

Polynomiality of the structure coefficients of double-class algebras

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Plan.

- I. Introduction: structure coefficients of an algebra
- II. Partitions
- III. Three polynomiality results
 1. $\mathcal{Z}(\mathbb{C}[\mathcal{S}_n])$
 2. Hecke algebra of $(\mathcal{S}_{2n}, \mathcal{B}_n)$
 3. $\mathcal{Z}(\mathbb{C}[GL_n(\mathbb{F}_q)])$
- IV. Structure coefficients of the algebra of double classes
- V. Conclusions and further applications

I. Introduction: structure coefficients of an algebra

- **Problem:** Let \mathcal{A} be an algebra over a field F with basis b_1, b_2, \dots, b_n . For two basis elements, say b_i and b_j , write:

$$b_i b_j = \sum_k c_{i,j}^k b_k,$$

where $c_{i,j}^k \in F$. The elements $c_{i,j}^k$ are called the **structure coefficients of \mathcal{A}** and there is **no explicit formula for them**, even in the particular algebras we will consider.

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- **Our work:**
 - 1- A framework in which one can obtain the form of the structure coefficients of the algebra of double-classes.¹
 - 2- A polynomiality property of these coefficients in some specific cases.

¹These coefficients "contain" structure coefficients of group algebras.

II. Partitions

- A **partition** λ is a list of integers $(\lambda_1, \lambda_2, \dots)$ where $\lambda_1 \geq \lambda_2 \geq \dots \geq 1$. The λ_i are called the parts of λ . The size of a partition λ (noted $|\lambda|$) is the sum of all its parts.
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The proper partitions will be used to index bases of the algebras considered in this talk.

III. Three polynomiality results

1. Center of the symmetric group algebra $\mathcal{Z}(\mathbb{C}[\mathcal{S}_n])$

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- For a permutation $\omega \in \mathcal{S}_n$, we define the **cycle-type** of ω , $ct(\omega)$, to be the partition of n with parts equal to the lengths of the cycles that appear in its decomposition.

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- The family $(S_\lambda(n))_{|\lambda| \leq n}$ indexed by proper partitions of size at most n forms a basis for $\mathcal{Z}(\mathbb{C}[\mathcal{S}_n])$, where,

$$S_\lambda(n) = \sum_{\substack{\omega \in \mathcal{S}_n, \\ ct(\omega) = \lambda \cup (\mathbf{1}^{n-|\lambda|)}}} \omega$$

- For λ and δ two proper partitions with size at most n ,

$$S_\lambda(n) \cdot S_\delta(n) = \sum_{\substack{\rho \text{ proper partition} \\ |\rho| \leq n}} c_{\lambda, \delta}^\rho(n) S_\rho(n),$$

the numbers $c_{\lambda, \delta}^\rho(n)$ are the **structure coefficients** of the center of the symmetric group algebra $\mathcal{Z}(\mathbb{C}[\mathcal{S}_n])$.

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- **Motivation (Cori [1975]:)** The structure coefficients of the center of the symmetric group algebra count the number of embeddings of certain graphs into orientable surfaces.

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- Theorem (Farahat and Higman [1958]):** Let λ , δ and ρ be three proper partitions, the function:

$$n \longmapsto c_{\lambda, \delta}^\rho(n)$$

defined for $n \geq |\lambda|, |\delta|, |\rho|$ is a polynomial in n .

Example: One can compute explicitly:

$$S_{(2)}(n) \cdot S_{(2)}(n) = \frac{n(n-1)}{2} S_{\emptyset}(n) + 3S_{(3)}(n) + 2S_{(2^2)}(n).$$

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- The polynomiality property for the structure coefficients of the center of the symmetric group algebra is interesting in the study of Plancherel measure on young diagrams.

III. Three polynomiality results
2. Hecke algebra of $(\mathcal{S}_{2n}, \mathcal{B}_n)$

- The **Hyperoctahedral** group \mathcal{B}_n is the subgroup of \mathcal{S}_{2n} consisting of all permutations of \mathcal{S}_{2n} which takes every pair of the form $\{2k - 1, 2k\}$ of $[2n]$ to another pair with the same form.

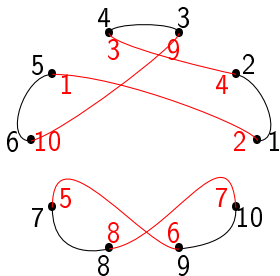
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- To each permutation ω of $2n$ we associate a graph $\Gamma(\omega)$.

Example: Take $\omega = 24931105867 \in \mathcal{S}_{10}$.

