

Approche combinatoire de l'analyse des k-arbres

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Génération
Aléatoire

A
M
L
E
S
P
R
O
B
L
E
M
E

k-trees: a class of graphs

Definition

A tree is:

- either a vertex,
- or a tree with one of its vertices connected to a new vertex.

k -trees: a class of graphs

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Definition - [Harary & Palmer-1968] ($k = 2$),
[Beineke & Pippert-1969]

A k -tree is:

- either a k -clique,
- or a k -tree with one of its k -cliques connected to a new vertex.

k-trees

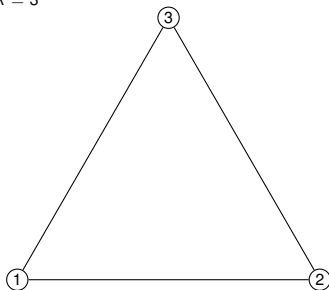
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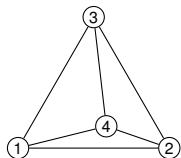
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$k = 3$

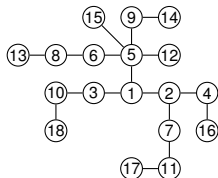


$k = 4$



k-trees

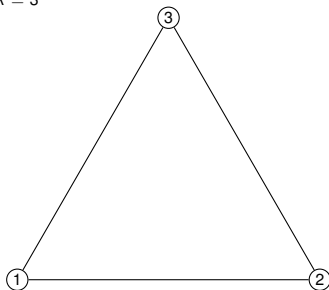
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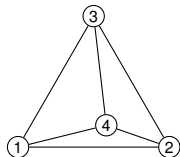
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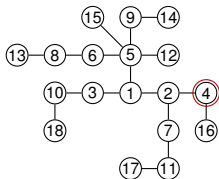


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k-trees

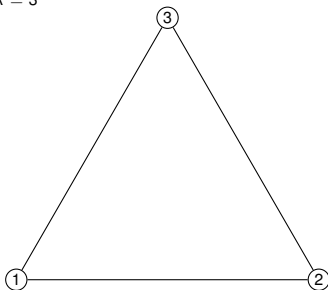
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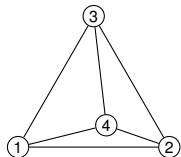
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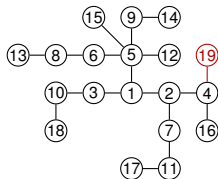


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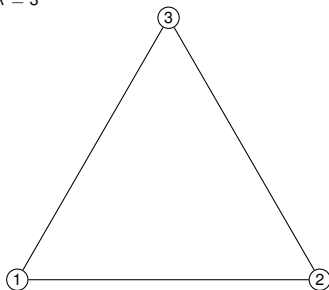
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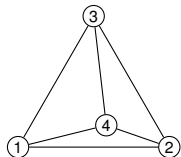
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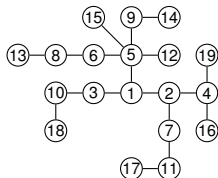


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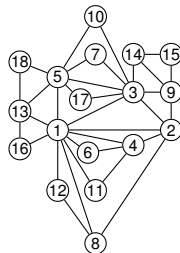


k-trees

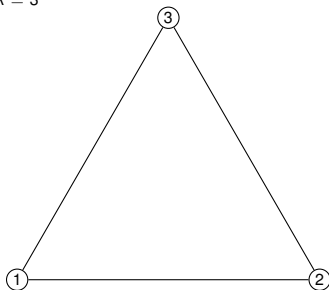
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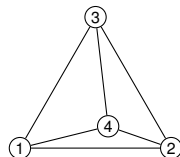
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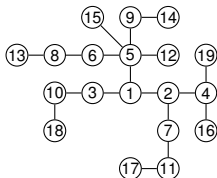


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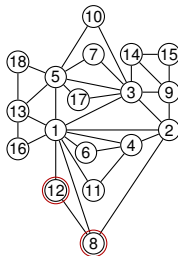


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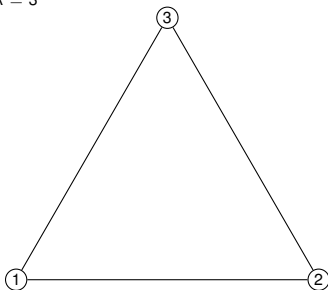
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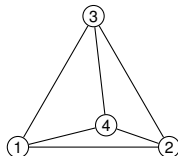
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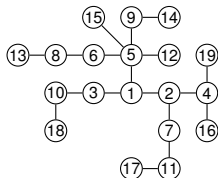


