

Symétrie et unimodalité de polynômes de descente via combinatoire de fractions continues

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Eulerian polynomials

The eulerian polynomials $A_n(t)$ can be defined by

$$\sum_{n=0}^{\infty} A_n(t) \frac{x^n}{n!} = \frac{1-t}{e^{(t-1)x} - t}.$$

The first values of $A_n(t) := a_{n,0} + a_{n,1}t + \cdots + a_{n,n-1}t^{n-1}$:

$$A_1(t) = 1,$$

$$A_2(t) = 1 + t,$$

$$A_3(t) = 1 + 4t + t^2,$$

$$A_4(t) = 1 + 11t + 11t^2 + t^3,$$

$$A_5(t) = 1 + 26t + 66t^2 + 26t^3 + t^4.$$

Unimodal and symmetric property

A polynomial $p(x) = a_l x^l + \cdots + a_n x^n$ is **symmetric** if $a_{l+i} = a_{n-i}$ for $0 \leq i \leq (n-l)/2$, **unimodal** if there is a k such that

$$a_l \leq a_{l+1} \leq \cdots a_k \geq a_{k+1} \geq \cdots \geq a_n.$$

Proposition

The polynomial $A_n(t)$ has only **real roots**. Hence $A_n(t)$ is both **unimodal** and **symmetric**. Moreover

$$A_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{des } \sigma} = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{exc } \sigma},$$

where \mathfrak{S}_n is the set of permutations on $\{1, \dots, n\}$ and

$$\text{des } \sigma = \#\{i \in [n-1] \mid \sigma(i) > \sigma(i+1)\},$$

$$\text{exc } \sigma = \#\{i \in [n] \mid \sigma(i) > i\}.$$

Some well-known facts

Let $p(x) = \sum_{k=0}^n a_k x^k$ be a polynomial with **nonnegative coefficients** and with **only real zeros**. Then

- (a) The sequence (a_0, \dots, a_n) is **log-concave**: $a_i^2 \geq a_{i-1} a_{i+1}$.
- (b) The sequence (a_0, \dots, a_n) is **unimodal**.
- (c) Furthermore, if $p(x) = \sum_{k=0}^n a_k x^k$ is symmetric, then

$$p(x) = \sum_{i=0}^{n/2} b_i x^i (1+x)^{n-2i} \quad \text{with } b_i \geq 0.$$

N.B. (a) \implies (b) and (c) \implies (b).

Foata-Schützenberger, Foata-Strehl in 1970's

Let $\sigma = \sigma_1 \dots \sigma_n \in \mathfrak{S}_n$ with $\sigma(0) = \sigma(n+1) = 0$. An integer $i \in [1, n]$ is

- a **peak** in σ if $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$;
- a **double descent** if $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$.

Proposition

Let $a_{n,k}$ be the number of permutations of $[n]$ with k peaks and without double descent. Then

$$A_n(t) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{n,k} t^k (1+t)^{n-1-2k}. \quad (1)$$

Example: If $n = 4$, $k = 2$, then $a_{n,k} = 8$:

2134, 3124, 4123, 1423, 2413, 3412, 1324, 2314

Background

Let (P, \prec) be a partial order of the set $[n] = \{1, 2, \dots, n\}$ and $\mathcal{L}(P)$ denote the set of linear extensions of P , i.e. permutations $w = (w_1, \dots, w_n)$ of $[n]$ such that if $i \prec j$ then i precedes j in the total order also. For each particular linear extension w , let $d(w) = |\{i: w_i > w_{i+1}\}|$ denote the number of so-called descents in P .

Example: if P is an **antichain**, then $W(P, t)$ is an **Eulerian polynomial**, and has real zeros only.

Conjecture (The Poset or Neggers-Stanley conjecture)

The polynomial $W_P(t) = \sum_{w \in \mathcal{L}(P)} t^{d(w)+1}$ has real zeros only.

This conjecture was **disproved** by **Brändén** for general posets in 2004, and by **Stembridge** for naturally labelled posets in 2007.

