

Tableaux escalier, PASEP et polynômes d'Askey Wilson

Sylvie Corteel (LIAFA - CNRS et Université Paris Diderot)

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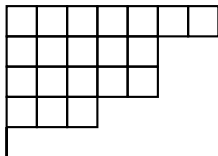
Coworkers

- ▶ M. Josuat-Vergès (Université Paris-Sud)
- ▶ P. Nadeau (U. Wien)
- ▶ L.K. Williams (Berkeley)
- ▶ D. Stanton (Minnesota)

Young diagram

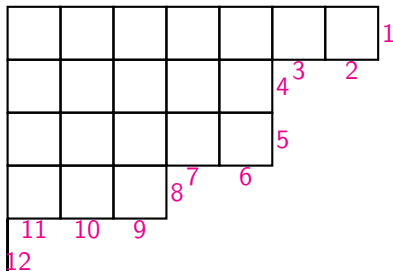
$\lambda = (\lambda_1, \dots, \lambda_k)$ with $\lambda_1 \geq \dots \geq \lambda_k \geq 0$

$\lambda = (7, 5, 5, 3, 0)$



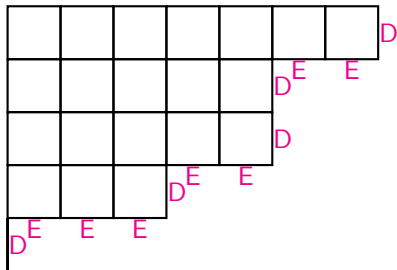
Length=number of rows + number of columns= $\lambda_1 + k$

Young diagram



Number the border : $1, 2, \dots, 12$

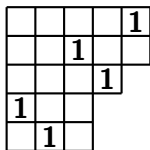
Young diagram



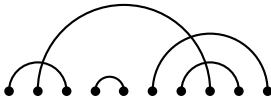
Code the border: *DEEDDEEDEEED*

Example 1

Any λ and one 1 per row and per column : Rook placements
The number of RP of length $2n$ is $(2n - 1)!!$.

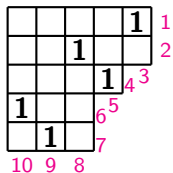


\mapsto

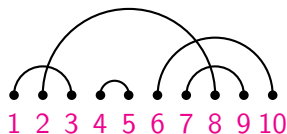


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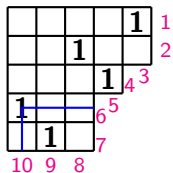


\mapsto

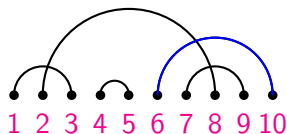


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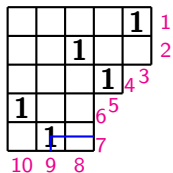


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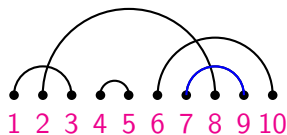


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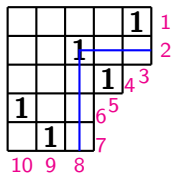


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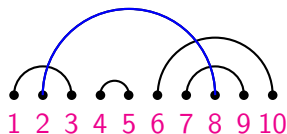


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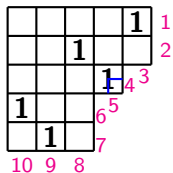


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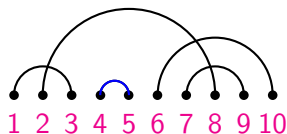


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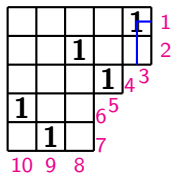


\mapsto

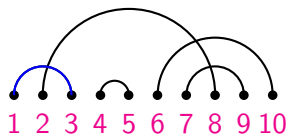


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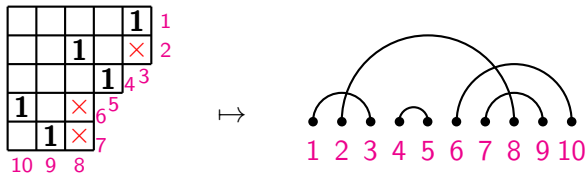
\mapsto



