

PROOF OF THE ASM-DPP CONJECTURE

(PDF + Roger Behrend + Paul Zinn-Justin)

- Physical Combinatorics

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- Physical Combinatorics
- Descending Plane Partitions

DPP

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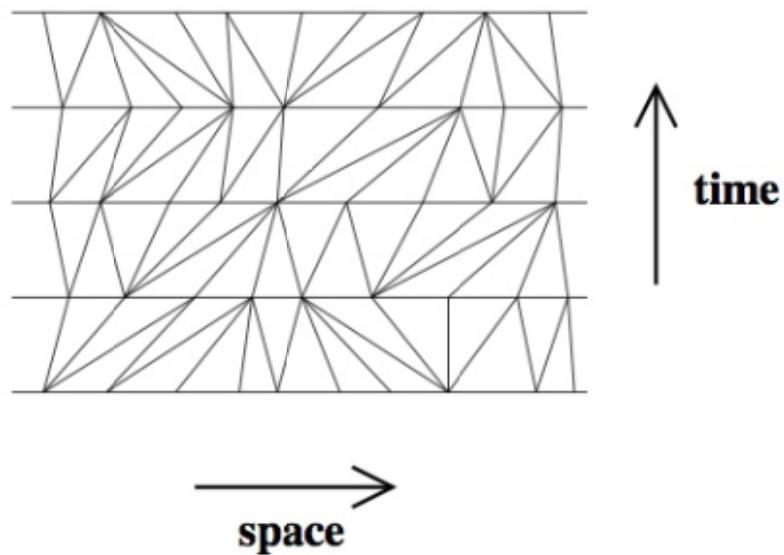
- Physical Combinatorics
- Descending Plane Partitions
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identity between refined enumerations

$$\det(\text{DPP}) = \det(\text{ASM})$$

DIGRESSION: 1+1 Dimensional Lorentzian quantum gravity (PDF + Emmanuel Grinber + Charlotte Kristjansen 'gg)

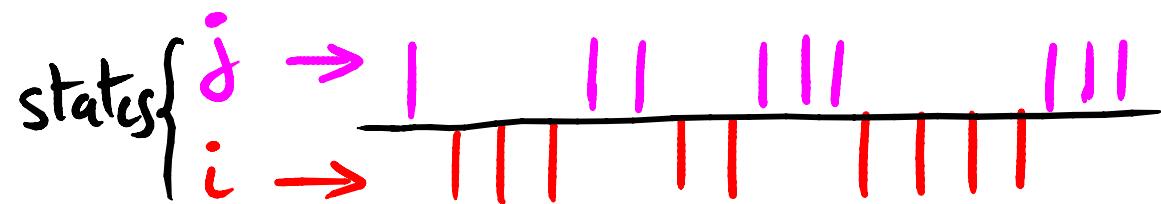


Triangulations that are

1. Random in space direction
2. Regular in time direction

⇒ TRANSFER MATRIX

$$T_{ij} = \binom{i+j}{i}$$



Include
 { curvature weight a / \perp or π
 { area weight g / \perp or T

Then

$$T_{i,j}(g,a) = (ag)^{i+j} \sum_{k=0}^{\min(i,j)} \binom{i}{k} \binom{j}{k} a^{-2k}$$

Generating Function

$$\sum_{i,j \geq 0} z^i w^j T_{i,j}(g,a) = \frac{1}{1 - ga(z+w) - g^2(1-a^2)zw}$$

END OF DIGRESSION

DPP = Arrays of positive integers

a_{11}	a_{12}	-----	a_{1,μ_1}
a_{22}	a_{23}	-----	a_{2,μ_2}
\vdots			
	a_{rr}	-----	a_{r,μ_r}

1. $a \geq b$

2. $\begin{matrix} a \\ \downarrow \\ b \end{matrix}$

3. $\lambda_i := \mu_i + i - 1 = \# \text{ parts in row } i$

$\lambda_i < a_{ii} \leq \lambda_{i-1}$

Vocabulary

- a_{ij} = part
- $a_{ij} \leq j-i$ = special part
- order n = $a_{ij} \leq n \quad \forall i, j$.