Conjecture (consequences of the POSET conjecture, **still open**)

- (1) $W_P(t) = \sum_{w \in \mathcal{L}(P)} t^{d(w)+1}$ is log-concave.
- (2) $W_P(t) = \sum_{w \in \mathcal{L}(P)} t^{d(w)+1}$ is unimodal.
- (3) If $W_P(t) = \sum_{w \in \mathcal{L}(P)} t^{d(w)+1}$ is symmetric of center $d/2$, then there are $w_{n,k} \geq 0$ such that

$$W_P(t) = \sum_{k=0}^{d/2} w_{n,k} t^k (1+t)^{d-2k}.$$

The descent polynomial of involutions

Let \mathcal{I}_n be the set of involutions on $[n]$ and consider

$$H_n(t) := \sum_{\sigma \in \mathcal{I}_n} t^{\text{des } \sigma} = h_{n,0} + h_{n,1}t + \cdots + h_{n,n-1}t^{n-1}.$$

- **Strehl** (1981) proved **Dumont's** conjecture: $\{h_{n,k}\}$ is **symmetric**.
- **Brenti** (2004) conjectured that $\{h_{n,k}\}$ is **log-concave**.
- **Guo-Z.** (2006) proved that $\{h_{n,k}\}$ is **unimodal**.
- **Barnabei-Bonetti-Silimbani** (2009) disproved Brenti's conjecture.

Conjecture (Guo-Z., 2006, **still open**)

$$H_n(t) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{n,k} t^k (1+t)^{n-1-2k} \quad a_{n,k} \in \mathbb{N}.$$

Problems related to continued fractions

Let $\sigma = \sigma(1) \dots \sigma(n) \in \mathfrak{S}_n$, define the statistics

$$(2-31) \sigma = \#\{(i, j) \mid 1 \leq i < j \leq n-1, \sigma(j+1) < \sigma(i) < \sigma(j)\},$$

$$(13-2) \sigma = \#\{(i, j) \mid 2 \leq i < j \leq n, \sigma(i-1) < \sigma(j) < \sigma(i)\}.$$

In 2008, [Brändén](#) (after the works of [Postnikov](#), [Williams](#) and [Corteel](#)) considered the refined Eulerian polynomials:

$$A_n(p, q, t) := \sum_{\sigma \in \mathfrak{S}_n} p^{(13-2)\sigma} q^{(2-31)\sigma} t^{\text{des } \sigma}$$

and proved, **modifying the Foata-Strehl action**, the identity:

$$A_n(p, q, t) := \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{n,k}(p, q) t^k (1+t)^{n-1-2k},$$

where $2^{n-1-2k} a_{n,k}(p, q) \in \mathbb{N}[p, q]$.

Conjecture (Brändén)

The polynomial $a_{n,k}(p, q)$ has a factor $(p + q)^k$ for all $0 \leq k \leq \lfloor (n - 1)/2 \rfloor$.

The first values of $a_{n,k}(p, q)$ are given by $a_{n,0}(p, q) = 1$ for $1 \leq n \leq 5$ and

$n \setminus k$	0	1	2
1	1		
2	1		
3	1	$p + q$	
4	1	$(p + q)(p + q + 2)$	
5	1	$(p + q)[(p + q)^2 + 2(p + q) + 3]$	$(p + q)^2(p^2 + pq + q^2 + 1)$

Derangement analogue

Let \mathcal{D}_n be the set of *derangements* in \mathfrak{S}_n . Consider the derangement analogue of the Eulerian polynomials:

$$B_n(t) = \sum_{\sigma \in \mathcal{D}_n} t^{\text{exc } \sigma} = B_{n,1}t + B_{n,2}t^2 + \cdots + B_{n,n-1}t^{n-1}.$$

Then

$$\sum_{n=0}^{\infty} B_n(t) \frac{x^n}{n!} = \frac{1-t}{e^{tx} - te^x}.$$

By analytical method, one can show

$$B_n(t) = \sum_{i=1}^{\lfloor n/2 \rfloor} b_{n,k} t^k (1+t)^{n-2k} \quad b_{n,k} \in \mathbb{N}. \quad (2)$$

No **combinatorial interpretation** for $b_{n,k}$ seems to be known hitherto.

