

# Formule exacte pour les cartes biparties simples

**Eric Fusy** (LIX, Ecole Polytechnique)  
travail en commun avec **Olivier Bernardi** (MIT)

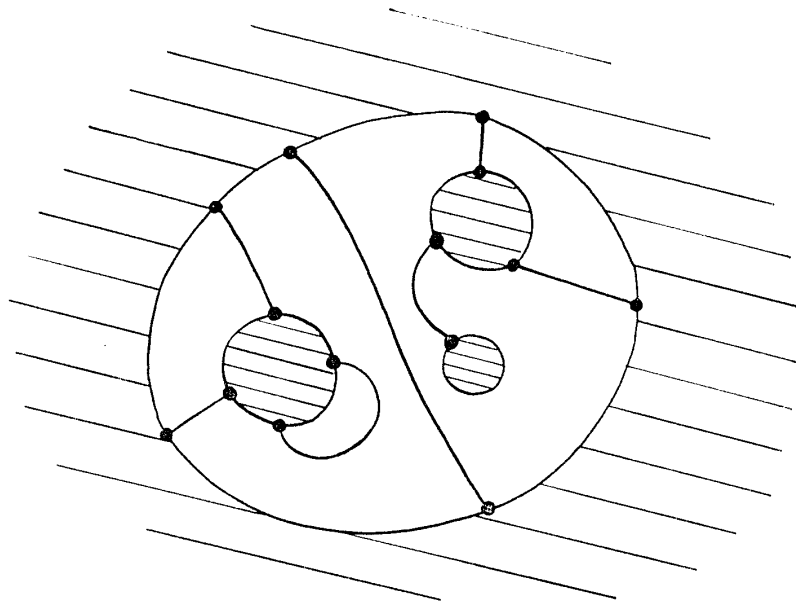
# Slicings formula

[Tutte'62] "a census of slicings":

The number of ways to "slice" the 2-sphere having boundaries of fixed even sizes ( $n_i$  boundaries of size  $2i$  for  $i$  in  $[1..k]$ ) is

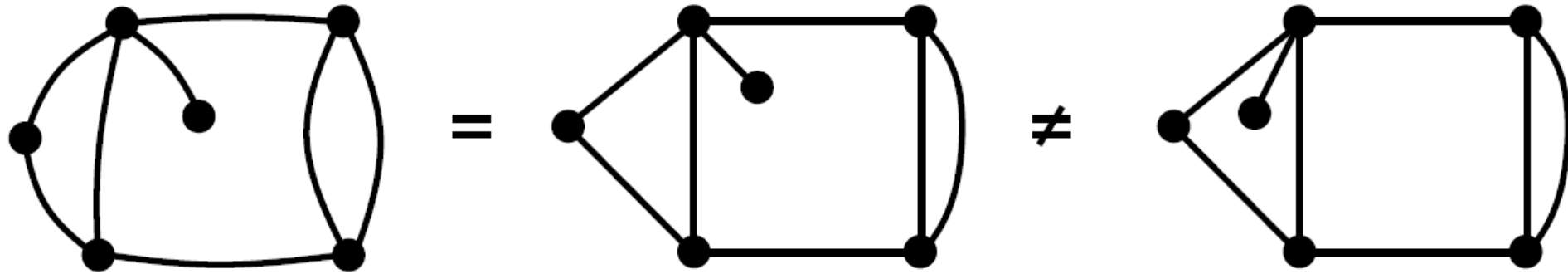
$$2 \frac{e!}{v!} \prod_{i=1}^k \frac{1}{n_i!} \binom{2i-1}{i-1}^{n_i}$$

where  $e = \sum_i i n_i$        $v = -2 + \sum_i (i-1) n_i$

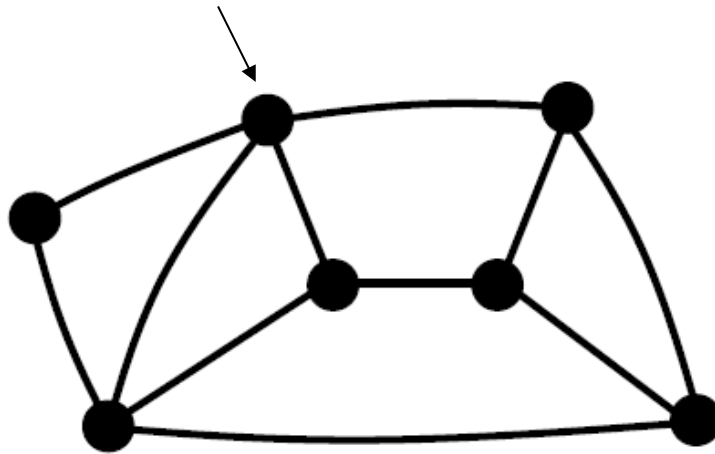


# Planar maps

- **Planar map:** connected planar graph embedded in the plane

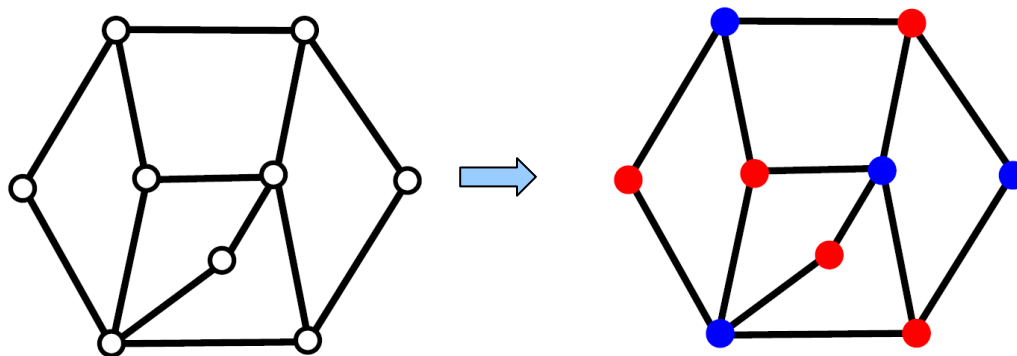


- **rooted map:** a corner in the outer face is distinguished



# Bipartite maps

- Bipartite map = all faces have **even** degree

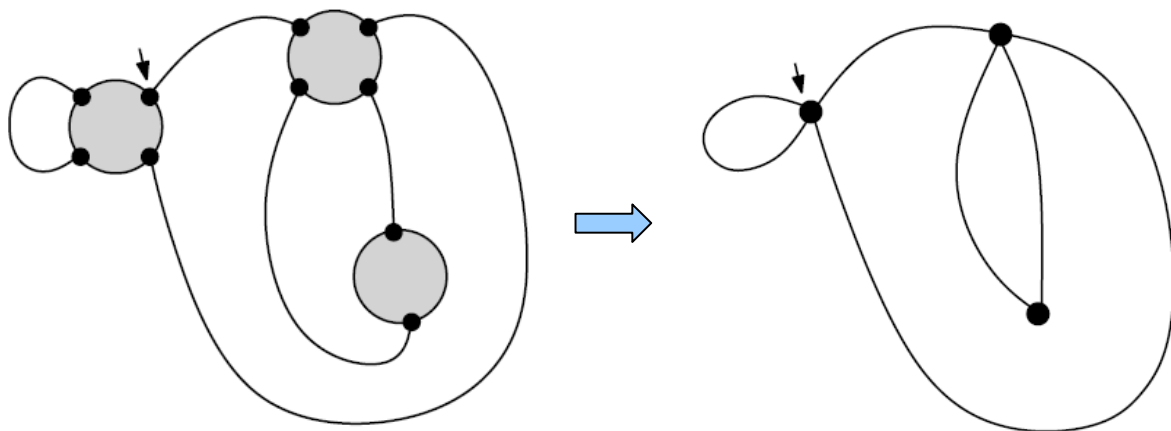


- Slicings formula



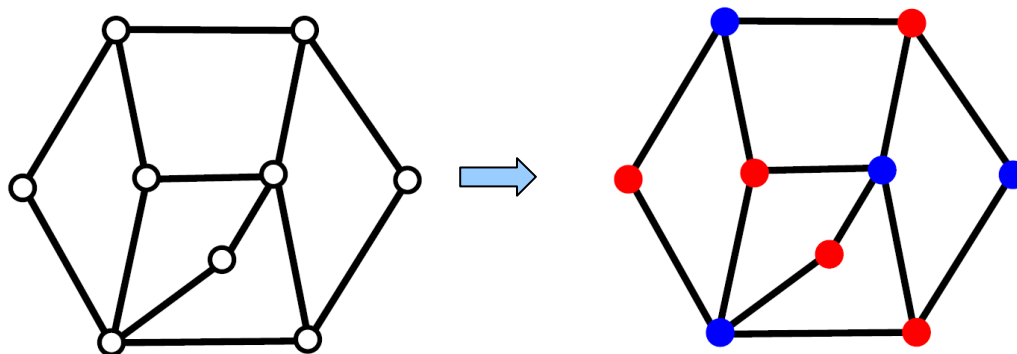
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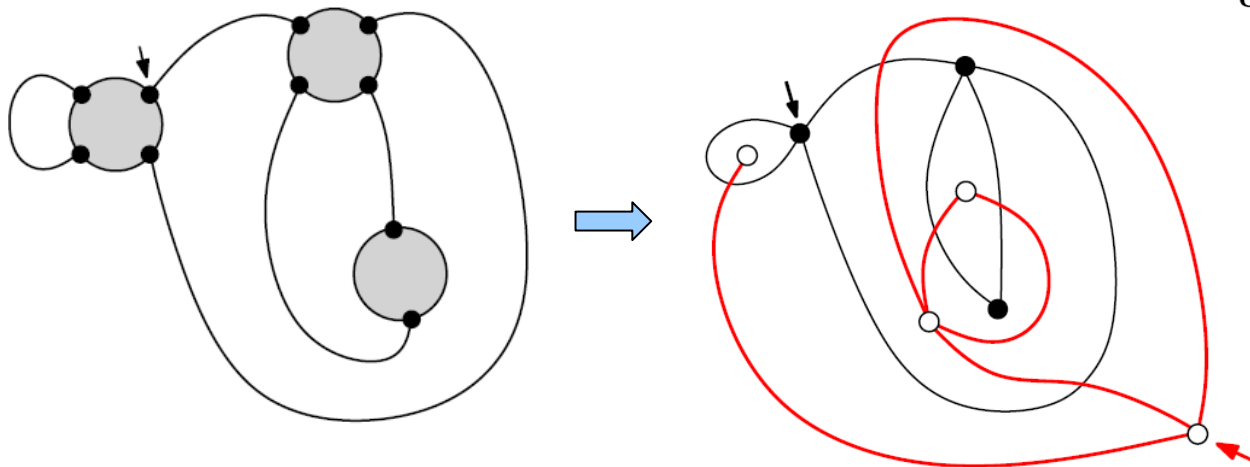


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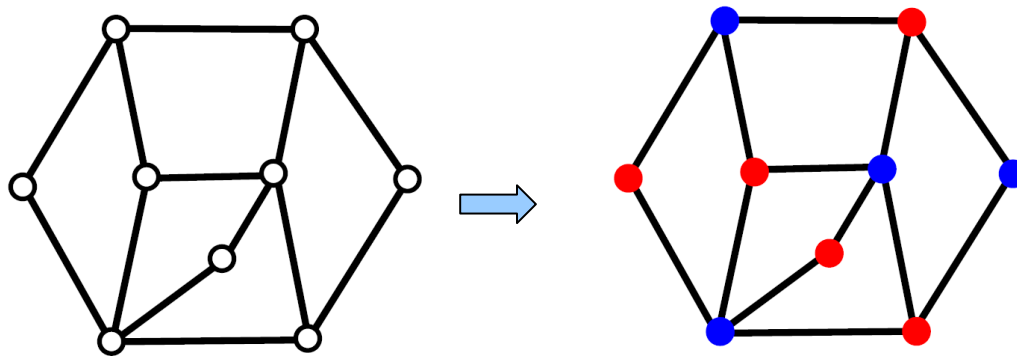
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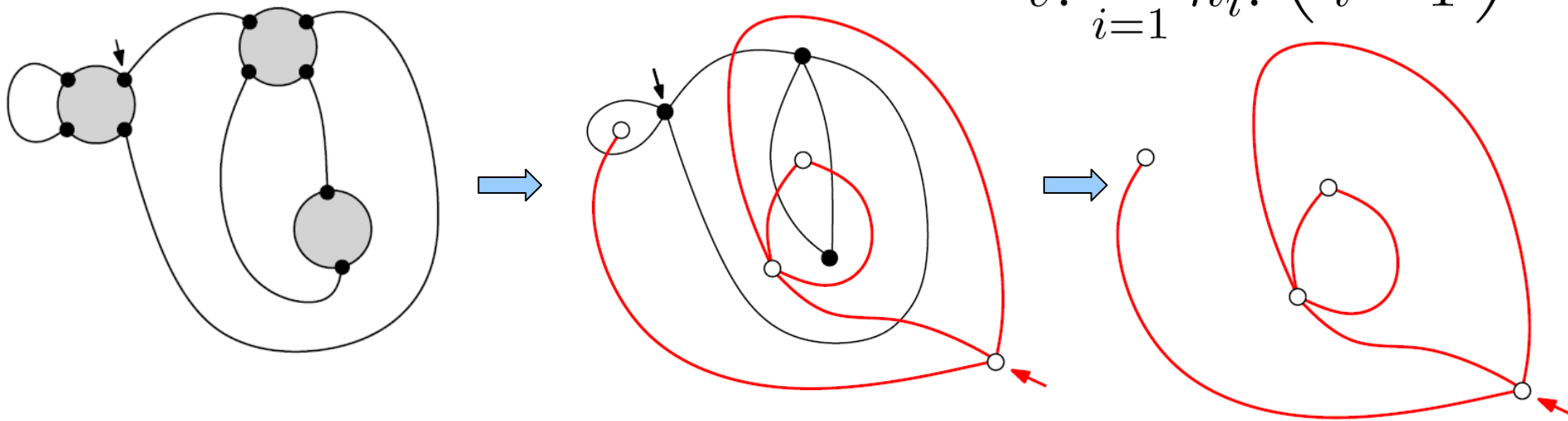


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# “Slicings formula” for simple maps

[Bernardi-F’11 bijectively], can be proved by substitution

- The number of **rooted simple bipartite maps** with  $n_i$  faces of degree  $2i$  for  $i$  in  $[2..k]$  equals

$$2 \frac{(e + n - 3)!}{(e - 1)!} \prod_{i=2}^k \frac{1}{n_i!} \binom{2i - 1}{i - 2}^{n_i}$$

where  $n$  is the number of faces and  $e$  is the number of edges

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- To be compared with  $2 \frac{e!}{v!} \prod_{i=1}^k \frac{1}{n_i!} \binom{2i - 1}{i - 1}^{n_i}$
- **Rk**: weighted sum over plane trees with  $n_i$  vertices of degree  $i$ , each having weight  $w_i$  is (cf **Lagrange inversion formula**):

$$\frac{m!}{p!} \prod_i \frac{1}{n_i!} w_i^{n_i} \quad \text{with} \quad \begin{aligned} m &= -2 + \sum_i i n_i \\ p &= m + 1 - \sum_i n_i \end{aligned}$$

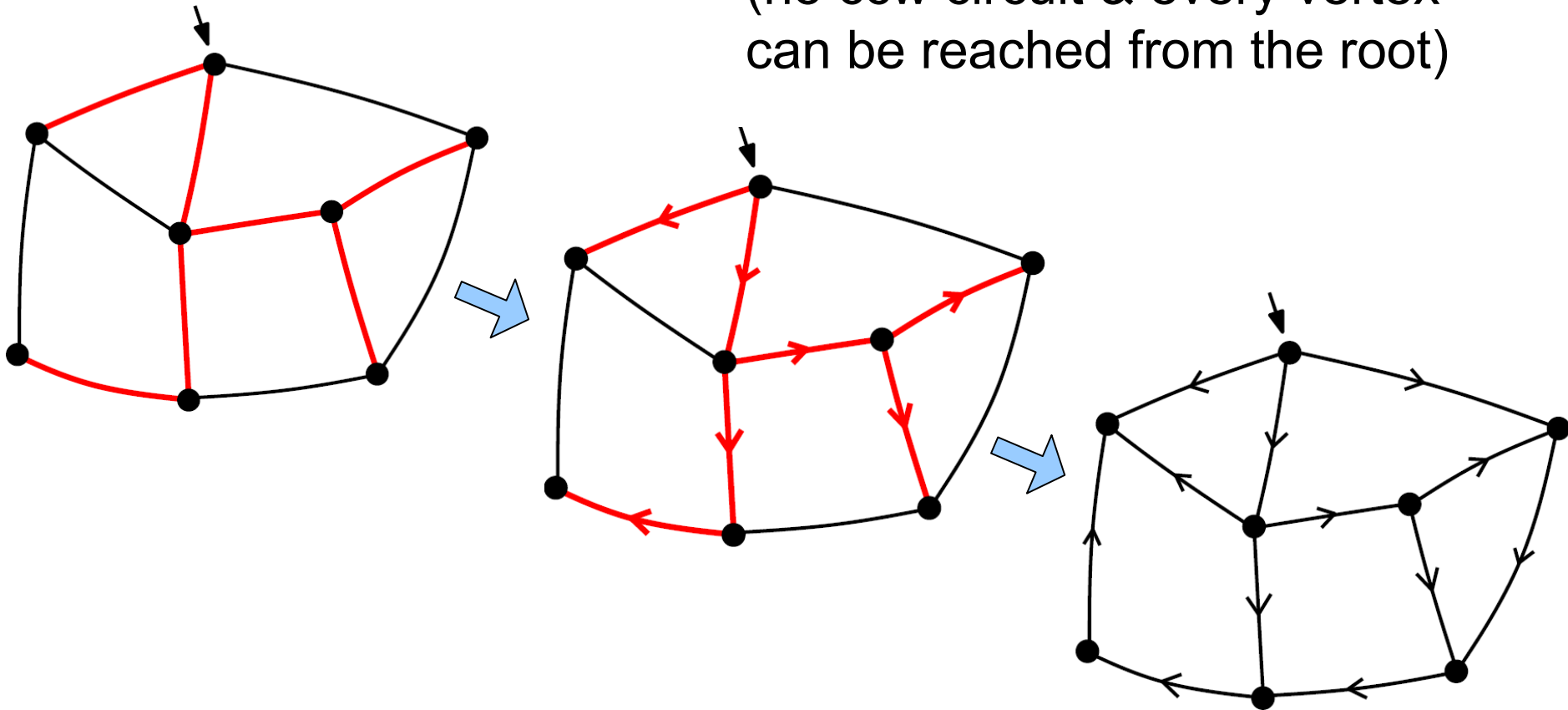
# Counting tree-rooted maps

**Def:** tree-rooted map = rooted planar map + spanning tree

[Mullin'67]: there are  $C_n C_{n+1}$  tree-rooted maps with  $n$  edges

Bijjective proof: [Bernardi'06, Bernardi-Chapuy'09]

**1st step:** tree-rooted map  $\iff$  minimal accessible orientation  
(no ccw circuit & every vertex can be reached from the root)



# Counting tree-rooted maps

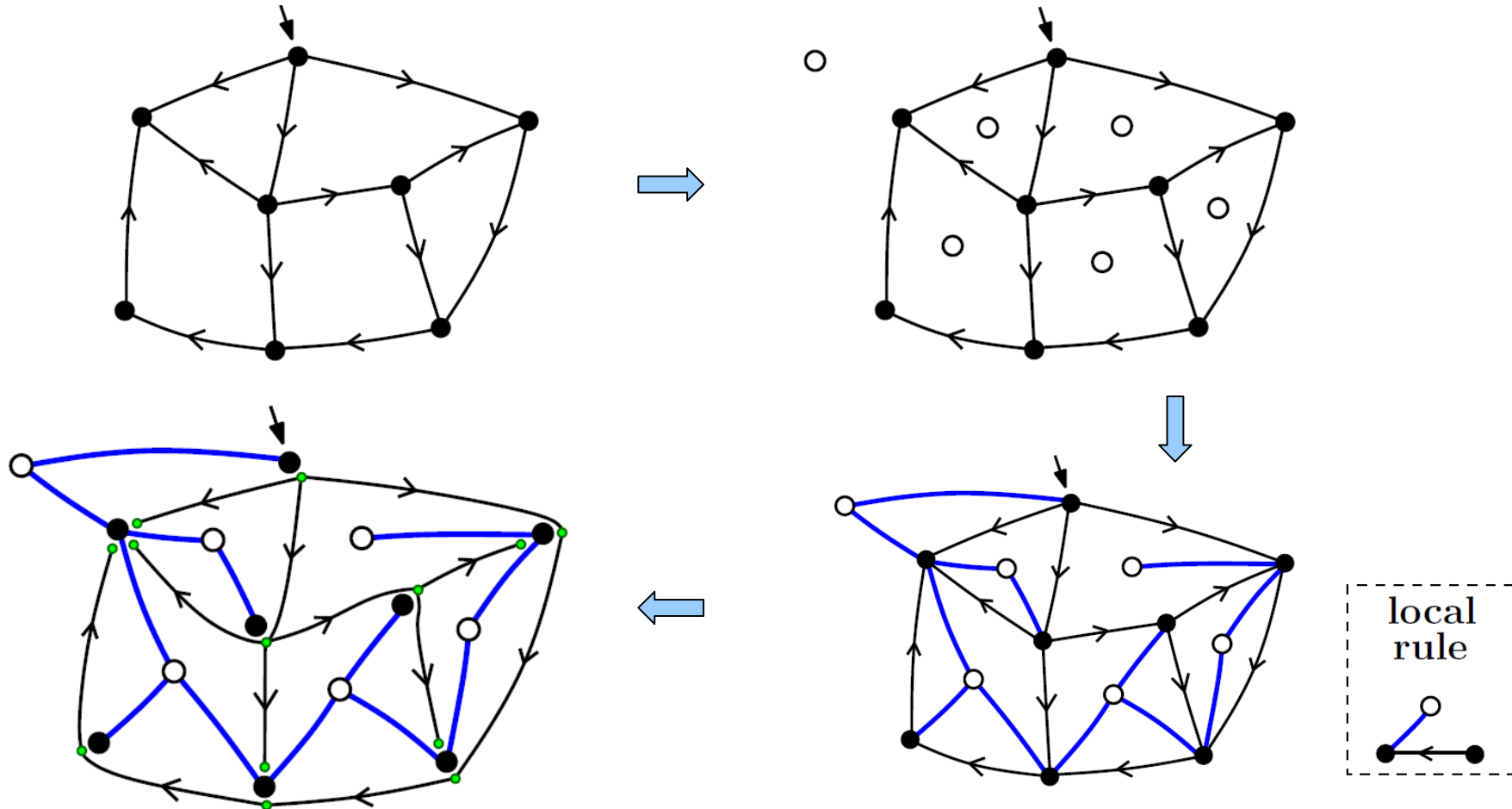
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2<sup>nd</sup> step: minimal accessible orientation

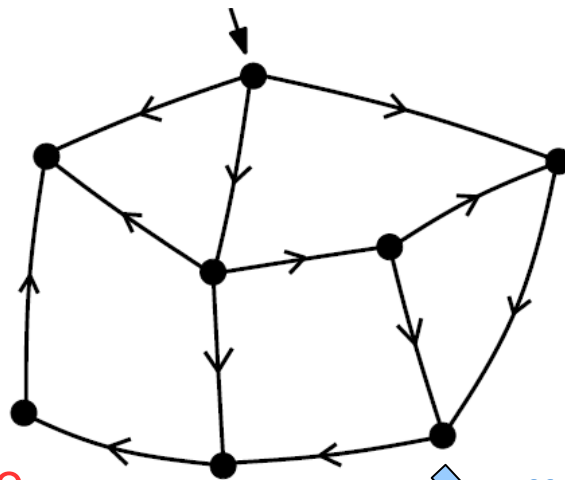


mobile + tree (folded around the mobile)



