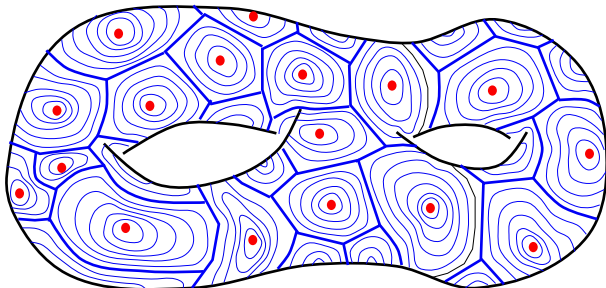


Combinatoire des graphes de Strebel

Bertrand Eynard, IPHT CEA Saclay,



Journées Combinatoires de Bordeaux, février 2014

1. Introduction

Quantum gravity

Idea: "2D quantum gravity" \Leftrightarrow "summing over surfaces"

$$\mathbb{E}(\text{"observable"}) = \int_{\mathcal{M}} d[S] e^{-\text{Energy}(S)} \text{observable}(S)$$

"Energy"=measure. "observable" can be "Area", "curvature", "geodesic distance between marked points", or anything ...

Claimed by physicists: \Leftrightarrow Liouville CFT (implies KdV, KPZ, GFF, SLE)

Ensemble: \mathcal{M} = "set of all Riemann surfaces", possibly of given topology, modulo automorphisms.

Given topology: $\mathcal{M} = \mathcal{M}_{g,n}$ = moduli space of Riemann surfaces of genus g with n marked points.

$\mathcal{M}_{g,n}$ is a complex manifold (orbifold) of dimension

$$\dim_{\mathbb{R}} \mathcal{M}_{g,n} = 2 \times (3g - 3 + n)$$

Un-marking points (forgetful maps):

$$\mathcal{M}_{g,n} \subset \mathcal{M}_{g,n+1} \subset \dots \subset \mathcal{M}_{g,n+k} \subset \dots$$



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Quantum gravity

Idea (in the 70's-90's): **Discretize**.

Family of discrete subsets $\mathbb{M}_{g,n} \subset \mathcal{M}_{g,n}$. $\mathbb{M}_{g,n}$ = set of maps of genus g with n marked faces.

Ex: $\mathbb{M}_{0,1}$ = set of rooted planar maps.

Forgetful maps $\mathbb{M}_{g,n} \subset \mathbb{M}_{g,n+1} \subset \dots \subset \mathbb{M}_{g,n+k} \subset \dots$

$$\mathbb{E}(\text{"observable"}) \stackrel{?}{=} \lim_{\text{continue } k \rightarrow \infty} \sum_{\mathbb{M}_{g,n+k}} e^{-\text{Energy}_k(S)} \text{observable}(S)$$

Remark on observables:

- some observables are "global" (ex: area) easily obtained by a derivative of the energy.
- some observables are "topological" independent of a metric (ex: average curvature, or Chern classes)
- some observables more subtle: geodesic distance between marked points. Need an "embedding" with a metric.

In the 80's: Witten's KdV conjecture

- maps \Leftrightarrow *formal* matrix models (BIPZ78)
 - matrix models \Leftrightarrow orthogonal polynomials (Dyson, Gaudin, Mehta,...60's). *True only for not formal.*
 - "double scaling limit" of orthogonal polynomials \Leftrightarrow KdV Tau-function (Kazakov,...80's) + KPZ scaling exponents. *Ansatz not proved before the 2000's.*
- led **Witten's to conjecture**:

Continuum limit of partition function of maps = KdV Tau-function

In 1992: M. Kontsevich proved that topological gravity = KdV:

$$\int_{\mathcal{M}_{g,n}} \text{Chern classes} = \mathcal{T}_{\text{KdV}}$$

Kontsevich's proof: *without using discretization* !

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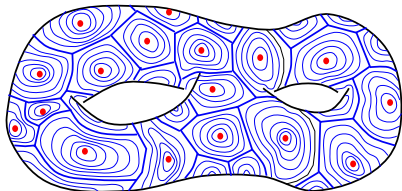
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Strebel Graphs

Kontsevich's method: Canonical bijection with **Strebel graphs**



Bijection : $\mathcal{M}_{g,n} \times \mathbb{R}_+^n \longleftrightarrow \bigoplus_{G \in \mathcal{G}_{g,n}} G \times \mathbb{R}_+^{\#\text{edges}}$

$\mathcal{G}_{g,n}$ = set of metric graphs of genus g , with n marked faces, and edge lengths ℓ_e .

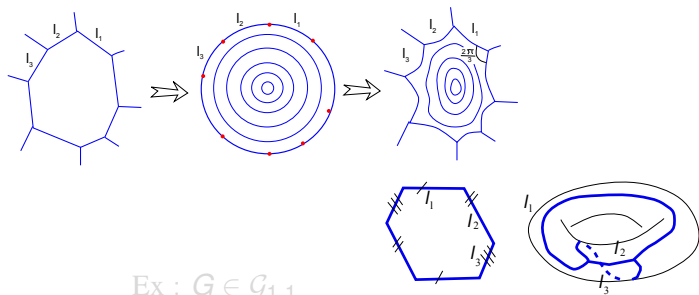
For each marked face L_f = perimeter: $L_f = \sum_{e \rightarrow f} \ell_e$.