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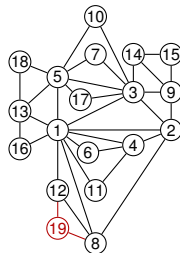


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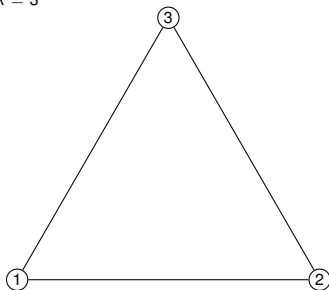
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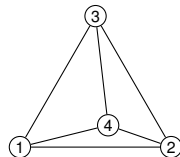
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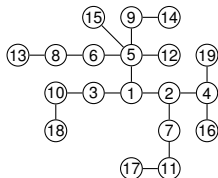


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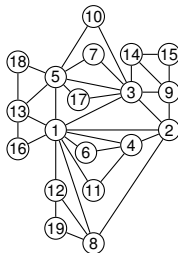


k-trees

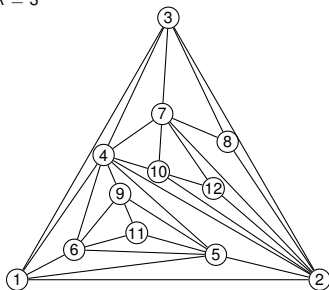
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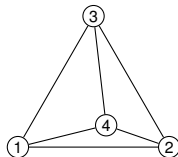
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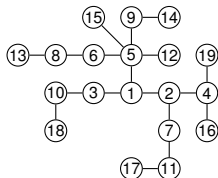


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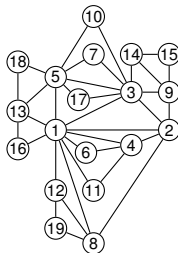


k-trees

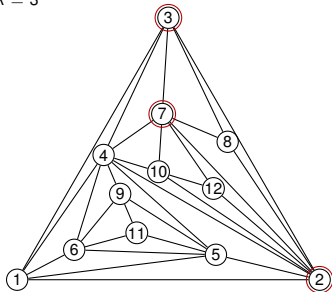
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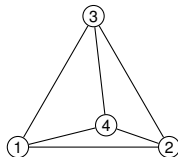
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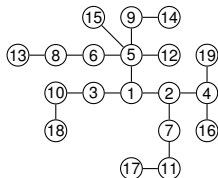


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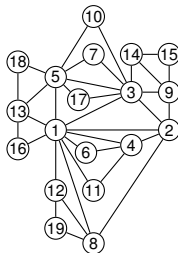


k-trees

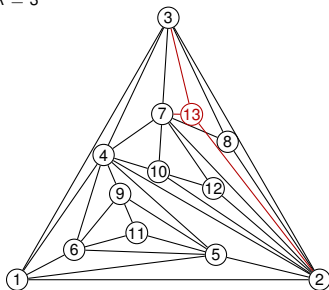
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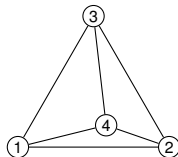
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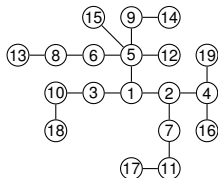


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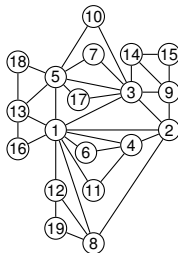


k-trees

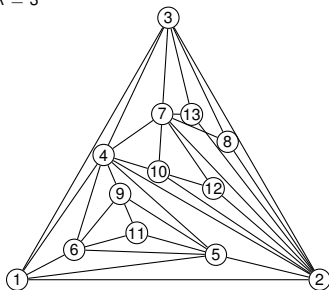
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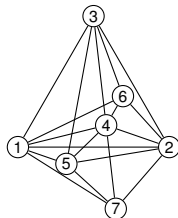
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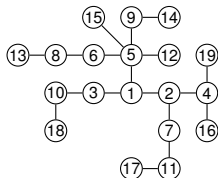