(p, q) -analogue of the (-1) -evaluation of $A_n(t)$ and $B_n(t)$

The Euler numbers E_n are defined by

$$\sum_{n \geq 0} E_n \frac{x^n}{n!} = \tan x + \sec x.$$

The Euler number E_n counts the **alternating permutations** in \mathfrak{S}_n . It is well known that

$$\begin{aligned} A_{2n}(-1) &= 0, & A_{2n+1}(-1) &= (-1)^n E_{2n+1}, \\ B_{2n}(-1) &= (-1)^n E_{2n}, & B_{2n+1}(-1) &= 0, \end{aligned} \quad (3)$$

It follows that $a_{2n+1,n} = E_{2n+1}$ and $b_{2n,n} = E_{2n}$. **Some q -analogues** of (3) have been recently studied by **Foata-Han, Josuat-Vergès**.

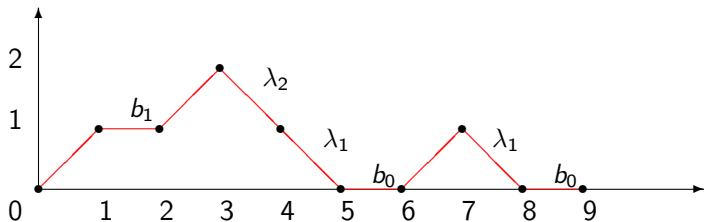
Two continued fractions of Euler, Rogers

$$\sum_{n=0}^{\infty} (n+1)!x^n = \frac{1}{1 - 2x - \frac{1 \cdot 2x^2}{1 - 4x - \frac{2 \cdot 3x^2}{\dots}}},$$

$$\sum_{n=0}^{\infty} n!x^n = \frac{1}{1 - x - \frac{1^2x^2}{1 - 3x - \frac{2^2x^2}{\dots}}}.$$

Combinatorial version by Flajolet, Françon-Viennot, Foata-Zeilberger, Biane

Consider the following Motzkin path γ :



The weight is $w(\gamma) = b_0^2 b_1 \lambda_1^2 \lambda_2$. Denote by \mathfrak{M}_n the set of Motzkin paths of length $n \geq 1$. Then

$$1 + \sum_{n \geq 1} \sum_{\gamma \in \mathfrak{M}_n} w(\gamma) x^n = \frac{1}{1 - b_0 x - \frac{1}{1 - b_1 x - \frac{\lambda_2 x^2}{\dots}}}. \quad (4)$$

Laguerre history: $h = (\gamma; p_1, \dots, p_n)$ with respect to the valuation

$$a_k = k + 1, \quad b_k = 2k + 2 \quad \text{for } k \geq 0; \quad c_k = k + 1 \quad \text{for } k \geq 1.$$

or $\lambda_k := a_{k-1}c_k = k(k + 1)$ for $k \geq 1$. The number of Laguerre histories of length n is $(n + 1)!$.

Restricted Laguerre history: $h = (\gamma; p_1, \dots, p_n)$ with respect to the valuation

$$a_k = k + 1, \quad b_k = 2k + 1 \quad \text{for } k \geq 0; \quad c_k = k \quad \text{for } k \geq 1.$$

or $\lambda_k := a_{k-1}c_k = k^2$ for $k \geq 1$. The number of Laguerre histories of length n is $n!$.

Theorem

There are *Bijections between \mathfrak{S}_n and the Laguerre histories of length n .*

Definition

For $\sigma \in \mathfrak{S}_n$, let $\sigma(0) = \sigma(n+1) = 0$. Then any entry $\sigma(i)$ ($i \in [n]$) can be classified according to one of the four cases:

- a *peak* if $\sigma(i-1) < \sigma(i)$ and $\sigma(i) > \sigma(i+1)$;
- a *valley* if $\sigma(i-1) > \sigma(i)$ and $\sigma(i) < \sigma(i+1)$;
- a *double ascent* if $\sigma(i-1) < \sigma(i)$ and $\sigma(i) < \sigma(i+1)$;
- a *double descent* if $\sigma(i-1) > \sigma(i)$ and $\sigma(i) > \sigma(i+1)$.

