k\}_{1 \leq k \leq m}$ be a family of representatives of the S_u given with a family of simplicial homeomorphisms $\{\psi_u : S_u \to T_{\mu(u)} \mid u\}$.

Let i, j, u and C be such that $\varphi_C(S_u) = B_{ij}$. Then $\psi_u \circ {\varphi_C}^{-1}|_{B_{ij}} : B_{ij} \to T_{\mu(u)}$ is a simplicial homeomorphism. Let us note that the number of simplicial homeomorphism between two triangulated surfaces is finite up to homotopy. Therefore, even if it means adding new representative of component of $M \setminus \bigcup_u S_u$, namely new M_i, we can consider that $\psi_u \circ {\varphi_C}^{-1}|_{B_{ij}}$ neither depends on u nor on C; let it be denoted by g_{ij}.

For each $k \in [1, m]$, let us choose a vertex x_k of the triangulation of T_k. For each pair i, j, let $k \in [1, m]$ be such that B_{ij} is mapped onto T_k by g_{ij}, we then define $x_{ij} = {g_{ij}}^{-1}(x_k)$.
Let t_i be a simplicial tree in M_i of root x_{i0} connecting x_{i0} to the other x_{ij}. Actually, we construct t_i in such a way that it is a star centered at x_{i0} and that, apart for the x_{ij}, it is included in the interior of M_i. Let us consider a spanning tree of the 1-skeleton of T_k. It induces via g_{ij}^{-1} a spanning tree of the 1-skeleton of each B_{ij}. Let s_i be a spanning tree of the 1-skeleton of M_i which contains these trees and t_i. By using φ_C^{-1} for each component C of $M \setminus \bigcup_u S_u$, the family $\{s_i\}_i$ defines a spanning tree of the 1-skeleton of M, let us denote it by s. For each C, let $x_C = \varphi_C^{-1}(x_{\iota(c)0})$. Let us choose one component C_0 of $M \setminus \bigcup_u S_u$, let us call it the *root component* of M. A priori, there is not any canonical choice for C_0. The union of the family $\{\varphi_C^{-1}(t_i)\}_C$, denoted by t, is called a *spine* of M and for any component C, x_C is called a *node* of t. Let us note that the spine of M is far from being canonical, however two spine of M have homeomorphic spaces of ends which actually are homeomorphic to the space of ends of M.

Even if it means adding new representatives of component, we can assume that if $C \neq C'$ then $x_C \neq x'_C$.

Since t is a tree containing x_{C_0}, $\pi_1(M, t)$ is canonically isomorphic to $\pi_1(M, x_{C_0})$. On the other hand, for each component C, there is a canonical homomorphism induced by φ_C^{-1} of $\pi_1(M_{\iota(c)}, x_{\iota(c)0})$ into $\pi_1(M, t)$. Let Γ_i denotes $\pi_1(M_i, x_{i0})$. Let us then consider the resulting homomorphism

$$j_C : \Gamma_{\iota(c)} \to \pi_1(M, x_{C_0})$$

Since S_u is incompressible, we have the following fact :

LEMMA 2. – *j_C is an injection.*

Step 1

Let Γ, Γ' be two discrete subgroups of hyperbolic elements of $\mathrm{Isom}\,\mathbb{H}^3$ such that $M = \mathbb{H}^3/\Gamma$ and $M' = \mathbb{H}^3/\Gamma'$ are treelike complete hyperbolic manifolds. Let $\Phi : \Gamma \to \Gamma'$ be an isomorphism.

This part is devoted to the proof of Lemma 6 i.e. the construction of a homotopy equivalence from M to M' which is k-lipschitz for some constant k.