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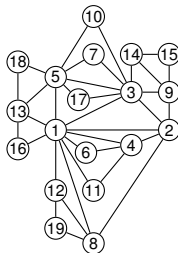


k-trees

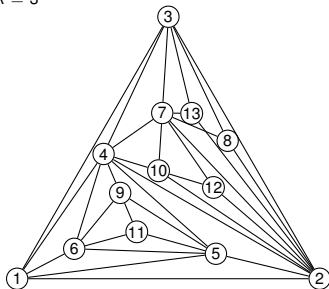
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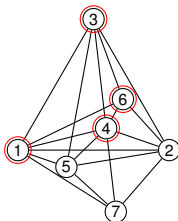
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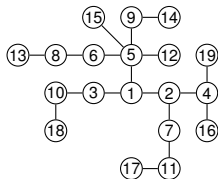


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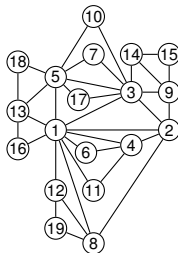


k-trees

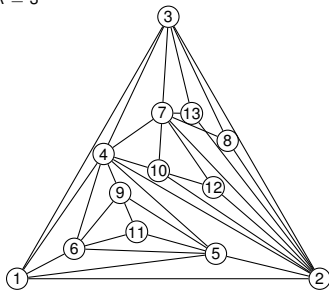
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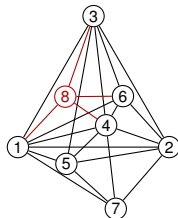
$k = 2$



$k = 3$



$k = 4$

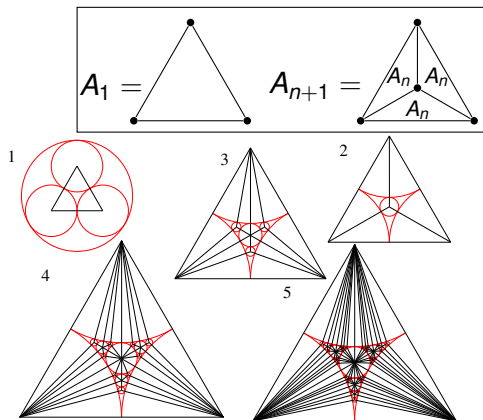




Combinatorial analysis of k -trees

bijection between planar 3-trees and ternary trees
estimating distances in planar 2-trees
BFS-profile of general k -trees

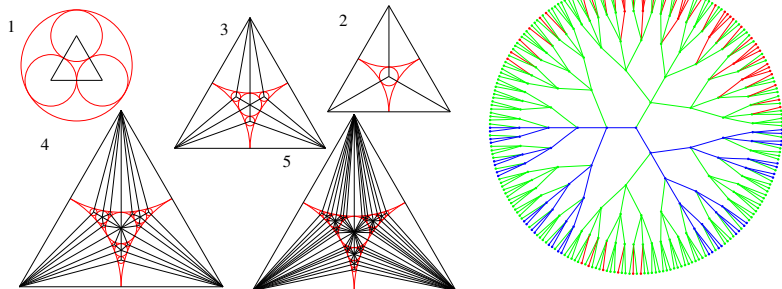
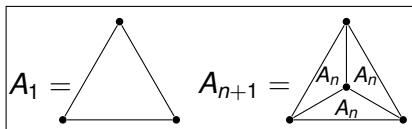
Apollonian networks (deterministic)



[Andrade et al. 05]

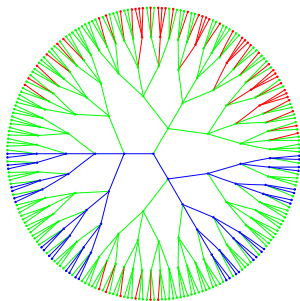
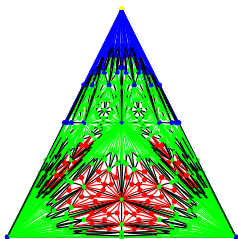
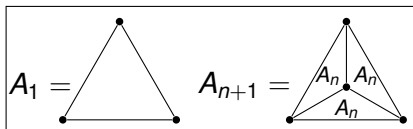
- Properties similar to “real world” graphs
- Inspired from the apollonian packings

Apollonian networks (deterministic)



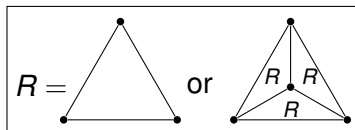
The (trivial) bijection with ternary trees can be used to study distances to an external vertex.