Let $\text{peak}^* \sigma$ (resp. $\text{valley}^* \sigma$, $\text{da}^* \sigma$, $\text{dd}^* \sigma$) denote the number of peaks (resp. valleys, double ascents, double descents) in σ . Clearly we have $\text{peak}^* \sigma = \text{valley}^* \sigma + 1$.

The generalized Eulerian polynomial defined by

$$A_n(p, q, t, u, v, w) = \sum_{\sigma \in \mathfrak{S}_n} p^{(2-13)\sigma} q^{(31-2)\sigma} t^{\text{des}\sigma} u^{\text{da}^*\sigma} v^{\text{dd}^*\sigma} w^{\text{valley}^*\sigma}.$$

Define the polynomial

$$a_{n,k}(p, q) = \sum_{\sigma \in \mathfrak{S}_{n,k}} p^{(2-13)\sigma} q^{(31-2)\sigma},$$

where $\mathfrak{S}_{n,k}$ is the subset of permutations $\sigma \in \mathfrak{S}_n$ with exactly k valleys and without double descents.

Theorem (Shin-Z.)

We have the expansion formula

$$A_n(p, q, t, u, v, w) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{n,k}(p, q)(tw)^k(u + vt)^{n-1-2k}. \quad (5)$$

Moreover, for all $0 \leq k \leq \lfloor (n-1)/2 \rfloor$, the following divisibility holds

$$(p + q)^k \mid a_{n,k}(p, q). \quad (6)$$

The above Theorem generalizes Brändén's result and (6) solves his conjecture.

Definition

For $\sigma \in \mathfrak{S}_n$, a value $x = \sigma(i)$ ($i \in [n]$) is called

- a *cyclic peak* if $i = \sigma^{-1}(x) < x$ and $x > \sigma(x)$;
- a *cyclic valley* if $i = \sigma^{-1}(x) > x$ and $x < \sigma(x)$;
- a *double excedance* if $i = \sigma^{-1}(x) < x$ and $x < \sigma(x)$;
- a *double drop* if $i = \sigma^{-1}(x) > x$ and $x > \sigma(x)$;
- a *fixed point* if $x = \sigma(x)$.

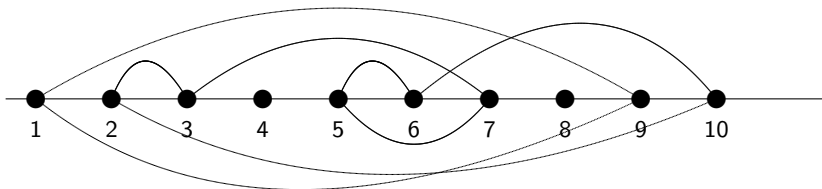
Let $c_{\text{peak}} \sigma$ (resp. $c_{\text{valley}} \sigma$, $c_{\text{da}} \sigma$, $c_{\text{dd}} \sigma$, $\text{fix} \sigma$) be the number of cyclic peaks (resp. valleys, double excedances, double drops, fixed points) in σ .

For $\sigma \in \mathfrak{S}_n$ define the crossing and nesting numbers by

$$\text{cros}(\sigma) = \sum_{i=1}^n \#\{j \in [n] \mid j < i \leq \sigma(j) < \sigma(i) \text{ or } j > i > \sigma(j) > \sigma(i)\},$$

$$\text{nest}(\sigma) = \sum_{i=1}^n \#\{j \in [n] \mid i < j \leq \sigma(j) < \sigma(i) \text{ or } i > j > \sigma(j) > \sigma(i)\},$$

Diagram of $\sigma = 9\ 3\ 7\ 4\ 6\ 10\ 5\ 8\ 1\ 2$: $\text{cros } \sigma = 4$ and $\text{nest } \sigma = 7$.