OBSERVABLES

- # parts = n
- # special parts
- # non-special parts

$n=3$

7 DPP's



DPP = Arrays of positive integers

a_{11}	a_{12}	-----	a_{1,μ_1}
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OBSERVABLES

parts = n

special parts

non-special parts

→ ③

→ █

$n=3$

7 DPP's

\emptyset

$\boxed{2}$ $\boxed{3}$

$\boxed{33}$

$\boxed{32}$

$\boxed{3}$ █

$\boxed{33}$
 $\boxed{2}$

ASM

- $n \times n$ matrices with elements $0, \pm 1$
- ± 1 alternate along rows and columns
- row and column sums = 1

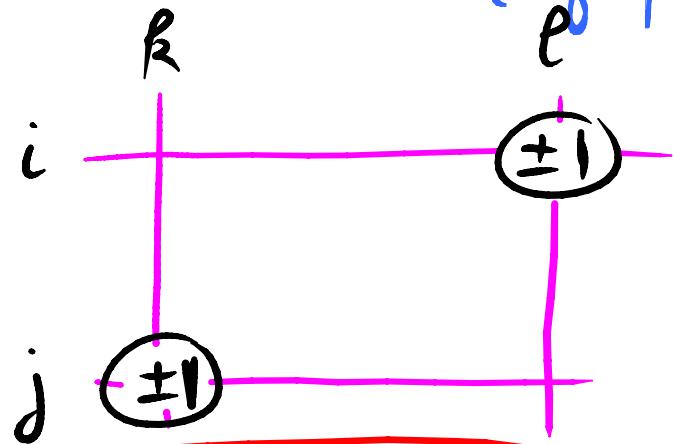
Generalize Permutation matrices (d -determinant
of Mills-Robbins
- Rumsey)

$n=3$: 7 ASM's

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

OBSERVABLES

- position of the 1 in the first row
- #(-1) number of -1's
- inversion number (of permutations)



$$\text{inv}(A) = \sum_{i < j} \sum_{k < l} A_{ie} A_{jl}$$

THE ASM-DPP CONJECTURE

JOURNAL OF COMBINATORIAL THEORY, Series A 34, 340–359 (1983)

Alternating Sign Matrices and Descending Plane Partitions

W. H. MILLS, DAVID P. ROBBINS, AND HOWARD RUMSEY, JR.

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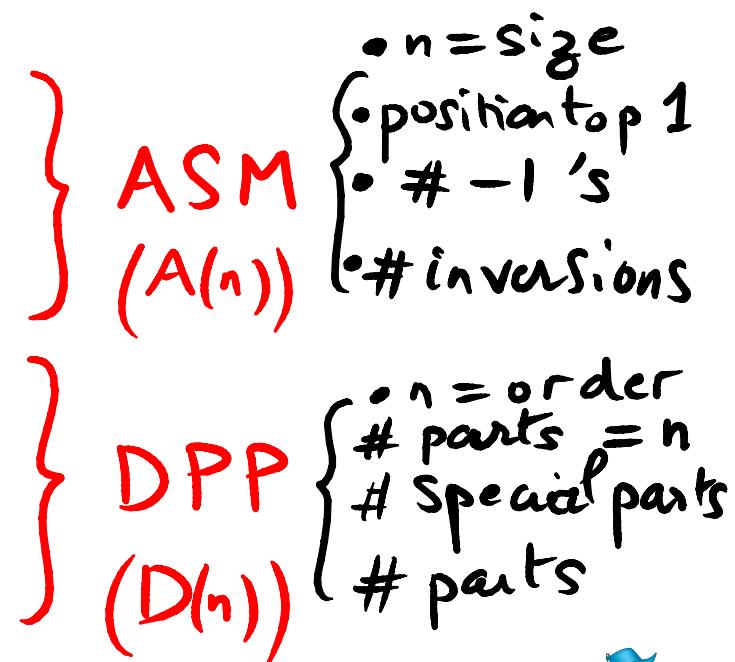
Conjecture 3. Suppose that n, k, m, p are nonnegative integers, $1 \leq k \leq n$.
Let $\mathcal{A}(n, k, m, p)$ be the set of alternating sign matrices such that

- (i) the size of the matrix is $n \times n$,
- (ii) the 1 in the top row occurs in position k ,
- (iii) the number of -1 's in the matrix is m ,
- (iv) the number of inversions in the matrix is p .

On the other hand, let $\mathcal{D}(n, k, m, p)$ be the set of descending plane partitions
such that

- (I) no parts exceed n ,
- (II) there are exactly $k - 1$ parts equal to n ,
- (III) there are exactly m special parts,
- (IV) there are a total of p parts.

Then $\mathcal{A}(n, k, m, p)$ and $\mathcal{D}(n, k, m, p)$ have the same cardinality.



Counting

- DPP : Andrews '79 \rightarrow formula $D(n)$
- ASM : Zeilberger '96
Kuperberg '96 $A(n) = D(n)$
6 vertex model



Izergin-Korepin
integrable lattice
model

Refinements ?