Remark: for 3-valent graphs: $\#\text{edges} = 6g - 6 + 3n$.

Higher valent vertices obtained when some $\ell_e \rightarrow 0$.

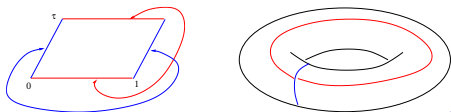
Strebel Graphs

Metric graph \mapsto Riemann surface



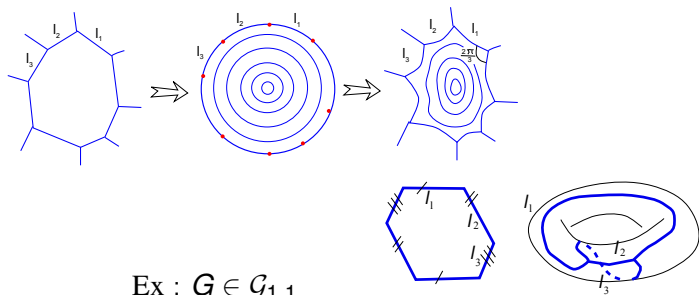
Perimeter $L = 2l_1 + 2l_2 + 2l_3$.

For fixed perimeter, an element of $\mathcal{M}_{1,1}$ is thus parametrized by $(l_1, l_2) \in \mathbb{R}_+^2$. Indeed, genus 1 Riemann surfaces are parametrized by a complex modulus $\tau \in \mathbb{C}$



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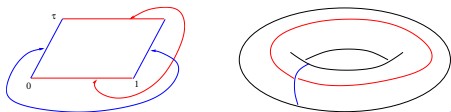
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Ex : $G \in \mathcal{G}_{1,1}$

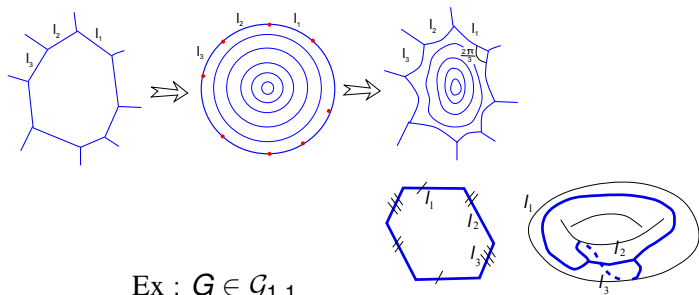
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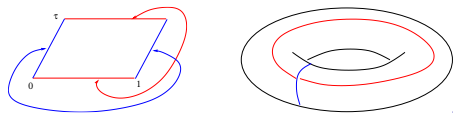
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Strebel's bijection

Riemann surface $(S, p_1, \dots, p_n) \mapsto$ Metric graph

Strebel's theorem: there exists a unique "metric" $\sqrt{|\Omega|^2}$ on S , singular (double poles) at the p_i 's, with residues $-L_i^2$:

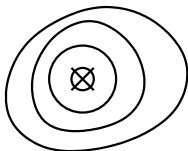
$$\Omega = \sum_{i=1}^n \frac{-L_i^2}{(x - p_i)^2} dx^2 + \text{at most simple poles}$$

such that all closed geodesics are singular.

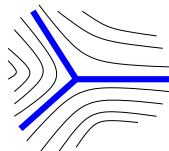
The lines $\text{Im } \Phi(x) = \text{constant}$ are geodesics:

$$\Phi(x) = \int^x \sqrt{\Omega}$$

near a pole p_i :



, near a zero :



Strebel's bijection

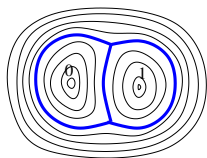
Example in $\mathcal{M}_{0,3}$.

Take the 3 marked points to be $(0, 1, \infty)$, chose 3 perimeter lengths L_0, L_1, L_∞ . Then the Strebel differential is:

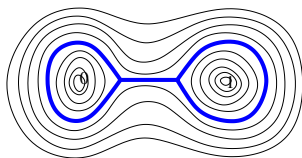
$$\Omega(x) = -\frac{L_\infty^2 x^2 - (L_\infty^2 + L_0^2 - L_1^2)x + L_0^2}{x^2(x-1)^2} dx^2 = -L_\infty^2 \frac{(x-a)(x-b)}{x^2(x-1)^2} dx^2$$

The vertices are located at the zeroes a, b of Ω :

$$a, b = \frac{1}{2L_\infty^2} \left(L_\infty^2 + L_0^2 - L_1^2 \pm \sqrt{L_0^4 + L_1^4 + L_\infty^4 - 2L_0^2 L_1^2 - 2L_0^2 L_\infty^2 - 2L_1^2 L_\infty^2} \right)$$



or



Remark: Ω =stress energy tensor of classical Liouville theory.