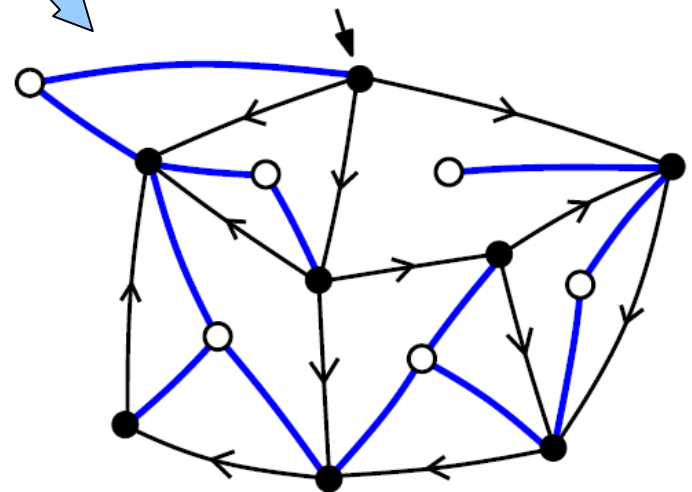
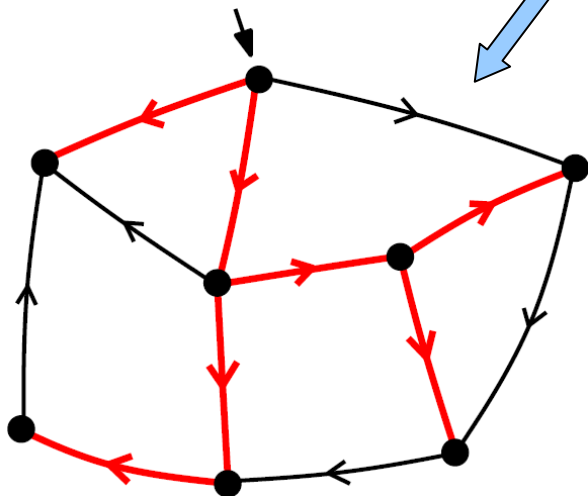
# Consequences

Two ways to encode a minimal accessible orientation:



canonical spanning tree

mobile



cf [Poulalhon, Schaeffer'03]

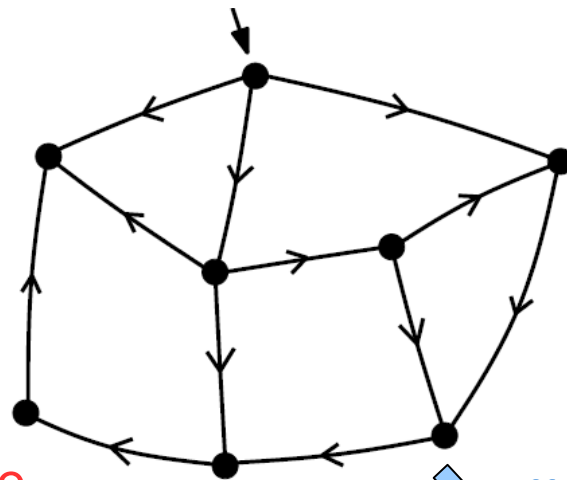
cf [Bouttier, Di Francesco, Guitter'04]

Strategy here:

map family  $\longleftrightarrow$  subfamily of minimal accessible orientations

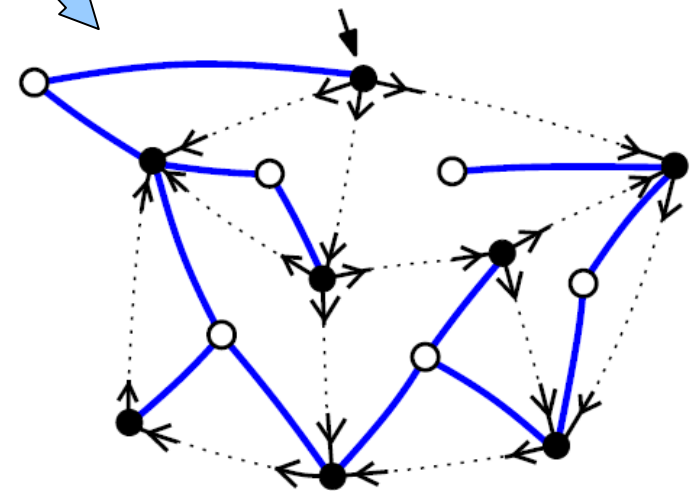
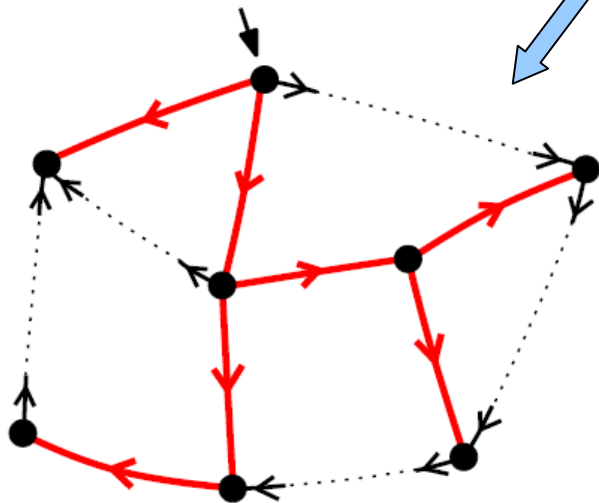
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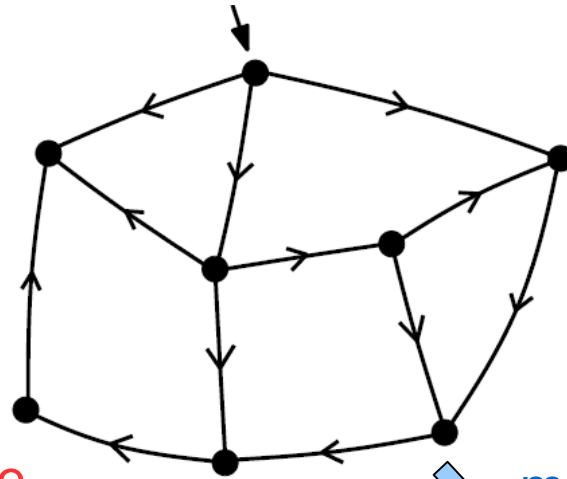
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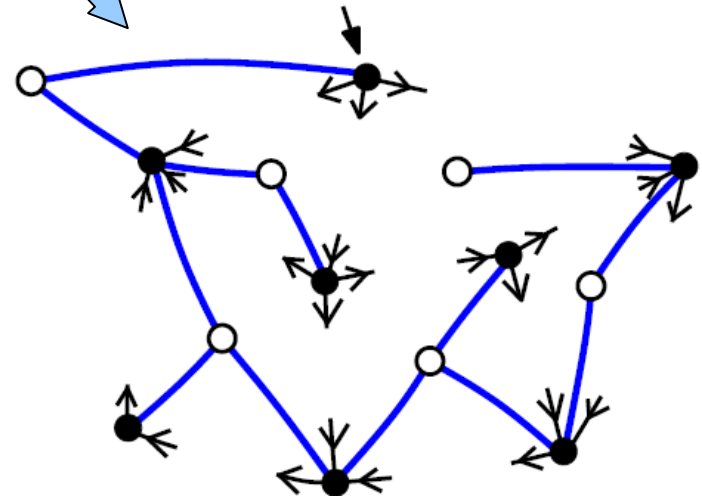
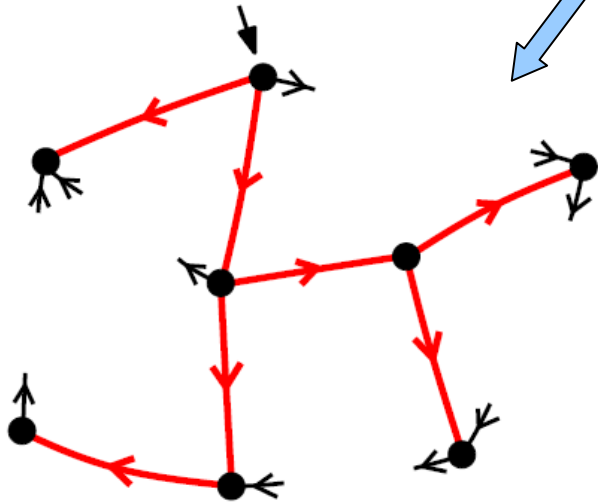
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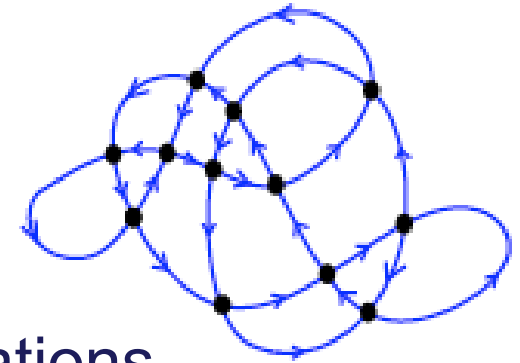
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# Outline

- Bijective proof of Tutte's slicings formula

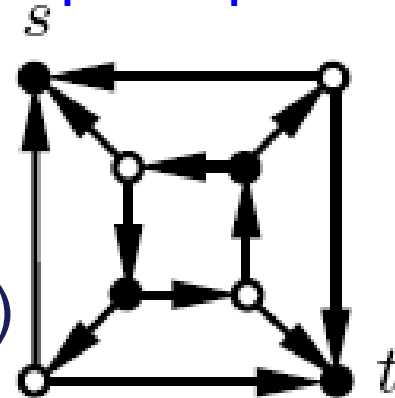
$$2 \frac{e!}{v!} \prod_{i=1}^k \frac{1}{n_i!} \binom{2i-1}{i-1}^{n_i}$$



canonical spanning trees & Eulerian orientations  
(recover Schaeffer's bijection [**Schaeffer'97**])

- Bijective proof of the “slicings formula for simple bip. maps”

$$2 \frac{(e+n-3)!}{(e-1)!} \prod_{i=2}^k \frac{1}{n_i!} \binom{2i-1}{i-2}^{n_i}$$



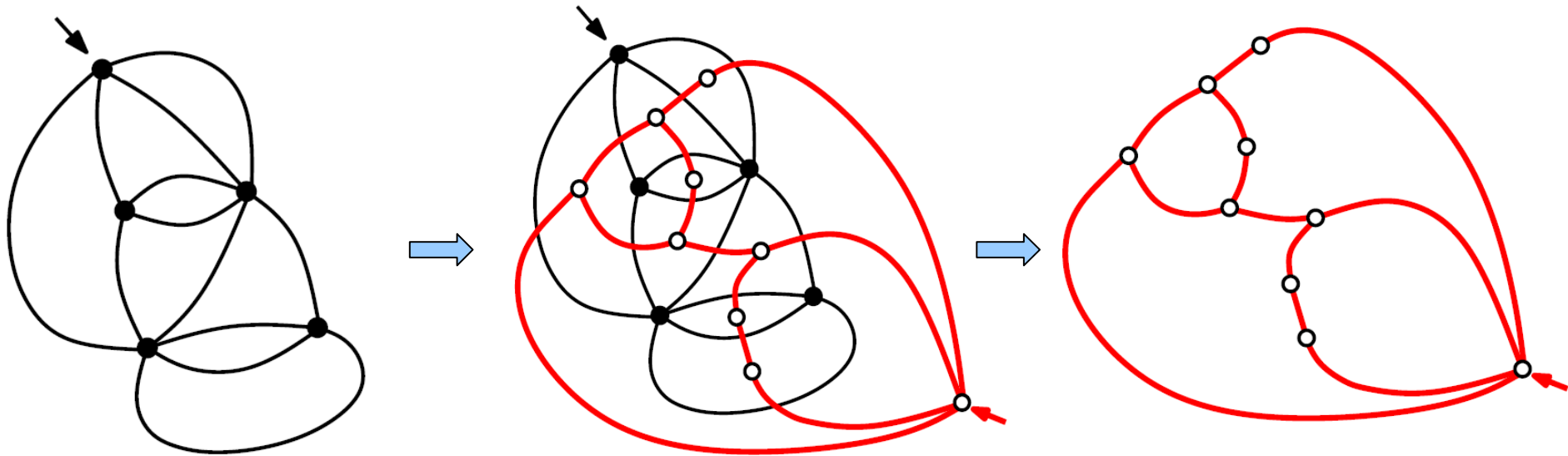
Mobiles & 2-orientations (indegree 2 at vertices)

- Generalization of the series expressions to higher girth  
(girth = length of a shortest cycle within the graph)

# **Part I: bijective counting of bipartite maps**



# Bipartite maps $\longleftrightarrow$ Eulerian maps



- Slicings formula**



number of rooted **eulerian** maps with  $n_i$  **vertices** of degree  $2i$  for  $i$  in  $[1..k]$  is

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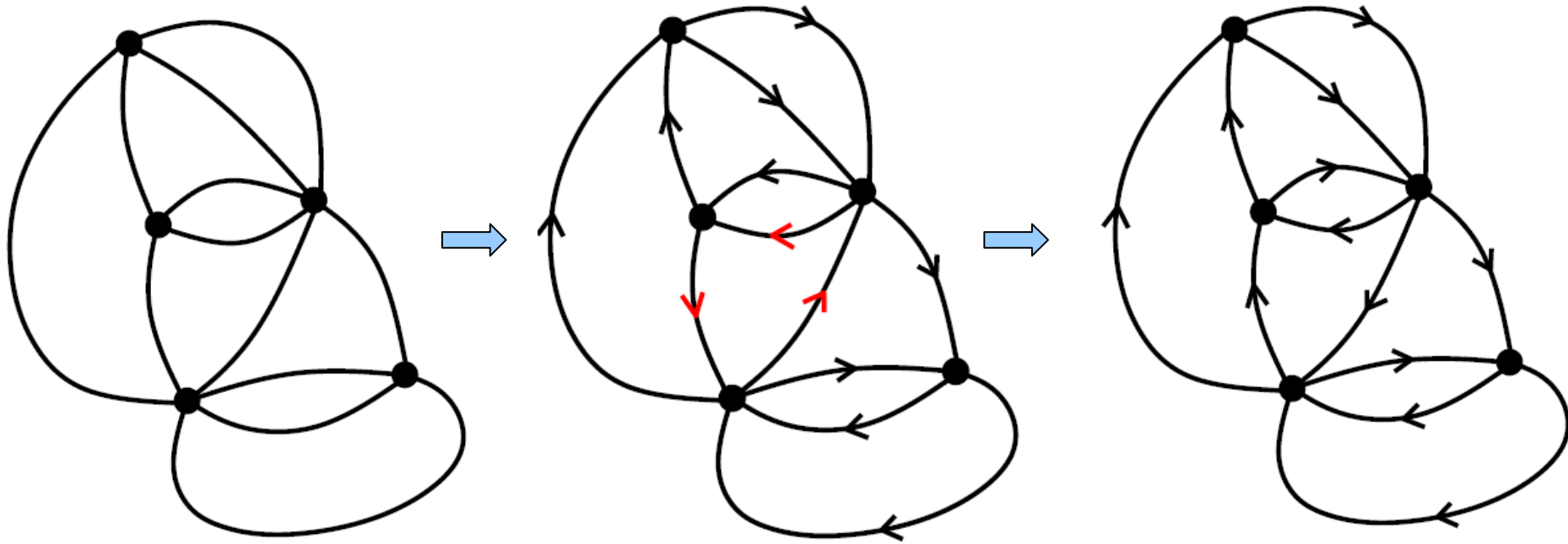


number of rooted **bipartite** maps with  $n_i$  **faces** of degree  $2i$  for  $i$  in  $[1..k]$  is

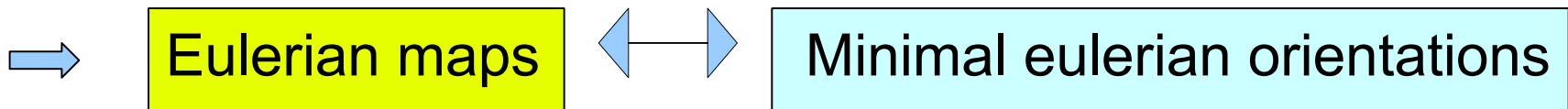
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# Eulerian orientations

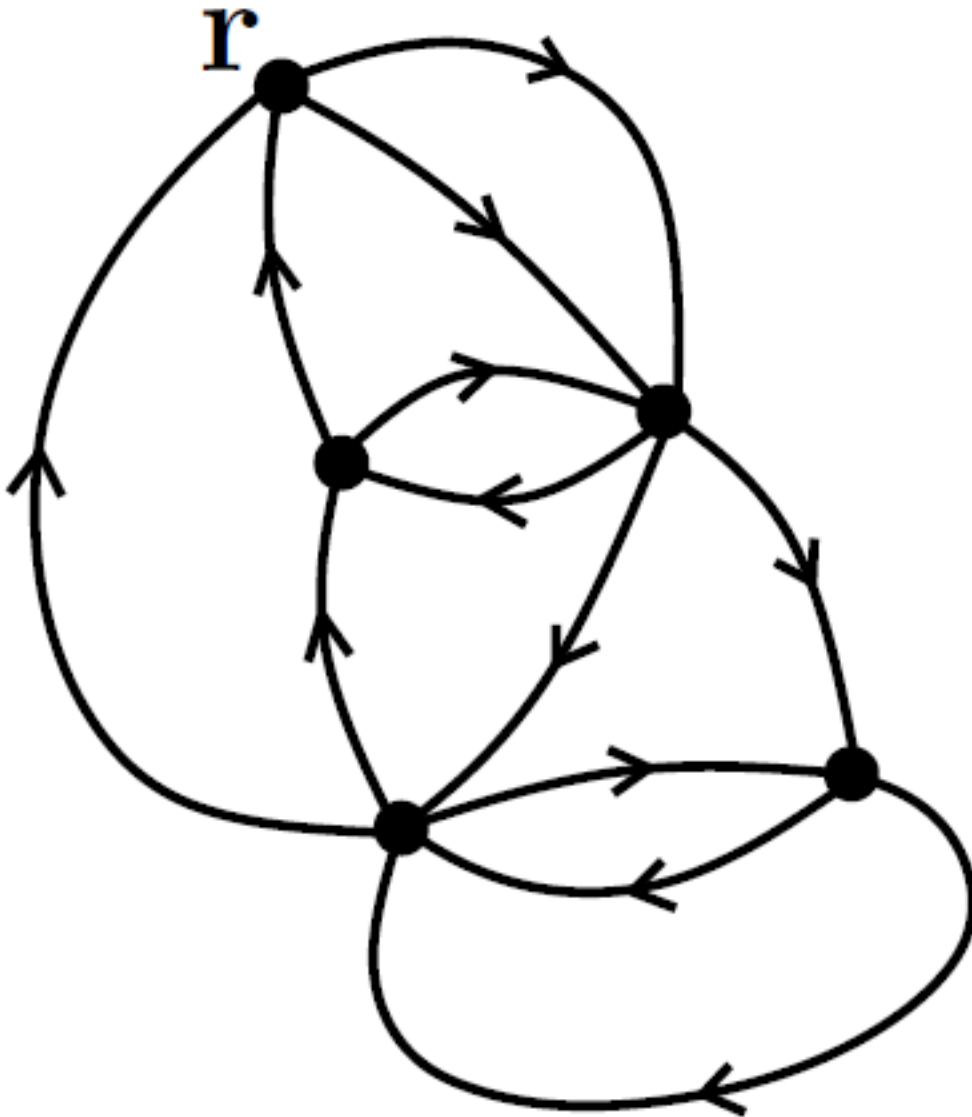
- Eulerian orientation:  $\text{indegree}(v) = \text{deg}(v)/2$  for each  $v$