Computation of the refined numbers

Strategy

of known (and manageable objects)

- DPP \rightarrow lattice paths (lattice fermions)
- ASM \rightarrow 6vertex model (integrable lattice model)

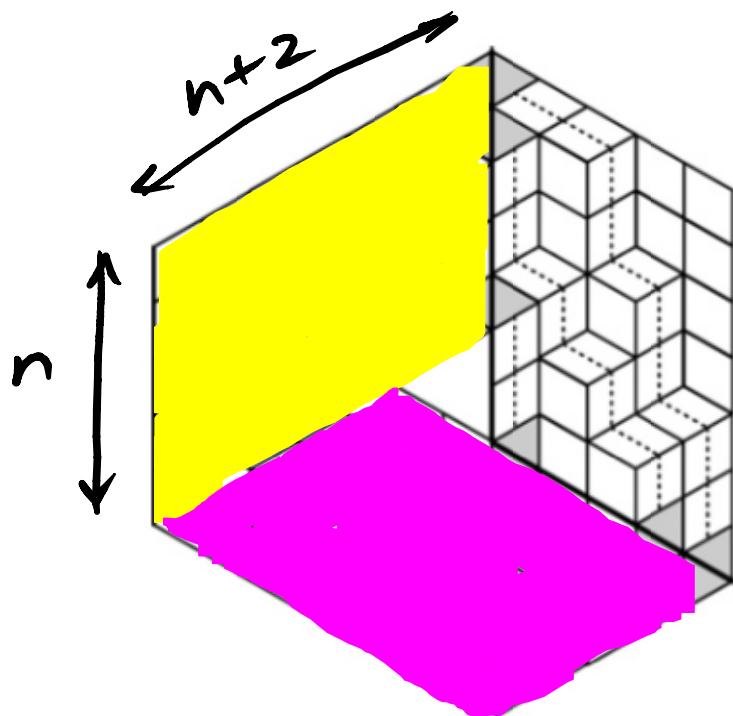
2. write refined generating functions
as determinants

3. Prove identity between determinants

DPP

From DPP to Lattice Paths

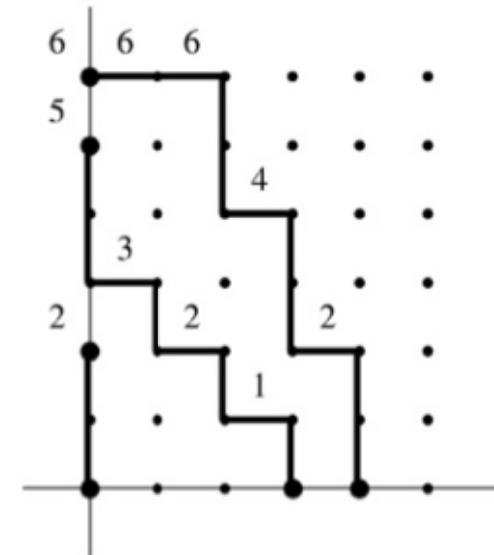
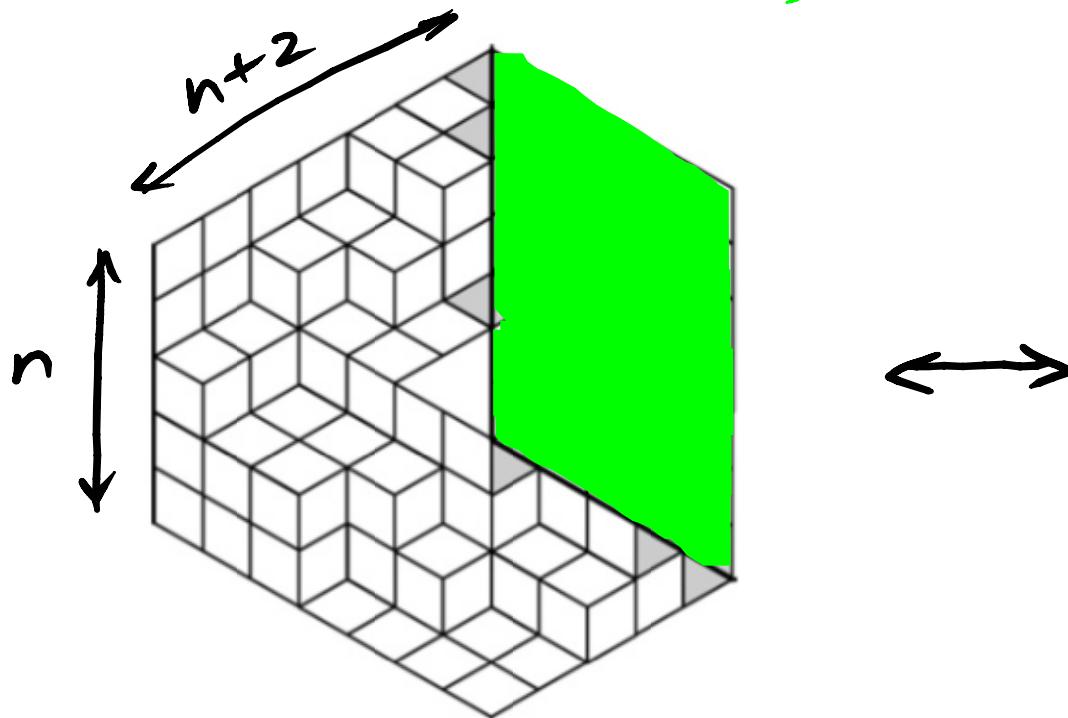
(Krattenthaler '06)



