

Indecomposable permutations, Hypermaps, and Subgroups of finite index of \mathbb{F}_2 .

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The sequence A003319 of OEIS

1, 3, 13, 71, 461, 3447, ...

First found by [Marshall Hall Jr](#) Canad J Math, Vol 1, (1949), see also:

The Theory of Groups, Macmillan (1959) page 105

Theorem The number $N_{n,r}$ of subgroups of index n in \mathbb{F}_r is given by $N_{1,r} = 1$, and :

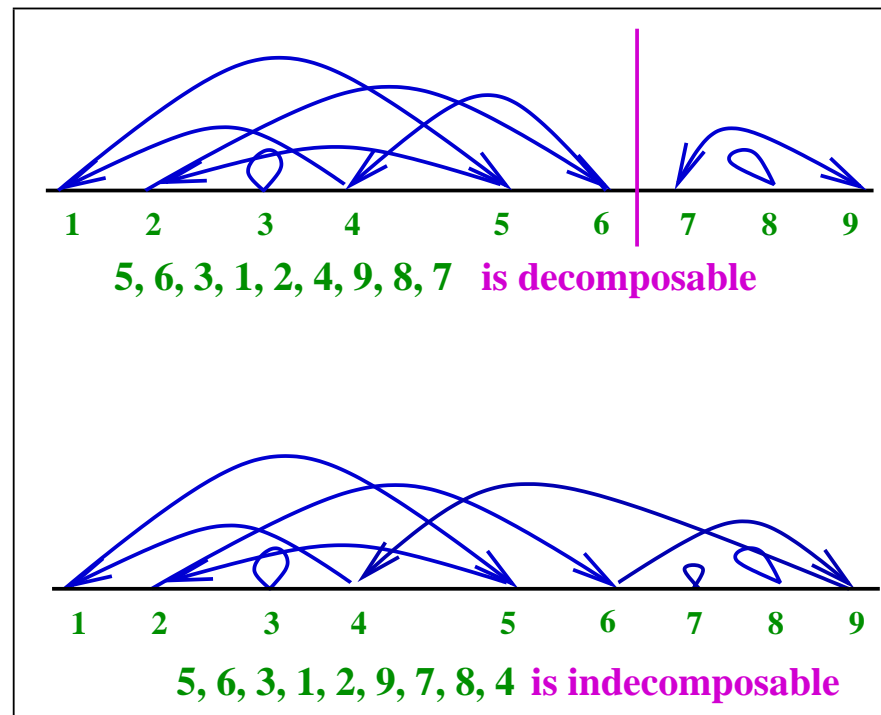
$$N_{n,r} = n(n!)^{r-1} - \sum_{i=1}^{n-1} (n-i)!^{r-1} N_{i,r}$$

denoting $c_n = N_{n,2}$ we have :

$$c_n = n(n!) - \sum_{i=1}^{n-1} (n-i)! c_i \quad c_2 = 3, c_3 = 13, c_4 = 71 \dots$$

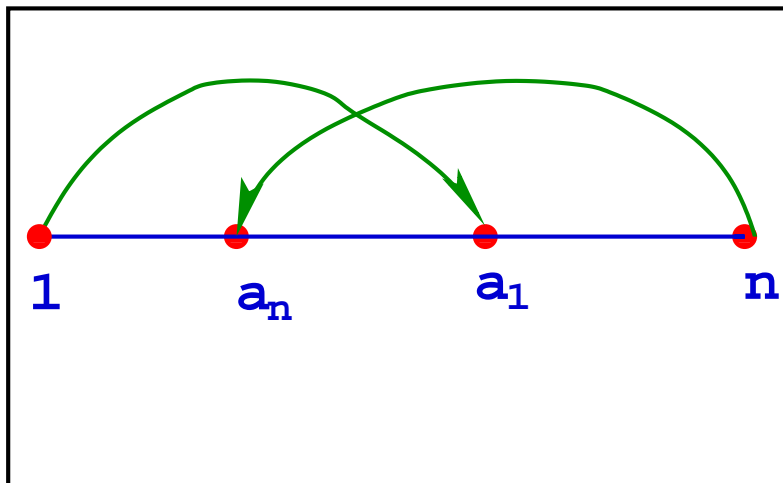
Indecomposable permutations

A permutation a_1, a_2, \dots, a_n is *decomposable* if there exists $p < n$ such that a_1, a_2, \dots, a_p is a permutation of $1, 2, \dots, p$



Elementary facts on indecomposable permutations

Remark If $a_1 > a_n$ then α is indecomposable.



Corollary Not less than one half of the permutations of \mathcal{S}_n are indecomposable

Almost all permutations are indecomposable

Proposition 1 Let c_n be the number of indecomposable permutations of \mathcal{S}_n then :

$$c_n = n! - \sum_{i=1}^{n-1} c_i (n-i)!$$

$$c_n = \sum_{i=1}^{n-1} i c_i (n-i-1)!$$

Proposition (Lentin, Comtet, 1969)

$$\frac{c_n}{n!} = 1 - \frac{2}{n} + O\left(\frac{1}{n^2}\right)$$

First values 1, 1, 3, 13, 71, 461, ...

Permutations by number of cycles, Stirling numbers

Cycle of α :

$$i_1, i_2 = a_{i_1}, \dots, i_k = a_{i_{k-1}}, \dots, i_j, \quad a_{i_j} = i_1$$

The number of permutations of \mathcal{S}_n having m cycles is the Stirling number $s_{n,m}$.

$$s_{n,m} = (n-1)s_{n-1,m} + s_{n-1,m-1} \quad s_{1,1} = 1 \quad s_{n,1} = (n-1)! \quad s_{n,n} = 1$$

1					
1	1				
2	3	1			
6	11	6	1		
24	50	35	10	1	
120	274	225	85	15	1

Indecomposable permutations of \mathbb{S}_4

2, 3, 4, 1	(1, 2, 3, 4)	4, 1, 2, 3	(1, 4, 3, 2)
2, 4, 1, 3	(1, 2, 4, 3)	4, 1, 3, 2	(1, 4, 2) (3)
2, 4, 3, 1	(1, 2, 4) (3)	4, 2, 1, 3	(1, 4, 3) (2)
3, 1, 4, 2	(1, 3, 4, 2)	4, 2, 3, 1	(1, 4) (2) (3)
3, 2, 4, 1	(1, 3, 4) (2)	4, 3, 1, 2	(1, 4, 2, 3)
3, 4, 1, 2	(1, 3) (2, 4)	4, 3, 2, 1	(1, 4) (2, 3)
3, 4, 2, 1	(1, 3, 2, 4)		