Apollonian networks (deterministic)

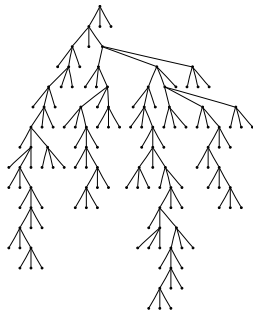
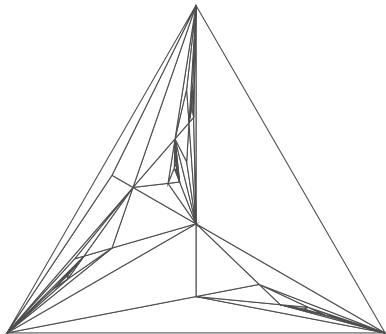


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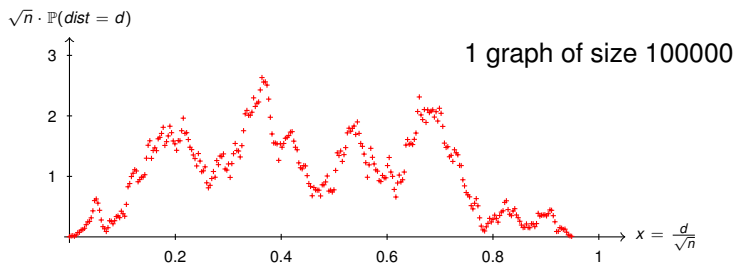
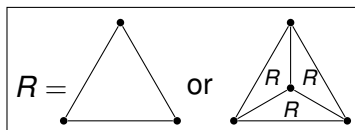
Random Apollonian networks (Planar 3-trees, Stack triangulations)



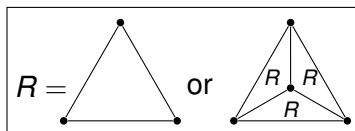
[Zhou et al. 05]



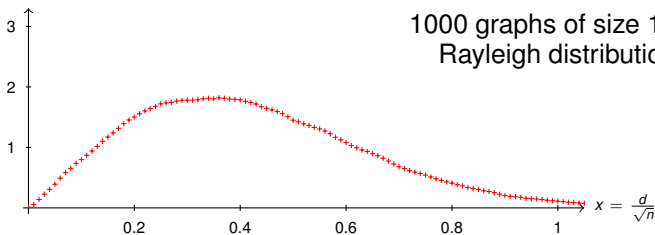
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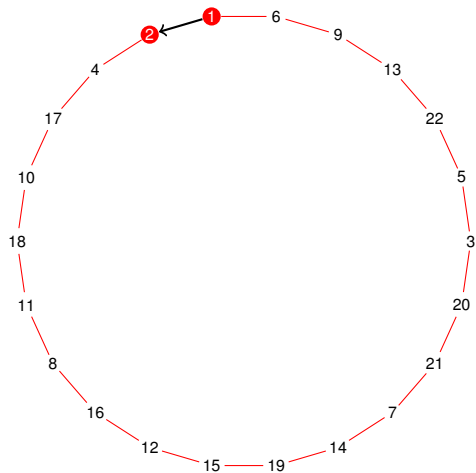
$\sqrt{n} \cdot \mathbb{P}(\text{dist} = d)$



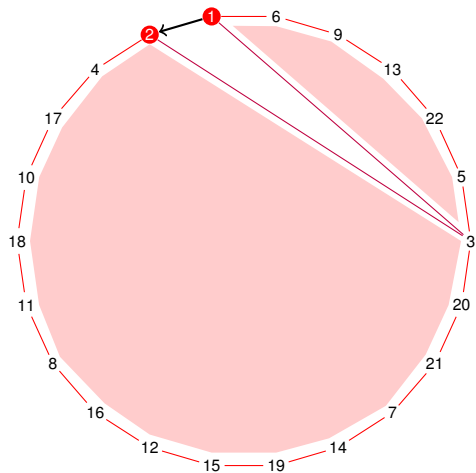
[A.D., Soria 07] [Bodini, A.D., Soria 08]

[Albenque, Marckert 08]

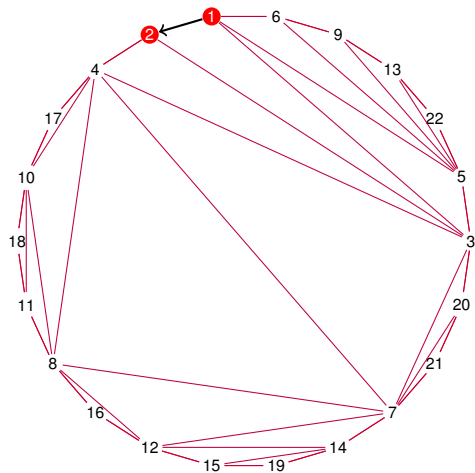
Planar 2-trees \equiv triangulations of a regular polygon