Rhombus Tilings of a cycl. symmetric
Hexagon $(n, n+2, n, n+2, n, n+2)$ with Δ hole

From DPP to Lattice Paths

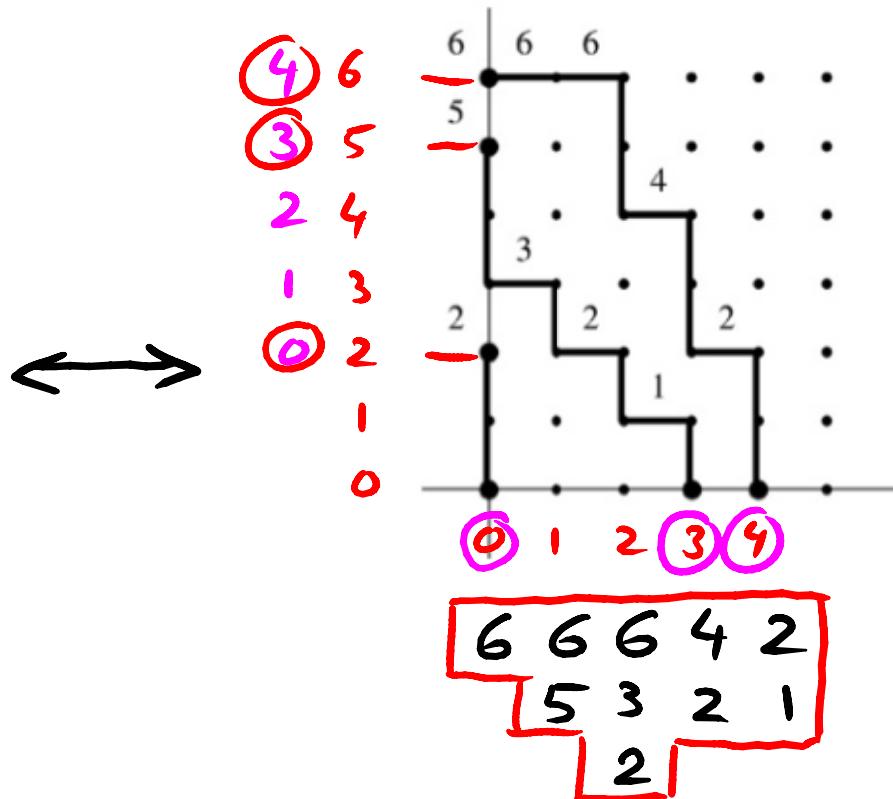
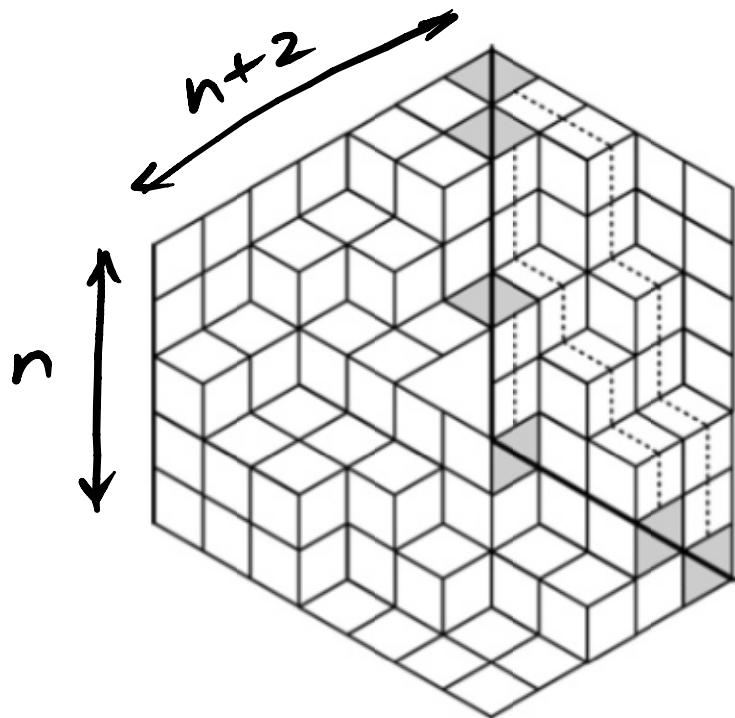
(Krattenthaler '06)