We thus have

$$\int \prod_i dL_i \int_{\mathcal{M}_{g,n}} \text{observable} = \sum_G \frac{1}{\#\text{Aut}(G)} \int_{\mathbb{R}_+^{\#\text{edges}}} \text{observable} \prod_e dl_e$$

Theorem Kontsevich 92: The measure $\prod_e dl_e$ is topological:

$$(3g-3+n)! 2^{5g-5+2n} \prod_e dl_e = \left(\sum_i L_i^2 c_1(T^*p_i) \right)^{3g-3+n} \prod_{i=1}^n dL_i$$

$\psi_i = c_1(T^*p_i)$ = Chern class of cotangent bundle at marked point p_i .

$$\psi_i = \sum_{e < e' \text{ around face } i} d \frac{l_e}{L_i} \wedge d \frac{l_{e'}}{L_i}$$

ψ_i is a 2-form on the cotangent space of $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$.

Laplace transform

Multiply by $e^{-\sum_{i=1}^n \lambda_i L_i}$ and integrate over ℓ_e .

$$e^{-\sum_i \lambda_i L_i} = e^{-\sum_{e=(e_+, e_-) \in \text{edges}} \ell_e (\lambda_{e_+} + \lambda_{e_-})}$$

For an observable polynomial in the ℓ_e 's:

$$\sum_G \int d\ell_e \prod_e \ell_e^{k_e} e^{-\sum_i \lambda_i L_i} = \sum_G \prod_{e=(e_+, e_-)} \frac{k_e!}{(\lambda_{e_+} + \lambda_{e_-})^{k_e+1}}$$

In principle, any measure $d\mu$ on $\mathcal{M}_{g,n}$ can be realized as a function of the edge lengths $f(\{\ell_e\}) \prod_e d\ell_e$, or in Laplace transform as a function of the λ_f 's.

In particular: discrete measures can be realized. Embedding $\mathbb{M}_{g,n} \subset \mathcal{M}_{g,n}$ with the discrete measure, can be realized by a suitable system of weights $w(G; \{\lambda_f\})$ on graphs $G \in \mathcal{G}_{g,n}$.

We shall consider the following set of graphs:

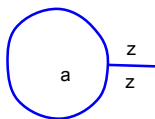
$\mathcal{G}_{g,n}$ = set of graphs of genus g , with n labeled marked faces.

- Marked face i carries a variable z_i .
- Unmarked faces carry a variable in a set $\{\lambda_1, \dots, \lambda_N\}$.
- an edge e between two faces e_+, e_- , has weight $\frac{1}{(\text{variable}(e_+) + \text{variable}(e_-))}$.
- 3-valent vertices: weight = 1.
- 1-valent vertices only for marked faces: weight = -1.

Weight of Feynman type (Wick's theorem type):

$$w(G) = N^{-\#\text{unmarked faces}} \prod_{e=(e_+,e_-) \in \text{edges}} \frac{1}{\lambda_{e_+} + \lambda_{e_-}} \prod_{v \in \text{vertices}} \text{weight}(v)$$

Example: $G \in \mathcal{G}_{0,1}$ with one marked face (exterior), and one unmarked face (interior):



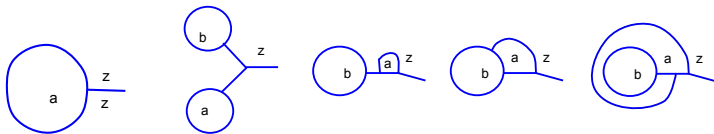
$$\rightarrow \text{weight} = \frac{-1}{N} \frac{1}{z+z} \frac{1}{z+\lambda_a}$$

Generating functions

Define :
$$W_{g,n}(z_1, \dots, z_n) = \sum_{G \in \mathcal{G}_{g,n}} w(G) - \delta_{g,0} \delta_{n,1} z_1$$

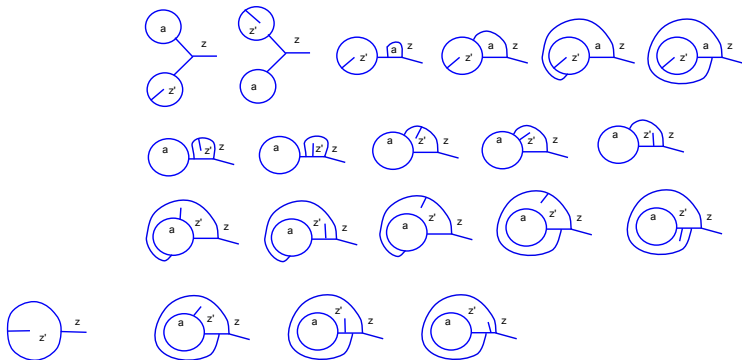
where we sum over graphs with n labeled marked faces, each having one 1-valent vertex, and with z_i in marked face i .