- An eulerian map admits a **unique minimal eulerian orientation**

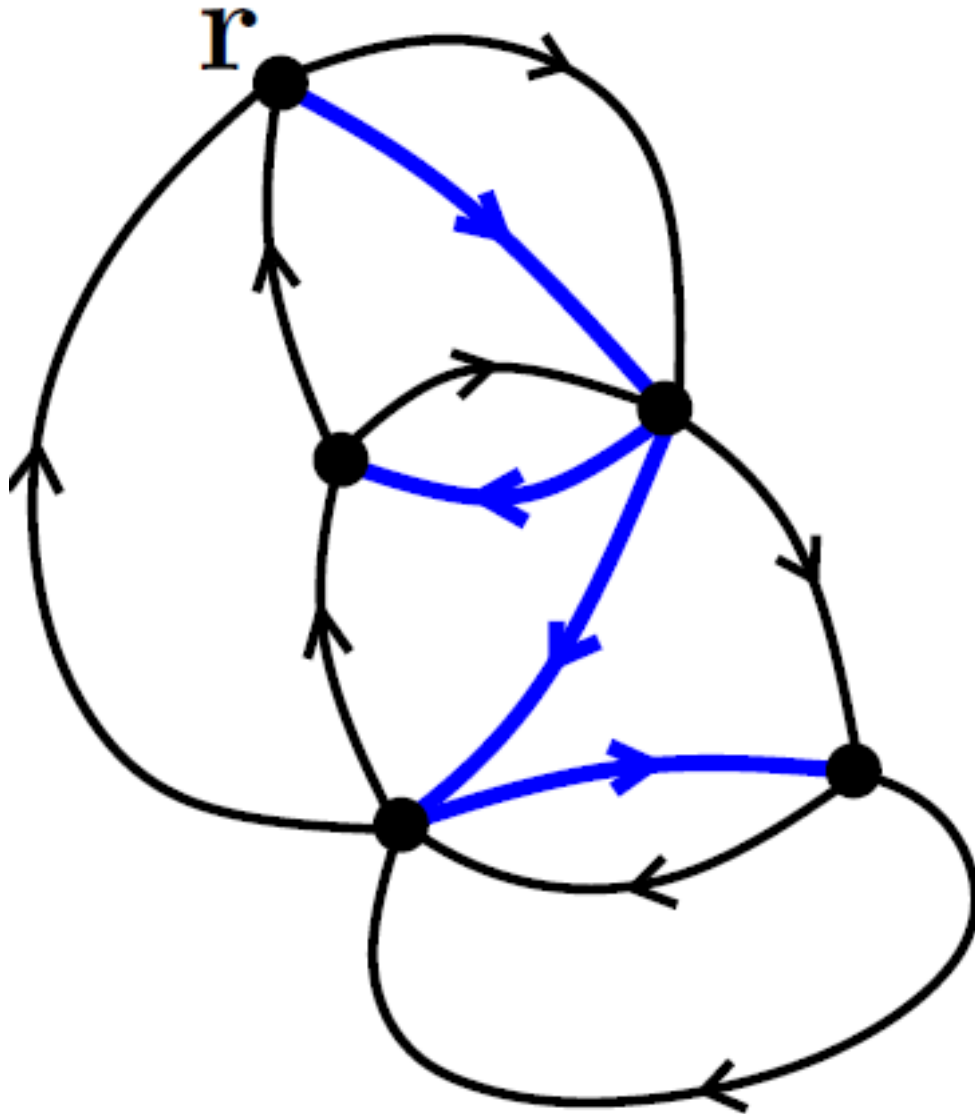


# Min. Eulerian ori. $\rightarrow$ Eulerian tree



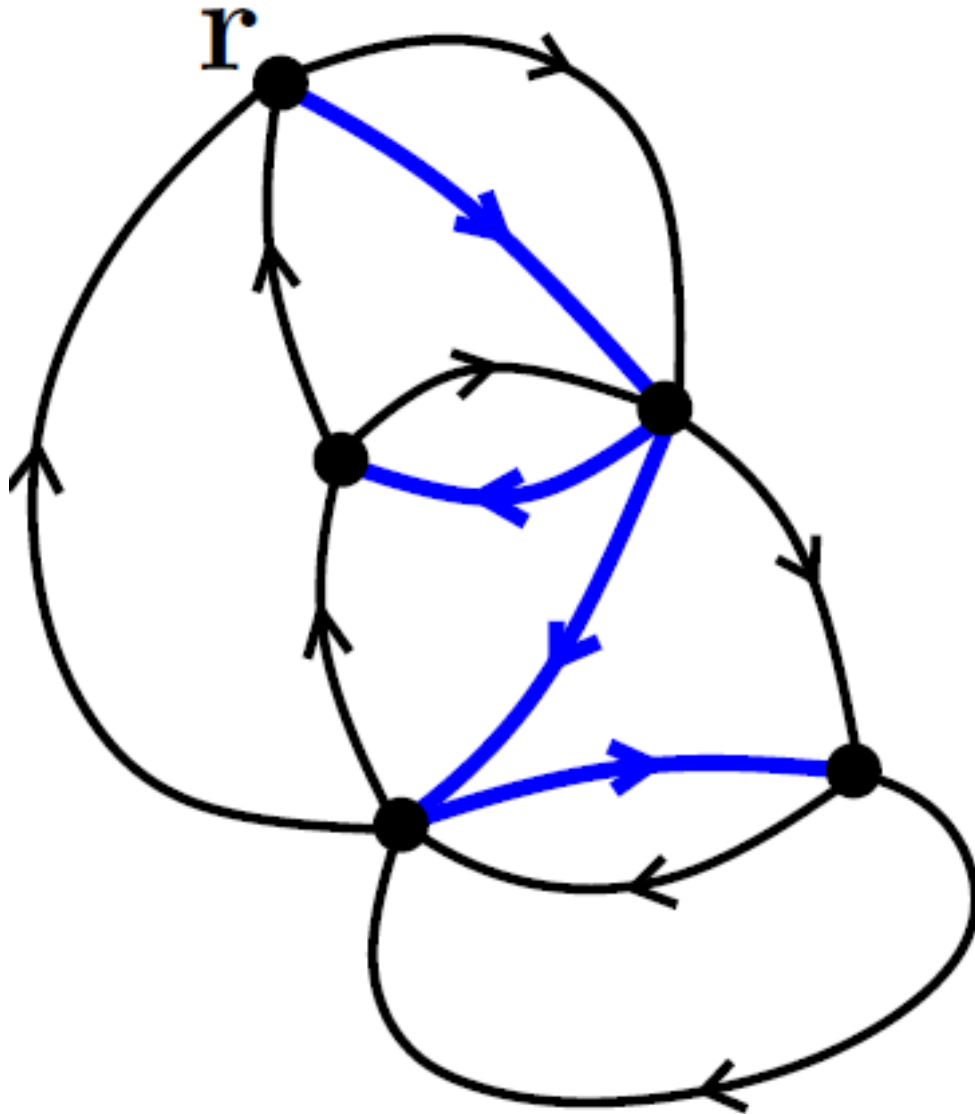
Compute the canonical spanning tree

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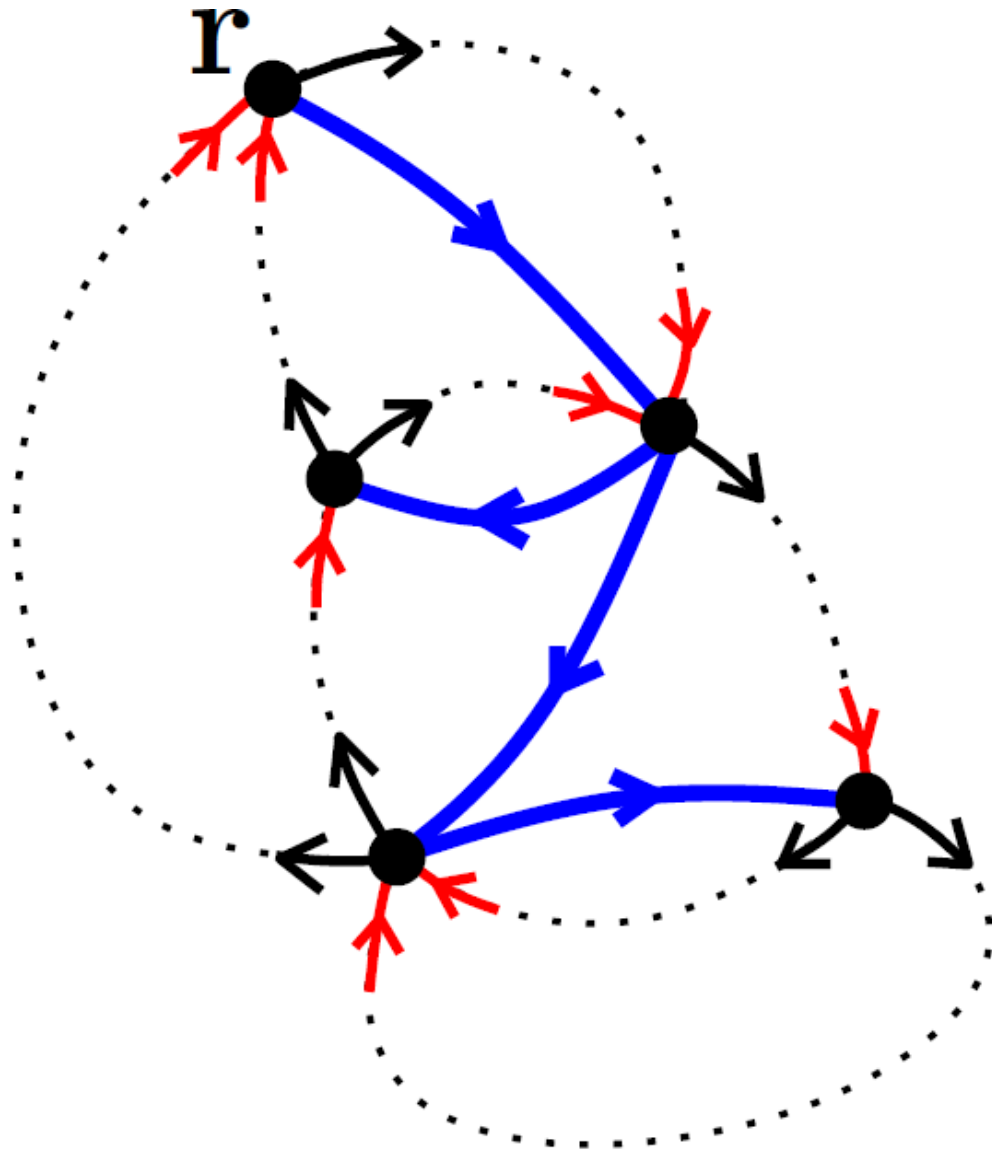
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# Min. Eulerian ori. $\rightarrow$ Eulerian tree



Cut each non-tree edge at its middle

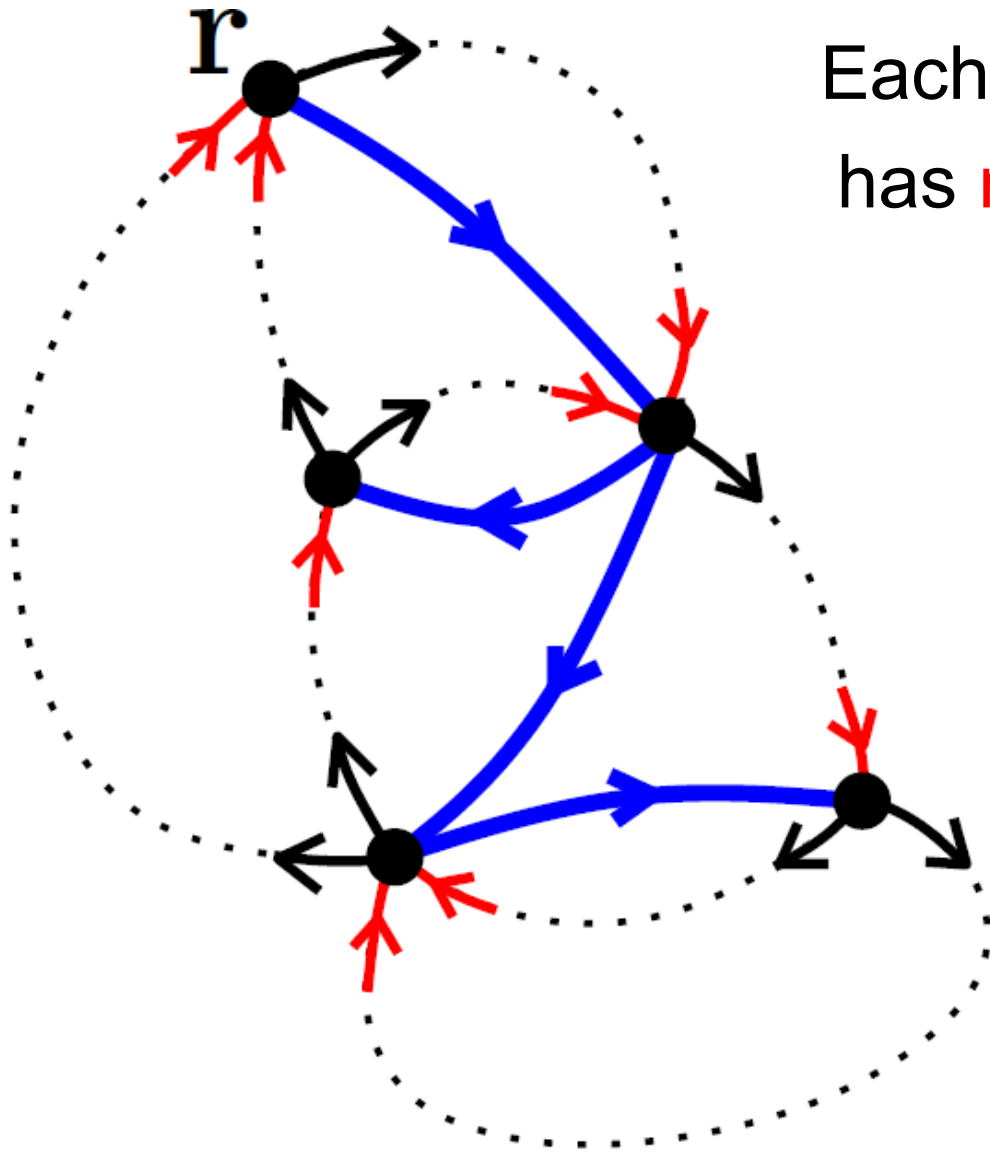
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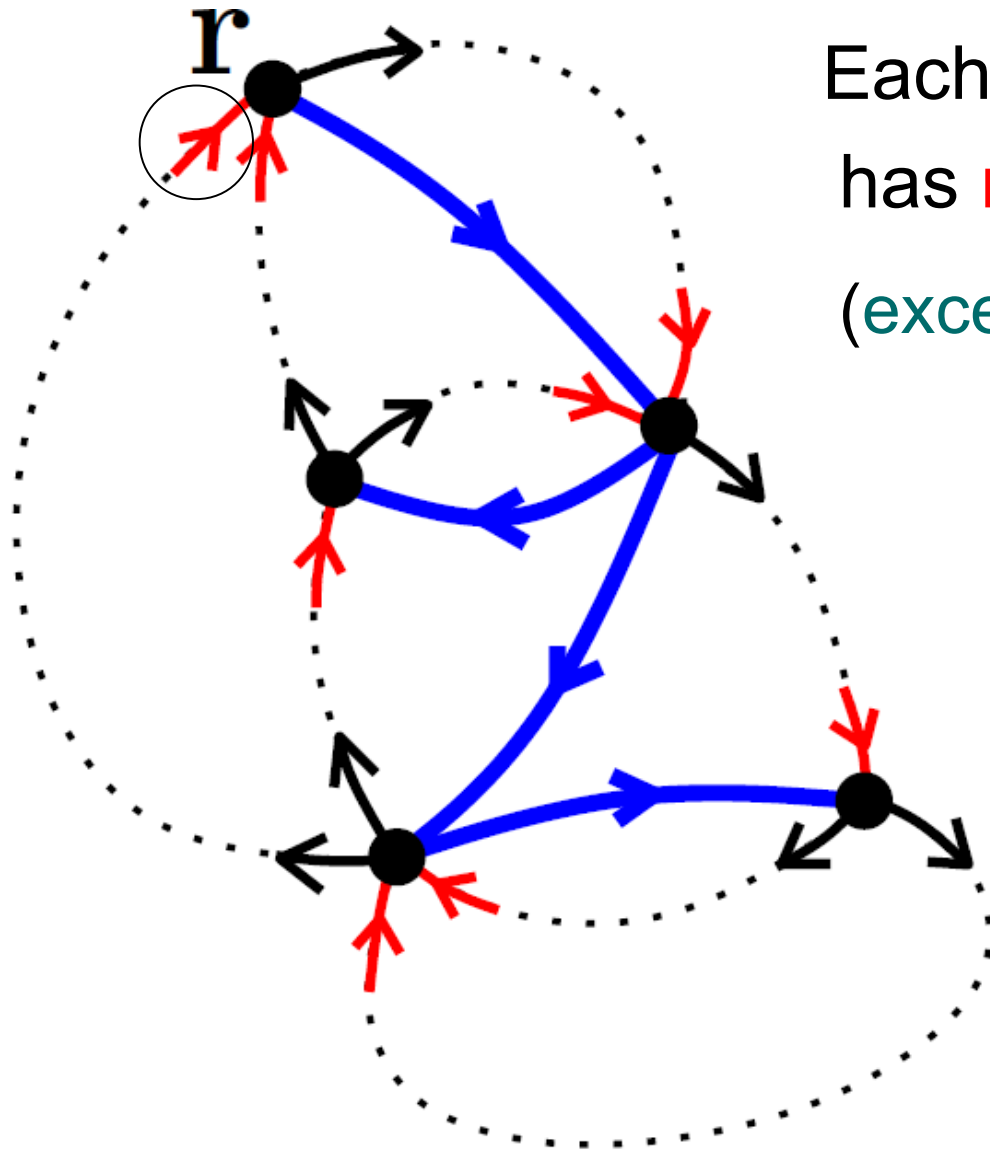
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# Min. Eulerian ori. $\rightarrow$ Eulerian tree

Each vertex  $v$  of degree  $2i$   
has red indegree =  $i-1$



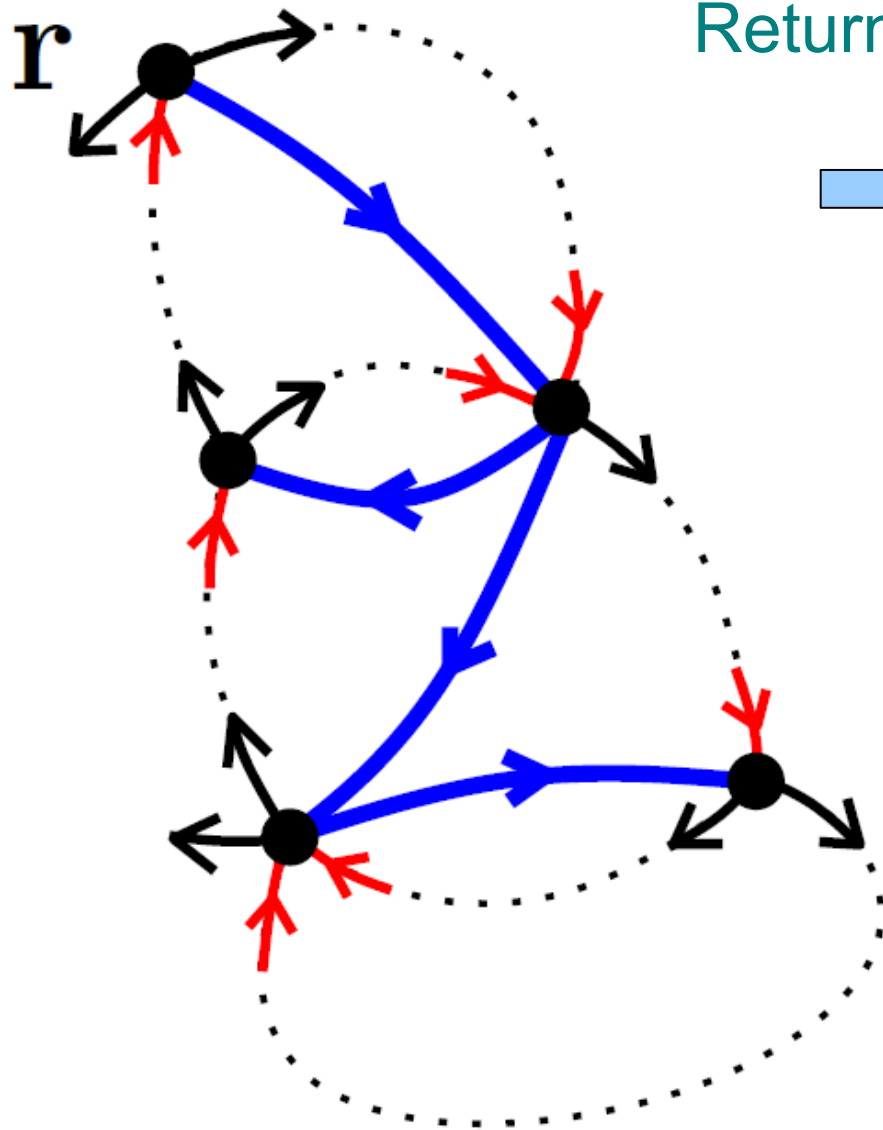
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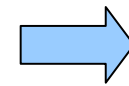
Each vertex  $v$  of degree  $2i$   
has red indegree =  $i-1$   
(except if  $v=r$ , red indegree= $i$ )



# Min. Eulerian ori. $\rightarrow$ Eulerian tree

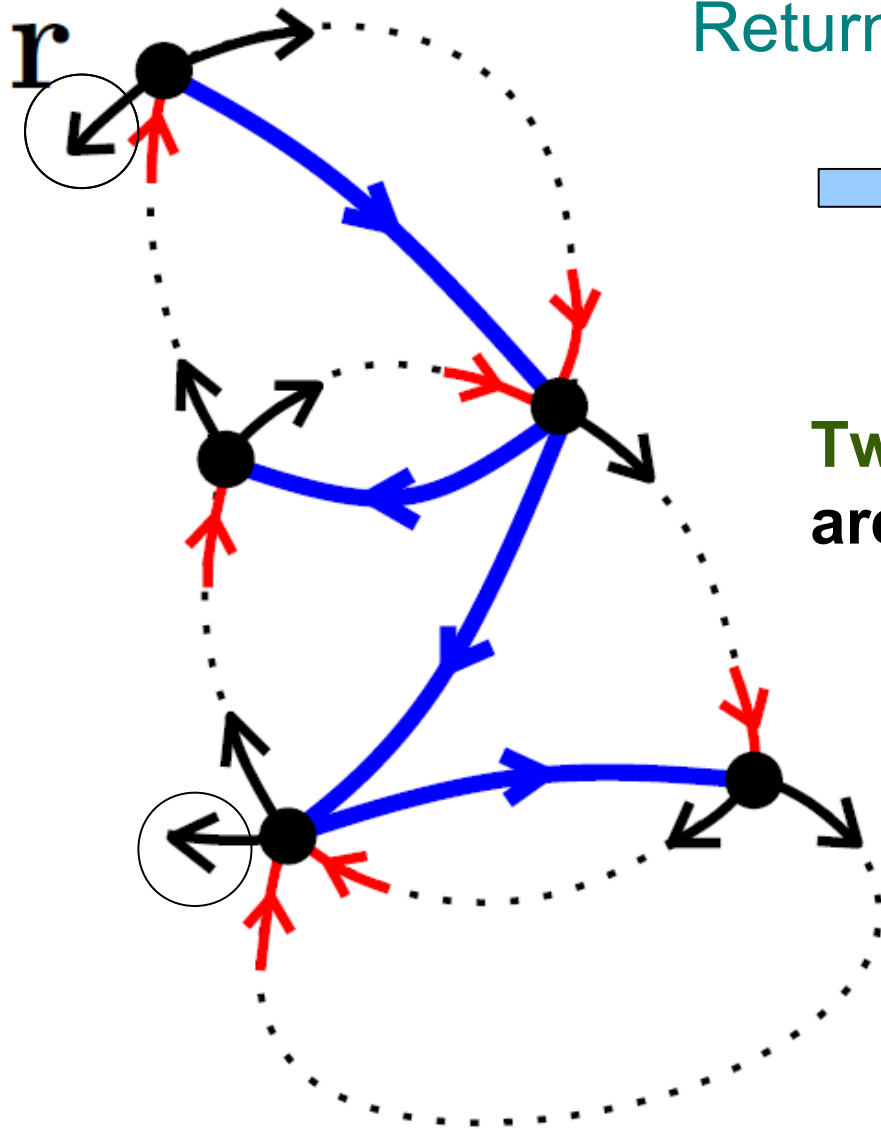


Return red ingoing leg at the root



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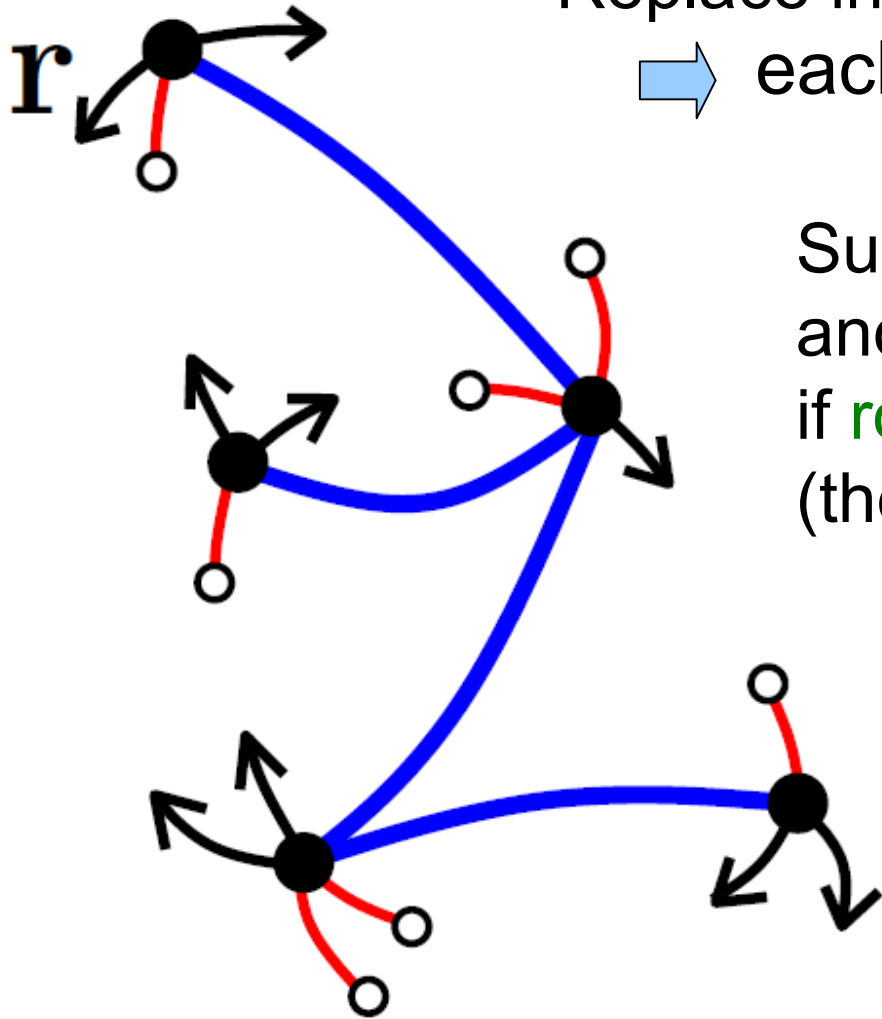
$\rightarrow$  each  $v$  of degree  $2i$   
has red indegree =  $i-1$