Rhombus Tilings of a Hexagon $(n, n+2, n, n+2, n, n+2)$ with Δ hole \leftrightarrow Lattice Paths

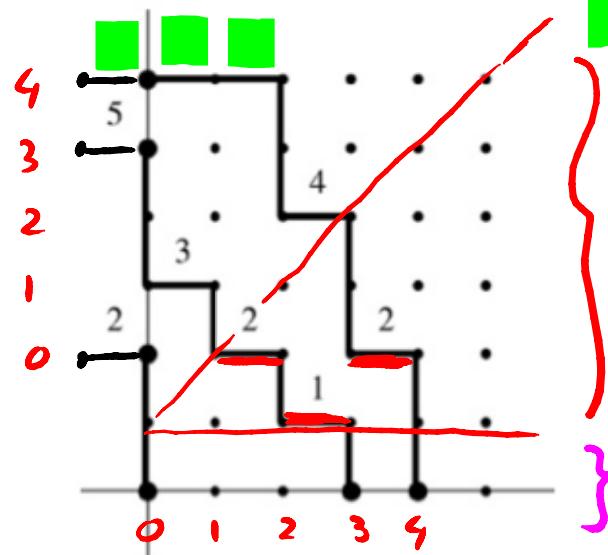
From DPP to Lattice Paths

(Krattenthaler '06)



Rhombus Tilings of a Hexagon $(n, n+2, n, n+2, n, n+2)$ with Δ hole \leftrightarrow Lattice Paths

horizontal steps — = non-special parts



= parts = n

horizontal steps —

= special parts



6	6	6	4	2
2	1			



6	6	6	4	
5	3	2	1	
2				

} horizontal steps here do not count

Lemma

$$\det_{n \times n} (I + M) = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \det(M_{i_1 \dots i_k}^{i_1 \dots i_k})$$

M's minor with rows $i_1 \dots i_k$

Here: $M_{ij} = \text{Part. fctn}(\text{path}(i,0) \rightarrow (0,j))$ cols $i_1 \dots i_k$

By Gessel-Viennot theorem

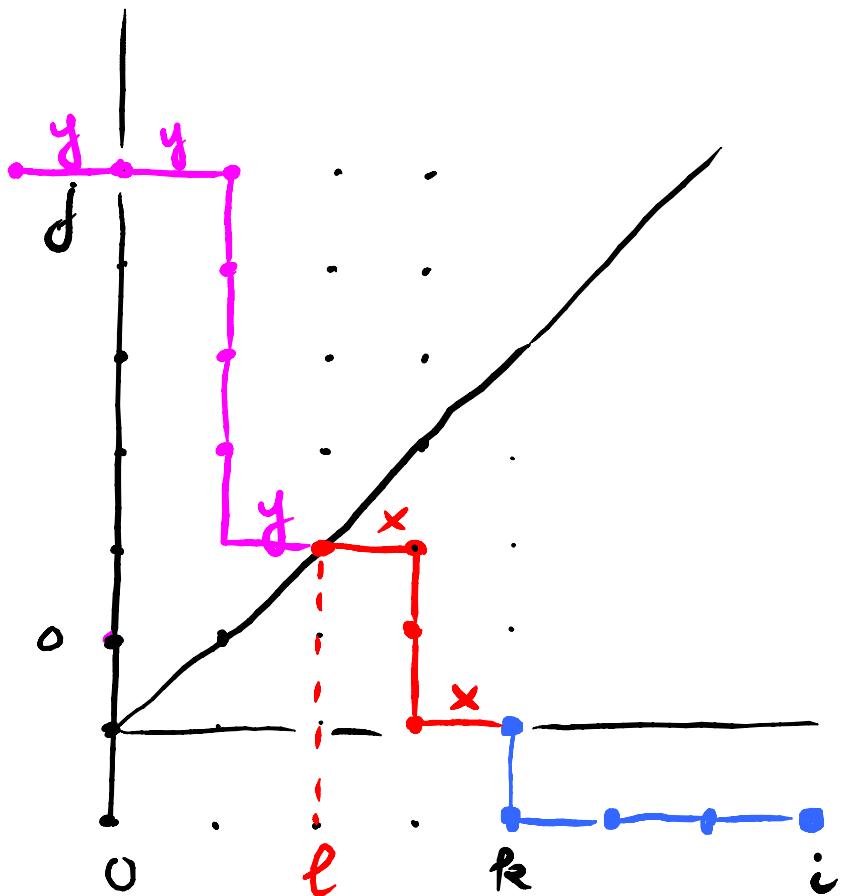
$\det(M_{i_1 \dots i_k}^{i_1 \dots i_k}) =$ Partition fctn for families of
 k non-intersecting paths starting at
 $(i_1, 0) \dots (i_k, 0)$, ending at $(0, i_1) \dots (0, i_k)$