Example I

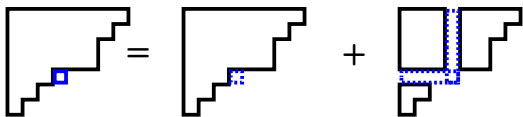
Any λ and one 1 per row and per column : Rook placements

The number of RP of length $2n$ is $(2n - 1)!!$.



$$\sum_{m \in \mathfrak{M}_{2n}} q^{cr(m)} = \langle W | (D + E)^{2n} | V \rangle$$

where $DE = qED + I$, $\langle W | E = 0$, $D | V \rangle = 0$.



(Touchard-Riordan, Penaud, Flajolet-Noy, Josuat-Vergès ...)

Moments of q -Hermite polynomials

Example II

Leroux (88) 0-1 tableaux : One 1 per column.

The number of tableaux of length n is B_n (the n^{th} Bell number)

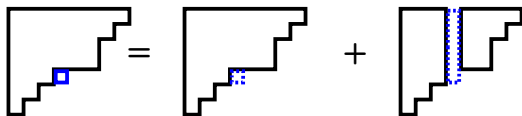
0-1 tableaux \mapsto Set partitions

	1		1	1	1
					2
		1	5 ⁴	3	
1		7 ⁶	8		
	9	8			

$$\mapsto \pi = (1, 3, 4, 8)(2)(5, 6)(7, 9)$$

$$\langle W | (yD + E)^n | V \rangle$$

with $DE = qED + D$, $\langle W | E = 0$, $D | V \rangle = |V\rangle$.



Moments of q -Charlier polynomials (de Medicis, Stanton, White 95)

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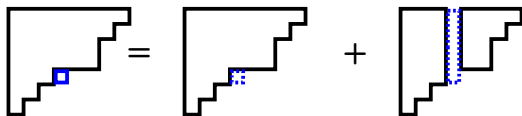
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	×		×	×	2
	×	1	5	4	3
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Moments of q -Charlier polynomials (de Medicis, Stanton, White 95)

Permutation tableaux (Postnikov 01, Williams 05)

Totally non negative part of the Grassmanian

Permutation tableau \mathcal{T} : a partition λ filled with 0's and 1's such that :

1. Each column contains at least one 1.
2. There is no 0 which has a 1 above it in the same column *and* a 1 to its left in the same row.

0	0	1	0	0	1	1
0	0	1	0	1		
0	1	1	1	1		
0	0	0				
1						

0	0	1	0	0	1	1
0	0	0	0	1		
0	1	0	1	1		
0	0	0				
1						

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0	0	1	0	1		
0	1	1	1	1		
0	0	0				
1						

0	0	1	0	0	1	1
0	0	0	0	1		
0	1	0	1	1		
0	0	0				
1						

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0	0	1	0	0	1	1
0	0	1	0	1		
0	1	1	1	1		
0	0	0				
1						

0	0	1	0	0	1	1
0	0	0	0	1		
0	1	0	1	1		
0	0	0				
1						

Number of permutation tableaux of length n is $n!$

Permutation tableaux and alternative tableaux

Restricted zero : lies below some 1

A permutation tableaux is uniquely defined by its topmost ones and rightmost restricted zeros.

(C. Nadeau 07)

0	0	1	0	0	1	1
0	0	1	0	1		
0	1	1	1	1		
0	0	0				
1						

Permutation tableaux and alternative tableaux

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		1			1	1
				1		
	1		1			
		0				
1						

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0	1		1			
0	0	0				
1						

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		1		1		
	1	1	1	1		
		0				
1						

Permutation tableaux and alternative tableaux

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		1			1	1
				1		
	1		1			
		0				
1						

					↑	
	↑			↑		
			←			
↑						

Alternative tableaux (Viennot 08, Nadeau 09)

Bijections with permutations

Postnikov (00)

Steingrimsson and Williams (06)

Burstein (07)

C. and Nadeau (07)

Viennot (08)

Descents, excedances, crossings, 31-2, RL-min, LR-Max.