Two black arrows (including  $r$ )  
are unmatched

# Min. Eulerian ori. $\rightarrow$ Eulerian tree

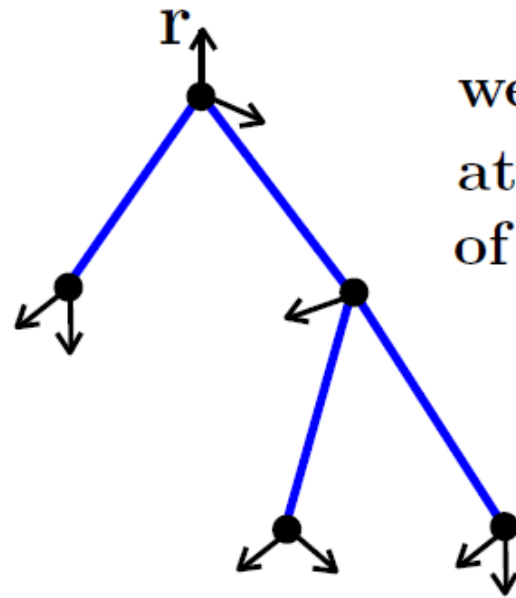
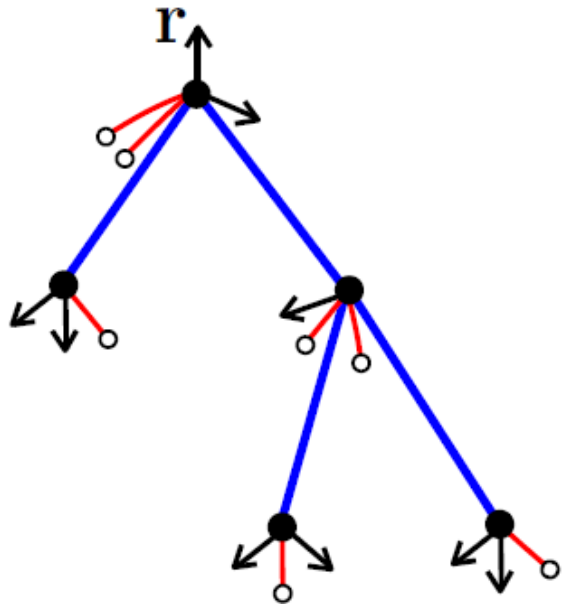
Replace ingoing red half-edges by "legs"  
 $\Rightarrow$  each  $v$  of degree  $2i$  has  $i-1$  legs

Such a tree is called "eulerian",  
and is called "balanced eulerian"  
if rooted at one unmatched arrow  
(there are two such arrows)



**Rk:** we get the same tree as in [Schaeffer'97]

# Counting balanced Eulerian trees



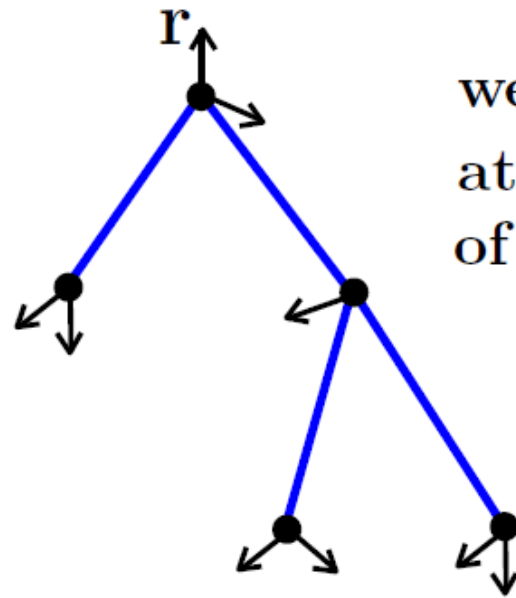
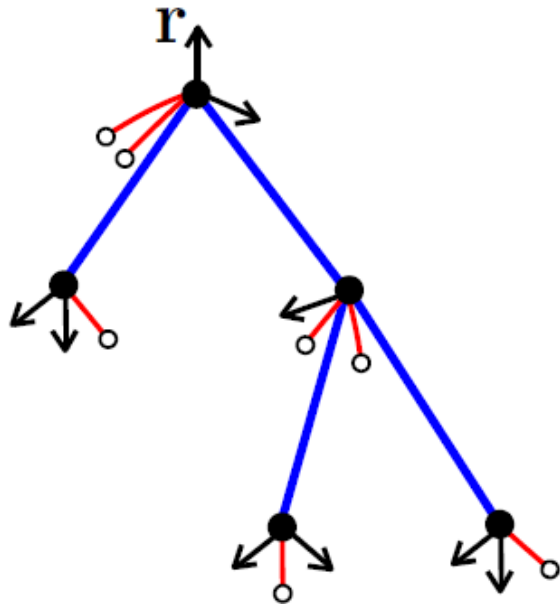
weight  $\binom{2i-1}{i-1}$   
at each node  
of degree  $i+1$

→ The number of **eulerian trees** with  $n_i$  nodes of degree  $2i$  is

$$\frac{e!}{(v-1)!} \prod_{i=1}^k \binom{2i-1}{i-1}^{n_i} \text{ where } \begin{aligned} e &= \sum_i i n_i \\ v &= -2 + \sum_i (i-1) n_i \end{aligned}$$

$2/v$  of these trees are balanced ( $v$  is the number of leaves)

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same as number of  
rooted bipartite maps  
with  $n_i$  faces of degree  $2i$

# **Part II: bijective counting of simple bipartite maps**

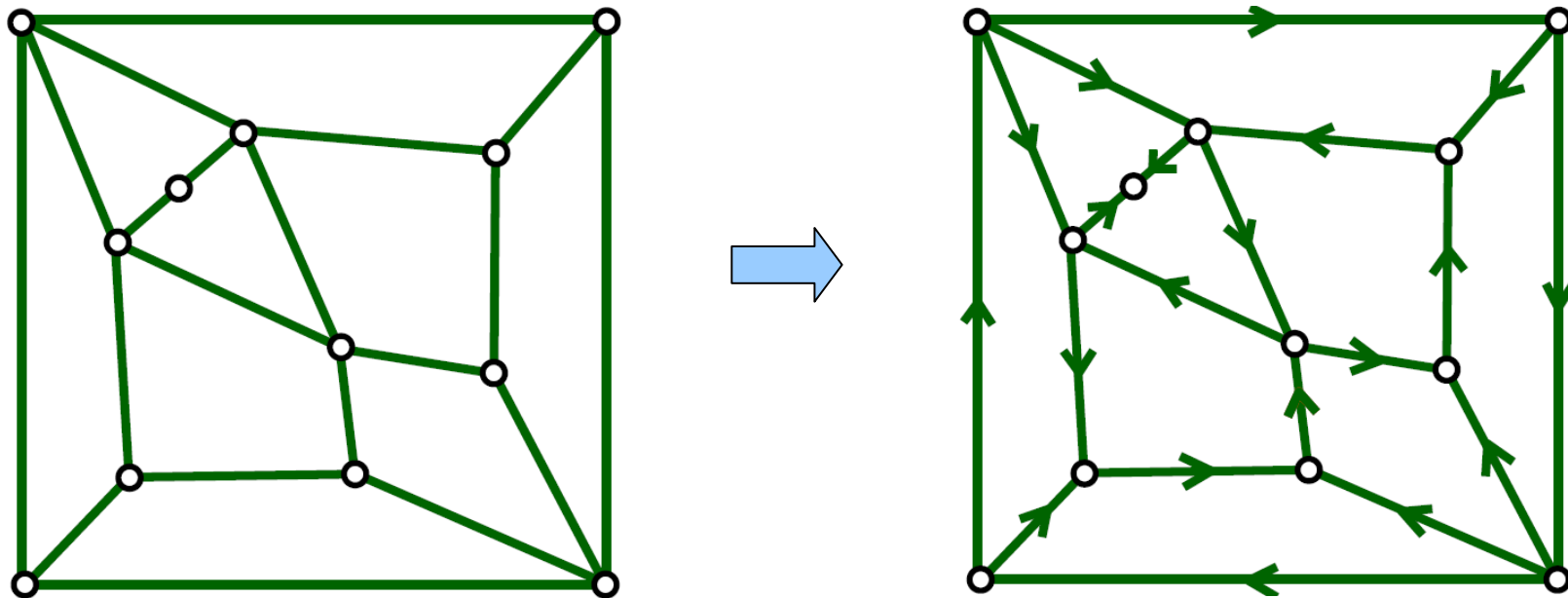
# Orientations for simple quadrangulations

[de Fraysseix, Ossona de Mendez'01]

A quadrangulation is simple iff it admits an orientation where:

inner vertices have indegree 2, outer vertices have indegree 1

Such an orientation is called a 2-orientation



# Orientations for simple quadrangulations

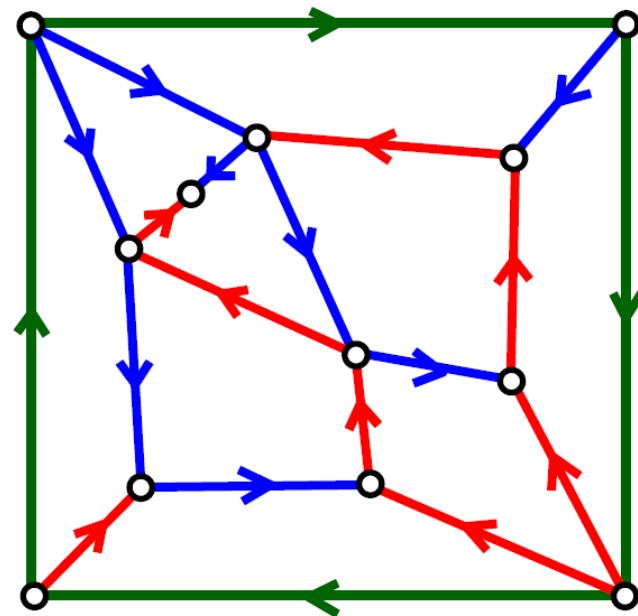
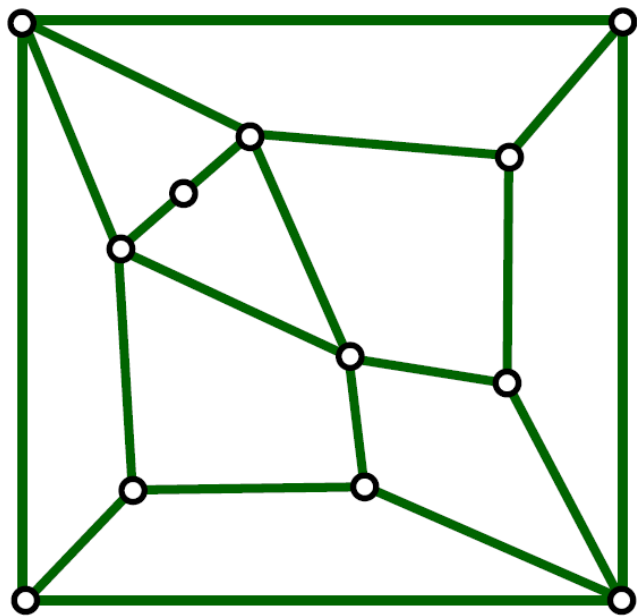
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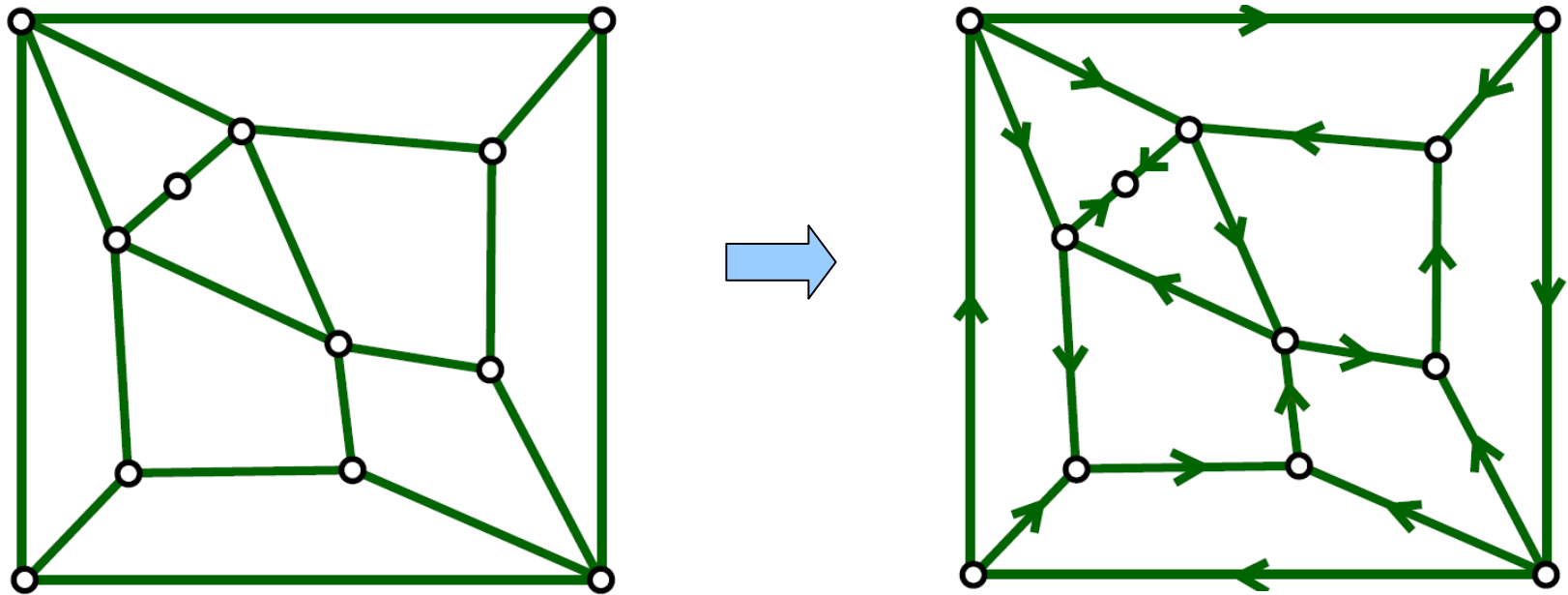


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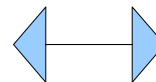


[Felsner'03]:

A simple quadrangulation has a unique minimal 2-orientation



Simple quadrangulations



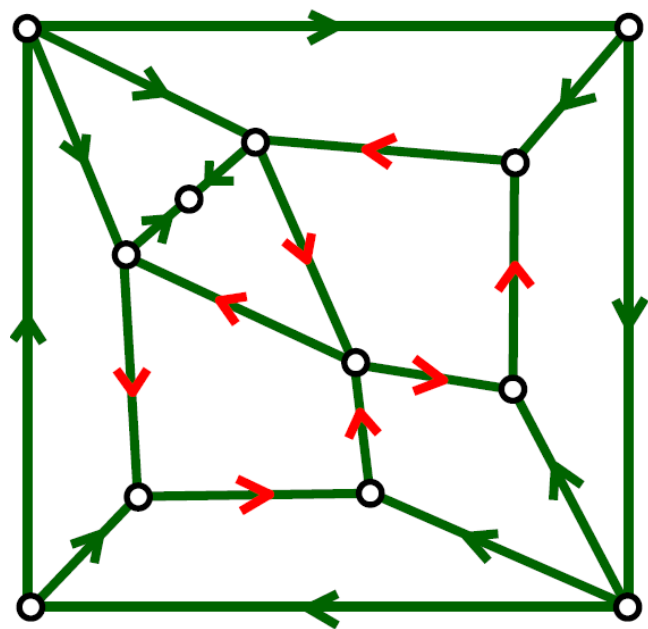
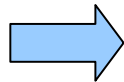
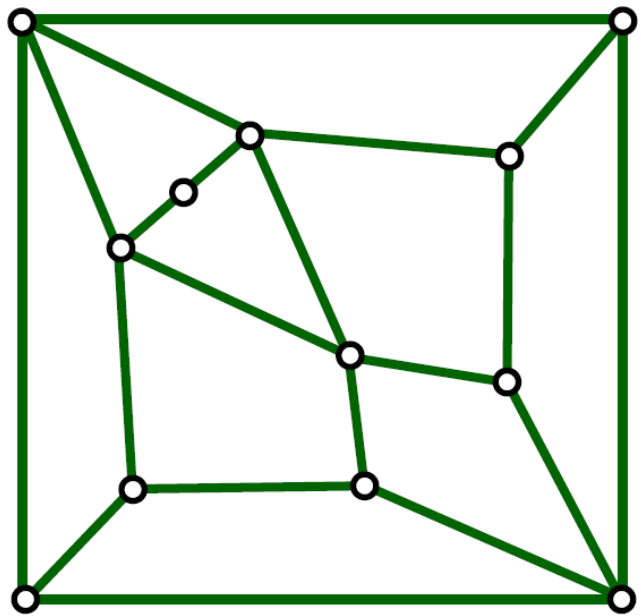
Minimal 2-orientations

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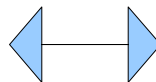


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Simple quadrangulations



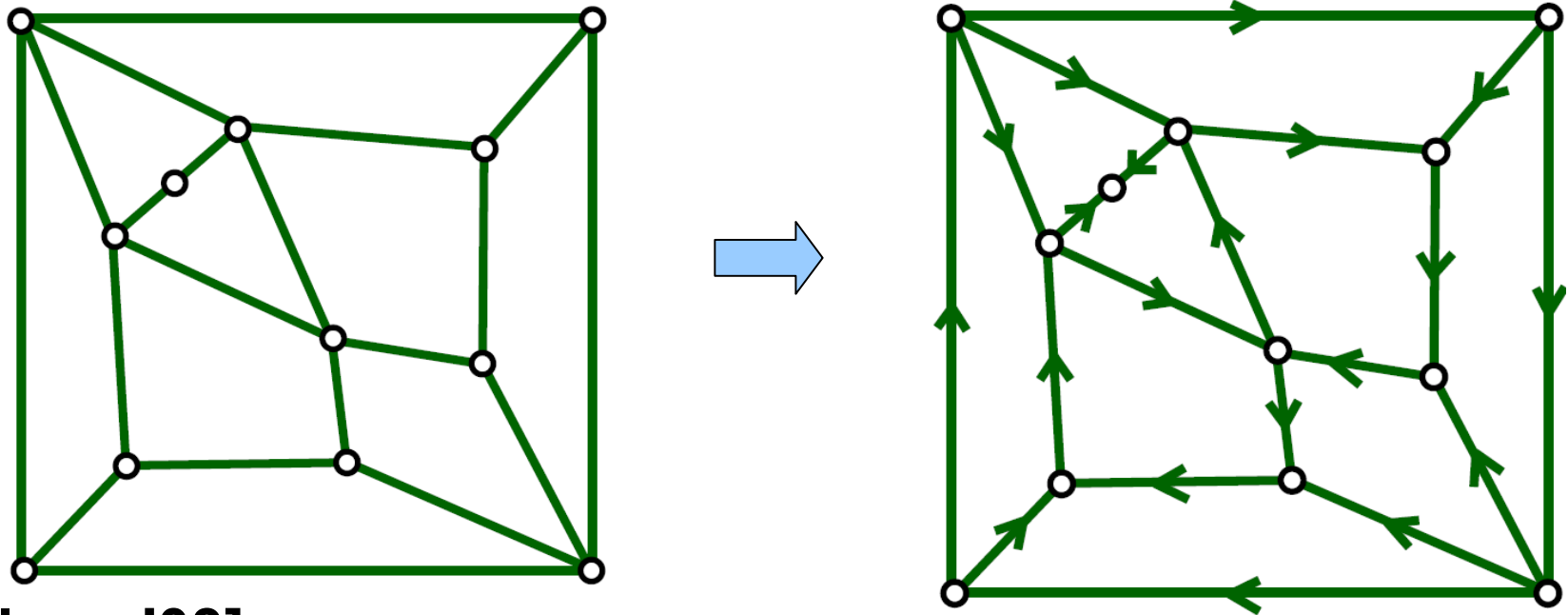
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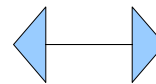


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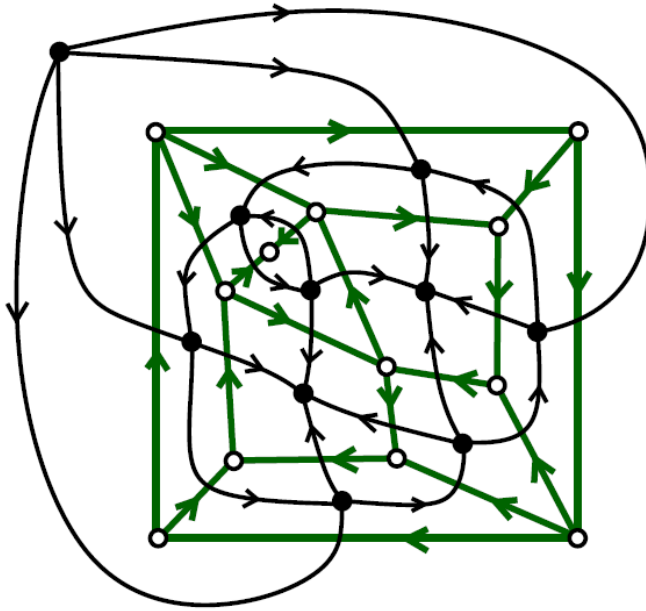
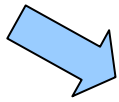
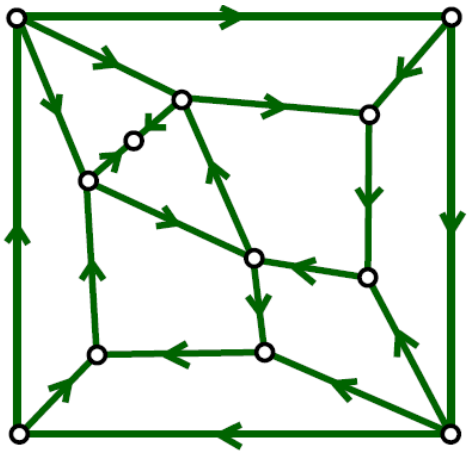