Partition Function for 1 path :

$$M_{ij} = \sum_{\text{paths } (i_0) \rightarrow (0j)} x^{\#(\leftarrow)} y^{\#(\rightarrow)} z^{\#(\overline{\rightarrow})}$$

x y z
↑ ↑ ↑
horizontal steps : lower wedge upper wedge

1 for simplicity



$$Z_{DPP}^{(n)}(x, y) = \det(I + M)$$

↑ ↑
 per special part per non-special part

$$M_{i,j} = \sum_{k=0}^i \sum_{e \geq 0} \binom{k}{e} x^{k-e} \binom{j+1}{e} y^{e+1}$$

Generating function:

$$f_{DPP}(z, w) = \sum_{i,j \geq 0} z^i w^j (I + M)_{i,j}$$

THM

$$f_{DPP}(z, w) = \frac{1}{1 - zw} + \frac{1}{1 - z} \frac{yz}{1 - xz - w - (y-x)zw}$$

weights : x / special part y / non-special part

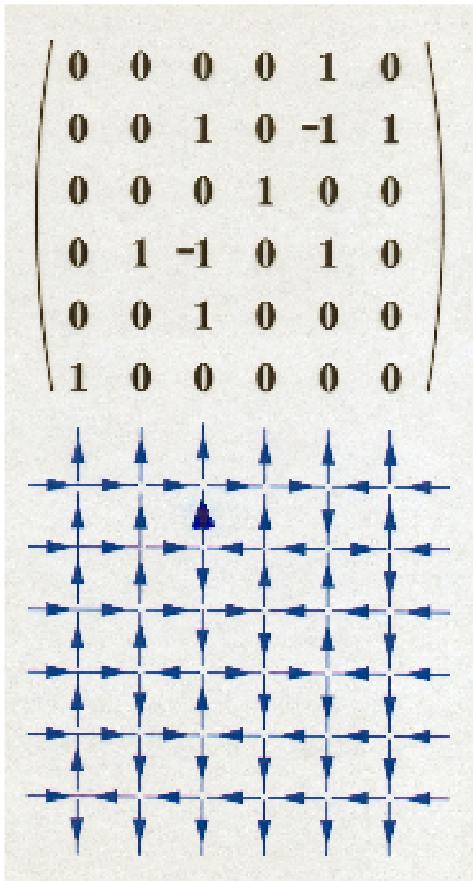
ASM

From ASM to 6 Vertex model with
Domain Wall Boundary conditions (Rupeborg)

$n \times n$
ASM

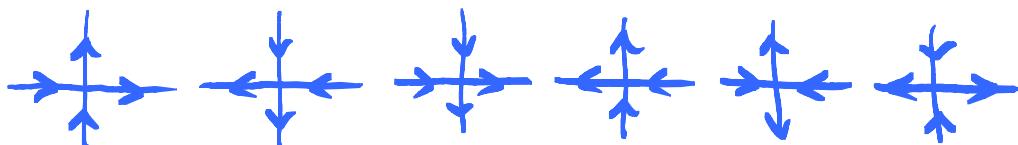


6 V+
DWBC
on
 $n \times n$ grid



Bijection :

0 0 0 0 1 -1



$$\underbrace{a_1}_{qz - q^{-1}w}, \underbrace{a_2}_{q^{-1}z - qw}$$

$$\underbrace{b_1}_{q^{-1}z - qw}, \underbrace{b_2}_{q^2z - q^{-2}w}, \underbrace{c_1}_{(q^2 - q^{-2})\sqrt{zw}}, \underbrace{c_2}_{(q^2 + q^{-2})\sqrt{zw}}$$