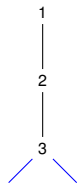
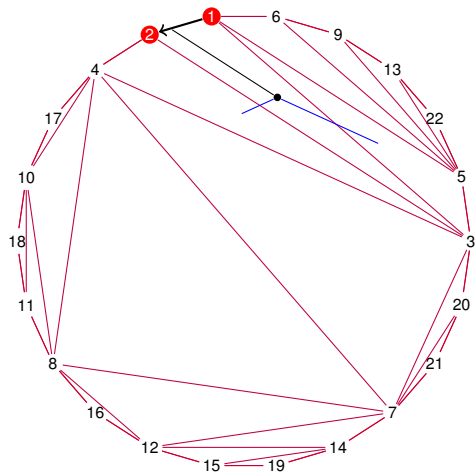
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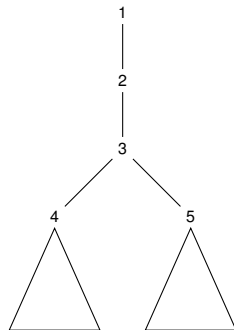
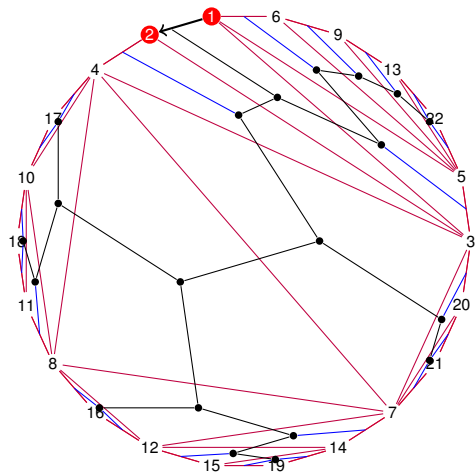
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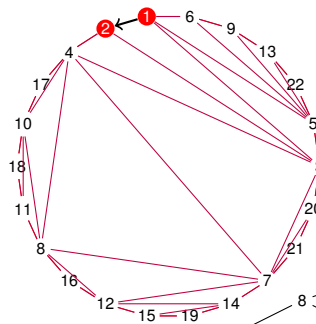
Planar 2-trees \equiv triangulations of a regular polygon



Planar 2-trees \equiv triangulations of a regular polygon

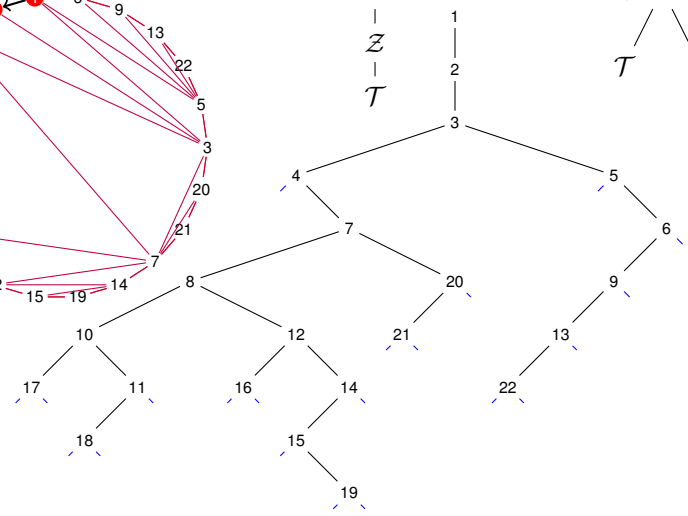


Planar 2-trees \equiv triangulations of a regular polygon

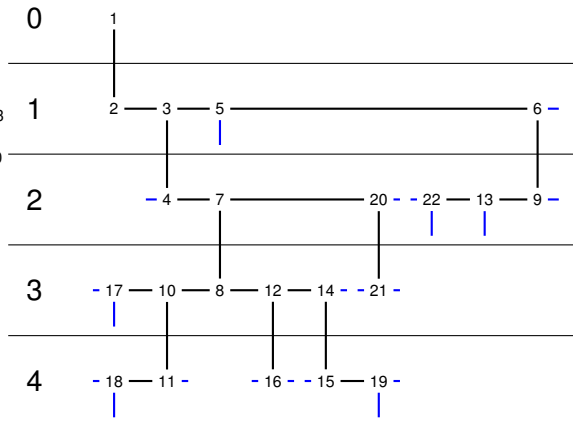
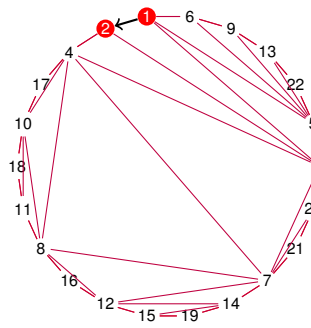