Simple quadrangulations

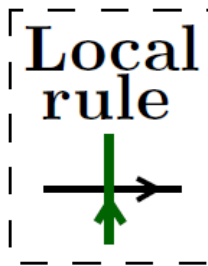
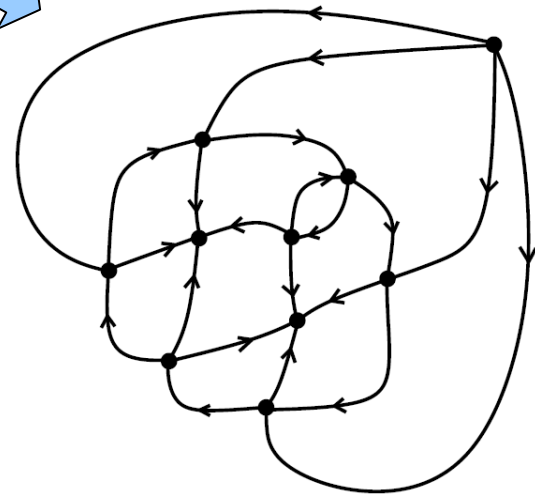
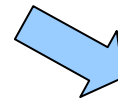


Minimal 2-orientations

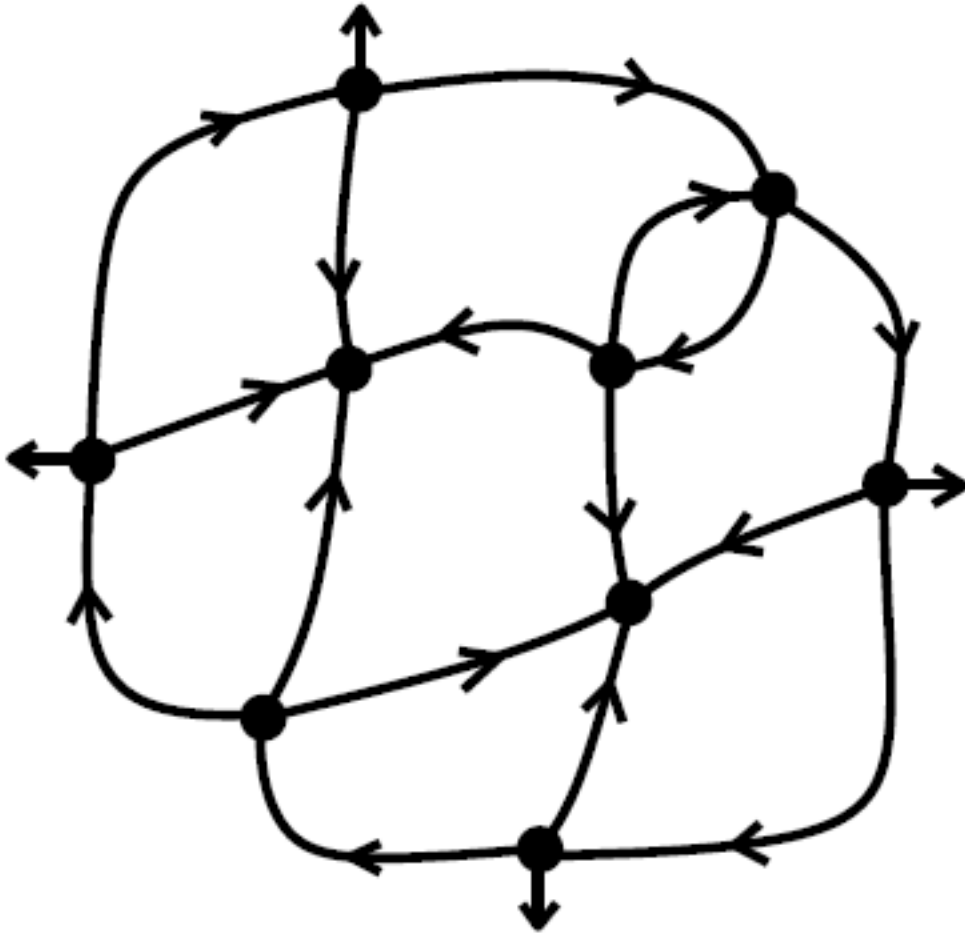
# Duality on 2-orientations



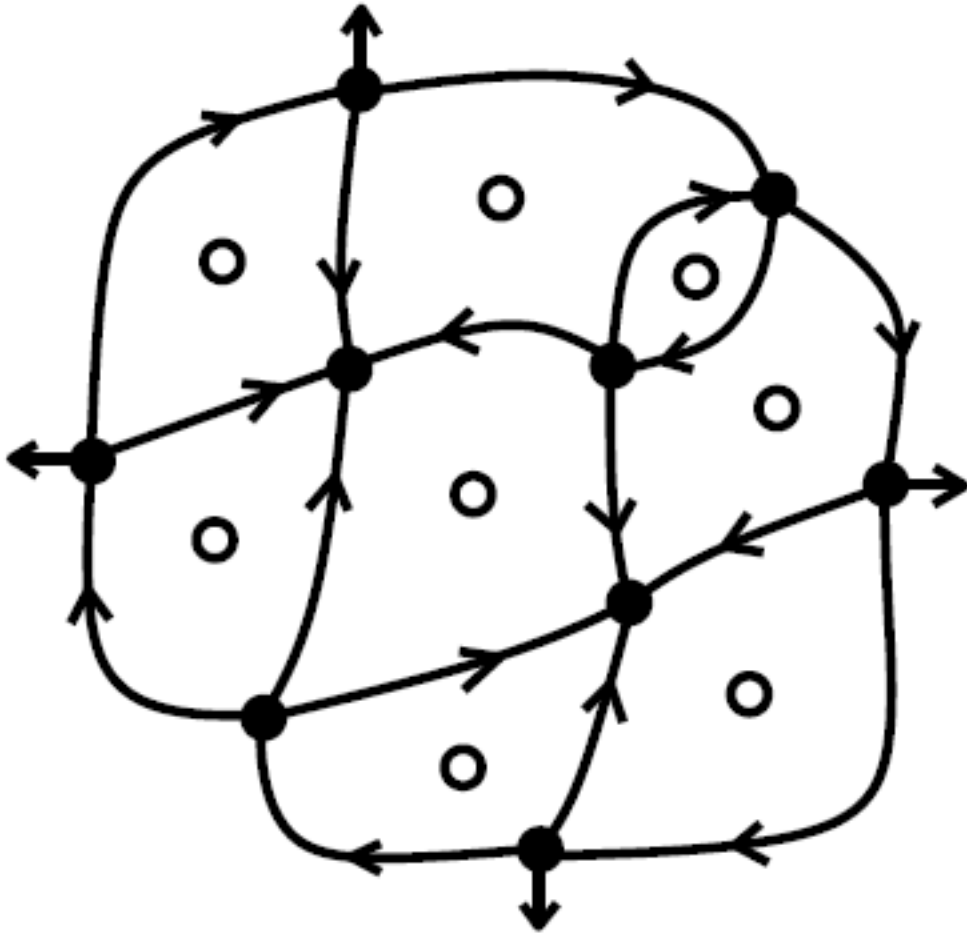
**mirror**



# Minimal dual 2-orientation $\rightarrow$ mobile

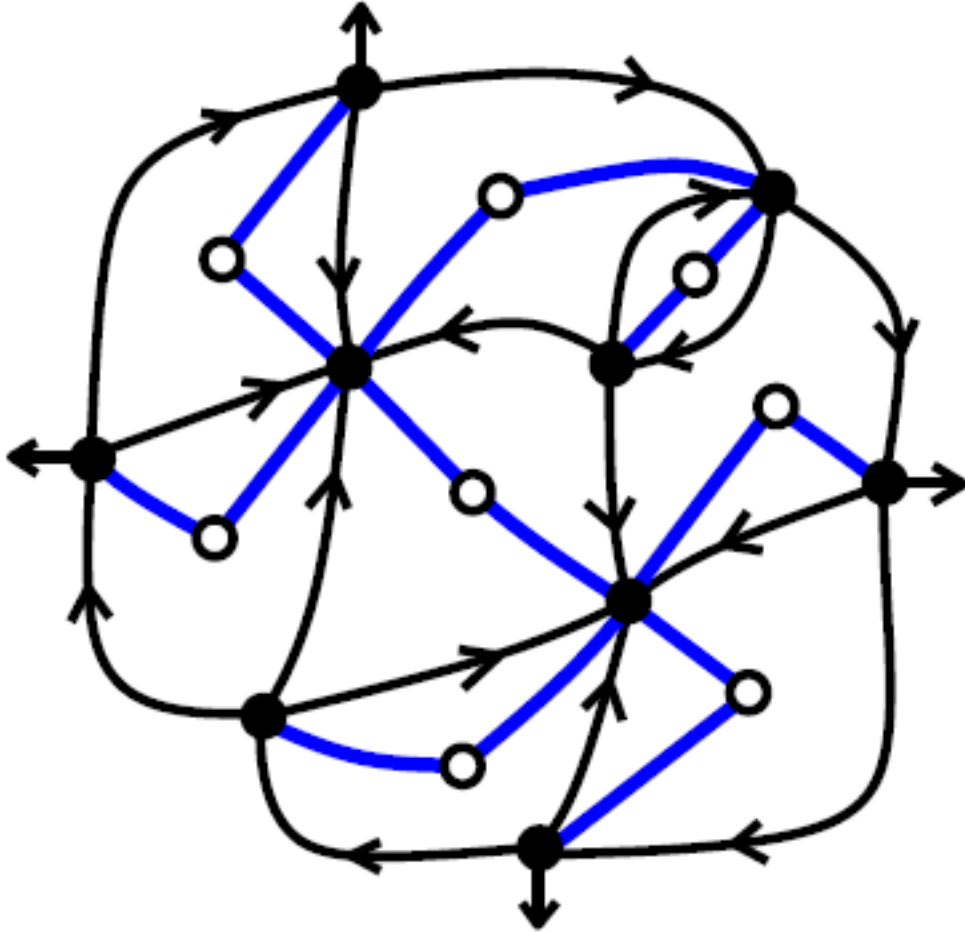


# Minimal dual 2-orientation $\rightarrow$ mobile

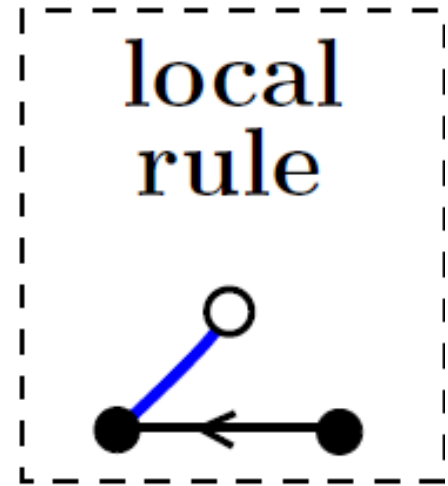


**Compute the mobile**

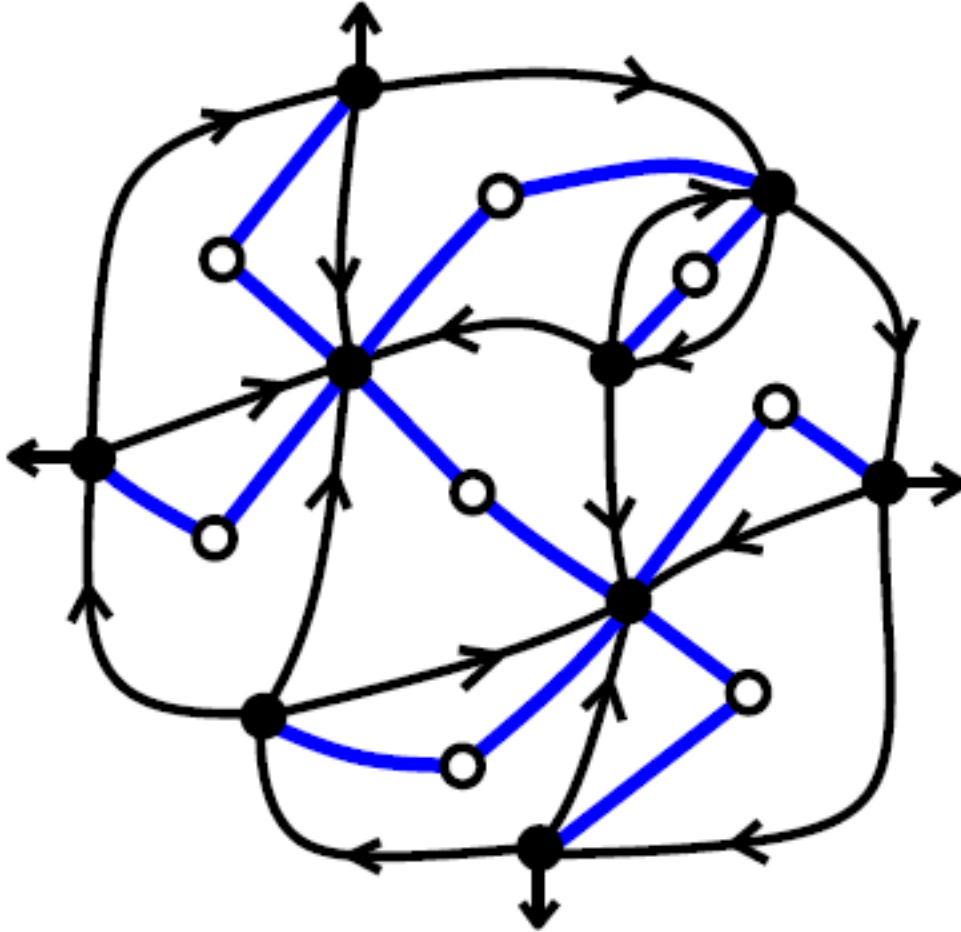
# Minimal dual 2-orientation $\rightarrow$ mobile



Compute the mobile



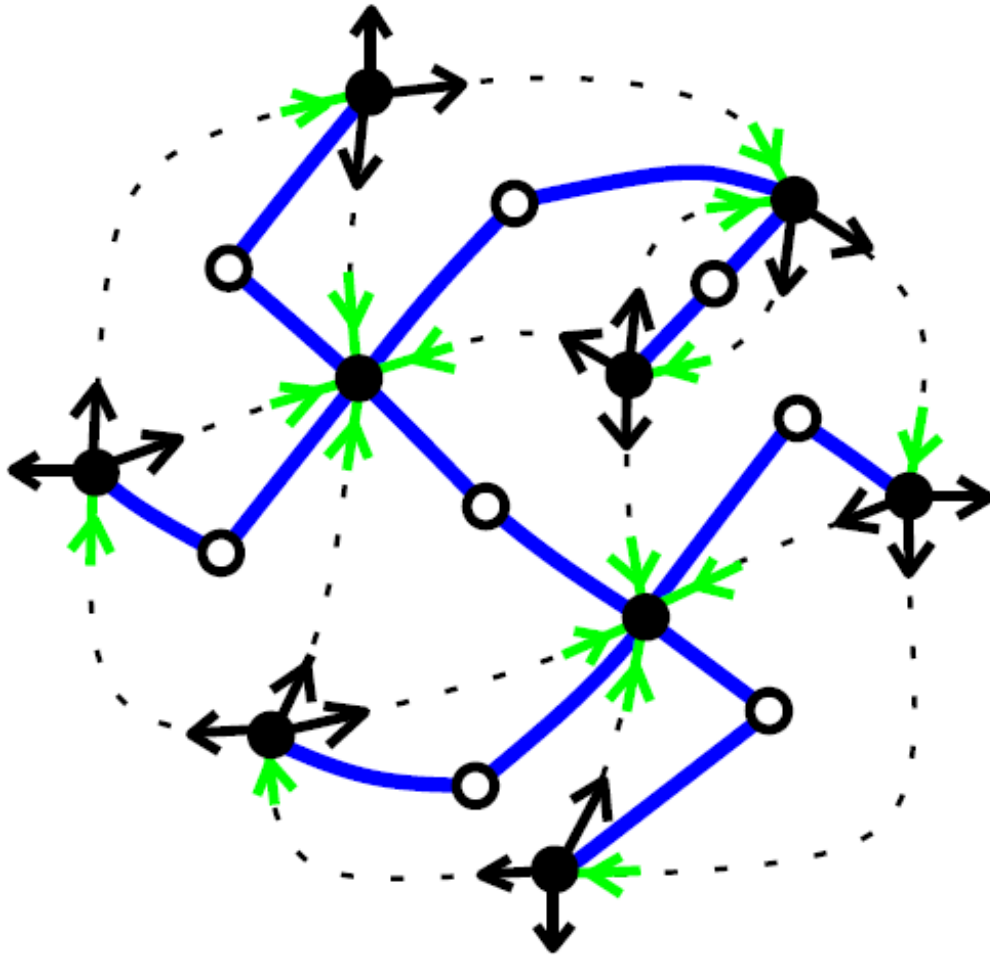
# Minimal dual 2-orientation $\rightarrow$ mobile



**Cut the non-mobile  
edges at their middle**

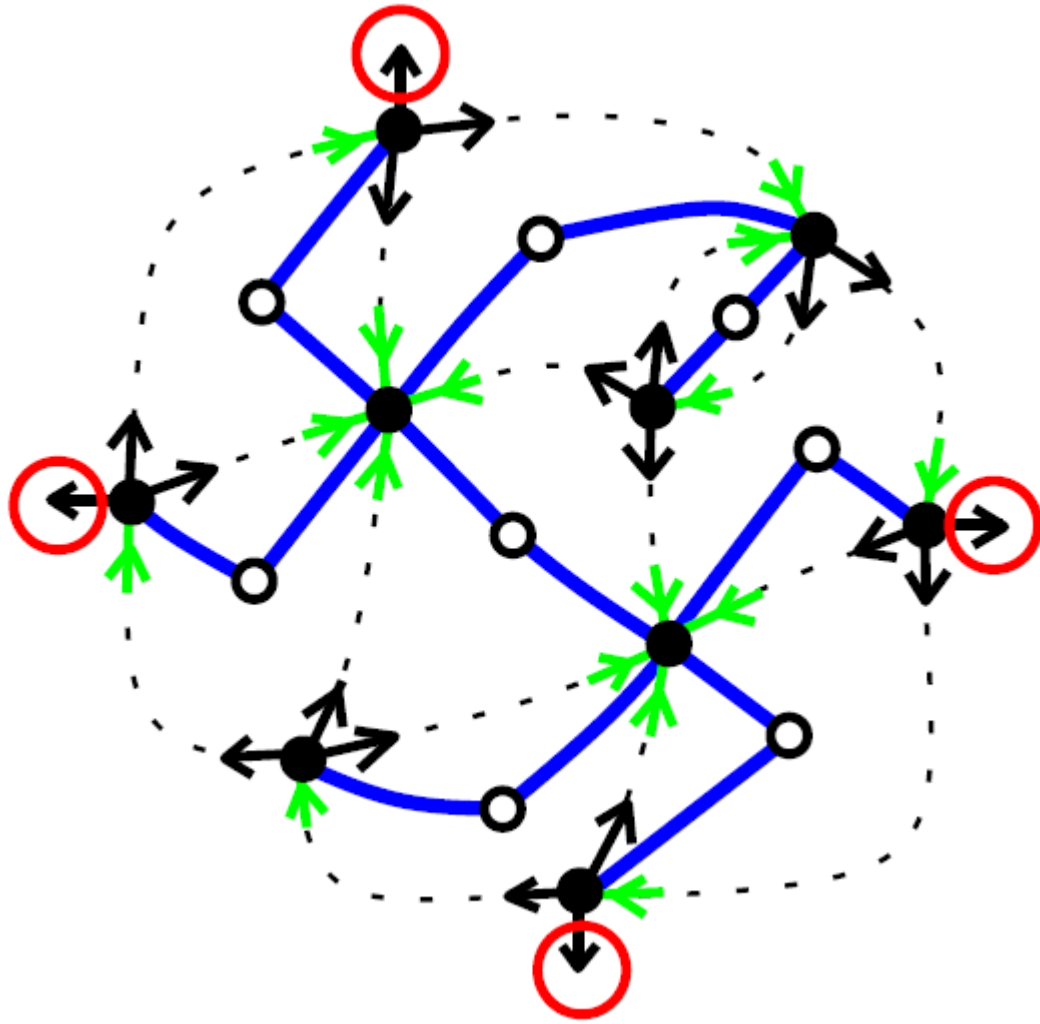


# Minimal dual 2-orientation $\rightarrow$ mobile



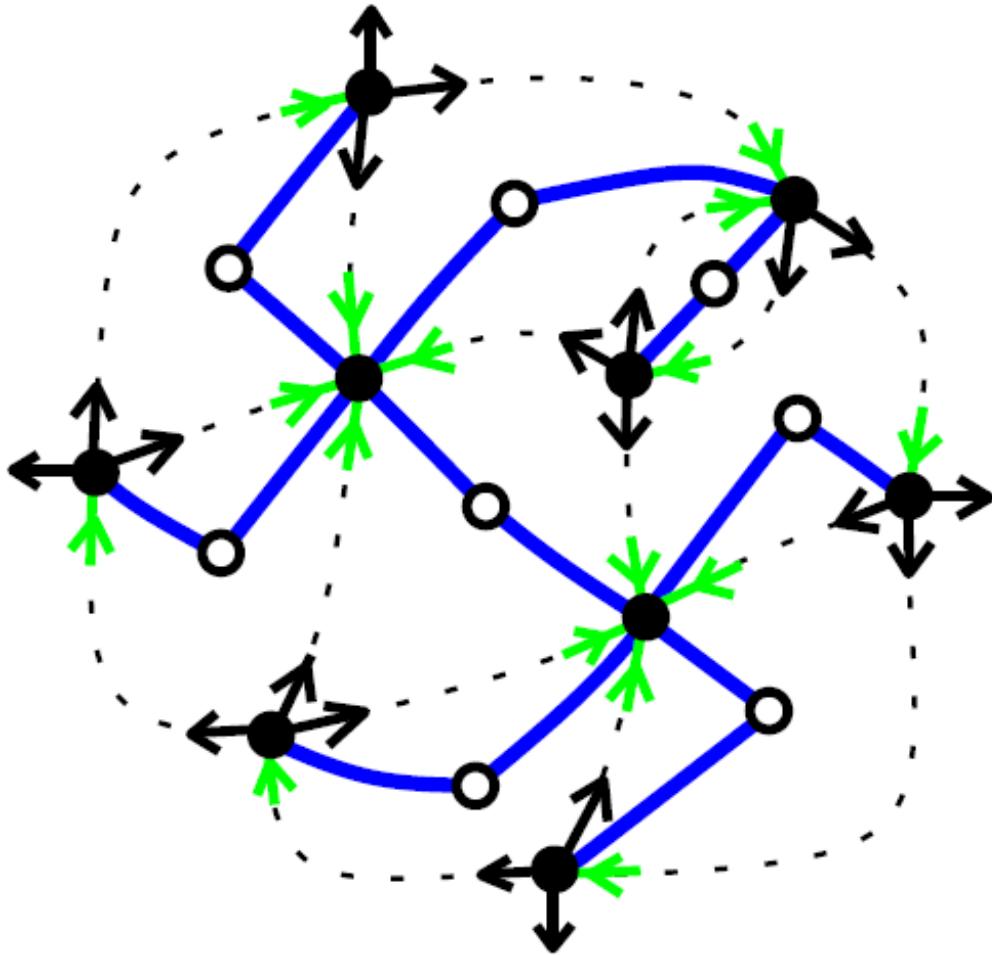
**Cut the non-mobile  
edges at their middle**