Enumeration of PT

- ▶ $u(\mathcal{T})$: number of unrestricted rows minus one
- ▶ $f(\mathcal{T})$: number of ones in the first row

0	0	1	0	0	1	1
0	0	1	0	1		
0	1	1	1	1		
0	0	0				
1						

$$f(\mathcal{T}) = 3$$

$$u(\mathcal{T}) = 4 - 1 = 3$$

$$\sum_{\mathcal{T} \text{ length } n+1} x^{u(\mathcal{T})} y^{f(\mathcal{T})} = \prod_{i=0}^{n-1} (x + y + i) = (x + y)_n.$$

(C. and Nadeau 07)

q-enumeration

$\text{wt}(\mathcal{T})$: number of ones minus number of columns

1	0	1	0	0	1	1
0	0	1	0	1		
0	1	1	1	1		

$$\text{wt}(\mathcal{T}) = 10 - 7 = 3$$

$$E_{k,n}(q) = \sum_{\mathcal{T}} \text{wt}(\mathcal{T}) = q^{n-k^2} \sum_{i=0}^{k-1} (-1)^i [k-i]_q^n \left(\binom{n}{i} q^{k-i} + \binom{n}{i-1} \right)$$

(Williams 05)

q -analogue of Eulerian numbers

$q = 0$ Narayana numbers, $q = -1$ Binomial numbers.

Moments of q -Laguerre polynomials (Kasraoui, Stanton, Zeng 09)

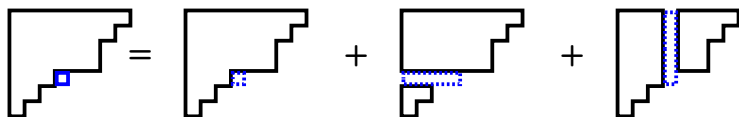
Tableaux of a given shape (Williams 05, Novelli, Thibon, Williams 08)

Three parameter enumeration

$$F_n(q, \alpha, \beta) = \sum_{\mathcal{T} \text{ length } n} W(\mathcal{T}); \quad W(\mathcal{T}) = q^{\text{wt}(\mathcal{T})} \alpha^{-f(\mathcal{T})} \beta^{-u(\mathcal{T})},$$

$$F_n(q, \alpha, \beta) = \langle W | (D + E)^{n-1} | V \rangle, \quad \text{where}$$

$$DE = qED + D + E;$$



$$\alpha \langle W | E = \langle W |; \quad \beta D | V \rangle = | V \rangle \quad \langle W | | V \rangle = 1.$$

(C, Williams 06)

Compute $F_n(q, \alpha, \beta)$ (Josuat-Vergès 09)

PT and Partially asymmetric exclusion process

Model : n sites that are empty or occupied

The sites are delimited by $n + 1$ positions ($n - 1$ positions in between sites, left border and right border).

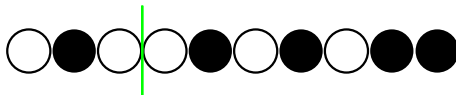


PT and Partially asymmetric exclusion process

Model : n sites that are empty or occupied

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- ▶ First a position is chosen at random

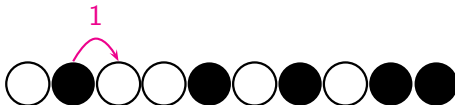


PT and Partially asymmetric exclusion process

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- ▶ First a position is chosen at random
- ▶ A particle hops to the right with probability 1

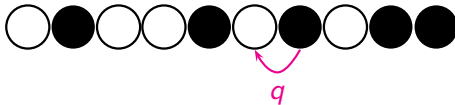


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- ▶ First a position is chosen at random
- ▶ A particle hops to the right with probability 1
- ▶ A particle hops to the left with probability q

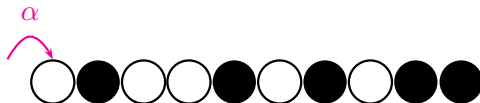


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- ▶ First a position is chosen at random
- ▶ A particle hops to the right with probability 1
- ▶ A particle hops to the left with probability q
- ▶ A particle enters with probability α