Definition

For $\sigma \in \mathfrak{S}_n$, let $\sigma(0) = 0$ and $\sigma(n+1) = n+1$. The corresponding number of peaks, valleys, double ascents, and double descents of permutation σ is denoted by

$$\text{peak } \sigma, \quad \text{valley } \sigma, \quad \text{da } \sigma, \quad \text{dd } \sigma.$$

Moreover, a **double ascent** $\sigma(i)$ of σ ($i \in [n]$) is said to be a **foremaximum** if $\sigma(i)$ is a *left-to-right maximum* of σ , i.e., $\sigma(j) < \sigma(i)$ for all $1 \leq i < j$. Denote the number of foremaxima of σ by $\text{fmax } \sigma$.

For instance, $\text{fmax}(42157368) = 2$. Note that $\text{peak } \sigma = \text{valley } \sigma$ for any $\sigma \in \mathfrak{S}_n$.

Theorem (Shin-Z.)

There is a bijection Φ on \mathfrak{S}_n such that for all $\sigma \in \mathfrak{S}_n$ we have

$$\begin{aligned} &(\text{nest, cros, drop, cda, cdd, cvalley, fix})\sigma \\ &= (2\text{-}31, 31\text{-}2, \text{des, da} - \text{fmax, dd, valley, fmax})\Phi(\sigma). \end{aligned}$$

In particular, Φ transforms (drop, nest, cros) to (des, 2-31, 31-2).

- Corteel gave a similar bijection Φ_C on \mathfrak{S}_n which transforms (ndes, 31-2, 2-31) to (wex, nest, cros),
- Steingrímsson-Williams gave another one which transforms (ndes, 31-2, 2-31) to $(n + 1 - \text{wex, nest, cros})$.

Note that $\text{wex} = n - \text{drop}$.

Corollary

We have

$$\sum_{\sigma \in \mathfrak{S}_n} p^{\text{nest } \sigma} q^{\text{cross } \sigma} t^{\text{wex } \sigma} = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{n,k}(p, q) t^{k+1} (1+t)^{n-1-2k}.$$

Consider the common enumerator polynomial

$$B_n(p, q, t, u, v, w, y)$$

$$:= \sum_{\sigma \in \mathfrak{S}_n} p^{\text{nest } \sigma} q^{\text{cross } \sigma} t^{\text{drop } \sigma} u^{\text{cda } \sigma} v^{\text{cdd } \sigma} w^{\text{cvalley } \sigma} y^{\text{fix } \sigma} \quad (7)$$

$$= \sum_{\sigma \in \mathfrak{S}_n} p^{(2-31) \sigma} q^{(31-2) \sigma} t^{\text{des } \sigma} u^{\text{da } \sigma} v^{\text{dd } \sigma} w^{\text{valley } \sigma} y^{\text{fmax } \sigma}. \quad (8)$$

When $u = 0$ and $t = v = 1$ we can write

$$B_n(p, q, 1, 0, 1, w, y) := \sum_{k, j \geq 0} b_{n, k, j}(p, q) w^k y^j, \quad (9)$$

where $b_{n, k, j}(p, q)$ is a polynomial in p and q with non negative integral coefficients.

Let

$$\mathfrak{S}_{n,k,j} = \{\sigma \in \mathfrak{S}_n : \text{cvalley } \sigma = k, \text{ fix } \sigma = j, \text{ cda } \sigma = 0\}$$

$$\mathfrak{S}_{n,k,j}^* = \{\sigma \in \mathfrak{S}_n : \text{valley } \sigma = k, \text{ fmax } \sigma = j, \text{ da } \sigma = j\}$$

We derive from the above Theorem the following result.

Corollary

We have

$$b_{n,k,d}(p, q) = \sum_{\sigma \in \mathfrak{S}_{n,k,j}} p^{\text{nest } \sigma} q^{\text{cros } \sigma} = \sum_{\sigma \in \mathfrak{S}_{n,k,j}^*} p^{(2-31)\sigma} q^{(31-2)\sigma}.$$

In particular, when $j = 0$, we obtain

$$b_{n,k,0}(p, q) = \sum_{\sigma \in \mathfrak{S}_{n,k,0}} p^{\text{nest } \sigma} q^{\text{cros } \sigma} = \sum_{\sigma \in \mathfrak{S}_{n,k,0}^*} p^{(2-31)\sigma} q^{(31-2)\sigma}.$$

Theorem

$$B_n(p, q, t, u, v, w, y) \\ = \sum_{j=0}^n y^j \sum_{k=0}^{\lfloor (n-d)/2 \rfloor} b_{n,k,j}(p, q) (tw)^k (qu + tv)^{n-j-2k}.$$

Setting $y = 0$ in the above Theorem we obtain the following (p, q) -analogue of (-1) -evaluation of Eulerian polynomials.