$$\mathcal{K} \equiv \mathcal{Z} \\ | \\ \mathcal{Z} \\ | \\ \mathcal{T}$$

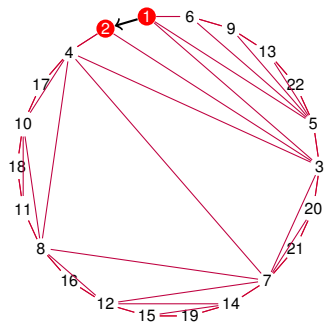
$$\mathcal{T} \equiv \mathcal{I} + \mathcal{Z} \\ \swarrow \quad \searrow \\ \mathcal{T} \quad \mathcal{T}$$



Planar 2-trees \equiv triangulations of a regular polygon



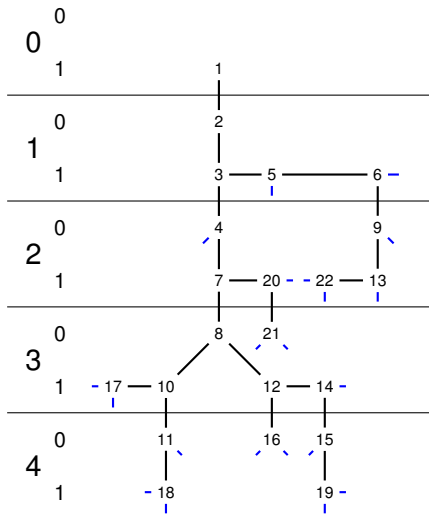
Planar 2-trees \equiv triangulations of a regular polygon



$$\mathcal{K} \equiv \mathcal{Z} \begin{array}{c} \mathcal{Z} \\ \mathcal{T}_{1,1} \end{array}$$

$$\mathcal{T}_{d,0} \equiv \mathcal{I} + \mathcal{Z}_d \begin{array}{c} \mathcal{T}_{d,1} \quad \mathcal{T}_{d,1} \end{array}$$

$$\mathcal{T}_{d,1} \equiv \mathcal{I} + \mathcal{Z}_d - \mathcal{T}_{d,1} \begin{array}{c} \mathcal{T}_{d+1,0} \end{array}$$



Planar 2-trees \equiv triangulations of a regular polygon

$$K_d(z, u) = \sum_{n,m} r_{n,m} u^m z^n$$

$r_{n,m}$: # of rooted k -trees with n total vertices
and m vertices at distance d

$$\frac{\partial}{\partial u} K_d(z, u) \Big|_{u=1} = \sum_n r_n z^n$$

r_n : # of vertices at distance d in all
rooted k -trees with n total vertices

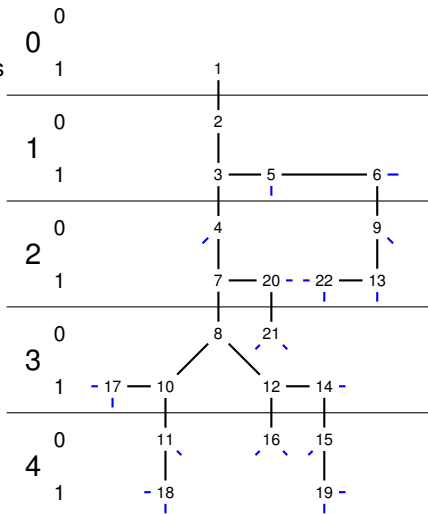
$$K_d(z, u) = z^2 T_{d-1,1}(z, u)$$

$$T_{d,1}(z, u) = 1 + z T_{d,1}(z, u) T_{d-1,0}(z, u)$$

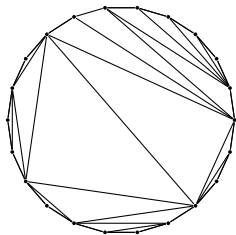
$$T_{d,0}(z, u) = 1 + z T_{d,1}^2(z, u)$$