Indecomposable permutations by number of cycles

Let $c_{n,m}$ be the number of indecomposable permutations of \mathcal{S}_n having m cycles, then:

$$c_{n,m} = s_{n,m} - \sum_{p=1}^{n-1} \sum_{i=1}^{\min(p,m)} c_{p,i} s_{n-p,m-i}$$

The probability of indecomposability depends heavily on the number of cycles

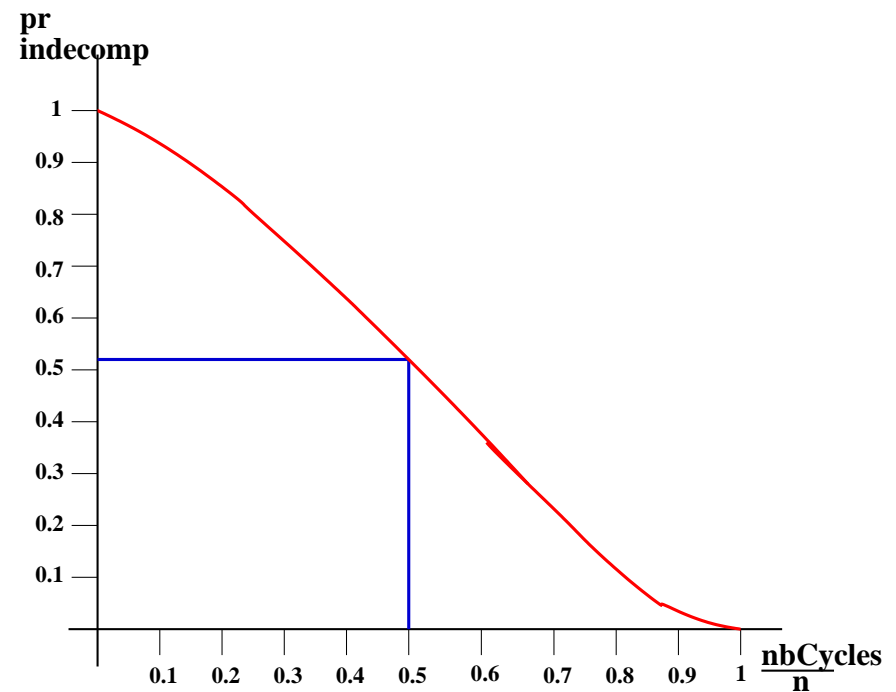
- Permutations with 1 cycle: $\frac{c_{n,1}}{s_{n,1}} = 1$
- Permutations with $n - 1$ cycles $\frac{c_{n,n-1}}{s_{n,n-1}} = \frac{2}{n(n-1)}$

Number of indecomposable permutations by number of cycles

						Stirling numbers:											
1						1						1					
1	0					1	1					1	1				
2	1	0				2	3	1				2	3	1			
6	6	1	0			6	11	6	1			6	11	6	1		
24	34	12	1	0		24	50	35	10	1		24	50	35	10	1	
120	210	110	20	1	0	120	274	225	85	15	1	120	274	225	85	15	1

Main result

The shape of the curve $\frac{c_{n,m}}{s_{n,m}}$ as a function of $\frac{m}{n}$ when n and m tend to ∞

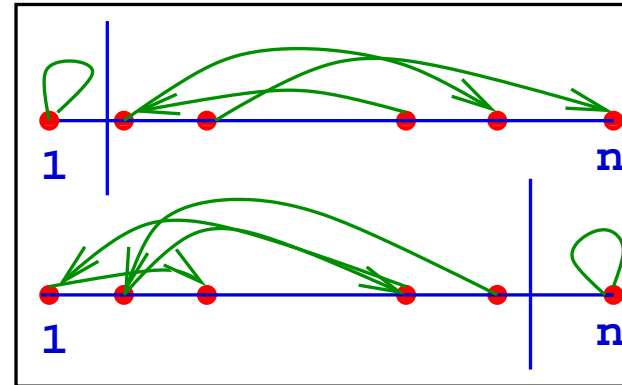


Sketch of the proof (1)

Lemma 1 If the following condition holds, then the permutation a_1, \dots, a_n is decomposable:

$$a_1 = 1 \text{ or } a_n = n$$

Moreover when n and m are sufficiently large almost all decomposable permutations **with** m **cycles** satisfy this condition.



The number of permutations of \mathcal{S}_n with m cycles satisfying the condition is :

$$2s_{n-1, m-1} - s_{n-2, m-2}$$

Sketch of the proof (2) : Moser-Wyman asymptotic formula

$s_{n,m}$ is asymptotically equivalent to:

$$\frac{b n!}{a^n \sqrt{n}} \frac{u^m}{m!}$$

a, b are functions of u , and u is given by:

$$\frac{m}{n} = \frac{u}{e^u - 1}$$

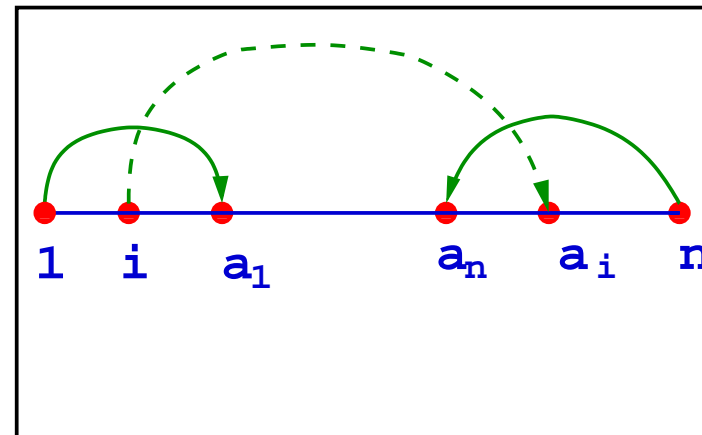
Hence the probability that a permutation satisfies the condition of Lemma 1 tends to:

$$\frac{2e^u - 1}{e^{2u}}$$

Sketch of the proof (3)

Lemma 2 If the following condition holds, then the permutation a_1, \dots, a_n is indecomposable:

$$\exists i, i \leq a_1 \text{ and } a_i > a_n$$



Sketch of the proof (4)