Example: graphs contributing to $W_{0,1}(z)$ with up to 5 edges:



$$W_{0,1}(z) = -z + \frac{1}{N} \sum_{a=1}^N \frac{-1}{2z(z + \lambda_a)} - \frac{1}{N^2} \sum_{a,b} \frac{1}{(2z)^3 (z + \lambda_a)(z + \lambda_b)} + \frac{1}{(2z)^2, (z + \lambda_a)^2 (z + \lambda_b)}$$

Other example: $W_{0,2}$: graphs with up to 7 edges

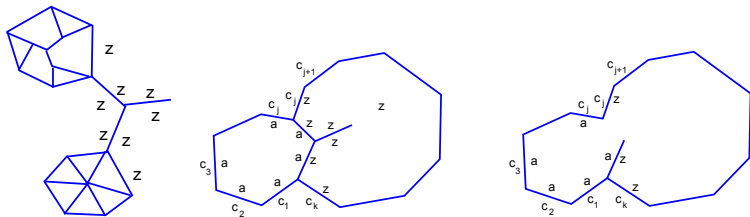


$$W_{0,2}(z, z') = \frac{1}{4zz'(z + z')^2} + 0 + \dots \quad (0.1)$$

Remark: huge cancellations.

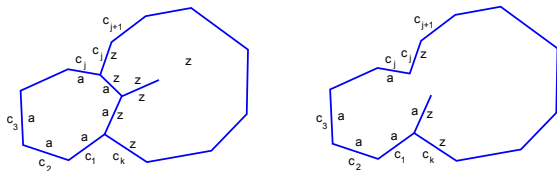
Tutte's recursion

Tutte's recursion = deleting edges.



- consider the 1-valent vertex of face 1,
- it is attached to an edge $\frac{1}{2z_1}$,
- which is attached to a 3-valent vertex.
- delete the right edge of the 3-valent vertex:
 - the face on the other side is the same face (z_1) and the graph gets disconnected, or $g \rightarrow g - 1$
 - the face on the other side is another face, marked or unmarked,

Tutte's recursion



Case: – the face on the other side is another face, marked or unmarked,

Lemma: if $F(z) = \prod_{i=1}^k \frac{1}{z+c_i}$, then

$$\frac{F(z) - F(a)}{z - a} = \sum_{j=1}^k \prod_{i=1}^j \frac{1}{z + c_i} \prod_{i=j}^k \frac{1}{a + c_i}$$

Tutte's recursion

It follows:

Theorem (Tutte's recursion formula):

$$\begin{aligned} & 2z(W_{g,n}(z, z_2, \dots, z_n) + z\delta_{g,0}\delta_{n,1}) \\ = & W_{g-1,n+1}(z, z, z_2, \dots, z_n) \\ & + \sum_{h+h'=g} \sum_{l \uplus l' = \{z_2, \dots, z_n\}} W_{h,1+|l|}(z, l) W_{h',1+|l'|}(z, l') \\ & + \frac{1}{N} \sum_{a=1}^N \frac{W_{g,n}(z, z_2, \dots, z_n) - W_{g,n}(\lambda_a, z_2, \dots, z_n)}{z^2 - \lambda_a^2} \\ & + \sum_{j=2}^n \frac{1}{2z_j} \frac{d}{dz_j} \frac{W_{g,n-1}(z, z_2, \dots, \hat{z}_j, \dots, z_n) - W_{g,n-1}(z_2, \dots, z_n)}{z^2 - z_j^2} \end{aligned}$$

Disc amplitude

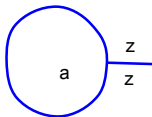
Planar maps with 1 marked face (rooted planar map):

$$2z(W_{0,1}(z) + z) = (W_{0,1}(z) + z)^2 + \frac{1}{N} \sum_{a=1}^N \frac{W_{0,1}(z) - W_{0,1}(\lambda_a)}{z^2 - \lambda_a^2}$$

Theorem: Solution (we assume $t_1 = \frac{1}{N} \sum_{a=1}^N \frac{1}{\lambda_a} = 0$):

$$W_{0,1}(z) = -z - \frac{1}{N} \sum_{a=1}^N \frac{1}{2z(z + \lambda_a)}$$

i.e. only these graphs contribute:



Annulus amplitude

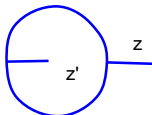
Planar maps with 2 marked faces:

$$\begin{aligned}2z W_{0,2}(z, z') &= 2 W_{0,1}(z) W_{0,2}(z, z') \\ &+ \frac{1}{N} \sum_{a=1}^N \frac{W_{0,2}(z, z') - W_{0,1}(\lambda_a, z')}{z^2 - \lambda_a^2} \\ &+ \frac{1}{2z'} \frac{d}{dz'} \frac{W_{0,1}(z) - W_{0,1}(z')}{z^2 - z'^2}\end{aligned}$$

Theorem: Solution (we assume $t_1 = \sum_{a=1}^N \frac{1}{\lambda_a} = 0$):

$$W_{0,2}(z, z') = \frac{1}{4zz'(z+z')^2}$$

i.e. only this graph contributes:



Planar maps with 3 marked faces:

$$\begin{aligned} 2z W_{0,3}(z_1, z_2, z_3) &= 2 W_{0,1}(z_1) W_{0,3}(z_1, z_2, z_3) + 2 W_{0,2}(z_1, z_2) W_{0,2}(z_1, z_2, z_3) \\ &+ \frac{1}{N} \sum_{a=1}^N \frac{W_{0,3}(z_1, z_2, z_3) - W_{0,3}(\lambda_a, z_2, z_3)}{z_1^2 - \lambda_a^2} \\ &+ \frac{1}{2z_2} \frac{d}{dz_2} \frac{W_{0,2}(z_1, z_3) - W_{0,2}(z_2, z_3)}{z_1^2 - z_2^2} \\ &+ \frac{1}{2z_3} \frac{d}{dz_3} \frac{W_{0,2}(z_1, z_2) - W_{0,2}(z_2, z_3)}{z_1^2 - z_3^2} \end{aligned}$$

Theorem: Solution (we assume $t_1 = \frac{1}{N} \sum_{a=1}^N \frac{1}{\lambda_a} = 0$):

$$W_{0,3}(z_1, z_2, z_3) = \frac{1}{8(2 - t_3) z_1^3 z_2^3 z_3^3}$$

where $t_k = \frac{1}{N} \sum_{a=1}^N \lambda_a^{-k}$.

Genus 1 maps with 1 marked face:

$$2z W_{1,1}(z) = 2 W_{0,1}(z) W_{1,1}(z) + W_{0,2}(z, z) + \frac{1}{N} \sum_{a=1}^N \frac{W_{1,1}(z) - W_{1,1}(\lambda_a)}{z^2 - \lambda_a^2}$$

Theorem: Solution (we assume $t_1 = \frac{1}{N} \sum_{a=1}^N \frac{1}{\lambda_a} = 0$):

$$W_{1,1}(z) = \frac{1}{16(2-t_3)z^5} + \frac{t_5}{16(2-t_3)^2 z^3}$$

where $t_k = \frac{1}{N} \sum_{a=1}^N \lambda_a^{-k}$.

Topological recursion

From Tutte's equation, one proves by recursion ($W_{g,n}$ = rational with poles at $z_i = 0$, expand at 0 instead of ∞):

Theorem (the $W_{g,n}$'s satisfy the topological recursion (EO07))

$$\begin{aligned} & W_{g,n}(z_1, z_2, \dots, z_n) \\ = & \operatorname{Res}_{z \rightarrow 0} K(z_1, z) \left[W_{g-1, n+1}(z, z, z_2, \dots, z_n) \right. \\ & \left. + \sum_{\substack{l+l'=g \\ l \uplus l' = \{z_2, \dots, z_n\}}} W_{h, 1+|l|}(z, l) W_{h', 1+|l'|}(z, l') \right] dz \end{aligned}$$

where
$$K(z_1, z) = \frac{z}{z_1} \frac{1}{z_1^2 - z^2} \left(2 - \sum_{k=0}^{\infty} t_{2k+3} z^{2k} \right)^{-1}, \quad t_k = \frac{1}{N} \sum_{a=1}^N \lambda_a^{-k}$$

This theorem computes recursively $W_{g,n}$ in terms of $W_{g',n'}$ with $2g' + n' - 2 < 2g + n - 2$.

Example of computation:

$$\begin{aligned}W_{1,1}(z_1) &= \operatorname{Res}_{z \rightarrow 0} K(z_1, z) W_{0,2}(z, z) dz \\&= \operatorname{Res}_{z \rightarrow 0} \frac{z dz}{z_1} \frac{1}{z_1^2 - z^2} (2 - t_3 - t_5 z^2 + \dots)^{-1} \frac{1}{4z^2(z+z)^2} \\&= \frac{1}{16(2-t_3)z_1^3} \operatorname{Res}_{z \rightarrow 0} \frac{dz}{z^3} \left(1 + \frac{z^2}{z_1^2} + O(z^4)\right) \left(1 + \frac{t_5}{2-t_3} z^2\right) \\&= \frac{1}{16(2-t_3)z_1^3} \left(\frac{1}{z_1^2} + \frac{t_5}{(2-t_3)}\right)\end{aligned}$$

→ agrees with known result 

Topological recursion and Virasoro constraints

By recursion on g and n , we find:

Lemma

$$W_{g,n} = \text{Polynomial} \in \mathbb{Q}[1/z_i, 1/(2 - t_3), t_k]$$

where $t_k = \frac{1}{N} \sum_a \lambda_a^{-k}$.