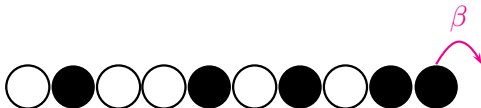


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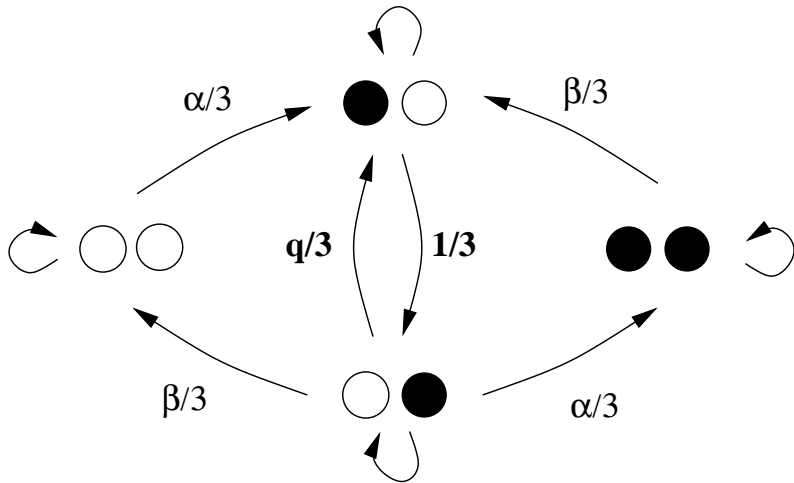
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- ▶ A particle hops to the left with probability q
- ▶ A particle enters with probability α
- ▶ A particles leaves with probability β



Markov chain $n = 2$



Stationary distribution of the PASEP chain

$$\begin{array}{c} \bigcirc \bigcirc \bullet \bigcirc \bigcirc \bullet \bullet \bigcirc \bigcirc \end{array} \quad \leftrightarrow \quad \tau = (0, 0, 1, 0, 0, 1, 1, 0, 0)$$

Let $P_n^{q,\alpha,\beta}(\tau)$ be the probability to be in state $\tau = (\tau_1, \dots, \tau_n)$.

Theorem. (Derrida et al. 93) The probability to be in state $\tau = (\tau_1, \dots, \tau_n)$ is

$$P_n(\tau) = \frac{\langle W | (\prod_{i=1}^n (\tau_i D + (1 - \tau_i) E)) | V \rangle}{Z_n}.$$

with $Z_n = \langle W | (D + E)^n | V \rangle$, D and E are infinite matrices, V is a column vector, and W is a row vector, such that

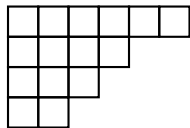
$$DE - qED = D + E$$

$$\beta D | V \rangle = | V \rangle$$

$$\alpha \langle W | E = \langle W |$$

Permutation tableaux

$$\tau = (0, 0, 1, 0, 0, 1, 1, 0, 0) \leftrightarrow \lambda(\tau) =$$



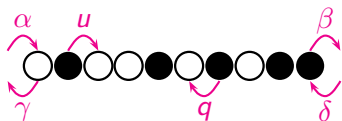
Theorem. Fix $\tau = (\tau_1, \dots, \tau_n) \in \{0, 1\}^n$, and let $\lambda := \lambda(\tau)$. The probability of finding the PASEP chain in configuration τ in the steady state is

$$\frac{\sum_{\mathcal{T}} W(\mathcal{T})}{Z_n}.$$

where the sum is on the permutation tableaux of **shape** λ .

(C. and Williams 2006)

General PASEP



Theorem. (Derrida et al. 93) The probability to be in state $\tau = (\tau_1, \dots, \tau_n)$ is

$$P_n(\tau) = \frac{\langle W | (\prod_{i=1}^n (\tau_i D + (1 - \tau_i) E)) | V \rangle}{Z_n}.$$

with $Z_n = \langle W | (D + E)^n | V \rangle$, D and E are infinite matrices, V is a column vector, and W is a row vector, such that

$$uDE - qED = D + E$$

$$\beta D | V \rangle = | V \rangle + \delta E | V \rangle$$

$$\alpha \langle W | E = \langle W | + \gamma \langle W | D$$

Theorem.