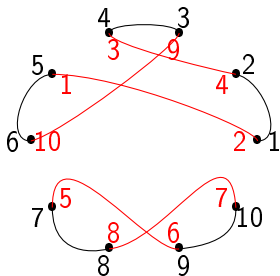


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- The **coset-type** of a permutation x of \mathcal{S}_{2n} is a partition of n with parts equal to half of lengths of the cycles of $\Gamma(x)$.

Example: $\text{coset-type}(\omega) = (3, 2)$.

- The **Hecke algebra** of $(\mathcal{S}_{2n}, \mathcal{B}_n)$ denoted by $\mathbb{C}[\mathcal{B}_n \backslash \mathcal{S}_{2n} / \mathcal{B}_n]$ is the sub-algebra of $\mathbb{C}[\mathcal{S}_{2n}]$ of elements invariant under the \mathcal{B}_n -double action. That means that:

$$x \in \mathbb{C}[\mathcal{B}_n \backslash \mathcal{S}_{2n} / \mathcal{B}_n] \iff hxh' = x \text{ for every } h, h' \in \mathcal{B}_n.$$

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- Proposition:** The family $(S'_\lambda(n))_{|\lambda| \leq n}$ indexed by proper partitions with size at most n forms a basis for $\mathbb{C}[\mathcal{B}_n \backslash \mathcal{S}_{2n} / \mathcal{B}_n]$, where

$$S'_\lambda(n) = \sum_{\substack{\omega \in \mathcal{S}_{2n} \\ \text{coset-type}(\omega) = \lambda \cup (\mathbf{1}^{n-|\lambda|)}}} \omega.$$

- For λ and δ two proper partitions with size at most n ,

$$S'_\lambda(n) \cdot S'_\delta(n) = \sum_{\substack{\rho \text{ proper partition} \\ |\rho| \leq n}} c'^{\rho}_{\lambda, \delta}(n) S'_\rho(n),$$

the numbers $c'^{\rho}_{\lambda, \delta}(n)$ are the **structure coefficients** of the Hecke algebra of $(\mathcal{S}_{2n}, \mathcal{B}_n)$.

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- **Motivation (Goulden and Jackson [1996])**: These coefficients count the number of embeddings of certain graphs into non-orientable surfaces.

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- Theorem (Dołęga and Féray [2012], T. [2013]):** Let λ , δ and ρ be three proper partitions, we have:

$$c'_{\lambda, \delta}{}^\rho(n) = \begin{cases} 2^n n! f_{\lambda, \delta}^\rho(n) & \text{if } n \geq |\rho|, \\ 0 & \text{if } n < |\rho|, \end{cases}$$

where $f_{\lambda, \delta}^\rho(n)$ is a polynomial in n .

Example: For every $n \geq 4$, we have:

$$S'_{(2)}(n) \cdot S'_{(2)}(n) = 2^n n! (n(n-1) S'_{\emptyset}(n) + 1 S'_{(2)}(n) + 3 S'_{(3)}(n) + 2 S'_{(2^2)}(n)).$$

III. Three polynomiality results
3. Center of $\mathbb{C}[GL_n(\mathbb{F}_q)]$

- Let q be a power of prime and F_q be the finite field of size q . A **polypartition** $\hat{\mu} = (\mu(P_1), \dots, \mu(P_r))$ over \mathbb{F}_q is a family of partitions indexed by monic irreducible polynomials of \mathbb{F}_q , all different from X . The size of $\hat{\mu}$ is:

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- A polypartition is called **proper** if its partition associated to the irreducible polynomial $X - 1$ is proper.
Example: $\hat{\mu} = (X - 1 : (2), X^2 - 2 : (1))$ is a proper polypartition of size 4 over \mathbb{F}_3 .

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- Theorem (Méliot [2013]):** Fix q , the coefficients $c_{\hat{\lambda}, \hat{\delta}}^{\hat{\rho}}(n)$ are polynomials in q^n .

IV. Structure coefficients of the algebra of double classes

- Let $(G_n, K_n)_n$ be a sequence where G_n is a group and K_n is a sub-group of G_n for each n .

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- Let $\overline{x_i}^n$ be the sum of the elements in $\overline{x_i}^n$. The algebra of double classes, denoted $\mathbb{C}[K_n \backslash G_n / K_n]$, is the algebra with basis the elements $\overline{x_i}^n$.

Example: The Hecke algebra of $(\mathcal{S}_{2n}, \mathcal{B}_n)$, $\mathbb{C}[\mathcal{B}_n \backslash \mathcal{S}_{2n} / \mathcal{B}_n]$, is an algebra of double classes.