Let us keep the above notations regarding the treelike decomposition of M. Let A_i denote the generating subset of Γ_i defined using s_i and the edges of the 1-skeleton of M_i not belonging to s_i. Let us consider the function m_i which associates to each faithful representation ρ of Γ_i the point $m_i(\rho) \in \mathbb{H}^3$ which minimizes the following function :

$$d_\rho(x) = \sum_{\alpha \in A_i} d_{\mathbb{H}^3}(x, \rho(\alpha)(x))$$

LEMMA 3. – *m_i is well defined and continuous regarding to the algebraic topology of the space of faithful representations of Γ_i.*

Proof. – For the existence of a minimum, we use the fact that $d_\rho(x)$ goes to infinity as x goes to infinity. Indeed, if d_ρ stays bounded around a point of $\partial\mathbb{H}^3$, this point is then a fixed point for all the elements of $\rho(\Gamma_i)$. Thus, $\rho(\Gamma_i)$ is a group of euclidian similarities. On the other hand, let us remark that we can assume that $\partial M_i \neq \emptyset$; for otherwise M would be just M_i (we assume that M is connected) and then M would be a closed hyperbolic manifold and Theorem 1 follows from Mostow's Theorem [Mos68]. Now, boundary components of M_i are incompressible surfaces of genus greater than two. And their fundamental groups are represented as groups of euclidian similarities, which is impossible. Unicity follows from the fact that d_ρ is strictly convex. $m_i(\rho)$ is well defined.
Let us fix a faithful representation ρ_0 of Γ_i. First, there exists a small neighborhood V of ρ_0 such that $m_i(V)$ is included in a compact set of $\mathbb{H}^3$. To see that, let us note that, in the completion of the ball model $\mathbb{H}^3 \cup S^2_\infty$, the axe of $\rho(\alpha)$ for a fixed $\alpha \in A_{\iota(C_i)}$ moves continuously with respect to ρ, regarding the Hausdorff topology, and so do the translation length along it. Thus, picking V small enough, the fix points of some pair of $\rho(\alpha)$ remain far from each other in S^2_∞. And the translation length along the axes remains large enough. Therefore, for $D > 0$ and $\rho \in V$, a point x far enough from $m_i(\rho_0)$ is at a distance greater than D from the axe of at least one of the $\rho(\alpha)$, which makes us sure that $d_{\mathbb{H}^3}(x, \rho(\alpha)(x))$ is large enough. Thus, $m_i(V)$ stays in a bounded neighborhood of $m_i(\rho_0)$, let W denote the closure of such a neighborhood. Let us observe that the variation, regarding ρ, of the sum of the translation distance functions of the $\rho(\alpha)$ for $\alpha \in A_{\iota(C_i)}$ restricted to the compact W is continuous for uniform topology. Thus, the variation of the point realizing the minimum of it i.e. $m_i(\rho)$ is also continuous, which completes the proof.
□

Let us pick an element $\tilde{x}_{C_0} \in \mathbb{H}^3$ in the fiber of x_{C_0}. Together with the covering map $\mathbb{H}^3 \to M$, this defines a holonomy map $H : \pi_1(M, x_{C_0}) \to \Gamma$. Let C be a component of $M \setminus \bigcup_u S_u$, let $\rho_C = \Phi \circ H \circ j_C$, this is a faithful representation of $\Gamma_{\iota(c)}$ in $\mathrm{Isom}\,\mathbb{H}^3$. Let $\tilde{x}'_C = m_{\iota(c)}(\rho_C)$.

LEMMA 4. – *There exists* $k_1 > 0$ *such that* $\forall C_1, C_2$ *such that* $\bar{C}_1 \cap \bar{C}_2 \neq \emptyset$ *:*

$$d_{\mathbb{H}^3}(\tilde{x}'_{C_1}, \tilde{x}'_{C_2}) \leq k_1$$

Proof. – Let us consider two components C_1 and C_2 such that $\bar{C}_1 \cap \bar{C}_2 \neq \emptyset$. Let $K = \pi_1(\bar{C}_1 \cup \bar{C}_2, x_{C_1})$. As above, there is a natural injection $j_K : K \to \pi(M, x_{C_0})$. For $i \in \{1, 2\}$, $\varphi_{C_i}^{-1}$ induces an injection j'_{C_i} of $\Gamma_{\iota(C_i)}$ into K. We have $j_{C_i} = j_K \circ j'_{C_i}$.
The function defined by

$$\rho \longmapsto d_{\mathbb{H}^3}(m_{\iota(C_1)}(\rho \circ j'_{C_1}), m_{\iota(C_2)}(\rho \circ j'_{C_2}))$$

from the space of faithful representations of K to $\mathbb{R}_+$ is continuous by Lemma 3. Obviously, it is invariant by conjugaison and then, it define a continuous function