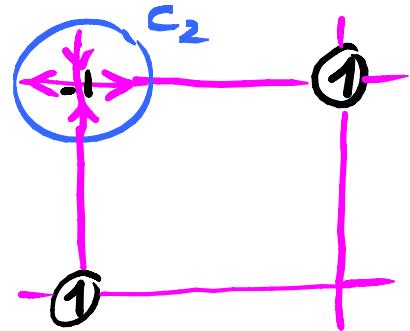
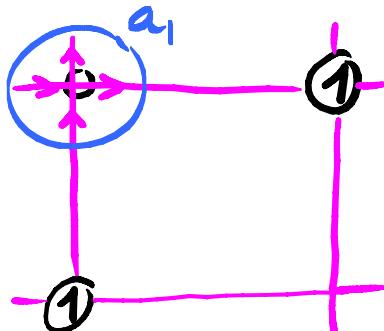
Refinements :

by symmetry : $\begin{cases} N_{a_1} = N_{a_2} = \frac{N_a}{2} & N_{b_1} = N_{b_2} = \frac{N_b}{2} \\ N_{c_1} = N_{c_2} + n & N_c = N_{c_1} + N_{c_2} \end{cases}$

- $\boxed{\#(-1) = N_{c_2} = \frac{N_c - n}{2}}$

- $\text{Inv}(A) = N_{a_1} + N_{c_2}$

$$\Rightarrow \boxed{\text{Inv}(A) - \#(-1) = N_{a_1}} \\ = \frac{N_a}{2}$$



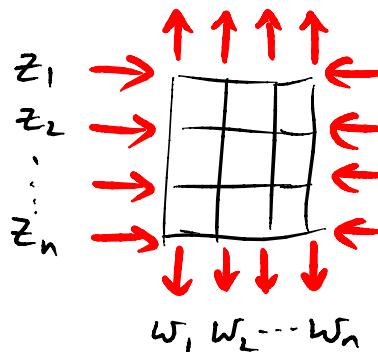
Partition function

$$Z_{\text{ASM}}^{(n)}(x, y) = \sum_{\substack{\text{configs of } n \times n \\ 6V \text{ DWBC}}} x^{\frac{N_c - n}{2}} y^{\frac{Na}{2}}$$

usually, one considers $Z_{6V}^{(n)}(a, b, c) = \sum_{\substack{\text{configs} \\ 6V \text{ DWBC}}} a^{N_a} b^{N_b} c^{N_c}$

$$Z_{6V}^{(n)}(a, b, c) = b^{n^2} \sum \left(\frac{c}{b}\right)^{N_c - n} \left(\frac{a}{b}\right)^{N_a}$$

$$\Leftrightarrow x = \left(\frac{c}{b}\right)^2 \quad y = \left(\frac{a}{b}\right)^2$$



$$Z_n = \frac{\sum_{\substack{\text{configs} \\ \text{on grid}}} \prod_{\text{Vertices} (i,j)} \text{weights}(z_i, w_j)}{\prod_i C(z_i, w_i)}$$

Partition function of 6V+DWBC

THM

$$Z_n = \frac{\prod_{i,j}^{i < j} a(z_i, w_j) b(z_i, w_j)}{\prod_{i < j} (z_i - z_j)(w_i - w_j)} \det \left\{ \frac{1}{a(z_i, w_j) b(z_i, w_j)} \right\}_{i \leq i, j \leq n}$$

(Izergin - Korepin)

recursion relation + symmetries (from commutation of Transfer matrices).

Homogeneous limit: $\begin{cases} z_i \rightarrow r & \forall i \\ w_j \rightarrow r^{-1} & \forall j \end{cases}$

$$a(z_i, w_j) \rightarrow q^r - q^{-1}r^{-1} = a(r, r^{-1})$$

$$b(z_i, w_j) \rightarrow q^{-1}r - qr^{-1} = b(r, r^{-1})$$

$$c(z_i, w_j) \rightarrow q^2 - q^{-2} = c(r, r^{-1})$$

$$Z_n(q, r) = (ab)^{n^2} \det_{0 \leq i, j \leq n-1} \left(\frac{(\frac{d}{du})^i}{i!} \frac{(\frac{d}{dv})^j}{j!} \left[\frac{c(u, v)}{a(u, v)b(v, u)} \right] \right|_{\substack{u=r \\ v=r^{-1}}} \right)$$

by Taylor expansion
around the limit

$$\frac{c}{a(u,v) b(u,v)} = \frac{1}{uv - q^{-2}} - \frac{1}{uv - q^2}$$

let $(A_+)_i,j = \left(\frac{d}{du} \right)^i \left(\frac{d}{dv} \right)^j \left. \frac{1}{uv - q^2} \right|_{\begin{array}{l} u=r^{-1} \\ v=r^{-1} \end{array}}$ $A_- = \text{idem } (q \rightarrow q^{-1})$