$$T_{0,1}(z, u) = 1 + z T_{0,1}(z, u) T(z)$$

$$T_{0,0}(z, u) = 1 + z T_{1,1}^2(z, u)$$

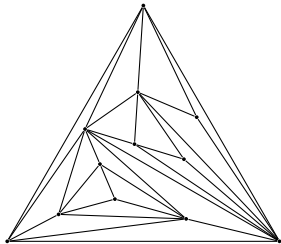


Generalization to planar k -trees



$k = 2$

binary trees
two sub-levels



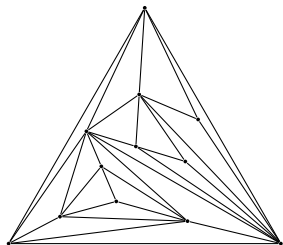
$k = 3$

ternary trees
three sub-levels

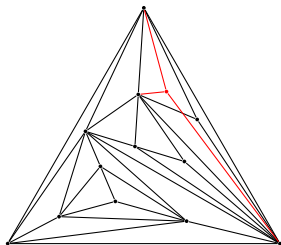
any k

k -ary trees
 k sub-levels

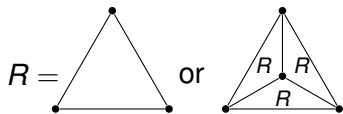
Generalization to (non-planar) k -trees



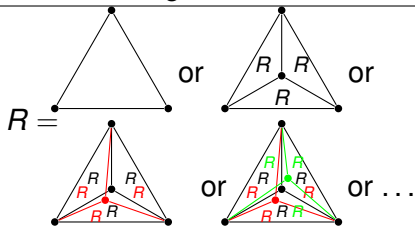
planar



general



$$\mathcal{T} = \mathcal{I} + \mathcal{Z} \star \mathcal{T}^k$$



$$\mathcal{T} = \text{SET}(\mathcal{Z} \star \mathcal{T}^k)$$

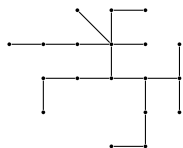
k-trees

Definition [Beineke, Pippert 69] ($k = 2$ [Harary, Palmer 68])

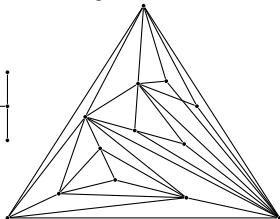
A k -tree is:

- either a k -clique,
- or a k -tree with one of its k -cliques connected to a new vertex.

$k = 1$



$k = 3$



Some graph theory NP-hard problems have linear time algorithms on partial k -trees [Arnborg, Proskurowski 89]

Generating function for distances

$$K(z) = \sum_n K_n z^n$$

$$K(z) = z^k T(z)$$

$$T(z) = \exp(zT^k(z))$$

$$K_d(z, u) = \sum_{n,m} r_{n,m} u^m z^n$$

$$K_d(z, u) = z^k T_{d,1}(z, u)$$

$$T_{d,i}(z, u) = \exp(zT_{d,i}^{k-i}(z, u) T_{d,i+1}^i(z, u))$$

$$T_{d,d}(z, u) = T_{d-1,0}(z, u)$$

$$T_{0,i}(z, u) = \exp(uzT_{0,i}^{k-i}(z, u) T_{0,i+1}^i(z, u))$$

$$T_{0,d-1}(z, u) = \exp(uzT_{0,d-1}^{k-1}(z, u) T(z))$$

$$\frac{1}{nK_n} [z^n] \left. \frac{\partial}{\partial u} K_d(z, u) \right|_{u=1} =$$

mean # of vertices at distance d

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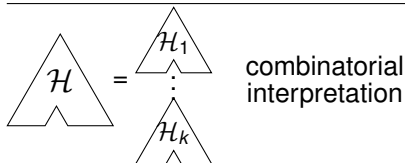
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mean # of vertices at distance d

$$\frac{\partial}{\partial u} K_d(z, u) \Big|_{u=1} = H^{d-2}(z) \frac{\partial}{\partial u} K_2(z, u) \Big|_{u=1}$$



$$H(z) = k!(zT^k(z))^k \prod_{i=1}^{k-1} \frac{1}{1 - izT^k(z)}$$

$$= 1 - c_k \sqrt{2(1 - kez)} + O(1 - kez).$$

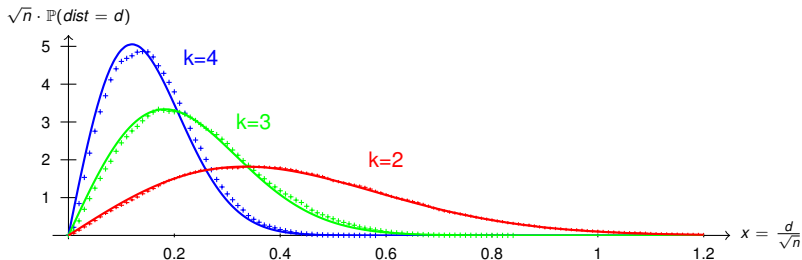
Semi-large power theorem for calculating $[z^n]H^{d-2}(z)$ in the range $x\sqrt{n}$.