$$\bar{d} : \mathrm{AH}(K) \to \mathbb{R}_+$$

Where $\mathrm{AH}(K)$ denotes the Teichmüller space of K endowed with the algebraic topology. Let us note that $\bar{C}_1 \cup \bar{C}_2$ is acylindrical. Therefore, by Thurston's Theorem [Thu86] $\mathrm{AH}(K)$ is compact and thus, $\bar{d}$ is bounded.
Finally, the number of such triple $(K, \Gamma_{\iota(C_1)}, \Gamma_{\iota(C_2)})$ arising like above is finite up to isomorphism and for each pair of intercepting components C_1 and C_2, $\Phi \circ H$ gives rise in a canonical way to a faithful representation of $K = \pi_1(\bar{C}_1 \cup \bar{C}_2, x_{C_1})$. The conclusion follows.
□

LEMMA 5. – *There exists* $k_2 > 0$ *such that* $\forall C$, $\forall \alpha \in A_{\iota(c)}$:

$$d_{\mathbb{H}^3}(\tilde{x}'_C, \rho_C(\alpha)(\tilde{x}'_C)) \leq k_2$$

Proof. – From Lemma 3, it follows that the function

$$\rho \longmapsto d_{\mathbb{H}^3}(m_{\iota(c)}(\rho), \rho(\alpha)(m_{\iota(c)}(\rho)))$$

is continuous on the space of faithful representation of $\Gamma_{\iota(c)}$ and invariant by conjugaison. We use again Thurston's Theorem.
□

We now construct an homotopy equivalence f between M and M' which is k-lipschitz for some constant k. First, we define f on the 1-skeleton $M^{(1)}$ of M.

- Image of the nodes of the spine of M :
  Let $p' : \mathbb{H}^3 \to M'$ a covering map associated to the hyperbolic structure of M'. For all component C, let $f(x_C) = p'(\tilde{x}'_C)$.
- Image of the spine of M :
  Let e be an simple path of t connecting x_{C_1} and x_{C_2} where C_1 and C_2 are components such that $C_1 \cap C_2 \neq \emptyset$. The length of e has a lower bound, which is independant of e. By Lemma 4 the distance between $\tilde{x}'_{C_1}$ and $\tilde{x}'_{C_2}$ is bounded in $\mathbb{H}^3$. We then choose $p'([\tilde{x}'_{C_1}, \tilde{x}'_{C_2}])$ for the image of e, where $[\tilde{x}'_{C_1}, \tilde{x}'_{C_2}]$ denotes the geodesic segment between $\tilde{x}'_{C_1}$ and $\tilde{x}'_{C_2}$. Clearly, $f|_e$ can be defined to be lipschitz for a constant not depending on e.
- Image of s :
  Let x be a point of s. As s is a tree containing t which is also a tree, x is connected to t within s by a unique path. We define $f(x)$ to be the image (defined in the previous item) of the other extremity of this path. Actually, we do not do more than retracting s to t. Again, the restriction of f on each edge of s can be construct to be lipschitz.
- Let us now consider an edge e of $M^{(1)}$ not belonging to s. Let x_1 and x_2 be its extremities. Let C be a component such that $e \subset \bar{C}$. Let $\alpha \in A_{\iota(c)}$ be the element of $\Gamma_{\iota(C)}$ associated to $\varphi_C(e)$. Let $\tilde{e}'$ denote the geodesic segment $[\tilde{x}'_C, \rho_C(\alpha)(\tilde{x}'_C)]$. The image of e by f is defined to be the concatenation $[f(x_1), f(x_C)]p'(\tilde{e}')[f(x_C), f(x_2)]$. The fact that f can be constructed to be lipschitz follows from Lemma 5.