# Minimal dual 2-orientation $\rightarrow$ mobile

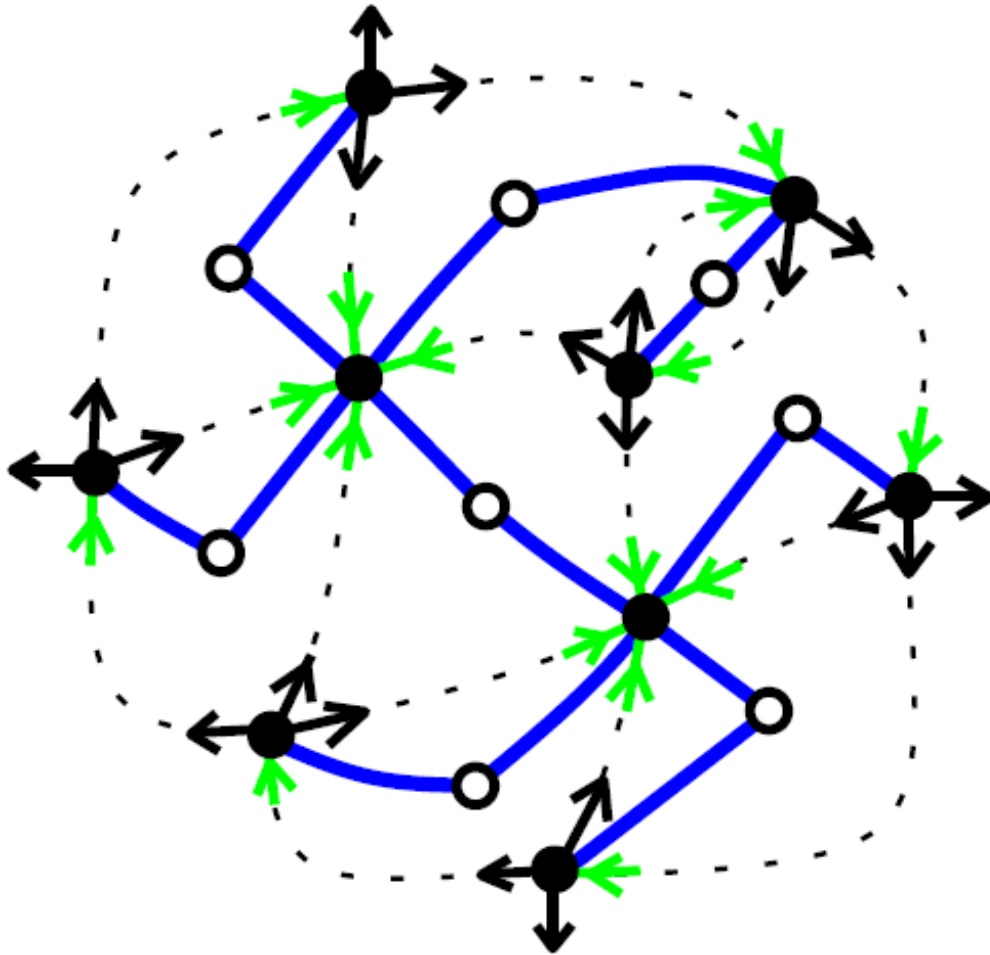


**Rk: there are 4  
unmatched arrows**

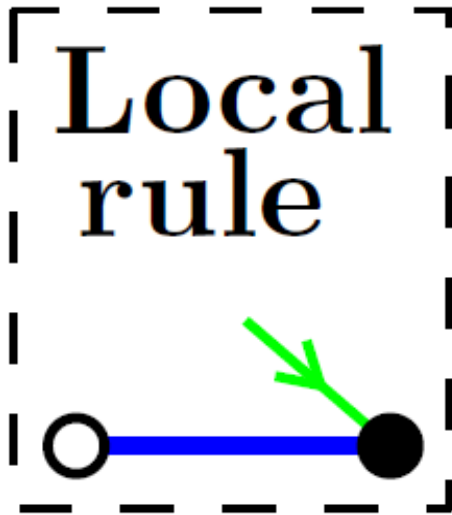
# Minimal dual 2-orientation $\rightarrow$ mobile



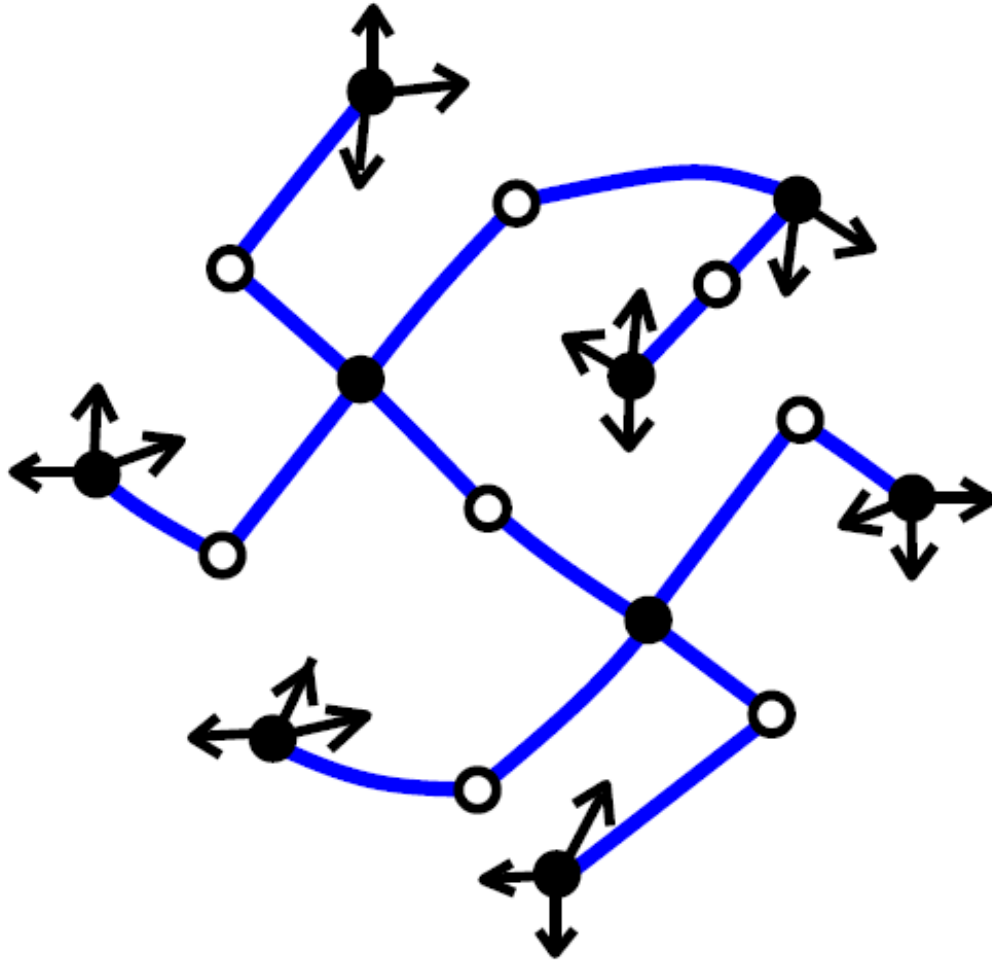
# Minimal dual 2-orientation $\rightarrow$ mobile



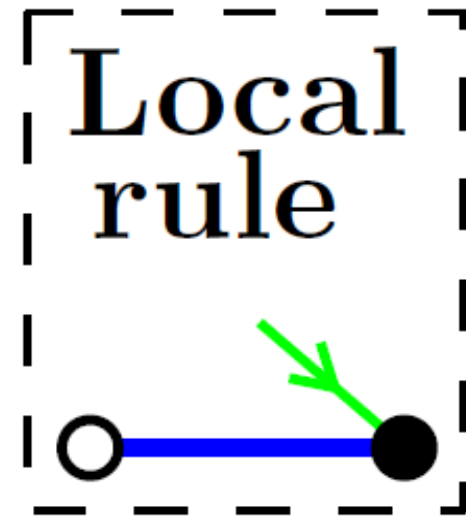
Ingoing half-edges  
can be deleted



# Minimal dual 2-orientation $\rightarrow$ mobile

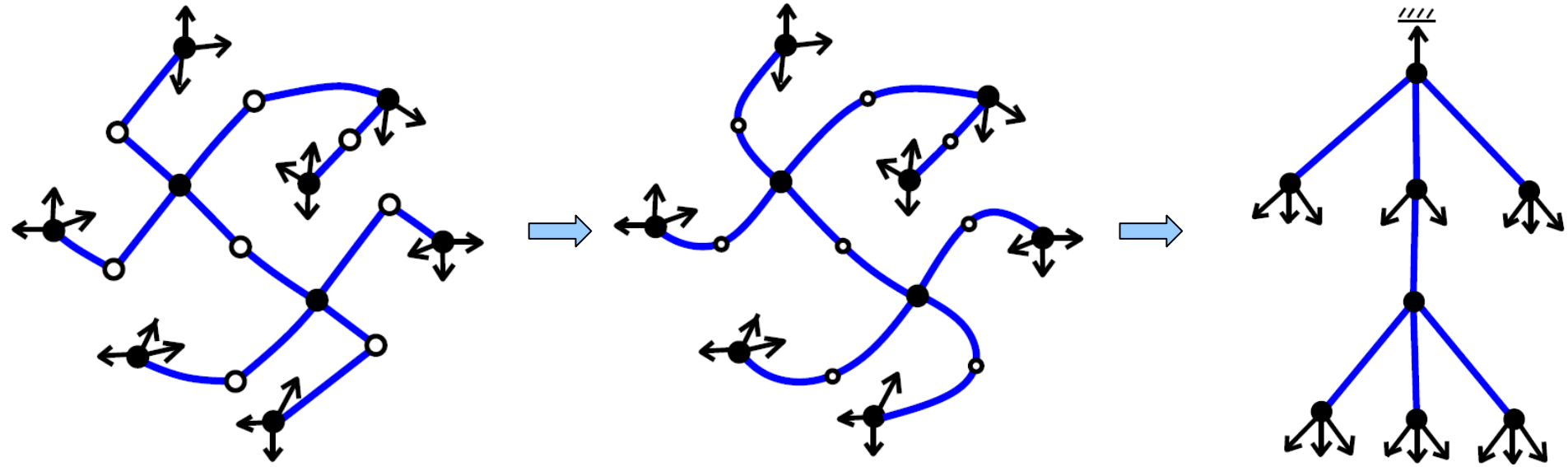


Ingoing half-edges  
can be deleted



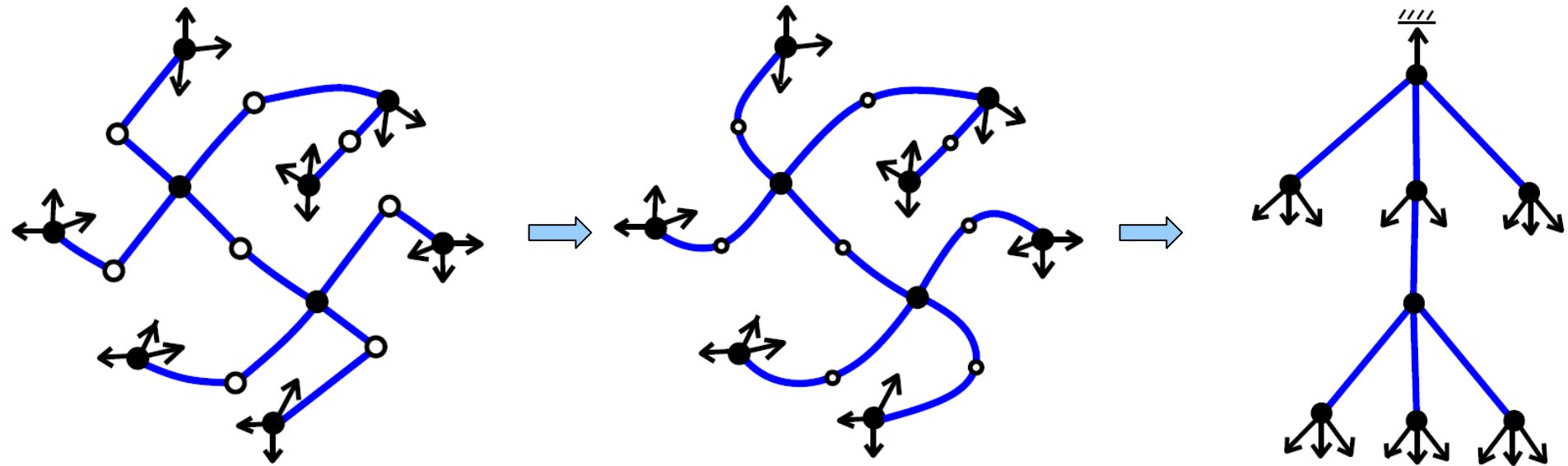
# Minimal dual 2-orientation $\rightarrow$ mobile

Construction gives a ternary tree (recover [Schaeffer'99])



# Minimal dual 2-orientation $\rightarrow$ mobile

Construction gives a ternary tree (recover [Schaeffer'99])



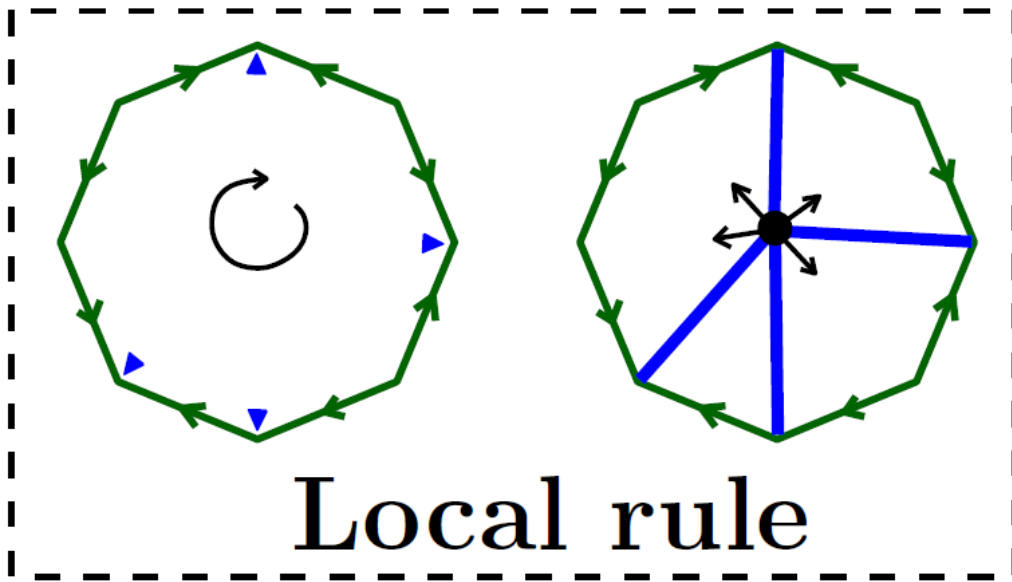
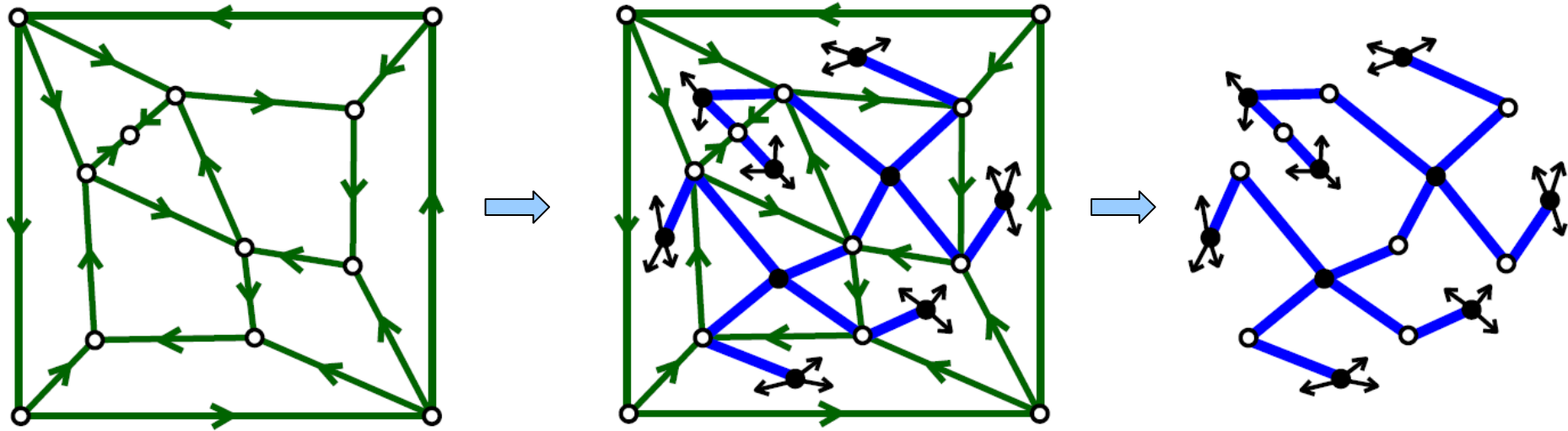
- There are  $\frac{(3n)!}{n!(2n+1)!}$  rooted ternary trees with  $n$  inner nodes

- A proportion  $4/(2n+2)$  of these rooted trees are balanced

$\Rightarrow$  there are  $\frac{4(3n)!}{n!(2n+2)!}$  rooted simple quad. with  $n+1$  faces

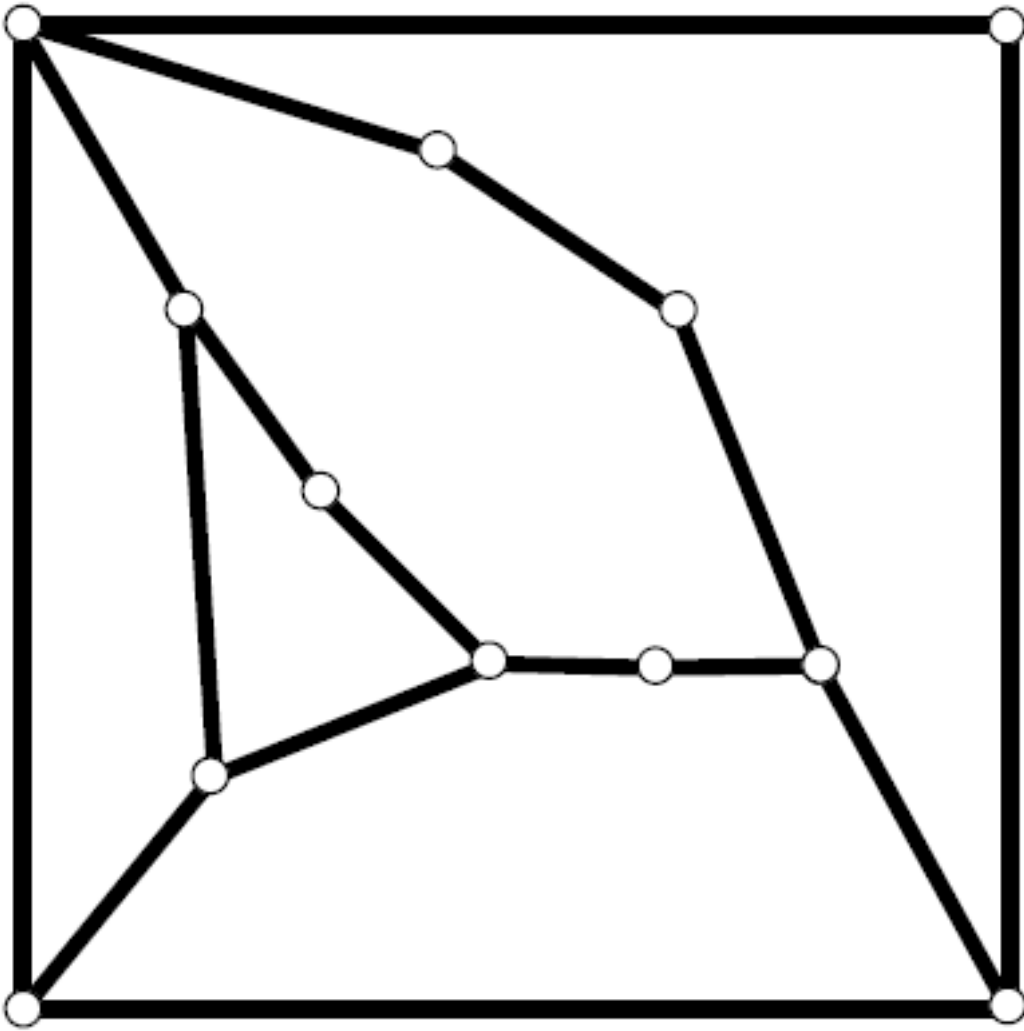
This is  $2 \frac{(e+n-3)!}{(e-1)!} \prod_{i=2}^k \frac{1}{n_i!} \binom{2i-1}{i-2}^{n_i}$  when  $n_2=n+1$   
 $n_i=0$  for  $i>2$