Lemma 3 The probability that a permutation of \mathcal{S}_n with m cycles satisfies no one of the two conditions:

$$a_1 = 1 \text{ or } a_n = n \quad \exists i, i \leq a_1 \text{ and } a_i > a_n$$

is:

$$O\left(\frac{\log(n - m)}{n - m}\right).$$

Main result

Theorem 1. The probability that a permutation of \mathcal{S}_n with m cycles is decomposable tends to:

$$(1 - e^{-u})^2$$

when n, m tend to infinity with $\frac{m}{n}$ constant, where:

$$\frac{m}{n} = \frac{u}{e^u - 1}$$

Probability for a permutation of S_{2m} with m cycles to be indecomposable

Computed values:

m	2	3	4	5	6	7	8	9	10	50	75
$\frac{c_{2m,m}}{s_{2m,m}}$	0.54	0.49	0.48	0.48	0.49	0.49	0.49	0.50	0.50	0.51	0.511

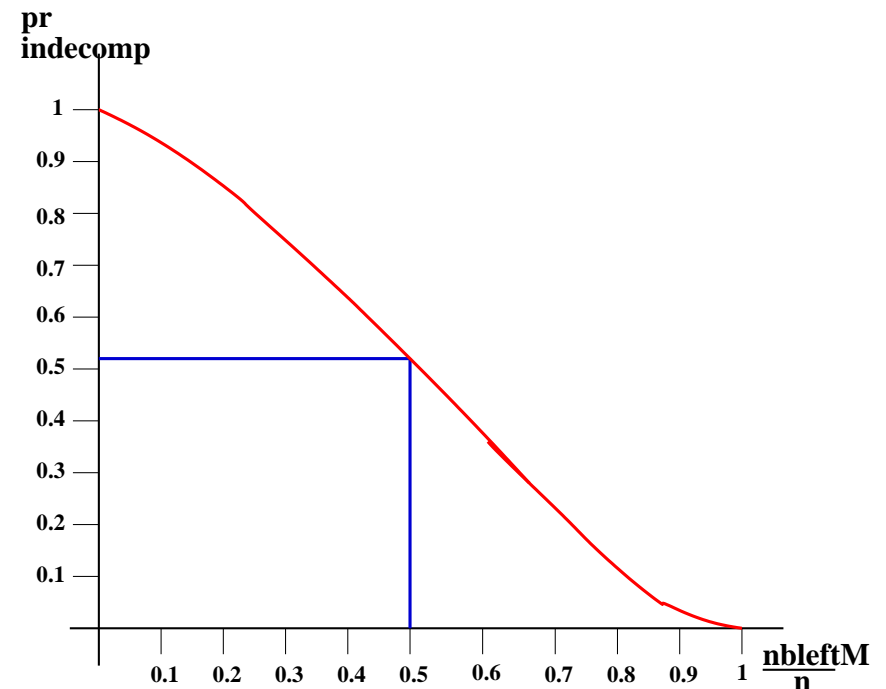
Value given by the Theorem:

About 51.1 % of the permutations of S_n with $\frac{n}{2}$ cycles are indecomposable

Permutations with m left-to-right maxima

Definition A left to right maximum in $a_1 a_2 \dots a_n$ is a_j such that $a_j > a_i$ for all $i < j$.

Remark The probability for a permutation of \mathcal{S}_n with m left-to-right maxima to be decomposable is the same as the probability for permutations with m cycles to be indecomposable



Indecomposable permutations of \mathbb{S}_4

2, 3, 4, 1	(1, 2, 3, 4)	4, 1, 2, 3	(1, 4, 3, 2)
2, 4, 1, 3	(1, 2, 4, 3)	4, 1, 3, 2	(1, 4, 2) (3)
2, 4, 3, 1	(1, 2, 4) (3)	4, 2, 1, 3	(1, 4, 3) (2)
3, 1, 4, 2	(1, 3, 4, 2)	4, 2, 3, 1	(1, 4) (2) (3)
3 2, 4, 1	(1, 3, 4) (2)	4, 3, 1, 2	(1, 4, 2, 3)
3, 4 1, 2	(1, 3) (2, 4)	4, 3, 2, 1	(1, 4) (2, 3)
3, 4, 2, 1	(1, 3, 2, 4)		

From cycles to left-to-right maxima

Proposition The number of permutations of \mathcal{S}_n with m cycles is equal to the number of those with m left-to-right maxima

Mapping F :

- Write the cycles of α with the largest element in the first position
- Order the cycles in increasing order of their first elements
- Remove parenthesis

Example

- $\alpha = 5, 3, 2, 6, 9, 7, 4, 8, 1 = (1, 5, 9)(2, 3)(4, 6, 7)(8)$
- $\alpha = (3, 2) (7, 4, 6) (8) (9, 1, 5)$
- $F(\alpha) = 3, 2, 7, 4, 6, 8, 9, 1, 5$

From cycles to left-to-right maxima

Proposition The number of indecomposable permutations of \mathcal{S}_n with m cycles is equal to the number of those with m left-to-right maxima

Proof: The permutation α is decomposable if and only if $F(\alpha)$ is

The probability of indecomposability decreases
while the number of cycles increases

Theorem 2.

$$\frac{c_{n,k}}{s_{n,k}} > \frac{c_{n,k+1}}{s_{n,k+1}}$$

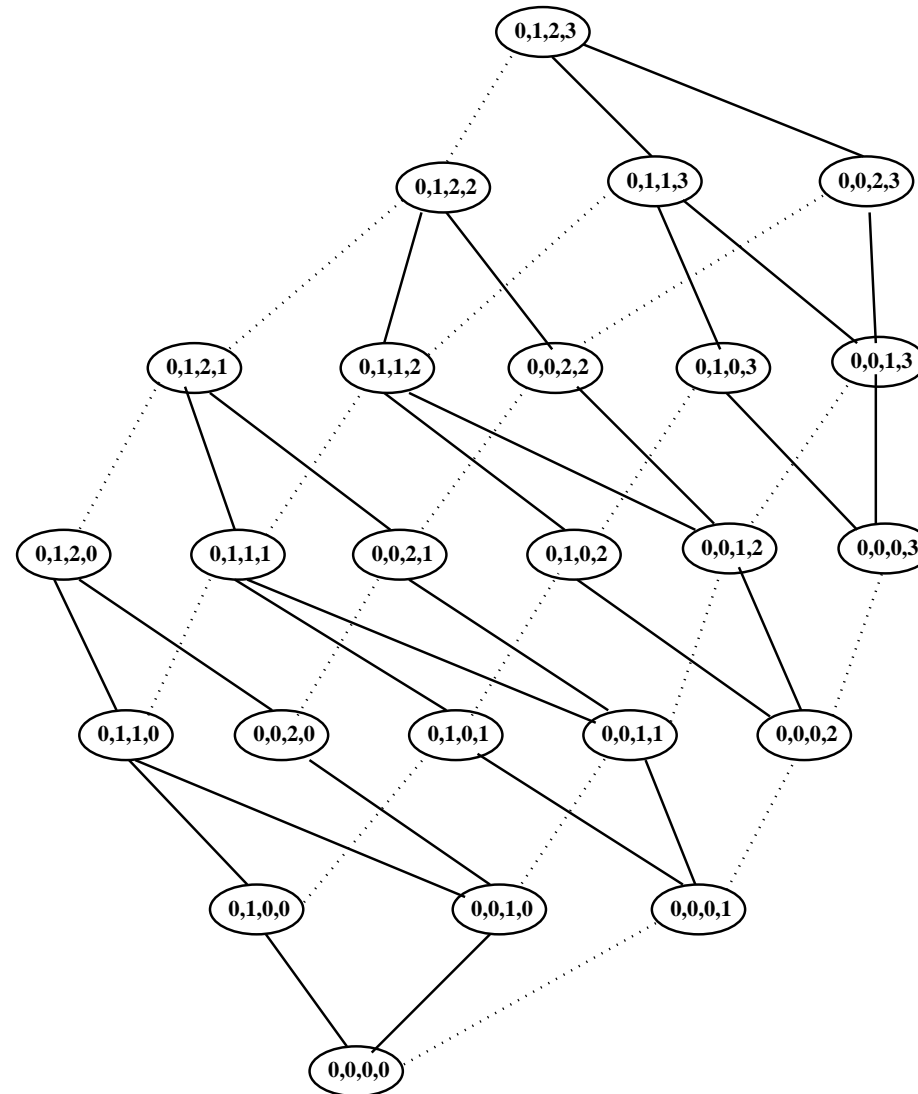
Proof:

- Consider the left-to-right maxima
- For any permutation α build the inversion tableau U_α such that:

$$U_\alpha[i] = |\{j | j < i, \alpha(j) > \alpha(i)\}|$$

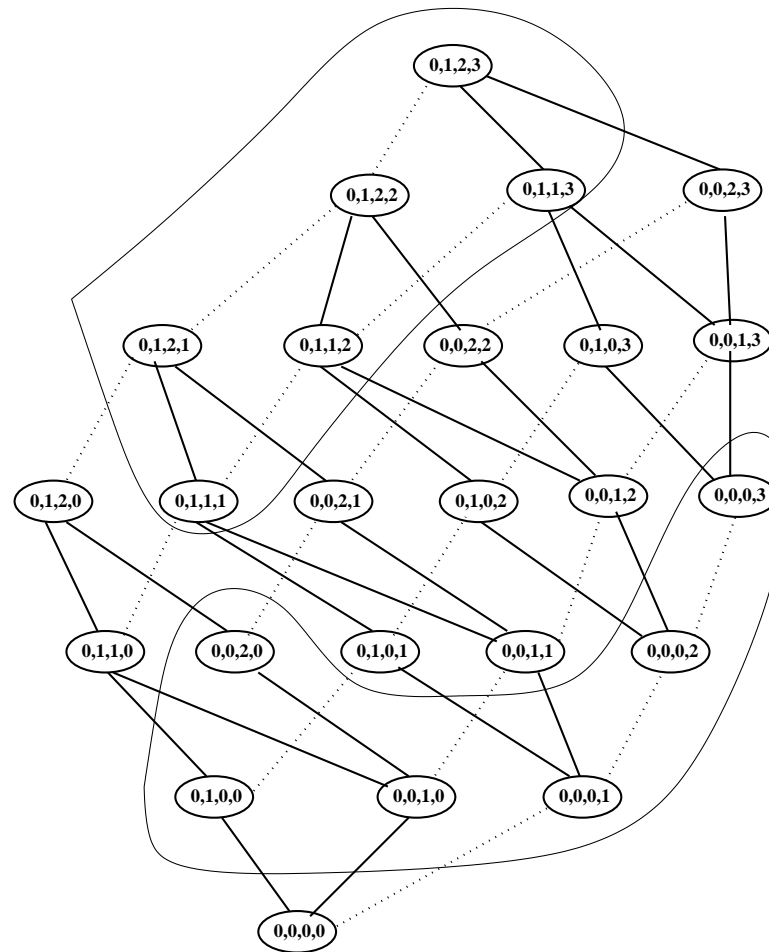
- Denote $\alpha \preceq \beta$ if $U_\alpha[i] \leq U_\beta[i]$ for all i

Description of the partial order \preceq



Division into slices

Divide the elements of \mathcal{S}_n into slices by counting the number of $U_\alpha[i]$ equal to 0. Then the number of permutations in slice k is equal to the Stirling number $s_{n,k}$.



Proof (continuation)

- The order \preceq satisfies the following:
If $\beta \preceq \alpha$ and α is decomposable then β also is
- Let a colouring of \mathcal{S}_n in two colors blue and red be such that such that if α is blue and $\beta \preceq \alpha$ then β is also blue. Then we have
- The numbers of red elements $r_{n,k}$ in slice k satisfy:

$$\frac{r_{n,k}}{S_{n,k}} > \frac{r_{n,k+1}}{S_{n,k+1}}$$

Hypermaps

A *Hypermap* (S, σ, α) :

σ, α are permutations acting on the set S , such that the group they generate acts transitively on S .

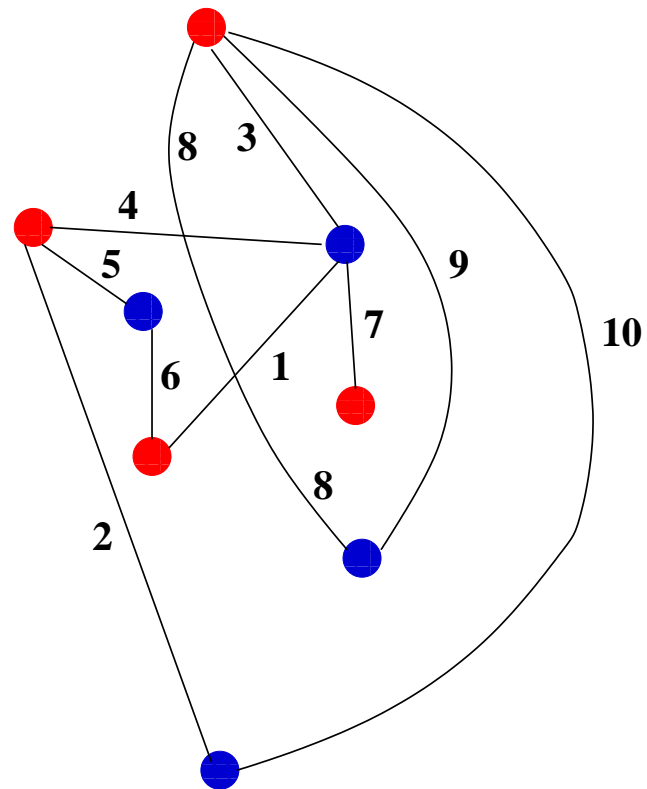
Graphical representation

A hypermap is represented by a **bicoloured map**, vertices are coloured **blue** and **red** and edges have endpoints of different colors.