Corollary

$$\sum_{\sigma \in \mathfrak{D}_n} p^{\text{nest } \sigma} q^{\text{cros } \sigma} t^{\text{exc } \sigma} = \sum_{k=0}^{\lfloor n/2 \rfloor} b_{n,k,0}(p, q) t^k (1 + qt)^{n-2k}.$$

Let \mathfrak{A}_n be the set of **alternating permutations**: $\sigma_1 > \sigma_2 < \dots$

Theorem (Shin-Z.)

For $n \geq 1$, we have

$$\begin{aligned}
 & \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\text{wex } \sigma} p^{\text{nest } \sigma} q^{\text{cros } \sigma} \\
 &= \begin{cases} 0 & \text{if } n \text{ is even,} \\ (-1)^{\frac{n+1}{2}} \sum_{\sigma \in \mathfrak{A}_n} p^{(2-13)\sigma} q^{(31-2)\sigma} & \text{if } n \text{ is odd;} \end{cases} \\
 & \sum_{\sigma \in \mathfrak{D}_n} (-1/q)^{\text{exc } \sigma} p^{\text{nest } \sigma} q^{\text{cros } \sigma} \\
 &= \begin{cases} (-1/q)^{\frac{n}{2}} \sum_{\sigma \in \mathfrak{A}_n} p^{(2-31)\sigma} q^{(31-2)\sigma} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}
 \end{aligned}$$

The $p = 1$ case is due to Josuat-Vergès.

Sketch of the proof

$$\begin{aligned} & \sum_{n \geq 1} A_n(p, q, t, u, v, w) x^{n-1} \\ &= \frac{1}{1 - (u + tv)[1]_{p,q}x - \frac{[1]_{p,q}[2]_{p,q}twx^2}{1 - (u + tv)[2]_{p,q}x - \frac{[2]_{p,q}[3]_{p,q}twx^2}{\dots}}}. \end{aligned}$$

For $0 \leq k \leq \lfloor (n-1)/2 \rfloor$, let $a_{n,k}(p, q, t, u, v)$ be the coefficient of w^k in $A_n(p, q, t, u, v, w)$, i.e.,

$$A_n(p, q, t, u, v, w) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{n,k}(p, q, t, u, v) w^k. \quad (10)$$

Substituting $x \leftarrow \frac{x}{(u+tv)}$ and $w \leftarrow \frac{w(u+tv)^2}{t}$ in the above formula

$$\sum_{n \geq 1} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{a_{n,k}(p, q, t, u, v)}{t^k (u+tv)^{n-1-2k}} w^k x^{n-1}$$

$$= \frac{1}{1 - [1]_{p,q}x - \frac{[1]_{p,q}[2]_{p,q}wx^2}{1 - [2]_{p,q}x - \frac{[2]_{p,q}[3]_{p,q}wx^2}{1 - [3]_{p,q}x - \frac{[3]_{p,q}[4]_{p,q}wx^2}{\dots}}}}.$$

The coefficient of $w^k x^{n-1}$ in the left-hand side is a polynomial in p and q with nonnegative integral coefficients. If we denote this coefficient by

$$P_{n,k}(p, q) := \frac{a_{n,k}(p, q, t, u, v)}{t^k (u + tv)^{n-1-2k}},$$

then

$$P_{n,k}(p, q) = a_{n,k}(p, q, 1, 1, 0) = a_{n,k}(p, q, 0, 1) = a_{n,k}(p, q).$$

Finally, as

$$(p + q) \mid [n]_{p,q} [n + 1]_{p,q} \quad \text{for all } n \geq 1,$$

each w appears with a factor $(p + q)$ in the expansion of the continued fraction, so the polynomial $P_{n,k}(p, q)$ is divisible by $(p + q)^k$.

Merci !