It is easy to check that the commutative diagram of Figure 1 commutes, making us sure that f is going to be an homotopy equivalence inducing Φ.
The construction of f on $M^{(2)}$, keeping the lipschitz property, is now easy. Each 2-simplex s comes from an M_i for some i, and its boundary γ_s is mapped by f to a piecewise geodesic loop γ'_s of uniformly bounded length. As this loop is null homotopic, it is lifted in $\mathbb{H}^3$ by a piecewise geodesic loop $\tilde{\gamma}'_s$ of the same length. This last loop bounds a disk D of bounded "size" in $\mathbb{H}^3$. We define $f(s)$ to be the image $p'(D)$ of this disk in M'. The "size" of s is bounded bellow in the sense that the lengths of all its edges are bounded below. So, $f|_s$ can be constructed to be lipschitz. In order to do that in a precise manner, one can define D to be a union of some totally geodesic triangle of bounded length and do a similar decomposition of s. Details are left to the reader.
The construction of f on $M^{(3)}$ follows the same ideas.

Let us verify that f is k-lipschitz for some constant k. Let $x, y \in M$. let γ be a geodesic segment between x and y. Let $x_0 = x, x_1, ..., x_l = y$ be a decomposition of γ in small geodesic segments

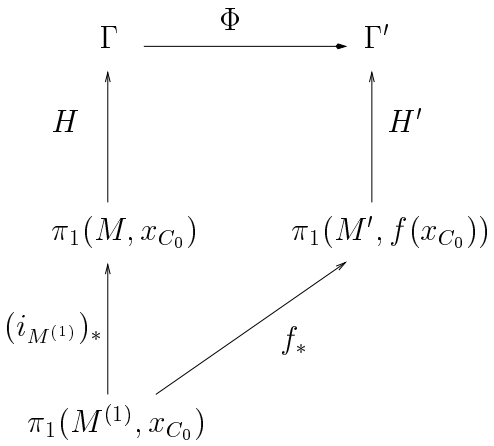

Figure 1:

$[x_i, x_{i+1}]$ such that each $[x_i, x_{i+1}]$ lies in a simplex of the triangulation of M. We have :

$$d(x,y) = \sum_{i=0}^{l-1} d(x_i, x_{i+1})$$

By construction, the restriction of f on each simplex is k-lipschitz. Thus

$$d(x,y) \geq \sum_{i=0}^{l-1} \frac{1}{k}\, d(f(x_i), f(x_{i+1}))$$

And then

$$d(x,y) \geq \frac{1}{k}\, d(f(x), f(y))$$

Eventually, we have the following result :

LEMMA 6. –
Let M and M' be complete hyperbolic 3-manifolds without boundary. Let us assume that M is treelike. Then every isomorphism between $\pi_1(M)$ and $\pi_1(M')$ is induced by a k-lipschitz homotopy equivalence from M to M' for some $k > 0$.

Step 2

LEMMA 7. – *Let M and M' be complete hyperbolic 3-manifolds without boundary. Let us assume that M is treelike. Let $f : M \to M'$ and $g : M' \to M$ be k-lipschitz homotopy equivalences for some $k > 0$ such that $f \circ g$ induces the identity on the fundamental group of M. Then there exists $l > 0$ such that for all $x \in M$: $d_M(x, g \circ f(x)) < l$.*

Proof. – Let $x \in M$. Let γ be a non null homotopic loop containing x. Let $l(\gamma)$ denotes the length of γ. Let us note that there exists a number $l_1 > 0$ independent of x such that we can always find such a γ such that $l(\gamma) < l_1$.
$g \circ f(x)$ belongs to $g \circ f(\gamma)$. By the assumptions, $g \circ f(\gamma)$ is freely homotopic to γ. Let

$H : [0,1]^2 \to M$ be an homotopy from $g \circ f(\gamma)$ to γ.
Let us assume that γ is separated from $g \circ f(\gamma)$ by several pairwise disjoint surfaces of the decomposition of M : S_{u_1} ... S_{u_m}. Then, on each S_{u_i}, H draws a loop γ_i which is freely homotopic to γ. Since each M_i is acylindrical and γ is non null homotopic, it can not be more than one such S_{u_i}. Therefore there exist two components C_1 and C_2 such that $\bar{C_1} \cap \bar{C_2} \neq \emptyset$, $\gamma \cap \bar{C_1} \neq \emptyset$ and $g \circ f(\gamma) \cap \bar{C_2} \neq \emptyset$. Since diameters of components of $M \setminus \bigcup_u S_u$ are bounded, there exists a number $l_2 > 0$ independent of x such that $d_M(\gamma, g \circ f(\gamma)) < l_2$. Let us recall that, $l(\gamma) < l_1$ and thus $l(g \circ f(\gamma)) < k^2 l_1$.
Finally $d_M(x, g \circ f(x)) < \frac{l_1}{2} + \frac{k^2 l_1}{2} + l_2$.
□