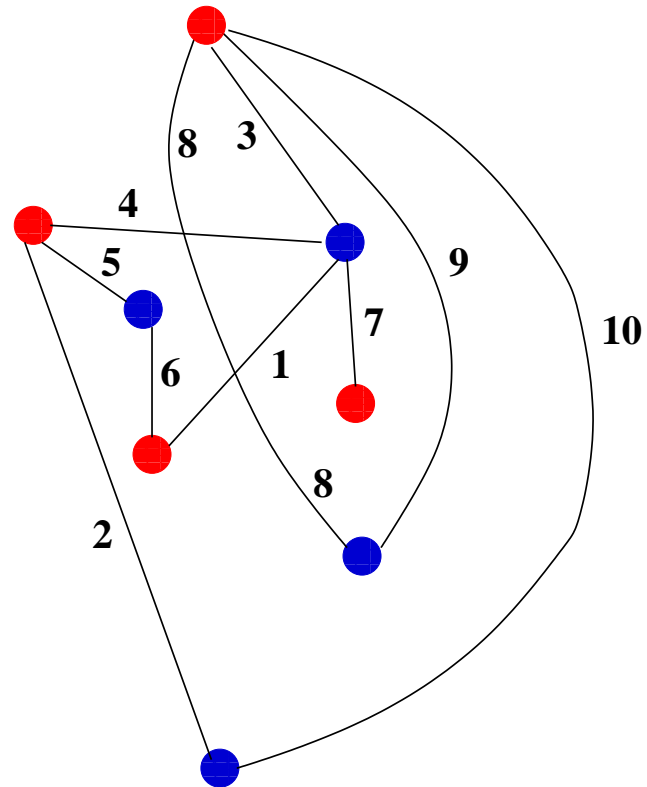
The edges are numbered by the integers $\{1, 2, \dots, n\}$ so that they represent the two permutations.

- To each *cycle of σ* corresponds a **blue vertex** such that when going around this vertex in the *trigonometric positive direction*, the edges are met in the same order as they appear in the corresponding cycle of σ .
- To each *cycle of α* corresponds a **red vertex** such that the edges met around it satisfy the same property as above with respect to α .

A hypermap on a set of 10 points

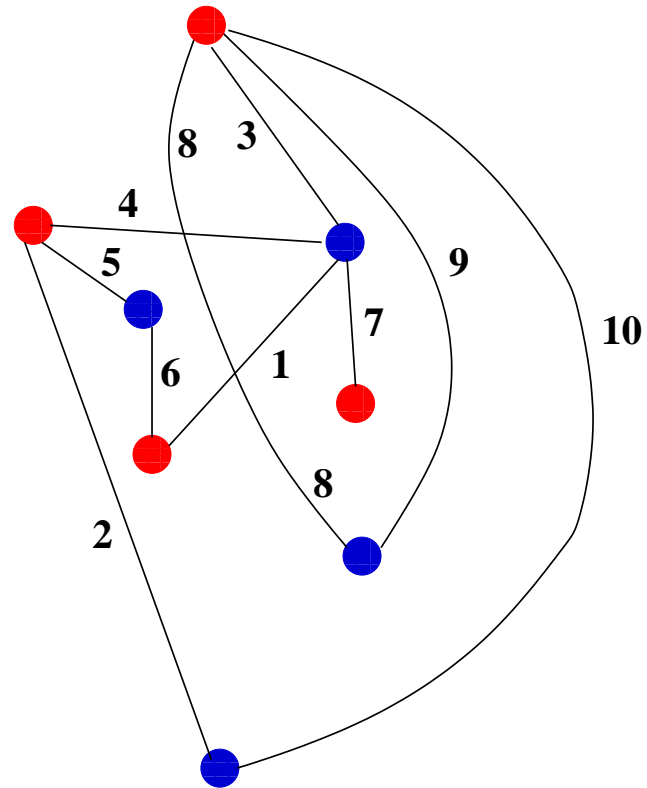


A hypermap on a set of 10 points



$$\sigma = (1, 7, 3, 4)(2, 10)(5, 6)(8, 9)$$

A hypermap on a set of 10 points



$$\sigma = (1, 7, 3, 4)(2, 10)(5, 6)(8, 9)$$

$$\alpha = (1, 6)(2, 5, 4)(3, 9, 10, 8)(7)$$

Rooted hypermaps

Two hypermaps are *isomorphic* if one can be obtained from the other by relabelling the set of points.

Or if (S, σ, α) and (S', σ', α') , acting S' are such that there exists a bijection ϕ of S onto S'

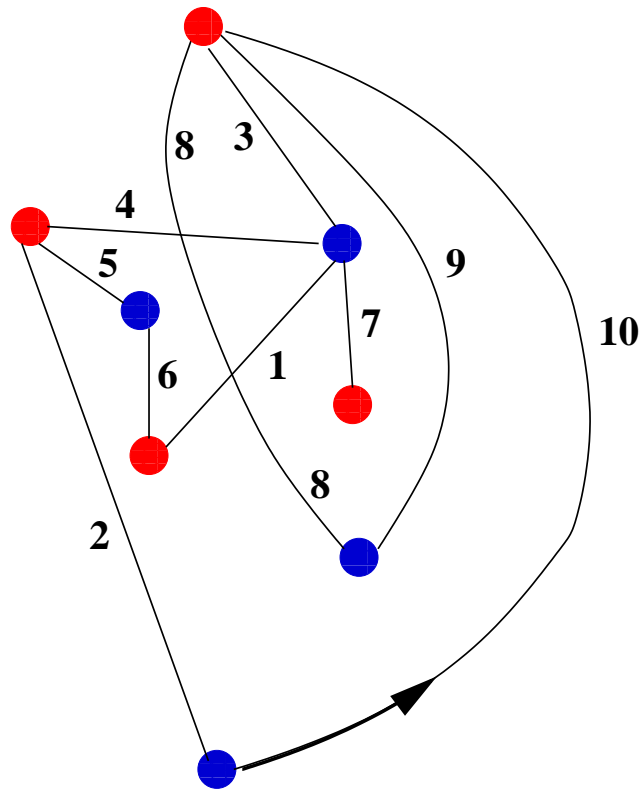
$$\sigma' = \phi\sigma\phi^{-1}, \quad \alpha' = \phi\alpha\phi^{-1}$$

A hypermap is *rooted* if an element r of S is distinguished.