Fix $\tau = (\tau_1, \dots, \tau_n) \in \{\bullet, \circ\}^n$, and The probability of finding the PASEP chain in configuration τ in the steady state is

$$\frac{\sum_{\mathcal{T}} W(\mathcal{T})}{Z_n}.$$

where the sum is on **staircase tableaux of type τ** .

(C. and Williams 2009)

Staircase tableaux

Staircase shape $(n, n - 1, \dots, 2, 1)$.

Fill some cells with α, β, γ or δ such that

- ▶ Cells of the diagonal are filled
- ▶ \uparrow for α or γ
- ▶ \leftarrow for β or δ

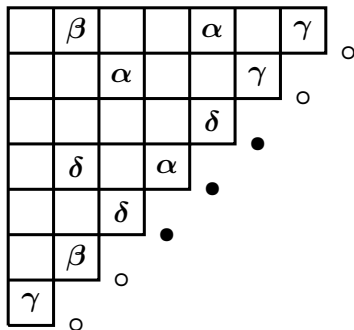
	β			α		γ
		α			γ	
				δ		
	δ		α			
		δ				
	β					
γ						

Staircase tableaux

Staircase shape $(n, n - 1, \dots, 2, 1)$.

Fill some cells with α, β, γ or δ such that

- ▶ Cells of the diagonal are filled
- ▶ \uparrow for α or γ
- ▶ \leftarrow for β or δ



Type : read the diagonal, put ○ if β or γ and • if α or δ .

Staircase tableaux

Staircase shape $(n, n - 1, \dots, 2, 1)$.

Fill some cells with α, β, γ or δ such that

- ▶ Cells of the diagonal are filled
- ▶ \uparrow for α or γ
- ▶ \leftarrow for β or δ

	β			α		γ	<i>E</i>
		α			γ		<i>E</i>
				δ			<i>E</i>
	δ		α				<i>D</i>
		δ					<i>D</i>
	β						<i>D</i>
γ							<i>E</i>

Weight of a staircase tableau

under \ right	α	β	γ	δ
α	u	u	u	q
β	q	u	q	q
γ	q	u	q	q
δ	u	u	u	q

u	β	u	u	α	q	γ
q	u	α	u	u	γ	
q	q	q	q	δ		
q	δ	u	α			
q	q	δ				
u	β					
γ						

$$W(\mathcal{T}) = \alpha^2 \beta^2 \gamma^3 \delta^2 q^9 u^8.$$

Staircase tableau of type $\bullet\bullet$

u	α	q	δ	q	δ	u	α	δ	α	γ	α	β	α	α	α
α		α		δ		δ		δ		δ		δ		δ	

Probabilité de $\bullet\bullet$

$$\frac{\alpha^2 u + \delta^2 q + \alpha \delta (u + q + \alpha + \beta + \gamma + \delta)}{Z_2}$$

Staircase and alternative tableaux

No γ , No δ

	β			α		α
		α			α	
				β		
	β		α			
		β				
	β					
α						

Staircase and alternative tableaux

No γ , No δ

	β			α		α
		α			α	
				β		
	β		α			
		β				
	β					
α						

Staircase and alternative tableaux

No γ , No δ

	β			α		α
		α			α	
				β		
	β		α			
		β				
	β					
α						

\leftrightarrow

β		α
	α	
β		

Staircase and alternative tableaux

No γ , No δ

	β			α		α
		α			α	
				β		
	β		α			
		β				
	β					
α						

\leftrightarrow

\leftarrow		\uparrow
	\uparrow	
\leftarrow		

Matrix Ansatz and staircase tableaux

under \ right	α	β	γ	δ
α	u	u	u	q
β	q	u	q	q
γ	q	u	q	q
δ	u	u	u	q

Staircase tableau of length $n + 1$. First diagonal:

- ▶ $\alpha \leftrightarrow \gamma$
- ▶ β : take off the first line βu^n
- ▶ δ : take off the first line δq^n

$$\alpha \langle W|EX|V \rangle = \gamma \langle W|DX|V \rangle + (\alpha\beta u^n - \gamma\delta q^n) \langle W|X|V \rangle$$

$$X \in \{D, E\}^n$$

Matrix Ansatz and staircase tableaux

$$\begin{aligned} \langle W|XDEY|V \rangle &= q \langle W|XEDY|V \rangle \\ &+ (\alpha\beta u^{n+m+1} - \gamma\delta q^{n+m+1}) \langle W|X(D+E)Y|V \rangle \end{aligned}$$

$$\beta \langle W|XD|V \rangle = \delta \langle W|XE|V \rangle + (\alpha\beta u^n - \gamma\delta q^n) \langle W|X|V \rangle$$

$$X \in \{D, E\}^n, Y \in \{D, E\}^m.$$

Proof by induction

Enumeration of staircase tableaux

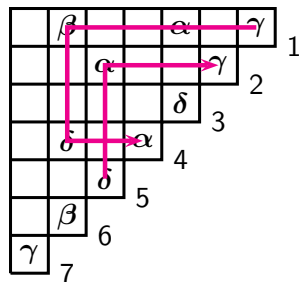
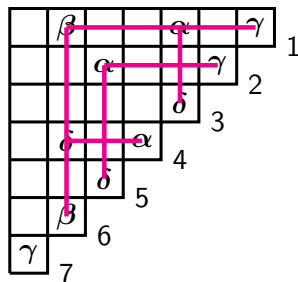
There exist $2^n n!$ staircase tableaux of type τ

There exist $4^n n!$ staircase tableaux of size n

$u = q = 1$,

$$Z_n = \prod_{i=0}^{n-1} (\alpha + \beta + \gamma + \delta + i(\alpha + \gamma))(\beta + \delta)$$

Bijection



(4, 5, 1, 6, 2, 3, 7)

First sign : type, second sign : label of the beginning of the path
(- - 4, - + 5, + - 1, + + 6, + + 2, - + 3, - - 7)

Enumeration formula $u = 1$

Theorem. (C. Stanton, Williams, 2010)

$$Z_n(y, a, b, c, d, q) = (abcd; q)_n \left(\frac{\alpha\beta}{1-q} \right)^n \frac{1}{2^n} \sum_{k=0}^n \frac{(ab, acy, ad; q)_k}{(abcd; q)_k} q^k \\ \times \sum_{j=0}^k q^{-(k-j)^2} (a^2 y)^{j-k} \frac{(1 + y + q^{k-j} ay + q^{j-k}/a)^n}{(q, q^{2j-2k+1}/a^2/y; q)_{k-j} (q, a^2 y q^{1-2j+2k}; q)_j}$$

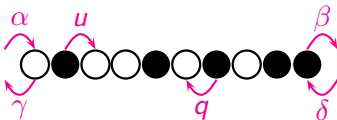
$$\alpha = \frac{1-q}{1-ac+a+c}, \quad \beta = \frac{1-q}{1-bd+b+d}, \\ \gamma = \frac{(1-q)ac}{1+ac+a+c}, \quad \delta = \frac{(1-q)bd}{1+bd+b+d}$$

Moments of the Askey Wilson polynomials

$$\frac{(1-q)^n}{2^n i^n \prod_{j=0}^{n-1} (\alpha\beta - \gamma\delta q^j)} Z_n(-1, ai, bi, ci, di, q)$$

Open problems

- ▶ Symmetry $\alpha \leftrightarrow \gamma$, $\beta \leftrightarrow \delta$, $q \leftrightarrow u$.



- ▶ Enumeration of staircase tableaux
- ▶ Combinatorics of staircase tableaux? What does q count?
- ▶ Lifted chain on staircase tableaux?
- ▶ Link with the double Grassmanian?
- ▶ Hopf algebra?

Merci

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