# Direct construction (without duality)

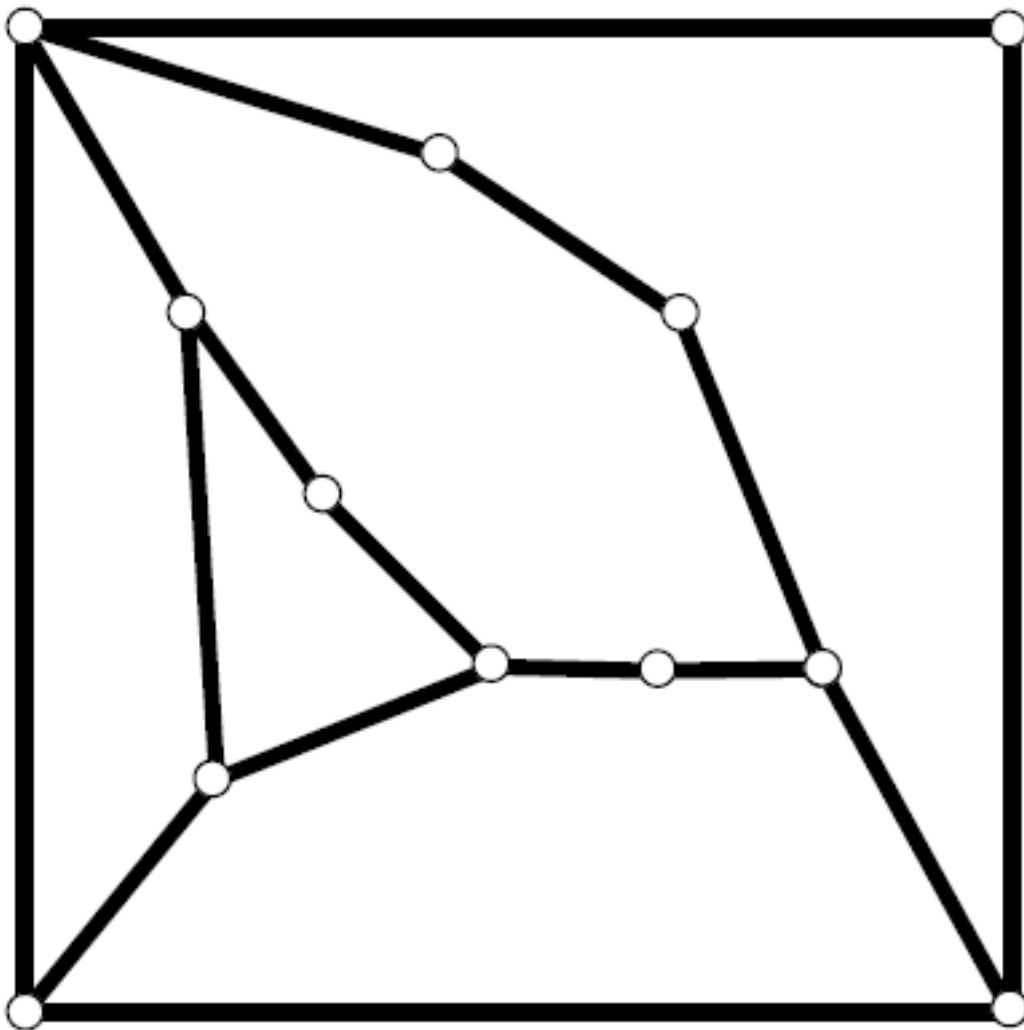




# Generalized 2-orientations

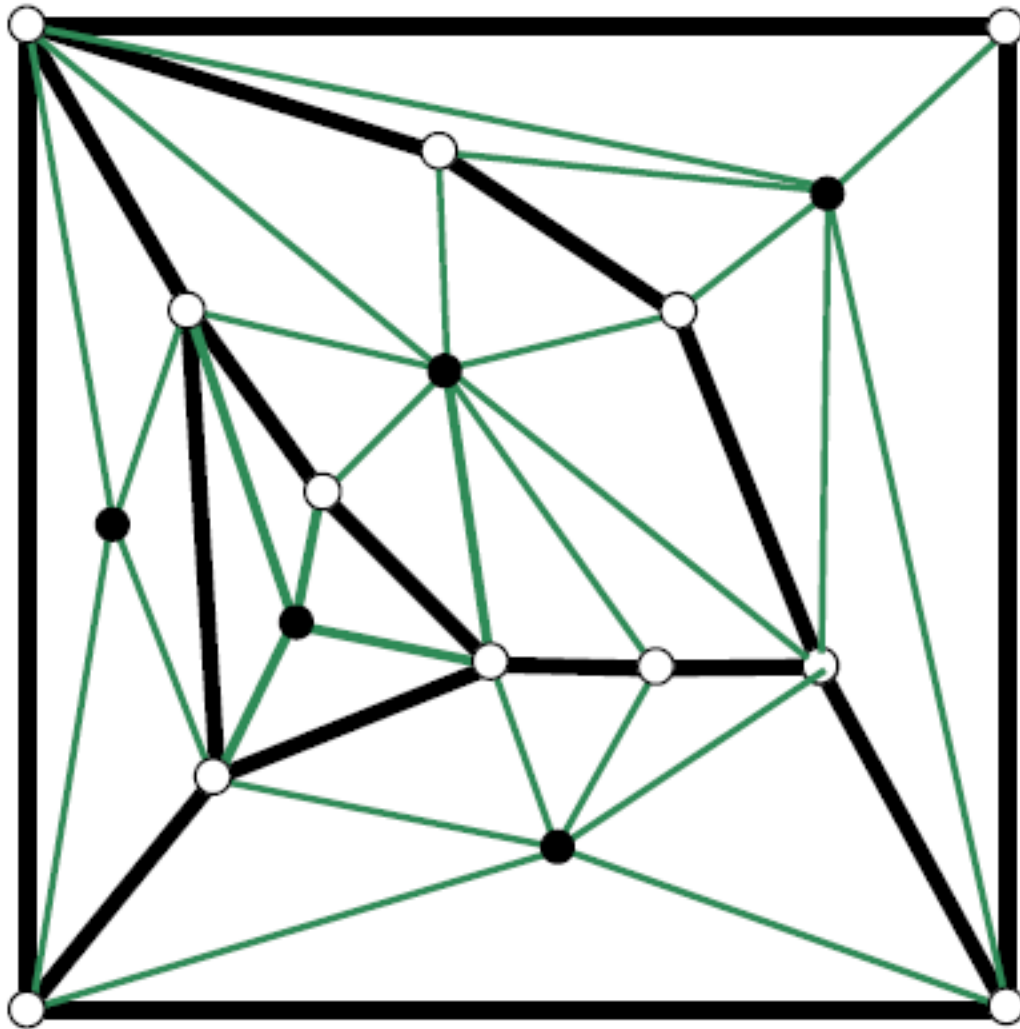


# Generalized 2-orientations



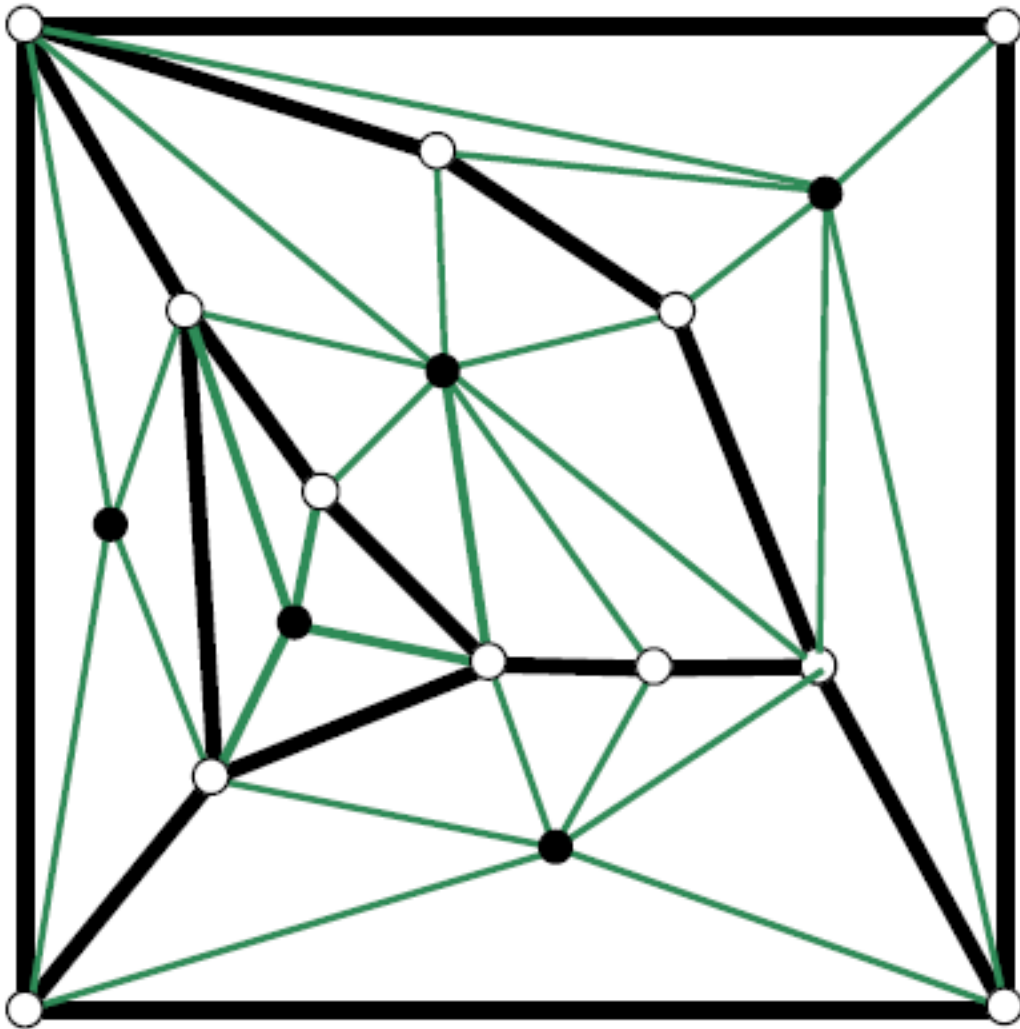
**insert a star into  
each inner face**

# Generalized 2-orientations

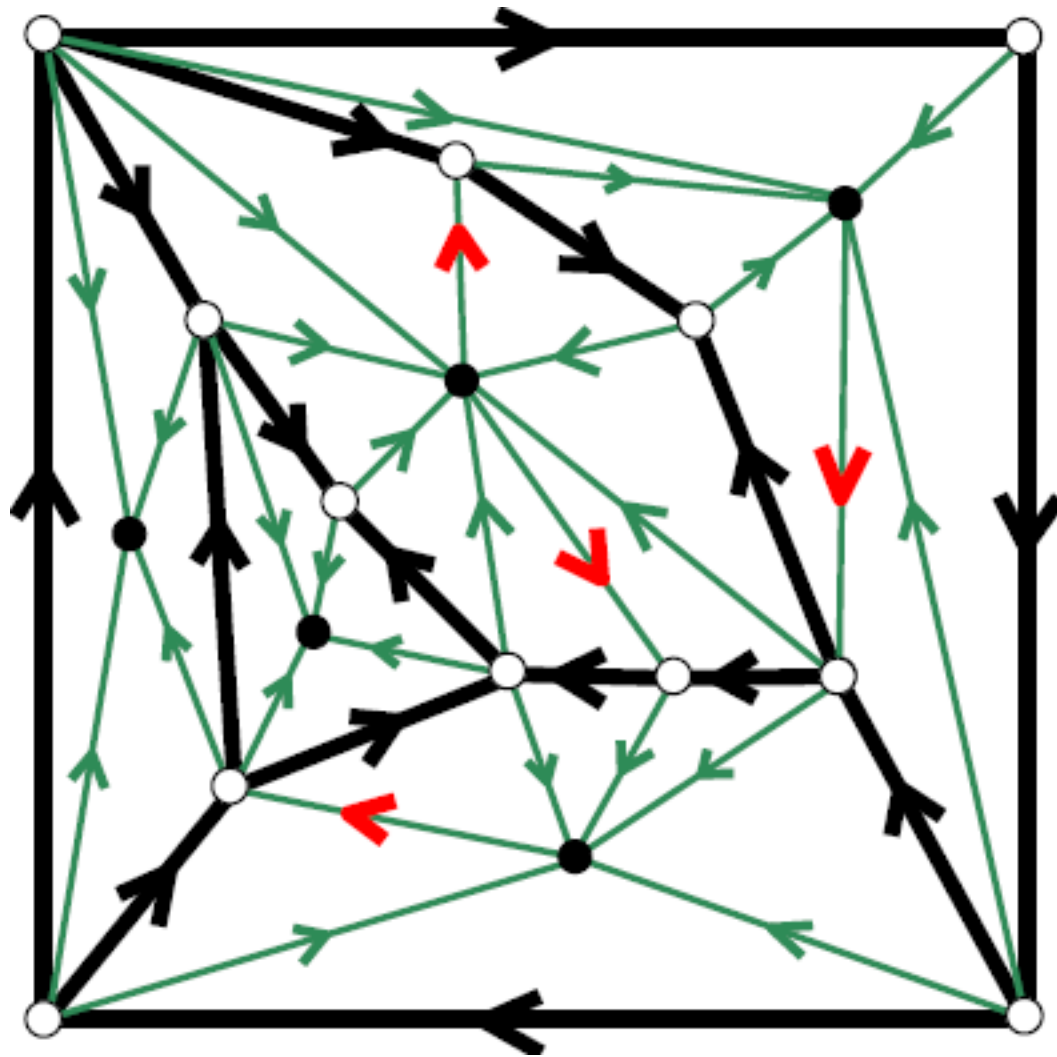


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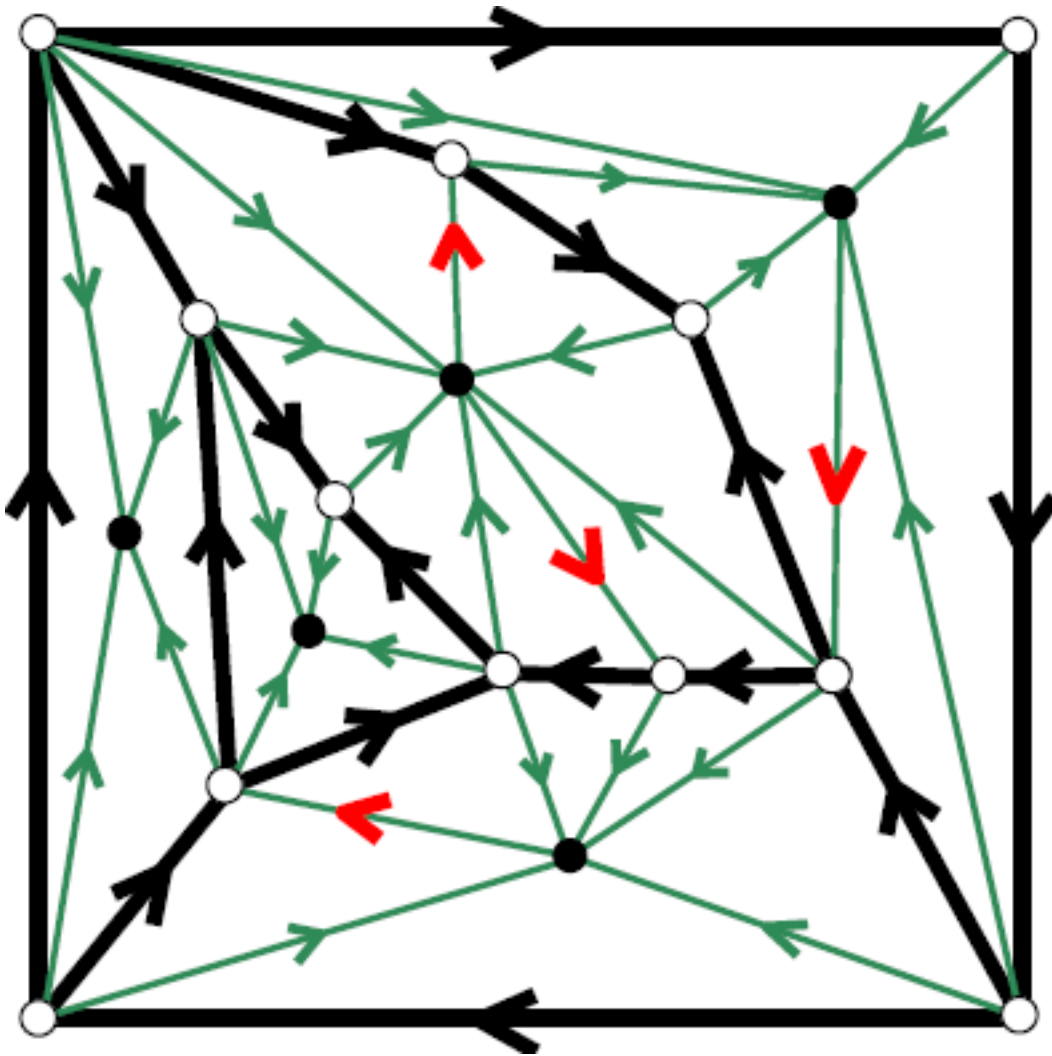


Outer white: indeg=1

Inner white:  $\text{indeg}=2$

Each black vertex of degree  $2i$  has outdeg  $i-2$

# Generalized 2-orientations

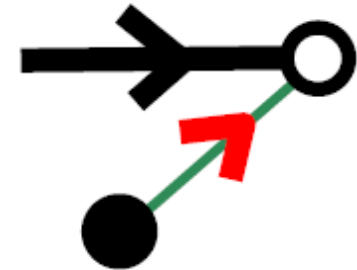


Outer white:  $\text{indeg}=1$

Inner white:  $\text{indeg}=2$

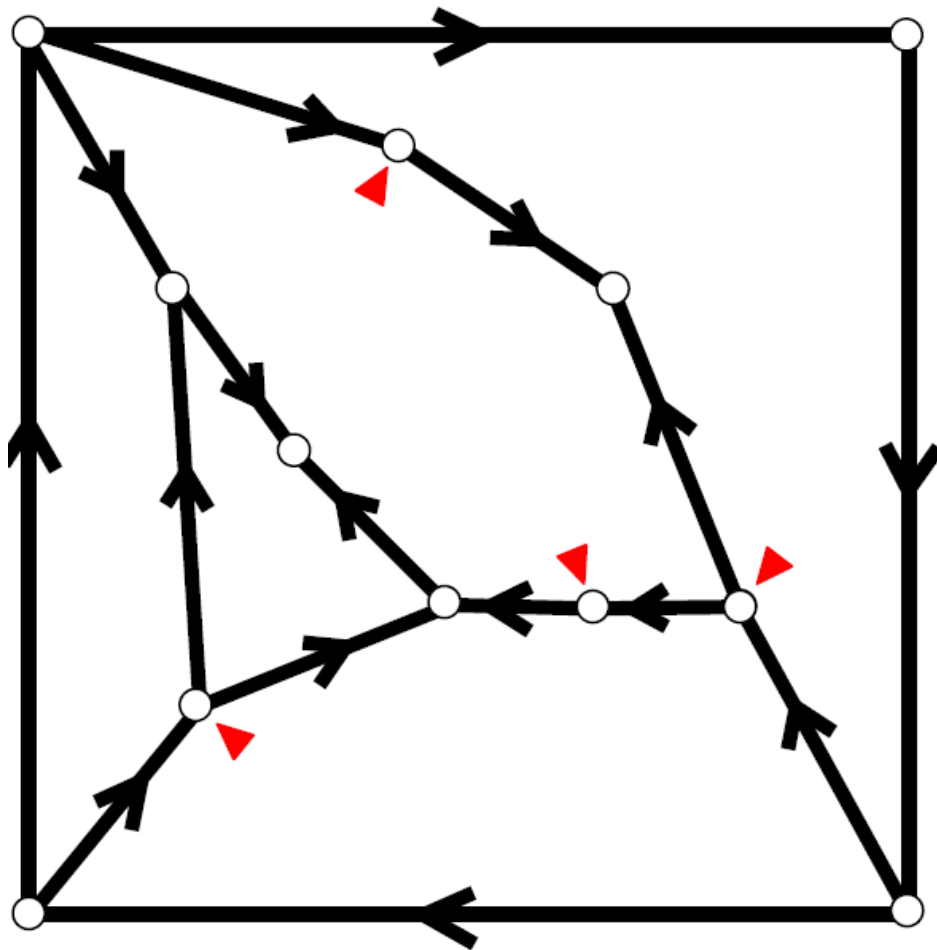
Each black vertex of degree  $2i$  has outdeg  $i-2$

For the **minimal** one:

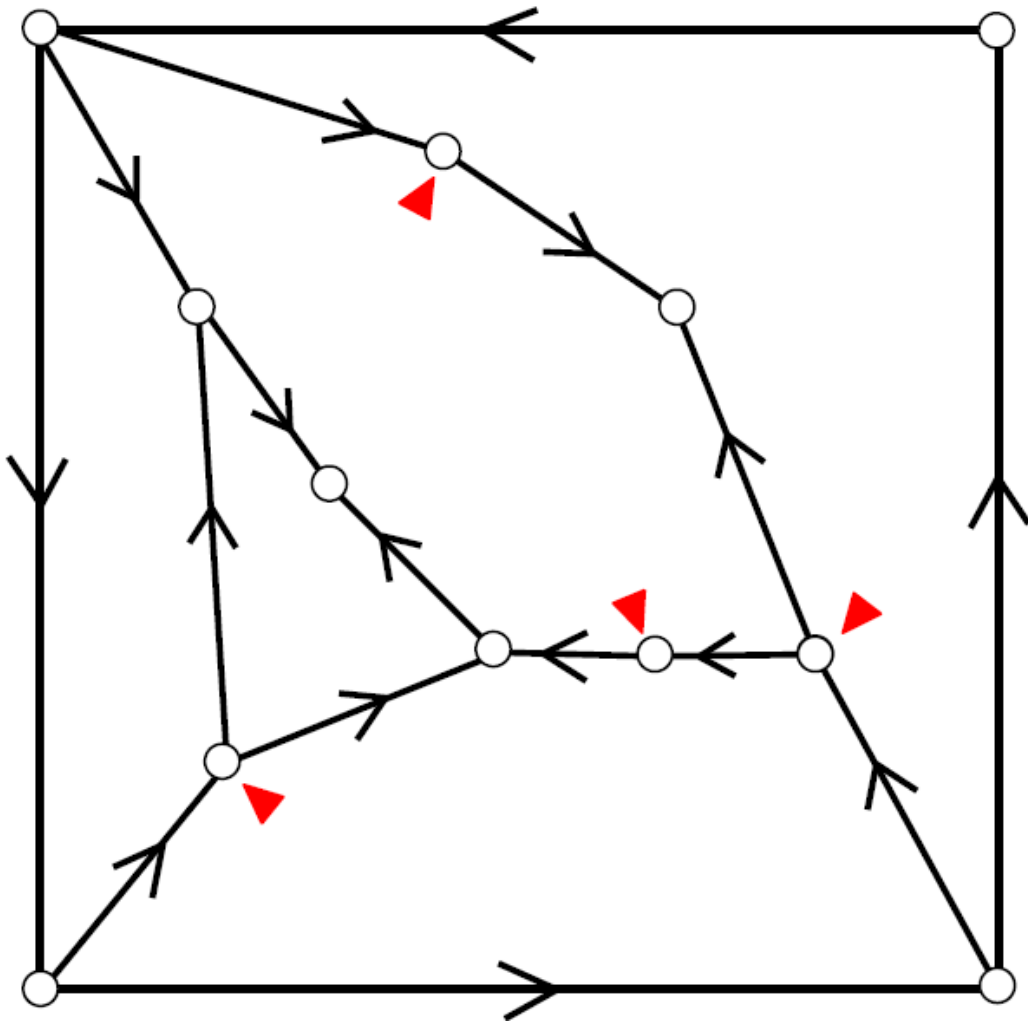


& Still accessible after deleting the stars

# Minimal 2-ori. $\rightarrow$ mobile

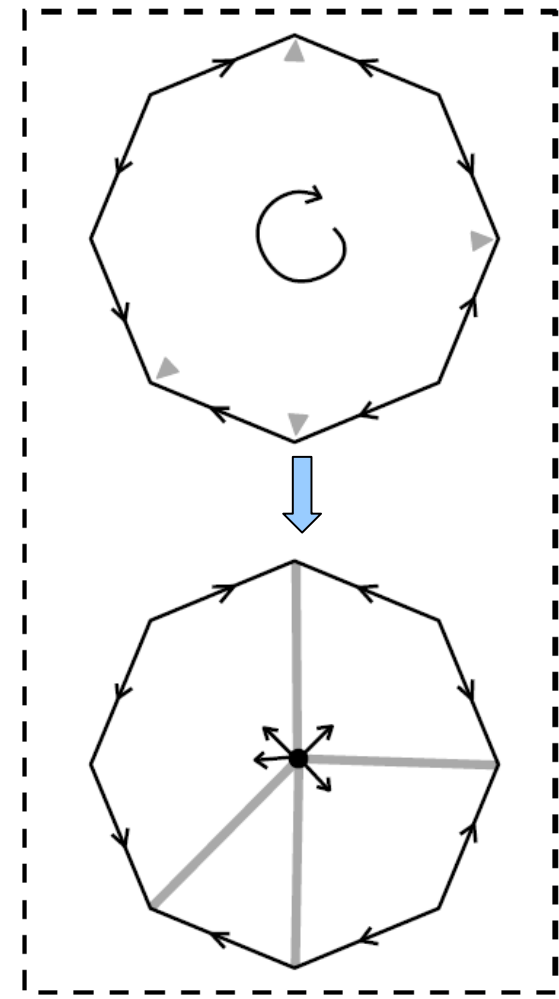
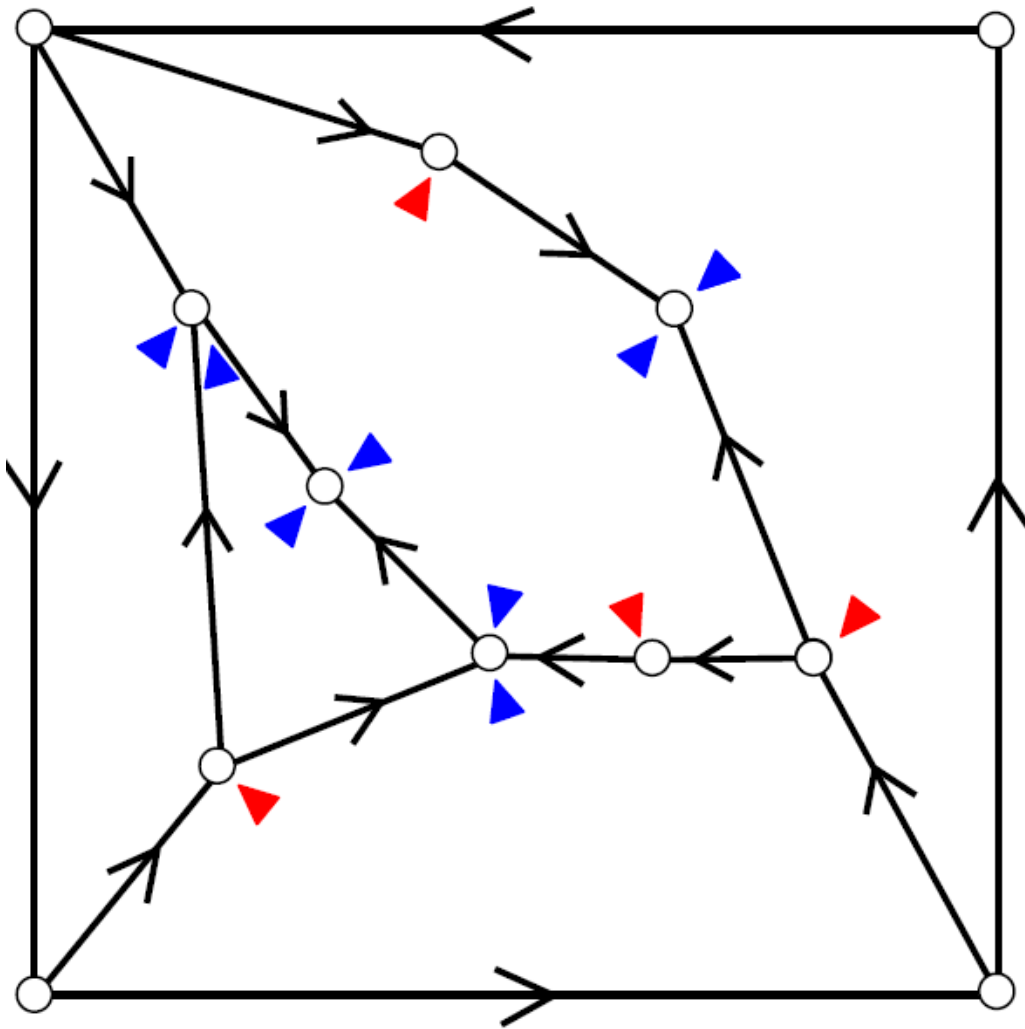