Two *rooted hypermaps* with roots r and r' are *isomorphic* if the above bijection satisfies also:

$$\phi(r) = r'.$$

A rooted hypermap on a set of 10 points



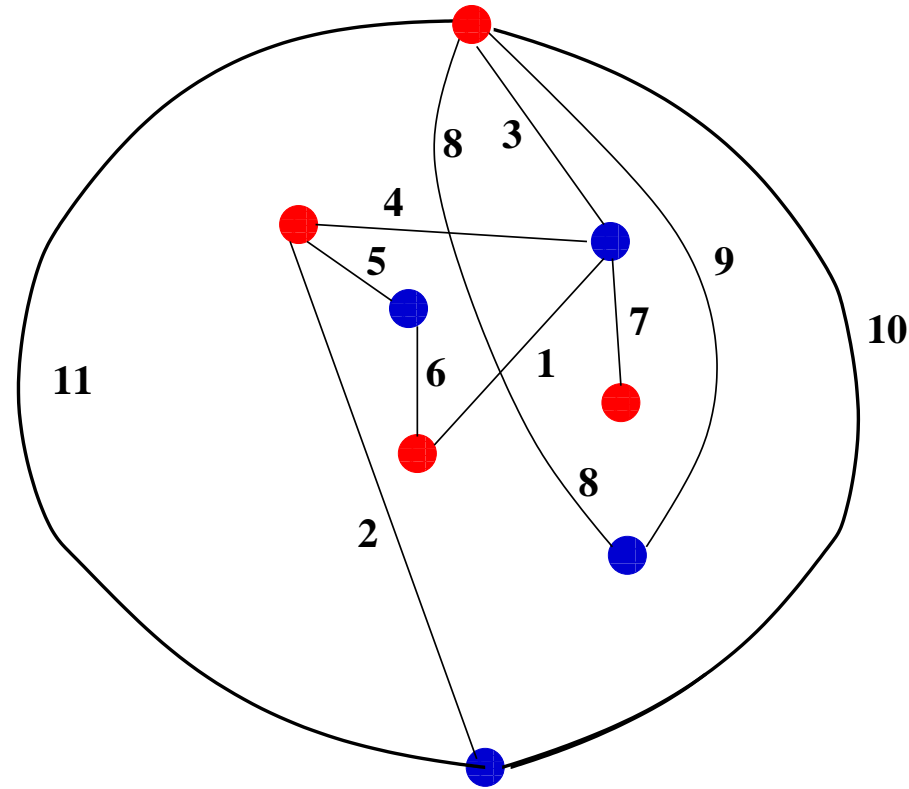
$$\alpha = (1, 6)(2, 5, 4)(3, 9, 10, 8)(7) \quad \sigma = (1, 7, 3, 4)(2, 10)(5, 6)(8, 9) \quad r = 10.$$

Doubly rooted hypermaps

A doubly rooted hypermap is given by two permutations σ, α on a set S of points, and two elements a, b of S such that the following conditions are satisfied:

- the group generated by σ, α is transitive on S ,
- $\alpha(a) = b, \quad \sigma(b) = a$

A doubly-rooted labelled hypermap on 11 points



$$\alpha = (1, 6)(2, 5, 4)(3, 9, 10, 11, 8)(7) \quad \sigma = (1, 7, 3, 4)(10, 2, 11)(5, 6)(8, 9) \quad a = 10 \quad b = 11.$$

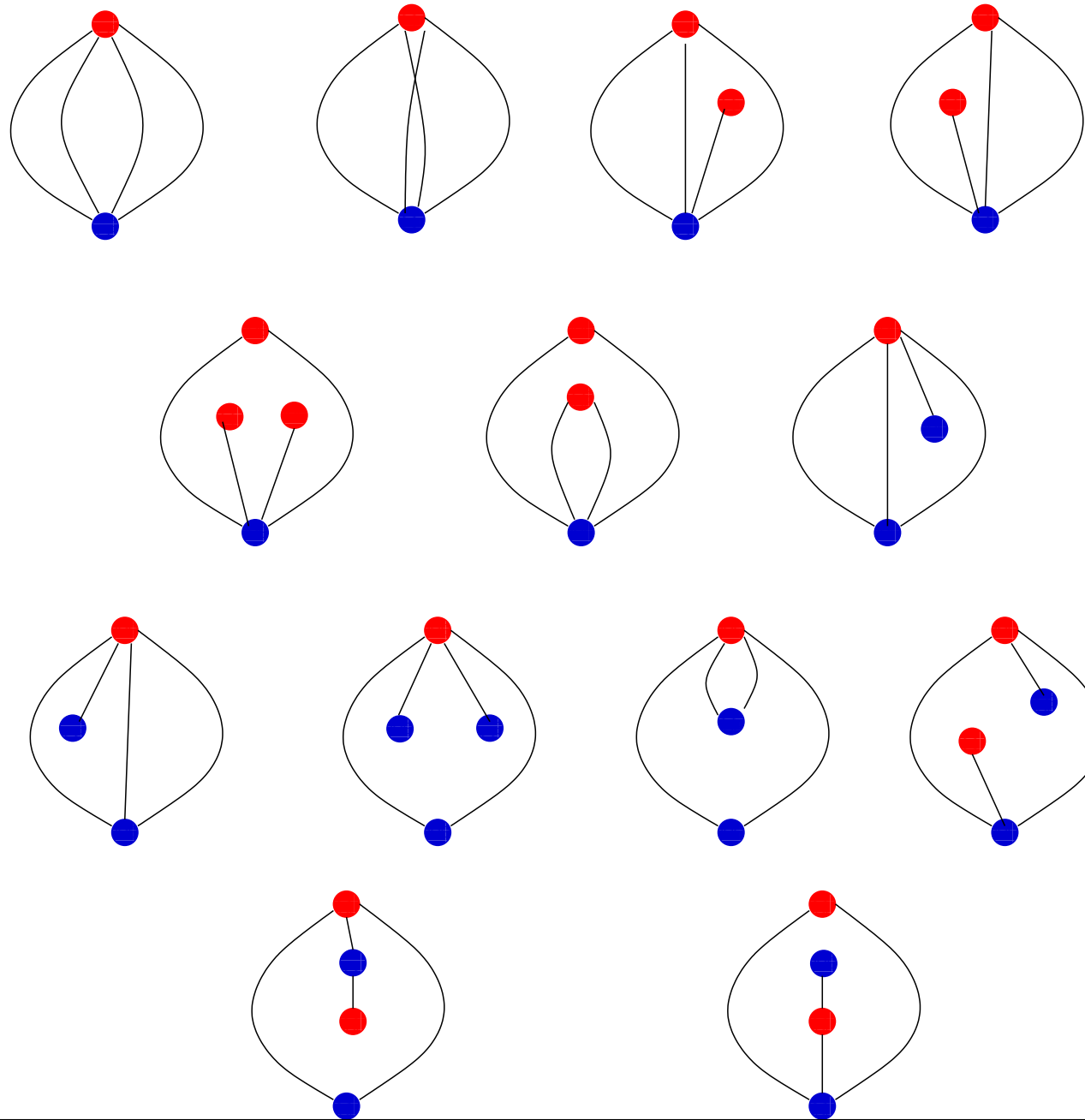
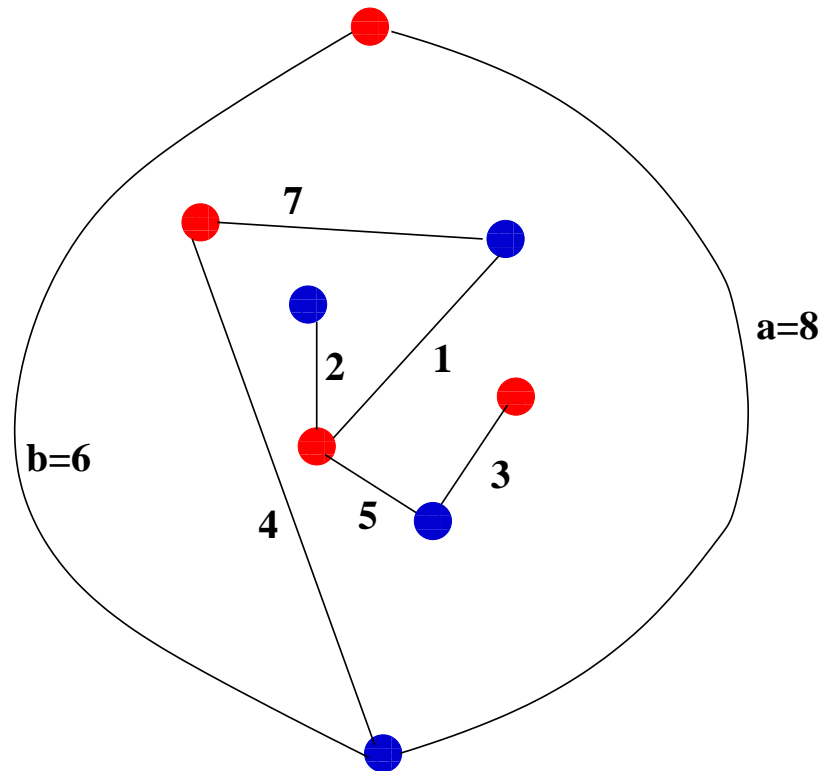