Instead of t_k 's, we prefer to use their Schur transforms \hat{t}_k 's:

$$\text{Def : } \sum_{k \geq 0} \hat{t}_k u^{-k} = -\ln \left(2 - \sum_{k \geq 0} (2k + 1)!! t_{2k+3} u^{-k} \right)$$

For example we have

$$\hat{t}_0 = -\ln(2 - t_3), \hat{t}_1 = \frac{3t_5}{2-t_3}, \hat{t}_2 = \frac{15t_7}{2-t_3} + \frac{9t_5^2}{2(2-t_3)^2}, \text{ etc...}$$

Topological recursion and Virasoro constraints

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Lemma

$$W_{g,n} = \text{Polynomial} \in \mathbb{Q}[1/z_i, 1/(2-t_3), t_k]$$

$$\text{where } t_k = \frac{1}{N} \sum_a \lambda_a^{-k}.$$

Instead of t_k 's, we prefer to use their Schur transforms \hat{t}_k 's:

$$\text{Def : } \sum_{k \geq 0} \hat{t}_k u^{-k} = -\ln \left(2 - \sum_{k \geq 0} (2k+1)!! t_{2k+3} u^{-k} \right)$$

For example we have

$$\hat{t}_0 = -\ln(2-t_3), \hat{t}_1 = \frac{3t_5}{2-t_3}, \hat{t}_2 = \frac{15t_7}{2-t_3} + \frac{9t_5^2}{2(2-t_3)^2}, \text{ etc...}$$

Lemma

$$W_{g,n} = \frac{(2 - t_3)^{2-2g-n}}{\prod_i 2z_i^3} \text{Polynomial} \in \mathbb{Q}[1/z_i^2, \{\hat{t}_k\}_{k \geq 1}]^{\text{deg}=3g-3+n}$$

(homogeneity $\text{deg}(1/z_i^2) = 1$, $\text{deg}(\hat{t}_k) = k$)

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$$W_{g,n} = \frac{(2 - t_3)^{2-2g-n}}{\prod_i 2z_i} \sum_{d_1 + \dots + d_n + \sum_{k \geq 1} kn_k = 3g-3+n} \prod_i \frac{(2d_i + 1)!!}{z_i^{2d_i+2}} \prod_{k \geq 1} \frac{\hat{t}_k^{n_k}}{n_k!} C_{g,n}(d_1, \dots, d_n; n_1, n_2, n_3, \dots)$$

Witten's Notation (intersection numbers):

$$C_{g,n}(d_1, \dots, d_n; n_1, \dots, n_{3g-3+n}) \stackrel{\text{def}}{=} \langle \tau_{d_1} \dots \tau_{d_n} \kappa_1^{n_1} \kappa_2^{n_2} \kappa_3^{n_3} \dots \rangle_g$$

We thus write:

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and we define $C_{g,n}(\{d_i\}; \{n_i\}) = \langle \prod_i \tau_{d_i} \prod_j \kappa_j^{n_j} \rangle_g = 0$ if $\sum_i d_i + \sum_k kn_k \neq 3g - 3 + n$.

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Topological recursion and Virasoro constraints

We thus write:

$$W_{g,n} = \frac{(2-t_3)^{2-2g-n}}{\prod_i 2z_i} \sum_{d_1, \dots, d_n} \prod_i \frac{(2d_i+1)!!}{z_i^{2d_i+2}} \langle \tau_{d_1} \dots \tau_{d_n} e^{\sum_k \hat{t}_k \kappa_k} \rangle_g$$

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Topological recursion and Virasoro constraints

- Example $(g, n) = (1, 1)$:

$$W_{1,1} = \frac{(2 - t_3)^{-1}}{2z} \left(\frac{1}{24} \frac{3}{z^4} + \frac{1}{24} \frac{\hat{t}_1}{z^2} \right)$$

i.e.

$$C_{1,1}(1; 0, 0, \dots) = \langle \tau_1 \rangle_1 = \frac{1}{24}$$

$$C_{1,1}(0; 1, 0, \dots) = \langle \tau_0 \kappa_1 \rangle_1 = \frac{1}{24}$$

- Example $(g, n) = (0, 3)$:

$$W_{0,3} = \frac{(2 - t_3)^{-1}}{8 z_1^3 z_2^3 z_3^3}$$

i.e.

$$C_{0,3}(0, 0, 0; 0, 0, \dots) = \langle \tau_0 \tau_0 \tau_0 \rangle_0 = 1$$

Topological recursion and Virasoro constraints

Notational remark: Algebraic topology

$\langle \tau_{d_1} \dots \tau_{d_n} \kappa_1^{n_1} \kappa_2^{n_2} \kappa_3^{n_3} \dots \rangle_g$ is not only a notation, it is actually an integral of cohomology classes.

$$\langle \tau_{d_1} \dots \tau_{d_n} \kappa_1^{n_1} \kappa_2^{n_2} \kappa_3^{n_3} \dots \rangle_g = \int_{\mathcal{M}_{g,n}} \prod_{i=1}^n \tau_{d_i} \prod_k \kappa_k^{n_k}$$

This can be proved [Kontsevich 1992, E 2008] from $\prod_e d\ell_e \propto (\sum_i L_i^2 \psi_i)^{3g-3+n} \prod_i dL_i$, and Laplace transform.

The classes τ and κ are:

- $\tau_d = \psi^d$, where $\psi = c_1(T^*p) = 1$ st Chern class of the cotangent bundle at marked point p . τ_d is called "**Witten's class**".
- $\kappa_d =$ Schur transform of τ_{d+1} , is called "**Mumford's class**".