- The product $\overline{x_i}^n \cdot \overline{x_j}^n$ can be written as follows:

$$\overline{x_i}^n \cdot \overline{x_j}^n = \sum_{1 \leq r \leq l(n)} c_{i,j}^r(n) \overline{x_r}^n.$$

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- **There is no explicit formula for these coefficients.**
- **Goals:**
 1. The form of these structure coefficients under some conditions.
 2. Applications to the three specific cases: $\mathcal{Z}(\mathbb{C}[\mathcal{S}_n])$, the Hecke algebra of $(\mathcal{S}_{2n}, \mathcal{B}_n)$ and $\mathcal{Z}(\mathbb{C}[GL_n(\mathbb{F}_q)])$.

Under the following conditions:

1. For each integer $k \leq n$ there exists a subset K_n^k of K_n such that:

Hyp 1. K_n^k is isomorphic to K_{n-k} .

For $K_n = \mathcal{S}_n$, \mathcal{S}_n^k is the sub-group of permutations acting on the set $\{k+1, \dots, n\}$.

Hyp 2. For any two elements $x \in K_k$ and $y \in K_n^k$, we have: $x \cdot y = y \cdot x$.

If $x \in \mathcal{S}_k$ and $y \in \mathcal{S}_n^k$ then x and y act on disjoint sets, hence the commutativity.

Hyp 3. $K_{n+1}^k \cap K_n = K_n^k$ if $k \leq n$.

$\mathcal{S}_{n+1}^k \cap \mathcal{S}_n$ is the set of permutations which fix the elements $1, \dots, k$ and $n+1$, so it is \mathcal{S}_n^k .

Define

$$k(X) := \min_{\substack{k \\ X \cap G_k \neq \emptyset}} k$$

and say that $y \in K_n$ is (k_1, k_2) -minimal if $y \in K_m$ where

$$m = m_{k_1, k_2}(y) := k(K_n^{k_1} y K_n^{k_2}).$$

2. In addition to the hypotheses above, suppose that:

Hyp 4. for any element $z \in K_n$, we have, $K_{n+1}^{k_1} z K_{n+1}^{k_2} \cap K_n = K_n^{k_1} z K_n^{k_2}$.

Hyp 5. $m_{k_1, k_2}(x) \leq k_1 + k_2$.

Hyp 6. $y K_n^{k_1} y^{-1} \cap K_n^{k_2} = K_n^{m_{k_1, k_2}(x)}$, if y is (k_1, k_2) -minimal.

We ensure that these conditions are true in the case of \mathcal{S}_n but we do not have the time to show them explicitly here.

we have:

Theorem (T.): For $k_1 = k(\bar{x}_i^n)$, $k_2 = k(\bar{x}_j^n)$ and $k_3 = k(\bar{x}_r^n)$ there exists rational numbers $a_{i,j}^r(k)$ all independent of n such that:

$$c_{ij}^r(n) = \frac{|\bar{x}_i^n| |\bar{x}_j^n| |K_{n-k_1}| |K_{n-k_2}|}{|K_n| |\bar{x}_r^n|} \sum_{k_3 \leq k \leq \min(k_1+k_2, n)} \frac{a_{i,j}^r(k)}{|K_{n-k}|}$$

we have:

Theorem (T.): For $k_1 = k(\bar{x}_i^n)$, $k_2 = k(\bar{x}_j^n)$ and $k_3 = k(\bar{x}_r^n)$ there exists rational numbers $a_{i,j}^r(k)$ **all independent of n** such that:

$$c_{ij}^r(n) = \frac{|\bar{x}_i^n| |\bar{x}_j^n| |K_{n-k_1}| |K_{n-k_2}|}{|K_n| |\bar{x}_r^n|} \sum_{k_3 \leq k \leq \min(k_1+k_2, n)} \frac{a_{i,j}^r(k)}{|K_{n-k}|}$$

Application (The Hecke algebra of $(\mathcal{S}_{2n}, \mathcal{B}_n)$): Let λ be a proper partition of size at most n . The size of its associated double class $S'_\lambda(n)$ is:

$$|S'_\lambda(n)| = \frac{(2^n n!)^2}{z_{2\lambda} 2^{n-|\lambda|} (n - |\lambda|)!},$$

where, $z_\lambda = \prod_{i \geq 1} i^{m_i(\lambda)} m_i(\lambda)!$.

Let δ and ρ be two proper partitions with size at most n , we have:

$$c'_{\lambda, \delta}{}^\rho(n) = 2^n n! \frac{z_{2\rho}}{z_{2\lambda} z_{2\delta}} \sum_{|\rho| \leq k \leq |\lambda| + |\delta|} a_{\lambda\delta}^\rho(k) 2^{k-|\rho|} \frac{(n - |\rho|)!}{(n - k)!}. \quad \text{Polynomial!}$$

Relation between structure coefficients of centers and double-classes algebras:

- Let (G, \cdot) be a finite group and $(G^{opp}, *)$ its opposite group.