Then generating function $(A_+) = \frac{1}{(r^{-1}+z)(r^{-1}+w) - q^2}$

THM $A_+ = -\frac{1}{r^2 - q^2} [U(\alpha, \beta)]^{-1} U(\alpha', \beta')$

$U_{ij}(\alpha, \beta) = \binom{\alpha}{i} \alpha^i \beta^j$ is upper triangular, and:

$\alpha = \frac{1 - q^2 r^2}{r}$ $\beta = \frac{q^2 - q^{-2}}{r^2 - q^2}$ $\alpha' = -q^2 r^2 \beta$ $\beta' = -\frac{1}{\alpha}$

Proof:

1. generating function $(U(\alpha, \beta)) = \frac{1}{1 - \beta w(1 + \alpha z)}$
2. $U(\alpha, \beta)^{-1} = U(-\frac{1}{\beta}, -\frac{1}{\alpha})$

$$\text{set } V = U^t(\alpha, \beta) \quad U = U(\alpha', \beta')$$

$$\bar{V} = V(q \rightarrow q^{-1}) \quad \bar{U} = U(q \rightarrow q^{-1})$$

then:

$$A_+ = \frac{1}{q^2 - r^{-2}} V^{-1} U$$

$$A_- = \frac{1}{q^{-2} - r^{-2}} \bar{V}^{-1} \bar{U}$$

$$\det(A_- - A_+) = \det(A_-) \det \left[I - \frac{q^{-2} - r^{-2}}{q^2 - r^{-2}} \bar{U}^{-1} \bar{V} V^{-1} U \right]$$

$$\propto \det \left[I - \frac{q^{-2} - r^{-2}}{q^2 - r^{-2}} (\bar{V} V^{-1})(U \bar{U}^{-1}) \right]$$

$$U \bar{U}^{-1} = U(-1, 1)$$

$$\bar{V} V^{-1} = U^t(-y, x)$$

Collecting all prefactors, we get:

$$Z_{ASM}^{(n)}(x, y) = \det((1-v)I + v G)$$

$$v = \frac{r^{-2} - q^{-2}}{q^2 - q^{-2}} \quad G = U^E\left(\frac{-y}{x}, x\right) \cup (-1, 1)$$

$$G_{ij} = \sum_{k \geq 0} \binom{i}{k} y^k \binom{j}{k} x^{i-k}$$

\Rightarrow generating function of $(1-v)I + v G = f_{ASM}(z, w)$

THM

$$f_{ASM}(z, w) = \frac{1-v}{1-zw} + \frac{v}{1-zx-w-(y-x)zw}$$



Final identity:

$$Z_{DPP}^{(n)} = \det(I + M) = \det((1-v)I + vG) = Z_{ASM}^{(n)}$$

Proof:

note that

$$\begin{aligned} & (1-z)(1-(1-v)w) f_{DPP}(z, w) - (1+(y-xv-1)z)(1-w) f_{ASM}(z, w) \\ &= (\underbrace{xv(1-v) - y(1-v) - v}_{=0}) \times \text{rational fraction}(z, w) \end{aligned}$$

$$\left(x = \left(\frac{q^2 - q^{-2}}{q^{-1}r - qr^{-1}} \right)^2, y = \left(\frac{qr - q^{-1}r^{-1}}{q^{-1}r - qr^{-1}} \right)^2, v = \left(\frac{r^{-2} - q^{-2}}{q^2 - q^{-2}} \right), 1-v = \left(\frac{q^2 - r^{-2}}{q^2 - q^{-2}} \right) \right)$$

+ Remark: let $A = (A_{ij})_{i,j \geq 0}$ $F = \sum A_{ij} z^i w^j$
 then $(1-\lambda z)(1-\mu w) F(z,w)$ is the generating
 function for $(I-\lambda S) A (I-\mu S^t)$ where

$S_{i,j} = \delta_{i,j+1}$ "shift" matrix strictly lower
 triangular \Rightarrow the determinant is unchanged.

Proof completed!

CONCLUSION

MRR PROVED

→ method of generating fctns
for the matrix of which we take the det

- Bijection ASM - DPP ? (-TSSCPP ?)
- More refinements ?

TO DO LIST

1. Write the paper
2. Generalizations : DPP with symmetries
 \leftrightarrow ASM with symmetries