Figure 1: The 13 unlabelled doubly-rooted hypermaps on 4 points

From indecomposable permutations to doubly-rooted hypermaps

Proposition Let θ be an indecomposable permutation of \mathcal{S}_n , and $\alpha = F(\theta)$, $a = n$, $b = \theta^{-1}(n)$ then (θ, α, a, b) is a doubly rooted hypermap.

The hypermap obtained from $\theta = (1, 7)(2)(3, 5)(4, 6, 8)$



$$F(\theta) = 2, 5, 3, 7, 1, 8, 4, 6 = (1, 2, 5)(3)(4, 7)(6, 8)$$

Main result

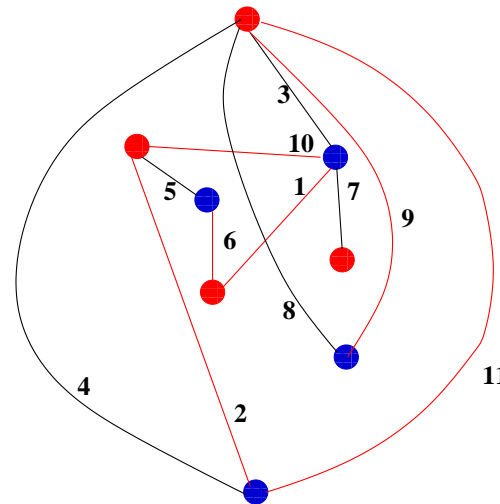
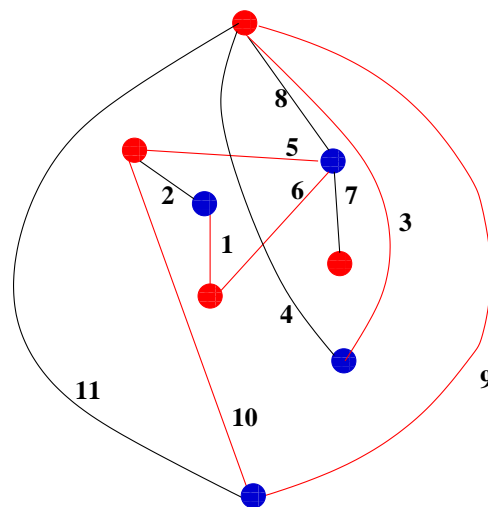
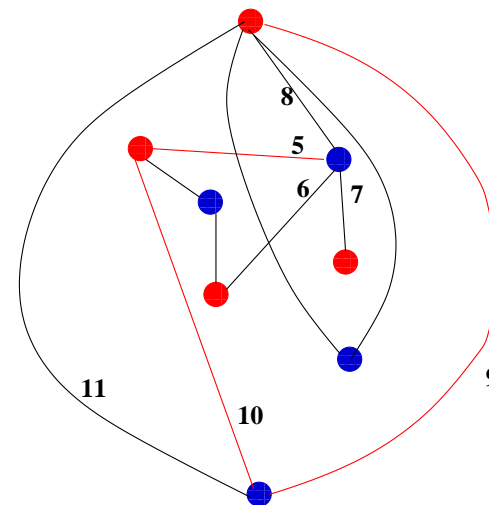
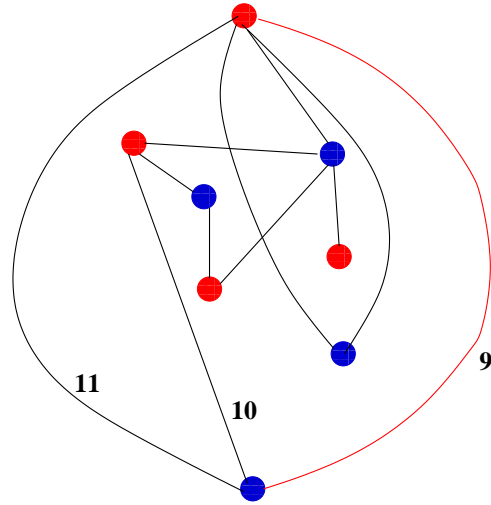
Theorem 3. For any doubly-rooted hypermap $(S, \sigma, \alpha, a, b)$ there exists a unique labeling ϕ of S such that $\phi(a) = n$ and

$$F(\phi\sigma\phi^{-1}) = \phi\alpha\phi^{-1}$$

Proof:

Ossona de Mendez and Rosenstiehl have proposed a bijection between indecomposable permutations and rooted hypermaps. This bijection uses the left-to-right maxima of the permutation. The algorithm which builds an indecomposable permutation from a doubly-rooted hypermap (σ, α, a, b) can be modified in order to give a proof of the theorem.