Topological recursion and Virasoro constraints

The topological recursion (implied by Tutte's recursion) implies some relations among the coefficients

$$C_{g,n} = \langle \tau_{d_1} \cdots \tau_{d_n} \prod_k \kappa_K^{n_k} \rangle_g:$$

Theorem (Virasoro)

The intersection numbers satisfy "Witten's Virasoro conjecture" [EHX1997].

This theorem was proved recently [Mulase, Safnuk 2006], [Liu, Xu...], using Mirzakhani's results from hyperbolic geometry, proof long and difficult.

Here we find a combinatorial proof from Tutte's recursion.

Topological recursion and Virasoro constraints

Case where all $t_k = 0$.

In that case we have

$$W_{g,n}(z_1, \dots, z_n) = \frac{2^{2-2g-n}}{\prod_i 2z_i} \sum_{d_1, \dots, d_n} \prod_{i=1}^n \frac{(2d_i + 1)!!}{z_i^{2d_i+2}} \langle \tau_{d_1} \dots \tau_{d_n} \rangle_g$$

Define:

$$\psi_{\hbar}(z) = e^{\sum_{g,n} \frac{\hbar^{2g-2+n}}{n!} \int_{\infty}^z \dots \int_{\infty}^z W_{g,n} dz_1 \dots dz_n}$$

Using Tutte's equations, one can prove that

Theorem

$\psi(z)$ satisfies Airy equation with respect to $x = z^2$:

$$\hbar^2 \frac{d^2}{dx^2} \psi = x \psi$$

This is another formulation of KdV Witten's conjecture.

3. Higher spin

Witten's r-spin conjecture

$\mathcal{M}_{g,n}^{1/r}$ = moduli space spin r bundles over Riemann surfaces of genus g with n marked points.

→ analogous of colored maps in the discrete version, Potts model, Tutte's chromatic polynomials.

Witten's conjecture: integrals of Chern classes over $\mathcal{M}_{g,n}^{1/r}$, make a Tau-function for the r-KdV hierarchy.

Proved [Faber, Shadrin, Zvonkine 2006].

But the Virasoro conjecture yet unproved, also many Faber-Pandarpande conjectures yet unproved.

Witten's r-spin conjecture

Kontsevich's r-spin combinatorial model:

Consider a "potential" $V(z) = \sum_{k=1}^{r+1} \frac{v_k}{k} z^k$.
Define for $2 \leq m \leq r+1$:

$$\mathcal{V}_m(z_1, \dots, z_m) = - \sum_{i=1}^m \frac{V'(z_i)}{\prod_{j \neq i} (z_i - z_j)} = - \sum_{j \geq m} v_j \sum_{k_1 + \dots + k_m = j-m} \prod_i z_i^{k_i}$$

and
$$\mathcal{P}(z_1, z_2) = - \frac{1}{V_2(z_1, z_2)} = \frac{z_1 - z_2}{V'(z_1) - V'(z_2)}$$

Case $r = 2$ and $V(z) = z^3/3$ gives back Strebel's graphs

$$\mathcal{V}_3(z_1, z_2, z_3) = 1 \quad , \quad \mathcal{P}(z_1, z_2) = \frac{1}{z_1 + z_2}$$

Witten's r-spin conjecture

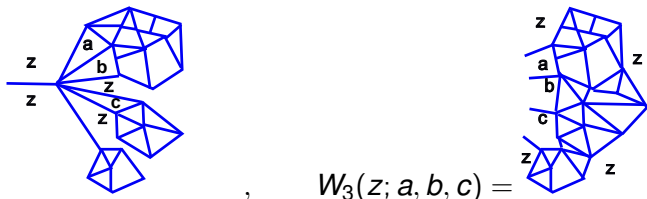
$$W_{g,n}(z_1, \dots, z_n) = \sum_G \frac{1}{\#\text{Aut}} \prod_{v=\text{vertex}} \mathcal{V}_{\text{valence}(v)}(\{\lambda_f\}_{f \rightarrow v}) \prod_{e=\text{edges}} \mathcal{P}(\lambda_{e_+}, \lambda_{e_-})$$

(Most often we shall assume that

$$\sum_{a=1}^N \frac{\lambda_a^k}{V''(\lambda_a)} = 0 \quad \forall k \in [0, r-2] \quad)$$

- The case $r = 2$, $V(z) = z^3/3$ is the Strebel-graphs model.
- The case $r \geq 3$ and $V(z) = z^{r+1}/(r+1)$ is the r-spin Strebel graphs model.