Main Result

Theorem [A.D., Soria 09] (RAN [Bodini, A.D., Soria 08])

Given a rand. k -tree G over n vert., the **distance** between the **root vertex r** and a **random vertex v** of G has asymptotic mean value of order \sqrt{n} and is **Rayleigh distributed** in the **range $x\sqrt{n}$** :

$$\sqrt{n} \cdot \lim_{n \rightarrow \infty} \mathbb{P}(D(r, v) = \lfloor x\sqrt{n} \rfloor) = c_k^2 x e^{-\frac{(c_k x)^2}{2}}, \text{ with } c_k = k \sum_{i=1}^k \frac{1}{i}$$



Main Result

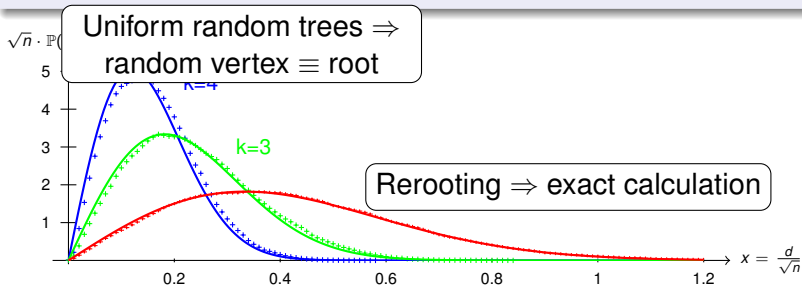
Theorem [A.D., Soria 09] (RAN [Bodini, A.D., Soria 08])

Given a rand. k -tree
root vertex r and a r
value of order \sqrt{n} ar

What about the
distance between 2
random vertices?

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as asymptotic mean
d in the range $x\sqrt{n}$:

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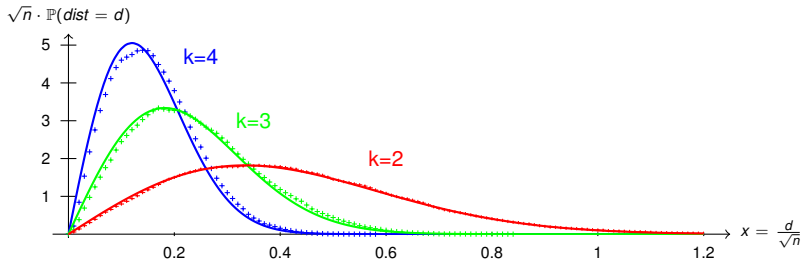


Main Result

Theorem [A.D., Soria 09]

Given a rand. k -tree G over n vert., the **distance** between the **two random vertices v, w** of G has asymptotic mean value of order \sqrt{n} and is **Rayleigh distributed** in the **range $x\sqrt{n}$** :

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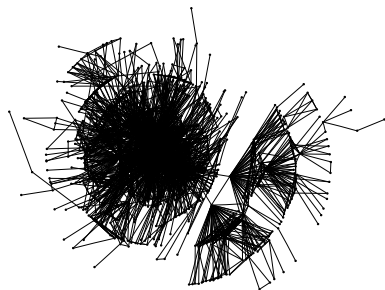


Increasing random model (joint work with Bodini, Hwang, Soria)



- Same bijection.
- $\mathcal{T} = \exp(\mathcal{Z} \star \mathcal{T}^k) \rightarrow \mathcal{T} = \exp(\mathcal{Z}^\square \star \mathcal{T}^k)$
- Probability model of original RAN.
- Power law degree distribution.
- BFS profile a gaussian around $\log(n)$.
- Extension to chordal graphs.

Increasing chordal graphs



$$\mathcal{T}_k = \text{SET}(\mathcal{Z}^\square \star \prod_{i=1}^{k+1} \mathcal{T}_i^{(k+1)_i})$$

1, 1, 3, 17, 153, 2005, 36435, 892589, ...

Non-increasing [Wormald 1985]:

1, 1, 4, 35, 541, 13302, 489287, 25864897, ...