Labelling the edges of a doubly rooted hypermap in the OMR bijection



Self-reciprocal hypermaps and quasi-maps

A doubly-rooted hypermap (σ, α, a, b) is self-reciprocal if it is isomorphic to (α, σ, b, a)

Proposition The number of self-reciprocal rooted hypermaps with $n - 1$ points is the same as the number of self-dual rooted hypermaps with $n - 1$ points, and is equal the number of indecomposable involutions of \mathcal{S}_n .

From hypermaps to subgroups of \mathbb{F}_2

- The free group \mathbb{F}_2 with two generators consists of the words on the alphabet a, b, a^{-1}, b^{-1} with the reductions:

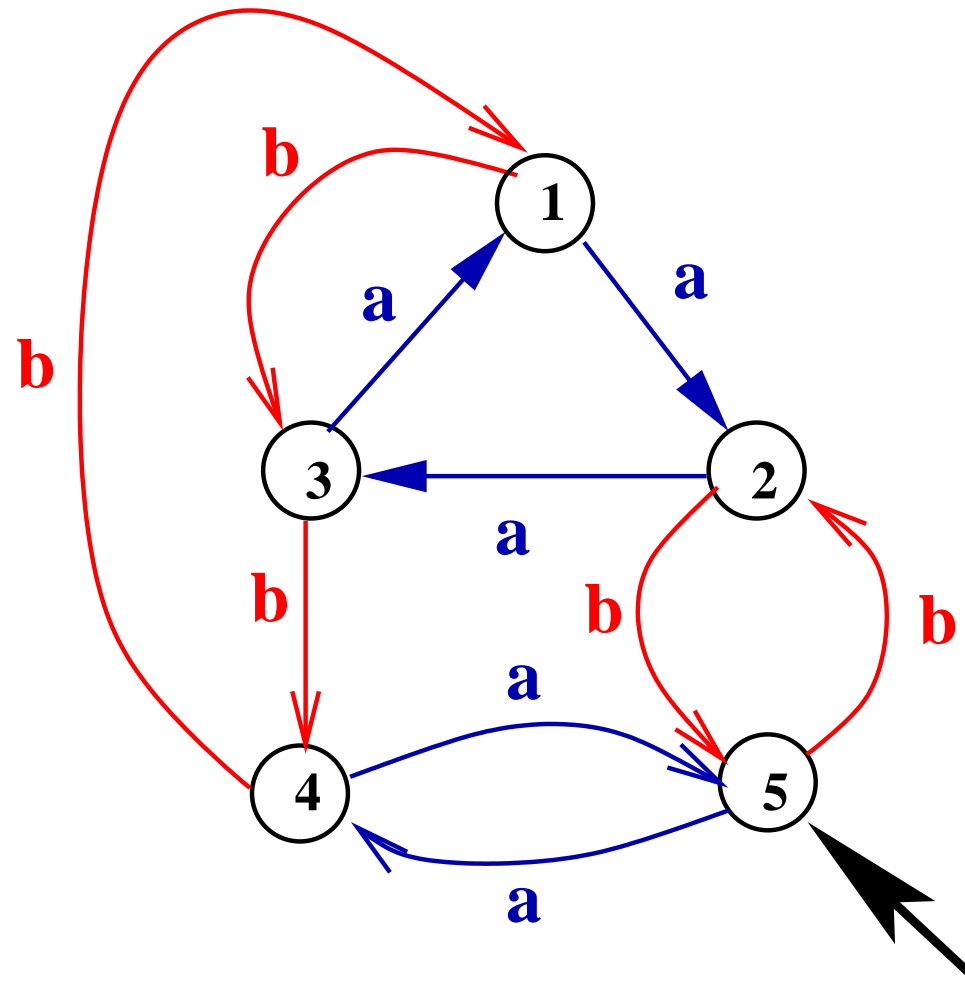
$$aa^{-1} = a^{-1}a = bb^{-1} = b^{-1}b = \infty$$

- To a rooted hypermap (S, σ, α, r) one associates the subgroup of \mathbb{F}_2 consisting of the words f such that $\phi(f)(r) = r$ where $\phi(a) = \sigma, \phi(b) = \alpha$

Representation by an automaton

- One can associate an automaton which set of states is S , the actions of a, b are those of σ, α and the initial and final state is r .
- The cosets correspond to the states of the automaton, the index of the group is the number of states

$$\sigma = (1, 2, 3)(4, 5) \quad \alpha = (1, 3, 4)(2, 5) \quad r = 5$$



Automorphisms of \mathbb{F}_2

- We have defined a bijection between indecomposable permutations of \mathcal{S}_n and subgroups of \mathbb{F}_2 of index $n - 1$.
- We know two natural actions on indecomposable permutations
 1. Take the inverse
 2. Take the mirror image and complement

$$b_i = n + 1 - a_{n+1-i}$$

- Are there automorphisms of \mathbb{F}_2 corresponding to those?

The answer is positive for the first with a well suited bijection (due to Sillke) and negative for the second until now.

Open problems

- Ricatti equation for the number of indecomposable permutations (Michel Marcus).

$$c_{n,k} = (n-1)c_{n-1,k} + \sum_{i=2}^{n-1} \sum_{j=1}^{\min(i,k-1)} c_{i,j} c_{n-i,k-j}$$

- Hetyei formula

$$c_n = \sum_{\theta \in \mathbb{S}_{n-2}} \prod_{i \in \text{Fix}(\theta)} (i+2)$$

where

$$\text{Fix}(\theta) = \{j \mid \theta(j) = j \text{ and } \forall i < j, \theta(i) < j\}$$

- Indecomposable matchings with k left to right maxima