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- Two pairs (a, b) and (c, d) are in the same double-class if and only if ab is conjugated with cd .

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- Let $C'_f = \{(a, b) \mid ab \in C_f\}$ be the double-class associated to $f \in G$ and \mathbf{C}'_f the sum of its elements. We have:

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- The structure coefficients of $\mathcal{Z}(\mathbb{C}[G])$ and the structure coefficients of the algebra of double classes of $(G \times G^{opp}, \text{diag}(G))$ are related by:

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- Theorem (T.):** Let $(G_n)_n$ be a sequence of finite groups such that its associated sequence of pairs $(G_n \times G_n^{opp}, \text{diag}(G_n))_n$ satisfying all necessary conditions given above. Let f, h and g be three elements of G_n and let $k_1 = k(C_f(n))$, $k_2 = k(C_h(n))$ and $k_3 = k(C_g(n))$. There exists rational numbers $a_{f,h}^g(k)$ independent of n such that:

$$c_{fh}^g(n) = \frac{|C_f(n)||C_h(n)||G_{n-k_1}||G_{n-k_2}|}{|G_n||C_g(n)|} \sum_{k_3 \leq k \leq \min(k_1+k_2, n)} \frac{a_{f,h}^g(k)}{|G_{n-k}|}.$$

Application to $\mathcal{Z}(\mathbb{C}[\mathcal{S}_n])$: Let λ be a proper partition of size at most n . The size of its associated conjugacy class $S_\lambda(n)$ is:

$$|S_\lambda(n)| = \frac{n!}{z_\lambda \cdot (n - |\lambda|)!}.$$

Let δ and ρ be two proper partitions with size at most n , we have:

$$c_{\lambda,\delta}^\rho(n) = \frac{z_\rho}{z_\lambda z_\delta} \sum_{|\rho| \leq k \leq |\lambda| + |\delta|} a_{\lambda\delta}^\rho(k) \frac{(n - |\rho|)!}{(n - k)!}. \quad \text{Polynomial!}$$

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Expected application to $\mathcal{Z}(\mathbb{C}[GL_n(\mathbb{F}_q)])$: Let $\hat{\lambda}$ be a proper polypartition of size at most n . The size of its associated conjugacy class $S_{\hat{\lambda}}(n)$ is:

$$|S_{\hat{\lambda}}(n)| = |S_{\hat{\lambda}}| \frac{|GL_n(\mathbb{F}_q)|}{|GL_{|\hat{\lambda}}(\mathbb{F}_q)| q^{(n - |\hat{\lambda}|)(2I(\lambda(X-1)))} |GL_{n - |\hat{\lambda}}(\mathbb{F}_q)|}.$$

Let $\hat{\delta}$ and $\hat{\rho}$ be two proper polypartitions with size at most n , we have:

$$\hat{c}_{\hat{\lambda}, \hat{\delta}}^{\hat{\rho}}(n) = \sum_{|\hat{\rho}| \leq k \leq |\hat{\lambda}| + |\hat{\delta}|} a_{\hat{\lambda} \hat{\delta}}^{\hat{\rho}}(k) q^{2n(k - |\hat{\rho}| + I(\rho(X-1)) - I(\lambda(X-1)) - I(\delta(X-1)))} \frac{(q^{-1})_{n - |\hat{\rho}|}}{(q^{-1})_{n - k}}. \quad \text{Polynomial!}$$

Proof's elements:

- 1- A **partial element** of G_n is a triplet $(hK_n^k, (x; k), K_n^k h')$, where $1 \leq k \leq n$, $x \in G_k$ and $h, h' \in K_n$.

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$$(h_1^i K_n^m, (x_1 h x_2; m), K_n^m h_1^{j'}), \text{ where, } m = \max(k_1, k_2, k(K_n^{k_1} h'_1 h_2 K_n^{k_2})),$$

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- 4- A projection from the algebra of partial elements to $\mathbb{C}[\mathcal{B}_n \setminus \mathcal{S}_{2n} / \mathcal{B}_n]$.
-

V. Conclusions and further applications

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Under technical conditions,

1. Form of the structure coefficients of double-class algebras.
2. Form of the structure coefficients of centers of group algebras.
3. We re-obtain the polynomiality property for $\mathcal{Z}(\mathbb{C}[\mathcal{S}_n])$, the Hecke algebra of $(\mathcal{S}_{2n}, \mathcal{B}_n)$ and $\mathcal{Z}(\mathbb{C}[GL_n(\mathbb{F}_q)])$.

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Further application:

- Superclasses of unitriangular groups...