Tutte's equations



$$, \quad W_3(z; a, b, c) =$$

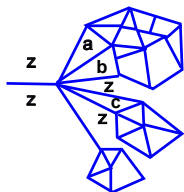
Define $W_0(z) = W_{0,1}(z) - z$ and for $k \geq 1$:

$$W_k(z; a_1, \dots, a_k) = \sum_{i=1}^k \frac{W_{0,1}(a_i) - W_{0,1}(z)}{(V'(a_i) - V'(z)) \prod_{j \neq i} (V'(a_i) - V'(a_j))}$$

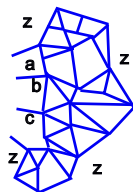
We have Tutte's recursion (planar rooted map case):

$$V''(z)W_{0,1}(z) = \sum_{k \geq 2} \sum_{L_1, \dots, L_k} \mathcal{V}_{k+1+\sum_i \#L_i}(\overbrace{z, \dots, z}^{k+1}, L_1, \dots, L_k) \prod_{i=1}^k W_k(z; L_i)$$

Tutte's equations



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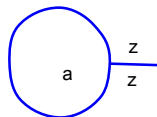
Example $r = 3$

$$\begin{aligned}
 V''(z)W_{0,1}(z) &= \nu_4(z, z, z, z)W_0(z)^3 \\
 &+ 2 \sum_a \nu_4(z, z, z, \lambda_a) W_0(z)W_1(z; \lambda_a) \\
 &+ \sum_{a,b} \nu_4(z, z, \lambda_a, \lambda_b) W_2(z; \lambda_a, \lambda_b) \\
 &+ \nu_3(z, z, z) W_0(z)^2 + \sum_a \nu_3(z, z, \lambda_a) W_1(z; \lambda_a)
 \end{aligned}$$

Tutte's equations

Solution:

$$\text{Disc : } W_{0,1}(z) = -z - \sum_a \mathcal{V}_3(z, z, \lambda_a) \mathcal{P}(z, z) \mathcal{P}(z, \lambda_a)$$



i.e. only these graphs contribute :

$$\text{Annulus : } W_{0,2}(z, z') = \frac{1}{V'''(z) V''(z') (z - z')^2} - \frac{1}{(V'(z) - V'(z'))^2}$$

Tutte's equations

Tutte's equations imply topological recursion :

Conjecture (the $W_{g,n}$'s satisfy the topological recursion (EO07))

$$\begin{aligned} & W_{g,n}(z_1, z_2, \dots, z_n) \\ = & \operatorname{Res}_{z \rightarrow V''^{-1}(0)} K(z_1, z) \left[W_{g-1, n+1}(z, \sigma(z), z_2, \dots, z_n) \right. \\ & \left. + \sum_{\substack{h+h'=g, l \uplus l' = \{z_2, \dots, z_n\}}} W_{h, 1+|l|}(z, l) W_{h', 1+|l'|}(\sigma(z)z, l') \right] dz \end{aligned}$$

$$\text{where } K(z_1, z) = \frac{V''(z_1)}{(z_1 - z)} \frac{1}{W_{0,1}(z) - W_{0,1}(\sigma(z))}$$

$$\text{and where } V'(\sigma(z)) = V'(z).$$

This implies the r-spin Virasoro conjecture.

Tutte's equations

Polynomiality

For $2g + n - 2 > 0$:

$$W_{g,n} \in \mathbb{Q}[z_i, 1/V''(z_i), 1/\hat{t}_{j,0}, \hat{t}_{j,k}]$$

where $\sum_k \hat{t}_{j,k} u^{-k} = \ln \left(\frac{1}{N} \sum_a \int_{\gamma_j} \frac{dz}{z-a} e^{-uV'(z)} \right) \quad j \in [1, r-1]$

$$W_{g,n} = \sum_{\{j_i, d_i\}, \{n_{j,k}\}} \prod_{i=1}^n \frac{z_i^{j_i}}{V''(z_i)^{d_i}} \langle W_r \prod_{i=1}^n \tau_{j_i, d_i} e^{\sum_{j,k} \hat{t}_{j,k} \kappa_{j,k}} \rangle_g$$

The coefficients are the r-spin intersection numbers

$$\langle W_r \prod_{i=1}^n \tau_{j_i, d_i} \prod_{j,k} \kappa_{j,k}^{n_{j,k}} \rangle_g$$

Tutte's equations

Choosing $V'(z) = z^r - r\epsilon^r z$ and taking the limit $\epsilon \rightarrow 0$, implies Faber-Pandarpande's conjectures.

Choosing $\hat{t}_{j,k} = 0$, and

Define:

$$\psi_{\hbar}(z) = e^{\sum_{g,n} \frac{\hbar^{2g-2+n}}{n!} \int_{\infty}^z \dots \int_{\infty}^z W_{g,n} dz_1 \dots dz_n}$$

Using Tutte's equations, one can prove that

Conjecture

$\psi(z)$ satisfies Airy equation with respect to $x = z^r$:

$$\hbar^r \frac{d^r}{dx^r} \psi = x \psi$$

This is another formulation of r -KdV Witten's conjecture.

5. Conclusion

Conclusion:

- Strebel's graphs are a very promising tool for studying "2d quantum gravity". Not only the topological part.
- The combinatorics of Strebel's graphs satisfies Tutte's equations \Rightarrow topological recursion
- From the Tutte's equations and topological recursion, we can prove many non-trivial topological gravity properties, like Virasoro or r-KdV.

Further prospects:

- study non-topological properties ? Example: the embedding into \mathbb{C} is function of ℓ_e 's, in particular, the location of vertices or centers of faces are functions of ℓ_e 's.

The end

Thank you for your attention

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This work was supported:

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