# Minimal 2-ori. $\rightarrow$ mobile



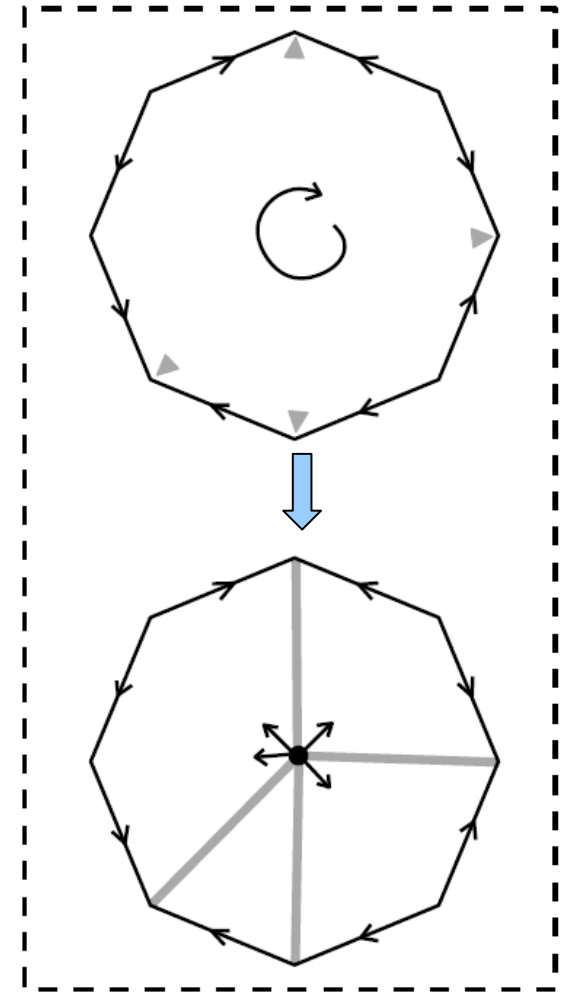
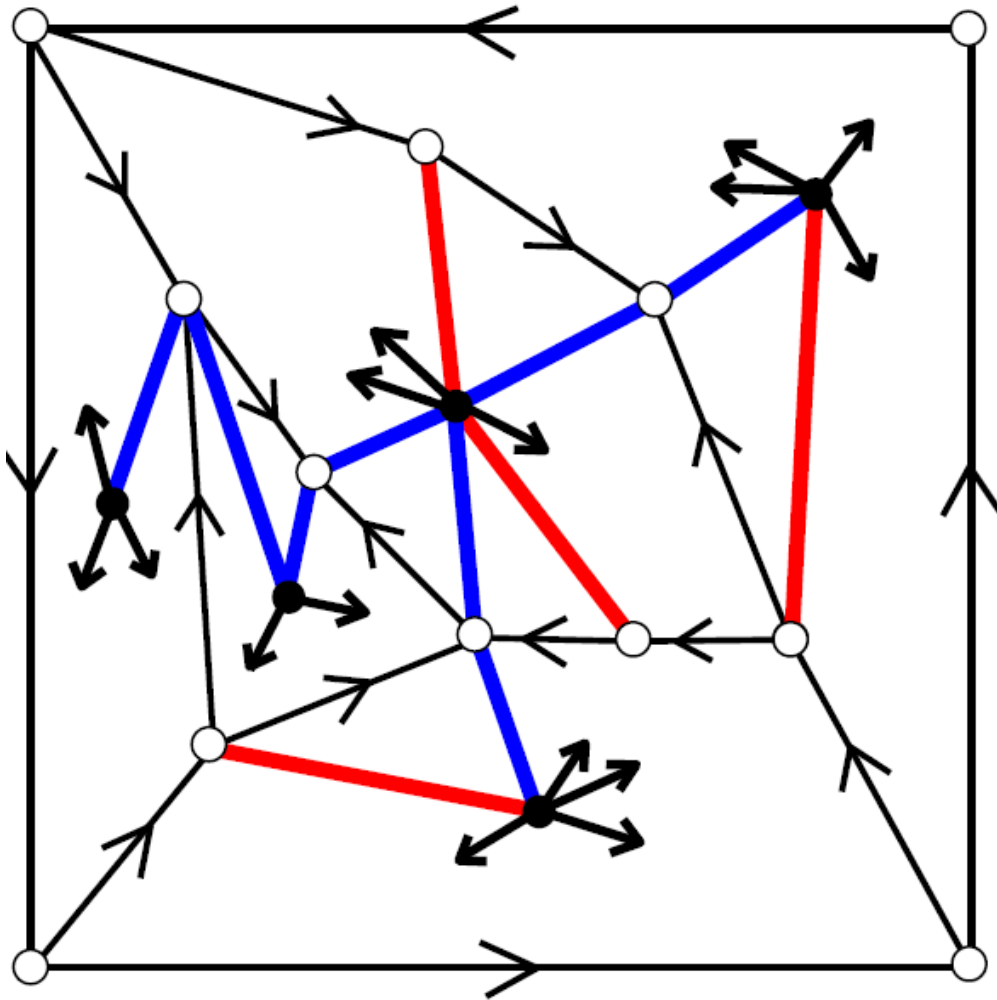


# Minimal 2-ori. $\rightarrow$ mobile



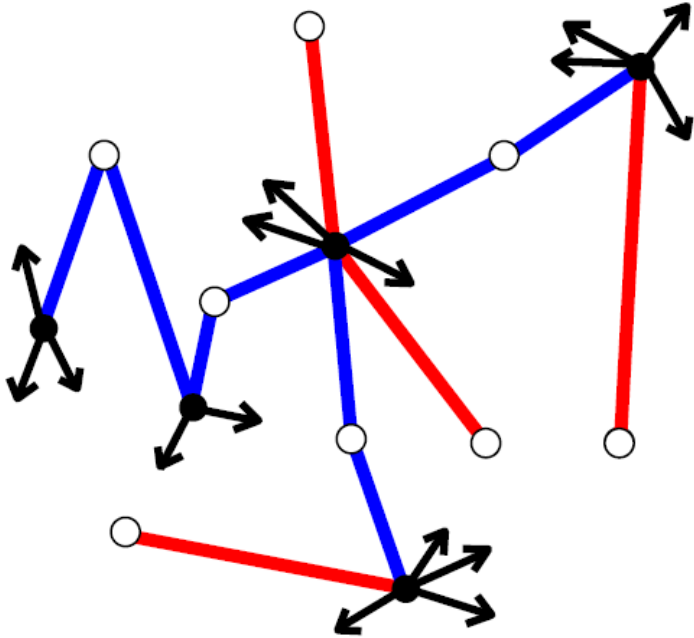
**Rule to draw mobile**

# Minimal 2-ori. $\rightarrow$ mobile



**Rule to draw mobile**

# Minimal 2-ori. $\rightarrow$ mobile

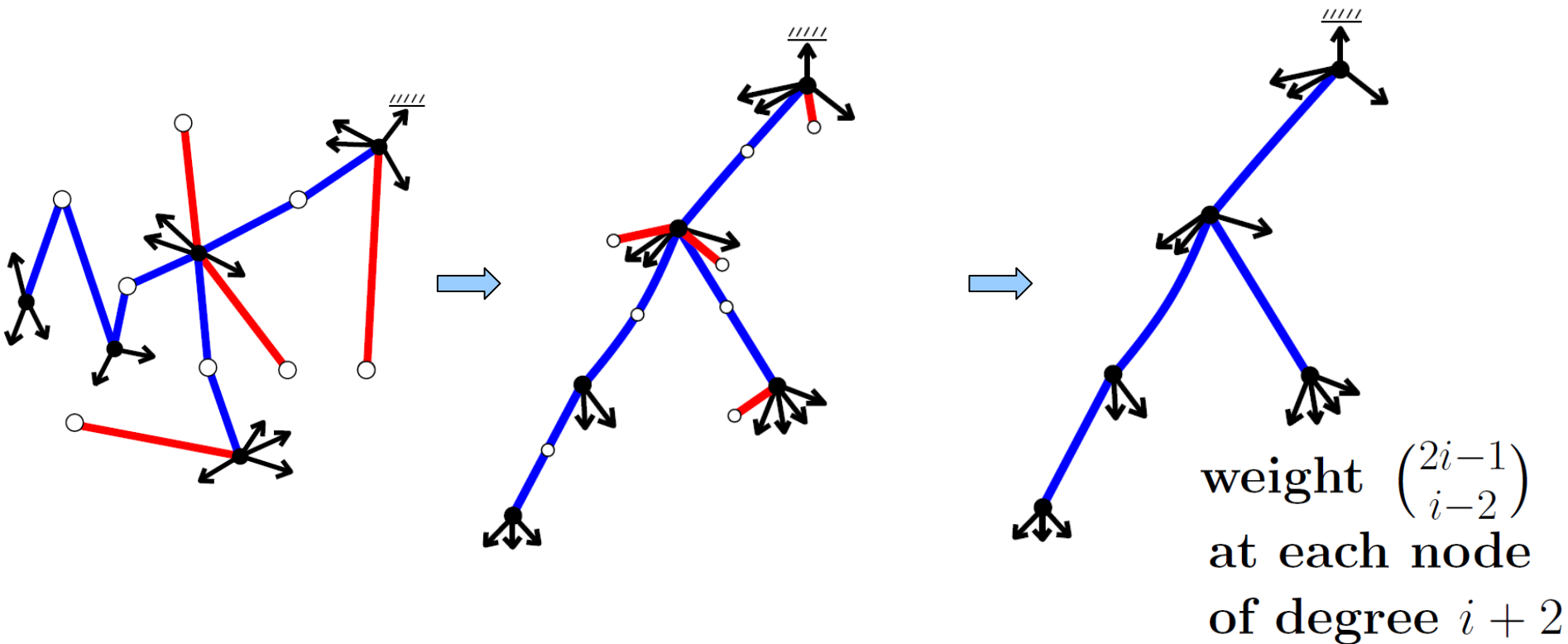


White vertices either have:

- degree 2 (middle blue edge)
- degree 1 (end of leg)

Each black vertex of degree  $2i$  has  $i-2$  legs

# Enumeration formula



& each mobile has 4 "balanced" rootings

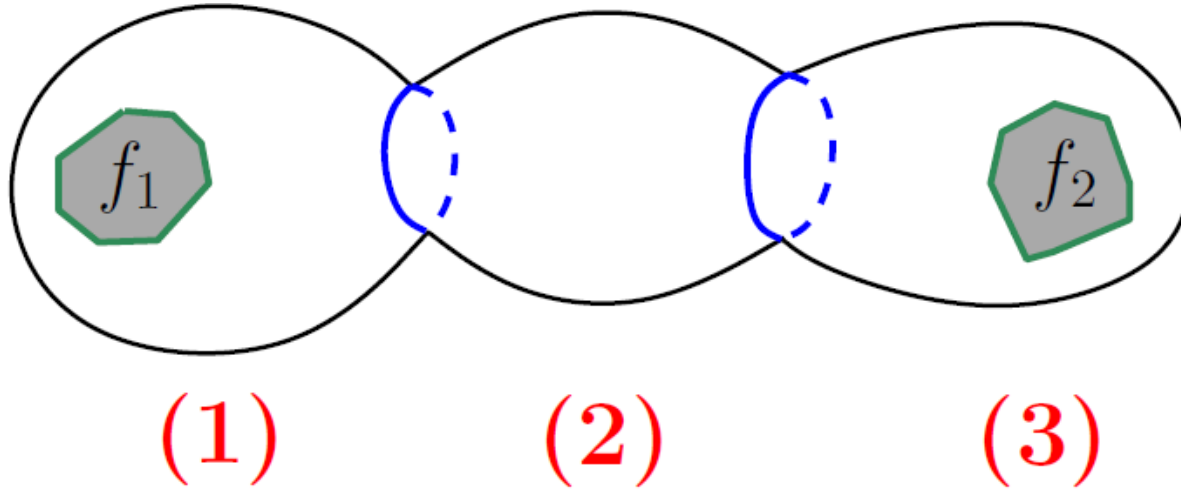
→ The number of simple rooted bipartite maps with  $n_i$  faces of degree  $2i$  is:

$$2 \frac{(e + n - 3)!}{(e - 1)!} \prod_{i=2}^k \frac{1}{n_i!} \binom{2i-1}{i-2}^{n_i}$$

Works for  $n_2 > 0$  only !

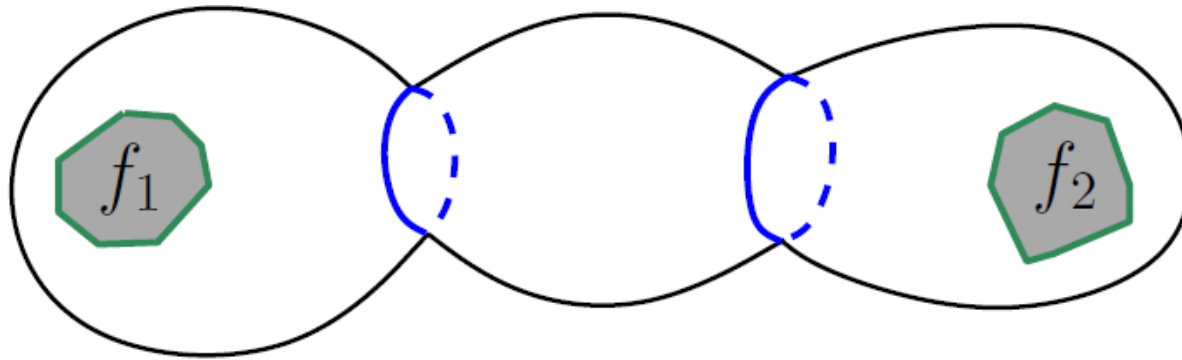
# Construction on the cylinder

- Two root-faces  $f_1$  of degree  $2a$ ,  $f_2$  of degree  $2b$
- Let  $2c = \text{length shortest cycle separating } f_1 \text{ from } f_2$



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(1)

(2)

(3)

Let  $R = t + \sum_{i \geq 2} \binom{2i-1}{i-2} x_i R^{i+1}$

series of (1\*2) =  $\binom{2a}{a-c} R^{a+c}$

series of (3) =  $\binom{2b}{b-c} R^{b-c}$

$$F_{a,b} = \sum_{c > 2} c \binom{2a}{a-c} \binom{2b}{b-c} R^{a+b} = \frac{(a+2)(b+2)}{2a+2b} \binom{2a}{a-2} \binom{2b}{b-2} R^{a+b}$$

$$\rightarrow 2 \frac{(e+n-3)!}{(e-1)!} \prod_{i=2}^k \frac{1}{n_i!} \binom{2i-1}{i-2}^{n_i}$$

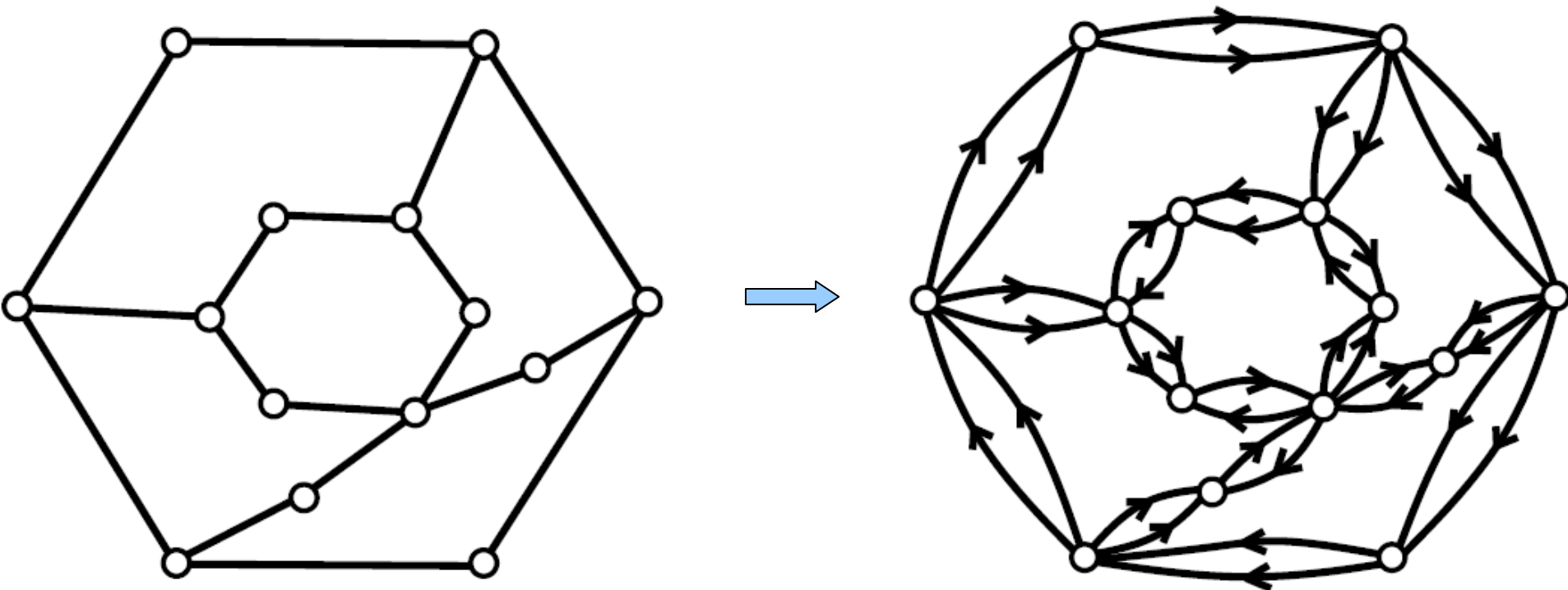
# **Part III: generalization to higher girth**

# $b/(b-1)$ -orientations for $b \geq 2$

A  $b/(b-1)$ -orientation on a  $2b$ -angulation  $G$  is an orientation of the planar map  $(b-1)^*G$  (each edge multiplied by  $b-1$ ) such that:

- inner vertices have indegree  $b$
- outer vertices have indegree  $b-1$

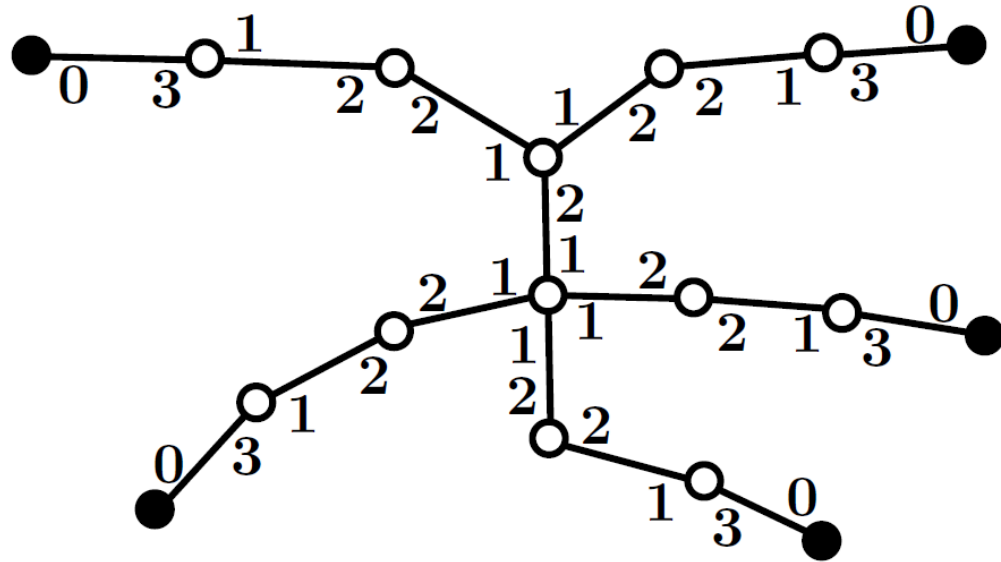
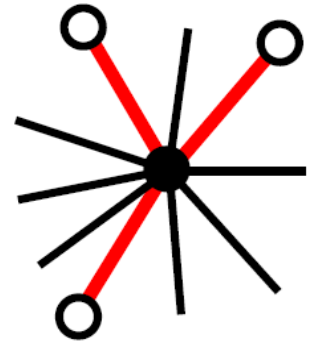
[Bernardi, F'10]: a  $2b$ -angulation can be endowed with a  $b/(b-1)$ -orientation iff it has girth  $2b$



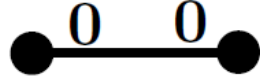


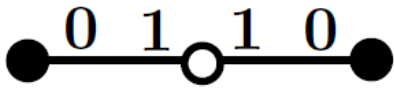
# Shape of the mobiles in girth $2b \geq 2$

- Each black vertex of degree  $2i$  has  $i-b$  legs
- There are "connectors" between black vertices: (trees with black leaves, white inner nodes, weights on half-edges)



a connector for  $b=4$

Connectors, for  $b = 1$ : 

$b = 2$ : 

$b = 3$ : **binary trees**