Step 3

DEFINITION 2. –
Let X be a metric space. $f : X \to X$ is said to be a *pseudo-isometry* if and only if there exist two numbers k and l such that for all $x, y \in X$,

$$k^{-1} d_X(x,y) - l \leq d(f(x), f(y)) \leq k d(x,y)$$

LEMMA 8. – *Let $M = \mathbb{H}^3/\Gamma$, $M' = \mathbb{H}^3/\Gamma'$ be two treelike complete hyperbolic 3-manifolds without boundary. Let $\Phi : \pi_1(M) \to \pi_1(M')$ be an isomorphism. Then there exists a homotopy equivalence f between M and M' which induces Φ and which lifts to a pseudo-isometry of $\mathbb{H}^3$.*

Proof. – Let $f : M \to M'$ and $g : M' \to M$ be k-lipschitz homotopy equivalences as constructed in the proof of Lemma 6 such that $g \circ f$ induces the identity on the fundamental groups. Let $F : M \times [0,1] \to M$ be a homotopy from $g \circ f$ to id_M. Let $\tilde{f}$, $\tilde{g}$ be lifts of f and g. By the covering homotopy theorem F lifts to a map $\tilde{F} : \mathbb{H}^3 \times [0,1] \to \mathbb{H}^3$ such that $\tilde{F}_0 = \tilde{g}\tilde{f}$. We have that $p\tilde{F}_1 = F_1 p = p$. Therefore $\tilde{F}_1 \in \Gamma$. Even if it means replacing $\tilde{g}$ with $\tilde{F}_1^{-1}\tilde{g}$ and $\tilde{F}$ with $\tilde{F}_1^{-1}\tilde{F}$, we can assume that $\tilde{F}_1 = \mathrm{id}_{\mathbb{H}^3}$. On the other hand, $\tilde{F}$ is Γ-equivariant. Indeed, let $\sigma \in \Gamma$.

$$\begin{aligned} p\tilde{F}(\sigma \times \mathrm{id}) &= F(p \times \mathrm{id})(\sigma \times \mathrm{id}) \\ &= F(p\sigma \times \mathrm{id}) \\ &= F(p \times \mathrm{id}) \\ &= p\tilde{F} \end{aligned}$$

Hence, there exists $\sigma' \in \Gamma$ such that $\tilde{F}(\sigma \times \mathrm{id}) = \sigma' \tilde{F}$. As $\tilde{F}_1 = \mathrm{id}_{\mathbb{H}^3}$, we have $\sigma = \sigma'$.
Now, we have that for all $x, y \in \mathbb{H}^3$:

$$d(x,y) \leq d(x, \tilde{g}\tilde{f}(x)) + d(\tilde{g}\tilde{f}(x), \tilde{g}\tilde{f}(y)) + d(\tilde{g}\tilde{f}(y), y)$$

Since f and g are k-lipschitz, $\tilde{f}$ and $\tilde{g}$ also are k-lipschitz. And by lemma 7, there exists $l > 0$ such that $d(x, \tilde{g}\tilde{f}(x))$ and $d(y, \tilde{g}\tilde{f}(y))$ are less than l. So, we have :

$$d(x,y) \leq l + k d(\tilde{f}(x), \tilde{f}(y)) + l$$

And then

$$k^{-1} d(x,y) - \frac{2l}{k} \leq d(\tilde{f}(x), \tilde{f}(y)) \leq k d(x,y)$$

□

Step 4

By Sullivan's Theorem [Sul81], if the action of Γ on the sphere at infinity is conservative then every Γ-equivariant pseudo-isometry is homotopic to an γ-equivariant isometry by a Γ-equivariant homotopy. On the other hand, by Theorem 5.11 of [MT98], the action of Γ on the sphere at infinity is conservative if and only if the injectivity radius of $\mathbb{H}^3/\Gamma$ is uniformly bounded. This last condition is obviously fulfilled by treelike manifolds. Theorem 1 is proved.
□

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